Revealing properties of an unknown function using scientific calculators

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Riassunto. Questo lavoro presenta alcune riflessioni sul risultato di un esperimento di insegnamento effettuato applicando un approccio tipo "ingegneria inversa" allo studio numerico del numero di Nepero e della funzione logaritmo. Questo esperimento mostra quanto sottili e profonde possono risultare le domande e le connessioni che emergono nel lavoro degli studenti che seguono questo approccio e come possano risultare impegnative per gli insegnanti che devono confrontarsi con esse. Questo genere di lavoro risulta comunque gratificante per gli studenti e utile al loro apprendimento. Inoltre, crediamo che questo approccio migliori la comprensione dei processi cognitivi degli studenti da parte dei loro insegnanti e delle ragioni che si nascondono dietro ai loro errori.

Abstract. This paper provides some reflections on the outcomes of a teaching experiment performed under a "reverse engineering approach", applied to a numerical investigation of Neper number and logarithm function. This experiment shows how subtle and deep can be the questions and the connections that arise from a group of students working under this approach, how challenging may be for teachers to deal with it, but also how rewarding it can be for students and for their understanding of mathematics. Moreover, we think that this approach improves teachers' understanding of students' cognitive processes and the reasons behind their mistakes.

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1. Introduction

These Native digital generation is accustomed to learning the functioning of how to use technological devices by doing and not by reading manuals. We tried to leverage this way of learning which is so natural for students, in order to let them discover some important features of a mathematical object they had not studied yet. In the activities described in this paper we asked them to build a coherent interpretation of the results of pressing a given sequence of keys on the keyboard of a scientific calculator, without knowing in advance the function they were calling i.e., using the calculator as a black box which transforms numbers in numbers. We have called this approach, that will be explained in detail later, "reverse engineering". It may be considered an extreme form of the black-box/white-box approach considered in (Drjivers, 1995).

We noticed that this reverse engineering approach awakens students' curiosity and solicits a great deal of discussion between them and their teacher. Consequently, the teacher is able to deal in a fashionable way with a topic (in this specific case, the logarithm function and the Neper number) for which students often show aversion. We have developed our reverse engineering method in order to provide a heuristic basis for developing good concept images (Tall & Vinner 1981) related to the concept of function. The reason why we decided to use the natural logarithm as the "mysterious function" is that it is: mysterious for students; mysterious for the character of one of its discoverers (John Napier); mysterious for the way we arranged to present it. We believe however that this approach can be very effective only if teachers are well prepared to adapt their teaching to answer the questions arisen by students or to clear their difficulties and possible misunderstanding (Drijvers, Doorman & al., 2010), (Bartolini Bussi & Mariotti 2008). This kind of activity has also a keen social dimension that has to be considered. In fact, students are required to cohoperate to fit the pieces of a sort of puzzle together.

Before going into the details of the presentation of our activities, we want to highlight the relationships between our reverse engineering method and some well-established theoretical frameworks of educational research.

Brousseau's theory of didactical situations (Brousseau 2002) recognizes particular importance to the moments of "breaking the didactical contract", in which a student is given the possibility of forming his own knowledge in a didactical situation prepared by the teacher but within which he/she is left to act freely. Our approach foresees a clear situation in which the traditional didactical contract is broken and can be profitably framed within Brousseau's theory.

The first phase of our educational path corresponds to the phase that Brousseau calls Action. In this phase, students work in pairs sharing a calculator (an artifact) and a "map" for their explorations, which consists of a series of activity sheets. At this stage, the role of the teacher is only that of arising interest in the problem and helping to avoid the "bewilderment" caused by the active role that students are asked to play in the construction of knowledge, a role that they are not used to playing.

The second phase of our activity corresponds in part to Brousseau's *Formulation*. In his theory, Formulation is more actively orchestrated by the teacher than in our activity. We ask students to provide written answers to open questions on their activity sheets. The activity consists, metaphorically, of building the pieces of the puzzle that will be assembled in the next phase. In our approach, even in this phase, the intervention of the teacher is limited to a simple control of the activity and clarification of the meaning of the instructions provided with the activity sheets. The reflection on the observations and the procedures, their verbal transposition and their answer to the questions are carried out through a dialogue within each pair of students. The main efforts, within couples, are: i) to develop practical strategies to deal with the proposed problems; ii) to refine the "primitive" language used in phase i) and iii) to share their observations with the rest of the class.

The third phase of our activity mixes Brousseau's *Formulation* and *Validation* phases and consists in the confrontation between pairs of students, mediated by the intervention of the teacher. She/he "presides over a scientific debate" and facilitates the construction of the theory made by the class.

Finally, in the fourth phase, the teacher reshapes the knowledge built in the previous phase and place it within the "institutional curriculum" according to Brusseau's phase of *Institutionalization*.

We belive that our activity could also be properly framed within Vygotsky's theory of education for which *learning is seen as a social process that takes place first of all between people and then it gets internalized*. An important concept developed in the context of Vygotsky's theory is the *Proximal Development Zone*, according to which the adequacy of a didactical activity has to be considered in the light of the potential development capacity of the girls and boys who participate in it. In this sense, it is very important to identify and develop suitable *scaffolding* (Wood) (or *webbing* in (Noss & Hoyles) capable of enabling boys and girls to make the most out of the theory-building activity that we intend to propose.

We believe that making solid scaffoldings for mathematical activities of "laboratorial kind" is a crucial prerequisite for getting good results in the knowledge building process. We have developed a "method" for helping the process of scaffolding construction that we have termed "global interdisciplinary laboratory" (Rogora &Tortoriello 2021). These laboratories are designed by a pool of researchers in didactics in collaboration with high school teachers of several disciplines. They are delivered in class by a pool of teachers *in co-presence* in order to facilitates the formation of a *"class – many teachers"* social group, instead of the usual *"class – single teacher"* one. In our opinion, this interdisciplinary work fosters a habit of collaboration between teachers and learners which is very effective for positioning children in the Proximal Development Zone.

The plan of the paper is the following. We start by describing the experimental setting in section "Experimental framework". Then, we provide the translation in English of the worksheets used in the activities in section "The activity sheets". We explain in more detail the didactical reasons behind the activities in section "Aim of the activities" and we provide a selection of student's answers together with our comments and considerations in section "Outcomes and comments". We end the paper with some final remarks and suggestions for future research.

2. Experimental framework

We made this educational experiment in two classes: a 9th grade class with 20 students, and a 10th grade class, with 21 students. Both classes belong to ITIS Galilei, a science-oriented public high school in Rome, Italy. Each experiment was carried out in two consecutive hours. We have distributed to students the worksheets, reproduced in the following section, which were tailored to provide specific educational stimuli. Students were paired off in couples and let be free to explore the activities and discussed their results together, following a cooperative learning approach. At the end of each activity, teacher discussed the outcomes with the class and summarized the results in a coherent framework.

Each pair of students was equipped with a scientific calculator CASIO fx-991 given by the teacher. No previous knowledge of this specific scientific calculator was assumed, and no help was given beyond what is written in the activity sheets. Students were familiar with pocket calculators only for performing the four arithmetic operations and square roots. They had no previous knowledge of the logarithm of a number.

The activities center on the use of the **ln** key[1], but the teacher never referred to it as to the logarithm key or the logarithm function. It was called the "mysterious function" throughout the activity. Ninth grade students were not yet exposed to the general concept of function while tenth grade students had already met it. This however made no substantial difference.

3. The Activity sheet

In italics we give suggestions and comments about how teachers should use the activity sheets and how to discuss the outcomes with their students. The parts in italics do not appear on student activity sheets.

Blank tables and boxes are provided for students' answers in the original Activity Sheets. In this section, they are filled with answers given by students. These answers are chosen among the "right answers" we got from students during the activities. Comments on "wrong" or "unexpected" answers are discussed in section "Outcome and comments". A blank version of these activity sheets can be downloaded at the internet address www.mat.uniroma1.it/people/rogora
The activity sheets start from the line below.

In order to specify which key to press, we use the code (n,m), where n is the number of the row and m that of the column on which the key is located. In the figure is shown the key for the mysterious function, i.e., key (3,6).

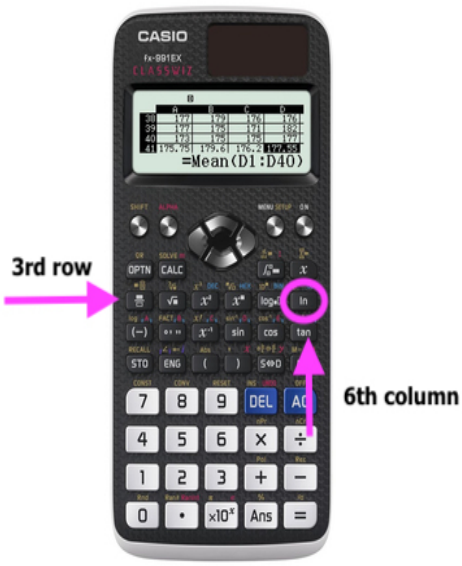

Figure 1. How to locate a key on the calculator keybord

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[1] On the keybord of CASIO fx-991 the main key we use for our activities is marked ln, for logarithm. However, it can also be read as "in". Since we did not want to provide students with any clue about the function we wanted to explore, we referred to the key as the "in" key. In order to stress the point, we always refer to the function as In throughout the paper.

3.1 First Activity

You can call the mysterious function pressing the key (3,6). Then you enter the number on which you want to compute the function. You do not need to close the parenthesis, but you can press key (5,4) if you want to do it.

Note: in order to make the calculator compute an expression you must press key (9,5) (the key marked with the equality sign) at the end of it.

Compute the value of the mysterious function for 9 numbers at your choice. Complete the following table by writing on the left column the numbers on which you have computed the function and on the right column what appears on the screen when you make the computation, even if it is not a number

Number	**Value**
6 (see footnote)[2]	1.791759469
0	Math ERROR
1	0
-1	Math ERROR
0.1	-2.302585093
0.2	-1.609437912
-0.1	Math ERROR
3	1.098612289
-3	Math ERROR

Did you find numbers for which the mysterious function returns an error message?

Find 5 of them and write them below

Number	0	-1	-0.1	-3	-0.2
Value	Math Error	Math Error	Math Error	Math Error	Math Error

Did you find numbers for which the mysterious function returns negative values?

Find 5 of them and write them below.

Number	½	1/3	1/4	1/5	0.23
Value	-0.69314718	-1.0986122	-1.3862943	-1.6094379	-1.469675

Starting from your findings, can you guess for which numbers the function returns a number (i.e. it does not return an error message)?

x>0

[2] As we said before, students’ activity sheets contain blank tables and blank box for students’ answers. Here we fill the blanks with answers given by students during the activity.

Starting from your findings, can you guess for which numbers the function returns negative values?

0<x<1

During the activity teacher goes around the tables. If students are not able to answer one or more of the proposed questions, teacher asks: what kind of numbers do you know? Have you tried with those? What is the smallest number you tried? Try with something smaller, possibly much smaller. Which is the smallest number with which you got a non-negative value? Which is the largest number for which you got an error message? Try something between them.

At the end of the activity, teacher discusses the results obtained by the students with them. During the discussion teacher recalls the various type of numbers that students know and those which they put in their answers.

Teacher remarks that the answers to the last two questions are just hypothesis and sums up the hypothesis formulated by the class.

3.2 Second Activity

Write a number for which the mysterious function returns a value between 0 and 1.

Number	**Value**
3	1.098612289

Write a number for which the mysterious function returns a value between 1 and 2.

Number	**Value**
3	1.098612289

Write a number for which the mysterious function returns a value between 0.95 and 1.

Number	**Value**
2.7	0.993251773

Write a number for which the mysterious function returns a value between 1 and 1.05?

Number	**Value**
2.8	1.029619417

Can you find a number for which the mysterious function takes a value closer to 1?

Teacher should use the last question as a stimulus to play a "turkey shoot game". Teacher updates a table on the blackboard with the pairs number/value that get closer and closer to 1. As a measure of nearness, it may be used either the absolute value of the difference between 1 and the value or the number of decimals up to which the proposed value coincides with 1 (=0.9999999999…). Teacher writes values which get closer to 1 from below and from above.

Each couple whose suggested pair updates the table skips one or two turns (up to teacher's discretion) before being allowed to make another suggestion. At the end of the game students copy the table that the teacher has written on the blackboard.

Number	Value (below 1)	Number	Value (above 1)
2	0.6931471806	4	1.386294361
2.7	0.955511445	3	1.098612289
2.71	0.9969486349		
2.718	0.9998963157	2.72	1.00063188
2.7182	0.9999698965	2.7185	1.000080258
2.71825	0.9999956485	2.71829	1.000003006
….	….	….	….
….	….	….	….
….	….	…..	….
….	….	….	….
		2.718281829	1

Explain the way you had found the numbers when you played the turkey shoot game

«Starting from 2.79, whose function gives 0.98955411936, we have added 0.02 to the number (to get 1 from its value). We have proceeded with the same method by adding, when the function gives a value less than 1, or diminishing, when the function gives a value greater than one. We reached in this way the number 2.718281829, whose value is 1.»

Teacher asks students to confront and discuss their explanations and leads them to the bisection algorithm to find a root of an equation.

3.3 Third Activity

Press key (3,2) and then key (8,2). Now press key (9,5). The symbol $\sqrt{2}$ will be displayed on the screen. Now press key (5,5). Copy what you got on the display.

1.414213562

Can you describe the purpose of key (5,5)?

«It gives the decimal expansion of $\sqrt{2}$ »

Compute $\sqrt{2}^2$ and $(1.414213562)^2$

$\sqrt{2}^2 = 2$	$(1.414213562)^2$ $= 2.000031194$

Do you want to modify the description of the purpose of key (5,5)? If your answer is "yes", write your new description below

«It gives an approximation of the decimal expansion of $\sqrt{2}$»

Press key (1,2) and then key (9,3). The Neper number **e** will be displayed. Now press key (5,5). Copy what you got on the display.

2.718281828

Compute the value of the mysterious function on the Neper number **e** and on the numbers 2.718281828 and 2.718281829.

Number	**Value**
2.718281828	0.9999999998
E	1
2.718281829	1

In your opinion, which of the claims below follows from your computations (you can choose more than an answer):

e is a finite decimal number
e is a periodic decimal number
e > 2.718281828
e < 2.718281829
none of the previous answers

None of the above answers follows from what we have seen. However, a) and b) are false while c) and d) are true: they follow from the fact that the mysterious function is continuous.

3.4 Fourth Activity

Play with the mysterious function to find a mathematical relation linking the following triples.

1. $\mathbf{In}(10), \mathbf{In}(2), \mathbf{In}(5)$
2. $\mathbf{In}(6), \mathbf{In}(2), \mathbf{In}(3)$
3. $\mathbf{In}(15), \mathbf{In}(3), \mathbf{In}(5)$

$\mathbf{In}(10) = \mathbf{In}(2) + \mathbf{In}(5)$	$\mathbf{In}(6) = \mathbf{In}(2) + \mathbf{In}(3)$	$\mathrm{In}(15) = \mathrm{In}(3) + \mathrm{In}(5)$

Can you find other triples which satisfy the same relation with the mysterious function?

$\mathrm{In}(20) = \mathrm{In}(4) + \mathrm{In}(5)$

Can you spot a general rule?

if $q = mxn$, then $\mathbf{In}(q) = \mathbf{In}(m) + \mathbf{In}(n)$

Play with the mysterious function to find a mathematical relation linking the following pairs:

1. $\mathbf{In}(9), \mathbf{In}(3)$
2. $\mathbf{In}(100), \mathbf{In}(10)$
3. $\mathbf{In}(25), \mathbf{In}(5)$

$\mathbf{In}(9) = 2 * \mathbf{In}(3)$	$\mathbf{In}(100) = 2 * \mathbf{In}(10)$	$\mathbf{In}(25) = 2 * \mathbf{In}(5)$

Can you find other pairs which satisfy the same relation with the mysterious function?

$$\mathbf{In}(64) = 2 * \mathbf{In}(8)$$

Can you spot a general rule?

$$\mathbf{In}(a^2) = 2 * \mathbf{In}(a)$$

3.4 Optional questions

It is very important to prepare optional questions in order to engage faster couples in activities which preserve them from getting bored and allow the rest of the class to proceed at its speed. We may suggest the following.

3.4.1 First activity – optional question

Can you find a number at which the mysterious function gives a value greater than 10? And a value greater than 20? Can you read this number? Do you think that the mysterious function is bounded from above, i.e. that the values that it takes are always less than a given number?

3.4.2 Second activity – optional question

Can you find a number on which the mysterious function gives 2 as a value using the algorithm you described before?

Second (or third activity) – optional question

Using key (5,5) compute

1

$1 + 1$

$1 + 1 + 1/2$

$1 + 1 + 1/2 + 1/6$

$1 + 1 + 1/2 + 1/6 + 1/24$

$1 + 1 + 1/2 + 1/6 + 1/24 + 1/120$

$1 + 1 + 1/2 + 1/6 + 1/24 + 1/120 + 1/720$

How are these numbers linked to **e**?

Which of the following numbers would you add to the last sum to get closer to **e**?

$\frac{1}{720\cdot 7}$ $\frac{1}{720+7}$ $\frac{1}{720^7}$

Can you suggest a rule to extend these sums in order to get closer and closer to **e**?

3.4.3 Third activity – optional question

Use the bisection algorithm to compute the number x such that $x^3 = 2$ and a number x such that $x^3 + 2x - 2 = 0$ with a precision up to the eight-decimal cipher.

3.4.4 Fourth activity – optional question

Your scientific calculator suffered a devastating mechanical shock. The only working keys are now the four arithmetic operations, the mysterious function key and the combination of keys which gives the number e. Are you able to display number 7 alone (i.e. not as a cypher of a more complicated number) using these keys only?

4. Aims of the activities

The activities proposed in this paper aim at providing many different educational stimuli, which will be discussed in more detail in the next section. In this section we discuss the goals we had in mind when we first designed our activities.

First of all, we intended to develop this "reverse engineering" approach for providing a working knowledge of, and a heuristic basis for, some of the main features of functions, in the realm of our mysterious function. For example, we think that the kind of activities we developed strongly suggests the idea that a numerical function acts like a "black box" (activated by a keystroke) which transforms numbers in numbers. Moreover, they show very clearly that, in order to "know" a function, like the mysterious function, it is not enough to know its values on a "simple" set of numbers (natural numbers, integers, rational numbers, ecc.) but it is necessary to use all types of numbers that have been taught to students.

In this way the concept images of function, domain, image, etc. that students have begun to train in their previous courses of study, are enriched, thanks to the use of the calculator, with experiences with more complicated numbers than those students usually use (integers, simple fractions, etc.). This practice helps to develop personal concept definitions of objects under investigation closer to formal concept definitions (Tall & Vinner 1981). Finally, all the notions that are usually introduced when talking about functions, like those of domain, intervals of positivity and negativity, intervals of increase and decrease, images and preimages, are not artificial notions, but natural concepts that cannot be avoided considering even when you play with very simple problems.

Our second general goal was that of exploring the possibilities for providing strong educational stimuli through scientific calculators-based activities which go beyond mere computations and data processing, encouraging students to play freely with functions and arithmetic operations in order to discover possible relations and properties of the objects progressively unveiled by their explorations.

In few words, we tried to encourage students to see the calculator as a "tool for investigation" that helps them in suggesting hypothesis, proving conjectures, discovering properties but also as an instrument that must be used in a critical way.

In this sense, the role of the teacher is fundamental. He/she organizes the personal experience of the students, the mathematical meaning of what He/she has observed and the use of symbols and concepts in an organic and effective way. He/she makes explicit the relationships between the mathematical meanings and the meanings constructed through the discussion in class that follows each group of activities. (Bartolini Bussi & al. 1995).

Finally, we were interested in finding out if and how the higher level of mathematics knowledge of the tenth-grade students could influence the outcomes and the attitude toward the activities compared to those of ninth-grade students and understanding the possible dangers or contraindications for a too early use of a scientific calculator in the class for exploring the concept of function.

5. Outcomes and comments

In this section we comment a selection of answers we got from our worksheets and quote some of those that, in our opinion, are the most interesting ones. We also make some more general didactical remarks.

5.1 First Activity

The time requested to complete the activity was ten minutes. Most of the pairs (in both classes) started computing the mysterious function at integer numbers. One couple (9th grade) computed it at π. Only after a while some couples asked for the permission to compute the logarithm of a decimal number. All made the correct guess for the domain of the function and for the interval on which the function is negative.

We have noticed that students tend to consider only natural numbers in order to fill the first column of the first table. An explanation of this is that even if teacher has introduced rational and real numbers and refers to them during the lesson, most of the exercises and problems proposed during the lessons use "very simple" numbers, especially integer ones. For this reason, students tend to believe that the only numbers to be used in the exercises should be "very simple" numbers. The possibility to easily process complicated numbers using a scientific calculator helps to widen, in our opinion, the set of numbers that students are ready to use in practice.

The second and third question of the first activity urge students to consider negative and rational numbers. We have noticed that most of the students consider only negative integers to get the MATH ERROR message and that it is more difficult for them to find the examples requested by the third question. What would have happened if we exchanged the order? This is the only point, in this activity, where teacher decided to provide some of the students with some hints.[3]

The request to guess the domain of the mysterious function was correctly answered by all the couples that answered question 2. The request to guess the domain of negativity of the mysterious function was correctly answered by all the couples which answered question 3. However, students did not feel the necessity to experiment on non-integer negative numbers to confirm the guess about the domain even after the discussion about the interval of negativity.

One of the couples in the ninth-grade group claimed that, in order to return a number, the mysterious functions should not be computed on a square root. The couple tried to compute it on $\sqrt{-1}$. Giving complete access to calculator (it was initially set up on "real numbers mode"), students may very well consider examples beyond teachers' imagination. We did not make any comment on this remark, except that it concerns numbers that they do not yet know.

5.2 Second Activity

The time requested to complete the whole activity was 40 minutes. No couple found difficulties in answering the first four questions. The turkey shot game solicited them to implement forms of iterative search which can easily be used to start a discussion which leads the class to discover the bisection method for finding a root of the equation $f(x) = 0$ in an interval $[a, b]$ such that f is continuous and $f(a) \cdot f(b) \leq 0$. We noticed also that all pairs contributed to the turkey shot game without the need to provide any specific hint. We were amazed by noticing how fast some students got a good approximation of the solution of the equation $ln(x) = 1$ applying their originally devised version of the bisection method, without knowing anything about logarithm and Nepier number **e**.

The last request, to write the strategy adopted to play the game, turned out to be the most difficult one. Very few pairs reported the strategy in a satisfactory way (what has been written by one of the couples has been used to fill the space provided for the answer in the activity sheet, see p. 64), but their efforts proved to be very rewarding for the teacher. Teacher was able to spot and discuss many interesting misconception and misuse of notation and, moreover, he/she found very effective to explain the bisection algorithm starting from student's answers. We give some miscellaneous excerpts of student's answers[4] below

5.2.1 Grade 9

we tried some numbers, the more we approached the request, we subtracted or added, depending if it was bigger or smaller and we got $2{,}718 = 0{,}999$ *(The misuse of the equality sign is a well-known and widespread mathematical mistake. They intended* $\mathbf{In}(0.999) \sim 2.718$*)*

Since 3 gave a cypher bigger than 1 we tried with 2. But with this cypher we got a result much too smaller than 1. 2,718719=1. (Same misuse of equality sign).

$\mathbf{In}(4) = 1{,}38$; $\mathbf{In}(2{,}5) = 0{,}91$; $\mathbf{In}(3) = 1{,}09$; $\mathbf{In}(2{,}7) = 0{,}993$ (This pair was not able to explain the procedure but just illustrated their effort to target the result. It is the only couple in this class that did make the misuse of the equality sign that we noticed before. However, also this pair used equality meaning approximate equality.).

5.2.2 Grade 10

We tried with numbers smaller than 3 since we found that the number diminishes if we diminish the number we used. (Here students observed a monotony property which use to give a motivation for their procedure)

We tried randomly. (When teacher asked, "what do you mean by randomly?" They said, "by repeated attempts").

We sorted the numbers we got, and we began to use the calculator to approach 1 better and better, until we got 2.718270 *and* 2.718289. *We agreed that the number we were looking for was between them and that*

3 The hints provided by the teacher are chosen from those suggested in italics in section "The activity sheet". In general, hints should conform to 9th Polya Commandment for mathematics teachers: «Do not give away your whole secret at once – let the students guess before you tell it – let them find out by themselves as much as feasible». (Polya, 2009)

4 We provide a literal translation of students' answers, without correcting any mistake, not even bad grammar.

we could further improve "up to infinity". (Comments like this trigger in a natural way many intriguing questions like: "*what does it mean to be close to a number*"; "*closer with respect to what"*?)

Many pairs described their procedure referring to "random choices" when they just meant that they were not able or did not want to describe the procedure they followed to complete the activity. They also used interchangeably "choose at random" and "choose arbitrarily".

We observed that in this activity, most of grade 9 students did not explain the procedure but simply copied the computations they made. They said that it was "too difficult and less enjoyable". Almost all grade 10 students tried to explain the procedure.

One grade 10 pair, which got $\mathbf{ln}(2.718281829) = 1$ observed that not even in this case the quest was over: «*the calculator cannot work on exact numbers but only with approximations*».

We noticed that many students, talking about the mysterious function $\mathbf{ln}$ tends to drop the function when report equality of computations: for example, they wrote $10 = 5 + 2$ meaning $\mathbf{ln}(10) = \mathbf{ln}(5) + \mathbf{ln}(2)$. Also, equality and approximations are not clearly distinguished: $e = 2.71 = 2.71828.$ or $3.14 = 3.141592654.$[5]

5.3 Third Activity

We already noticed, before doing the activities described in this paper, that many of the students of our classes do not have precise ideas about numbers and their approximations. For example, they often write "$\pi = 3.14$" or, when they use the scientific calculator of their smartphone, they write $\pi = 3.1415926536$. We further noticed that some of them write expressions like "$3.14 = 3.1415926536$", using the equality sign not for numerical equality but for a rougher (pseudo) equivalence relation, expressing only that two expressions refer to the same number.

The activity, which took 30 minutes (discussion included), was devised for making students aware of the difference between a number and its approximations and to appreciate the importance and the difficulty to get better approximations of the same number. Moreover, the activity has been designed to help students make sense of their first acquaintance with "an alien" of mathematics: the Neper number **e**. Students were urged to recognize that their preceding work partially unveiled it. In fact, all cyphers displayed by the calculator has been got by the students in their quest for a number which is transformed to 1 by the mysterious function, but still … they have not yet reached the number.

All students recognized that **e** is the number which they were looking for in the second activity since $\mathbf{ln}(e) = 1$. They were puzzled at first by the fact that when **e** is pressed, 2.718281828 is displayed, but $\mathbf{ln}(2.718281828) = 0.999999998$ while $\mathbf{ln}(\boldsymbol{e}) = 1$ and $\mathbf{ln}(2.718281829) = 1$. The discussion about the interpretation of these outcomes, together with what has been obtained about the square root of two, put students on the right track. They understood that: **e** is not 2.718281828, nor 2.718281829; it lies between the two and it is closer to the second; the scientific calculator knows a better approximation of **e** than the one it displays; we cannot use the scientific calculator to answer the question «*is **e** a finite decimal number?*»; when teacher inform students that **e** is not a finite decimal number, it is however still not possible to use the scientific calculator to answer the question: «*is **e** an infinite periodical decimal number?*».

After the discussion, teacher informs students about the nature of **e**, the properties it has in common with π and those it has in common with $\sqrt{2}$.

During the discussion, teacher asked students for their opinion about questions a-d at p. 65. They are ill-posed questions, since they cannot be answered with the use of a scientific calculator. However, we believe that this kind of ill posed questions may quite often result be very useful to make students understand more clearly the objects they are working on, and this may greatly help teacher have a glimpse inside student's head. For example, we found very interesting to discuss with the class the following answers/remarks that came out from the activity:

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[5] As Bartolini Bussi and Mariotti claim in their work (Bartolini Bussi &Mariotti 2008), p. 750, "*The use of signs in accomplishing a task has a twofold cognitive function: the subject produces signs related directly to accomplish the task and to communicate with the diverse partners collaborating in the task. In this second case, the production of signs is strictly related to the process of interpretation that allows exchange of information and consequently communication."*.

1. *«**e** is not periodical, nor finite but it is irrational since it cannot be transformed into a fraction».*
2. ***«e** is not finite; it is periodical».*
3. *«A* scientific *calculator may only output a finite number of cyphers. How can we know what possibly comes next?»*
4. *«**e** is a "mixed" periodical number because there is the repetition of the cyphers 1828 but this is not certain».*
5. *«**e** is not finite; we do not know».*
6. *«For sure **e** is decimal; then who knows».*

We got answers 1 and 2 from the ninth-grade students and the others from the other group. We do not report the discussions induced by these answers/comments because they turned out to be very class dependent.

5.4 Fourth Activity

The primary goal of this activity is, of course, to let students discover two of the main properties of the logarithm function: «*Know about the way of learning; the best way to learn anything is to discover it by yourself*» (Polya 2009).

We noticed however that the requests turned out to be more challenging than we expected, because of the difficulty to go from particular examples $\mathbf{In}(10) = \mathbf{In}(2) + \mathbf{In}(5)$; $\mathbf{In}(6) = \mathbf{In}(2) + \mathbf{In}(3)$; $\mathbf{In}(15) = \mathbf{In}(3) + \mathbf{In}(5)$ to the general rule $\mathbf{In}(ab) = \mathbf{In}(a) + \mathbf{In}(b)$. Some of the students wrote in fact $\mathbf{In}(c) = \mathbf{In}(a) + \mathbf{In}(b)$ and only when teacher guided them in providing counterexamples, they realized the necessity to constrain the letters and came up with the condition $c = a \cdot b$. Only one couple wrote the formula $\mathbf{In}(ab) = \mathbf{In}(a) + \mathbf{In}(b)$.

Teacher decided to split this activity into two parts and let the class arrive at the formula $\mathbf{In}(ab) = \mathbf{In}(a) + \mathbf{In}(b)$ before considering the next problem. She noticed that all the examples were given using only non-negative integers and the class quickly reacted by checking that the formula continues to hold for non-negative decimal numbers. Nobody tried to check the formula with negative numbers.

After this discussion, students had less problems with the next question. We noticed however that they preferred the answer «$\mathbf{In}(b) = 2 \cdot \mathbf{In}\ (a)$ with $b = a^2$» to the equivalent but shorter «$\mathbf{In}(a^2) = 2 \cdot \mathbf{In}\ (a)$». Teacher noticed again that all the examples were given considering only integer numbers. The class was again quick to check the formula also for positive decimal numbers. Teacher asked for further generalization and only after a while a pair suggested $\mathbf{In}(a^3) = 3 \cdot \mathbf{In}\ (a)$. When teacher insisted for generalizations, students had no difficulties to suggest the rule $\mathbf{In}(a^n) = n \cdot \mathbf{In}(a)$, for other natural numbers n, but they did not dare to try with negative exponent nor with rational or decimal exponent.

6. Final remarks and suggestions for further research

We found these activities, based on a sort of "reverse engineering method", particularly suited to take advantage of the potentialities of scientific calculators for learning/teaching some aspects of mathematics and particularly welcomed by students.

Using scientific calculators, computations with decimal numbers are easily performed. This may help to widen experiments with numbers and overcomes the pre-concept that mathematical practice only deals with "simple numbers". Moreover, arithmetic games with scientific calculators may be developed for enhancing arithmetic intuition, something that was traditionally got by exploring Pythagorean table and that many educators (not without good reasons) claim that scientific calculators contributed to destroy. This tool allows to develop also activities, like the one we suggest in this paper, which improve student's confidence in more complex functions, like the logarithm, and insight in the meaning of non-trivial mathematical objects, like number **e**.

Some interesting points have emerged during these activities. First of all, it was noticed in older students a more established awareness of the necessity of providing formalizations and formal arguments in their answers. For younger students, a higher propensity to gaming let formal explanations be considered as redundant: "I understood and that is the only important thing, even if I am not able to explain why, and I am not interested in doing it". The difficulties they have to describe the procedure they follow to get a result, like that devised for the second activity, may be addressed by teaching them how to write simple programs with scientific calculators.

We also observed in both classes that students overloaded the meaning of equality. We noticed that many students, talking about the mysterious function **In** tend to drop the function when report equality of computations: for example, they wrote 10=5+2 meaning $\mathbf{In}(10) = \mathbf{In}(5) + \mathbf{In}(2)$. Also, they mix arbitrarily equality and approximations, writing, for example: e=2.71=2.71828.

The last remark we want to make is that these activities can be easily extended to other mysterious functions with a more powerful graphic calculator, like CASIO Fx-Cg50. It suffices to save:

1. in a function variable, let us say Y1, the mysterious function you want to study;
2. in a numerical variable, let's say A, the value of the solution of Y1(X)=H, where H is a suitable number (H=1 in the activity described above).

We tried with $e^2 -> A$; $Y1 = \mathbf{In}(X)/\mathbf{In}(A)$ with a group of 12-th grade students and with a group of perspective teachers with great satisfaction of both students and teachers.

We plan to apply this "reverse engineering" approach also to the study of random numbers. We actually got the idea of this approach when thinking of how to introduce the idea of random number and random variable. We will report on this elsewhere.

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