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Fractional (s,p)-Robin–Venttsel' problems on extension domains

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Abstract. We study a nonlocal Robin–Venttsel'-type problem for the regional fractional p-Laplacian in an extension domain Ω with boundary a d-set. We prove existence and uniqueness of a strong solution via a semigroup approach. Markovianity and ultracontractivity properties are proved. We then consider the elliptic problem. We prove existence, uniqueness and global boundedness of the weak solution.

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Introduction

Aim of this paper is to study a parabolic problem for the regional fractional p-Laplacian with nonlocal Robin–Venttsel' boundary conditions in extension domains.

Nowadays the literature on fractional operators is huge, due to the fact that they describe mathematically many physical phenomena exhibiting deviations from standard diffusion, the so-called anomalous diffusion. This is an important topic not only in physics, but also in finance and probability [1, 29, 42, 44].

Several models appear in the literature to describe such diffusion, e.g. the fractional Brownian motion, the continuous time random walk, the Lévy flight as well as random walk models based on evolution equations of single and distributed fractional order in time and/or space [21, 26, 41, 44, 46].

In the present paper, we consider the following evolution problem for the regional fractional p-Laplacian with nonlocal dynamical Robin–Venttsel' boundary conditions.

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The problem can be formally stated as:

$$(\tilde{P})\begin{cases} \frac{\partial u}{\partial t}(t,x)+(-\Delta_p)^s_\Omega u(t,x)=f(t,x) & \text{in } (0,T]\times\Omega,\\ \frac{\partial u}{\partial t}+\mathcal{N}_p^{p'(1-s)}u+b|u|^{p-2}u+p\Theta_{p,\gamma}(u)=g & \text{on } (0,T]\times\partial\Omega,\\ u(0,x)=u_0(x) & \text{in } \overline{\Omega},\end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is an (ϵ,δ) domain satisfying suitable hypotheses (see Sect. 1.2 for details).

Here $(-\Delta_p)^s_\Omega$ denotes the regional fractional p-Laplacian (see (2.1)), $s\in(0,1)$, $p>1$, $\mathcal{N}_p^{p'(1-s)}u$ is the fractional normal derivative to be suitably defined, f, g, b, u_0 are given functions, T is a positive number and $\Theta_{p,\gamma}(u)$ is a nonlocal term which plays the role of a regional fractional p-Laplacian of order $\gamma\in(0,1)$ on the boundary (see (3.1)).

Boundary value problems for the regional fractional Laplacian with Dirichlet, Neumann, Robin or Venttsel'-type boundary conditions on Lipschitz domains are studied in [23–25], along with the physical motivations. The case of the fractional Laplacian with Robin boundary conditions in Lipschitz domains has been recently investigated in [12]. The results on the regional fractional p-Laplacian in piecewise smooth domains are more recent [22,51,52].

Venttsel'-type boundary value problems in irregular domains (possibly of fractal type) for $s=1$ are studied e.g. in [13,14,17,34–36,38]. The Robin–Venttsel' problem for the (linear) regional fractional Laplacian in irregular domains has been investigated recently in [15], where also a constructive approach is developed. The nonlinear case of the regional fractional p-Laplacian with local boundary conditions is studied in [16] in Koch-type cylinders.

In the present paper we generalize the results of [16] to the class of those extension domains which are (ϵ,δ) domains with boundary a d-set (see Sect. 1.2), and we focus on the regularity properties of the solution of the problem at hand. We remark that (ϵ,δ) domains can have a highly non-rectifiable boundary, possibly of fractal type.

There are very few regularity results for Venttsel' problems in fractal domains (see [34,39] for the local case), which are also crucial in view of the numerical approximation. To our knowledge, the proof of regularity results for fractional Venttsel' problems in irregular domains is still open and it is the goal of this paper.

A key point is to investigate if the presence of a "fractal fractional Laplacian" affects the regularity. In the parabolic case, this is achieved by proving regularity properties of the associated semigroup, namely its ultracontractivity. This result, in turn, deeply relies on a fractional logarithmic Sobolev inequality, suitably tailored for the problem at hand, whence it clearly appears the role of the fractal boundary. In the elliptic case, regularity results for the weak solution can be obtained via a powerful tool by Murthy and Stampacchia [43].

More precisely, we firstly focus on giving a rigorous formulation of the parabolic problem for the regional fractional p-Laplacian with dynamical boundary conditions in extension domains. This will be achieved by introducing a suitable notion of p-fractional normal derivative on irregular sets, via a generalized fractional Green formula, see Theorem 2.2.

We then consider the fractional energy functional $\Phi_{p,s}$ defined in (3.2), which is proper, convex and weakly lower semicontinuous, and the corresponding associated subdifferential $\mathcal{A}$. In Theorem 3.3 we prove, via a semigroup approach, existence and uniqueness of a strong solution for a suitable abstract Cauchy problem (P) for the operator $\mathcal{A}$. Then, via Theorem 3.6, we prove that problem $(\tilde{P})$ is the strong formulation of the abstract problem (P). In Theorem 3.10 we prove that the associated (nonlinear) semigroup is order-preserving and non-expansive on L^∞, and in Theorem 4.7 we prove that it is ultracontractive.

We then consider the elliptic problem. After proving existence and uniqueness of a weak solution in suitable functional spaces in Theorem 5.1, we prove its global boundedness in Theorem 5.4 via Lemma 5.2.

The plan of the paper is the following.

In Sect. 1 we introduce the extension domain Ω and we recall some preliminary results on fractional Sobolev spaces, embeddings and traces.

In Sect. 2 we introduce the regional fractional p-Laplacian and the notion of weak p-fractional normal derivative by proving a generalized p-fractional Green formula.

In Sect. 3 we introduce the energy functional $\Phi_{p,s}$ which is proper, convex and weakly lower semicontinuous and the associated subdifferential $\mathcal{A}$ which is the generator of a nonlinear C_0 semigroup. We prove existence and uniqueness of a strong solution for the corresponding abstract Cauchy problem and we give a strong interpretation. Moreover, we prove that the semigroup is Markovian.

In Sect. 4 we prove that the associated semigroup is ultracontractive. The proof, as usual, relies on a fractional logarithmic-Sobolev inequality adapted to the present case.

In Sect. 5 we consider the elliptic problem. We prove existence and uniqueness of the weak solution and its global boundedness.

1. Preliminaries

1.1. Functional spaces

Let $\mathcal{G}$ (resp. $\mathcal{S}$) be an open (resp. closed) set of $\mathbb{R}^N$. By $L^p(\mathcal{G})$, for $p > 1$, we denote the Lebesgue space with respect to the Lebesgue measure $\mathrm{d}\mathcal{L}_N$, which will be left to the context whenever that does not create ambiguity. By $L^p(\partial\mathcal{G})$ we denote the Lebesgue space on $\partial\mathcal{G}$ with respect to a Hausdorff measure μ supported on $\partial\mathcal{G}$. By $\mathcal{D}(\mathcal{G})$ we denote the space of infinitely differentiable functions with compact support on $\mathcal{G}$. By $C(\mathcal{S})$ we denote the space of continuous functions on $\mathcal{S}$.

By $W^{s,p}(\mathcal{G})$, for $0 < s < 1$, we denote the fractional Sobolev space of exponent s. Endowed with the following norm

$$\|u\|^p_{W^{s,p}(\mathcal{G})} = \|u\|^p_{L^p(\mathcal{G})} + \iint_{\mathcal{G}\times\mathcal{G}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y),$$

it becomes a Banach space. Moreover, we denote by $|u|_{W^{s,p}(\mathcal{G})}$ the seminorm associated to $\|u\|_{W^{s,p}(\mathcal{G})}$ and we define, for every $u, v \in W^{s,p}(\mathcal{G})$,

$$(u,v)_{s,p} := \iint_{\mathcal{G}\times\mathcal{G}} |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y).$$

In the following we will denote by $|A|$ the Lebesgue measure of a subset $A \subset \mathbb{R}^N$. For $f \in W^{s,p}(\mathcal{G})$, we define the trace operator γ_0 as

$$\gamma_0 f(x) := \lim_{r\to 0}\frac{1}{|B(x,r)\cap\mathcal{G}|}\int_{B(x,r)\cap\mathcal{G}} f(y)\,\mathrm{d}\mathcal{L}_N(y)$$

at every point $x \in \overline{\mathcal{G}}$ where the limit exists. The above limit exists at quasi every $x \in \overline{\mathcal{G}}$ with respect to the (s,p)-capacity (see Definition 2.2.4 and Theorem 6.2.1 page 159 in [2]). From now one we denote the trace operator simply by $f|_{\mathcal{G}}$; sometimes we will omit the trace symbol and the interpretation will be left to the context.

For $1 \leq q, r \leq \infty$, we introduce the space

$$\mathbb{X}^{q,r}(\Omega,\partial\Omega) := L^q(\Omega)\times L^r(\partial\Omega) = \{(f,g)\,:\, f\in L^q(\Omega) \text{ and } g \in L^r(\partial\Omega)\}, \tag{1.1}$$

endowed with the norm

$$\|(f,g)\|_{q,r} = \|(f,g)\|_{\mathbb{X}^{q,r}(\Omega,\partial\Omega)} := \|f\|_{L^q(\Omega)} + \|g\|_{L^r(\partial\Omega)}.$$

If $q = r < \infty$, we denote the above space simply by $\mathbb{X}^q(\Omega,\partial\Omega)$ and we endow it with the following norm:

$$\|(f,g)\|_q^q = \|(f,g)\|^q_{\mathbb{X}^q(\Omega,\partial\Omega)} := \|f\|^q_{L^q(\Omega)} + \|g\|^q_{L^q(\partial\Omega)}.$$

If $q = r = \infty$, we endow $\mathbb{X}^\infty(\Omega,\partial\Omega)$ with the following norm:

$$\|(f,g)\|_\infty := \max\left\{\|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\partial\Omega)}\right\}.$$

In the following, for a function f with well-defined trace $f|_{\partial\Omega}$ on $\partial\Omega$, we simply denote $\mathbf{f} = (f, f|_{\partial\Omega})$.

1.2. (ϵ, δ) domains and trace theorems

We now recall the definition of (ϵ,δ) domain. For details see [30].

Definition 1.1. Let $\mathcal{F} \subset \mathbb{R}^N$ be open and connected. For $x \in \mathcal{F}$, let $d(x) := \inf_{y\in\mathcal{F}^c} |x-y|$. We say that $\mathcal{F}$ is an (ϵ,δ) domain if, whenever $x, y \in \mathcal{F}$ with $|x-y| < \delta$, there exists a rectifiable arc $\gamma \in \mathcal{F}$ joining x to y such that

$$\ell(\gamma) \leq \frac{1}{\epsilon}|x-y| \quad \text{and} \quad d(z) \geq \frac{\epsilon|x-z||y-z|}{|x-y|} \text{ for every } z \in \gamma.$$

In this paper, we consider two particular classes of (ϵ,δ) domains $\Omega \subset \mathbb{R}^N$. More precisely, Ω can be a (ϵ,δ) domain having as boundary either a d-set or an arbitrary closed set in the sense of [31]. For the sake of simplicity, from now on we restrict ourselves to the case in which $\partial\Omega$ is a d-set (see [32]).

Definition 1.2. A closed nonempty set $\mathcal{M} \subset \mathbb{R}^N$ is a d-set (for $0 < d \leq N$) if there exist a Borel measure μ with $\operatorname{supp} \mu = \mathcal{M}$ and two positive constants c_1 and c_2 such that

$$c_1 r^d \leq \mu(B(x,r) \cap \mathcal{M}) \leq c_2 r^d \quad \forall\, x \in \mathcal{M}. \tag{1.2}$$

The measure μ is called d-measure.

In the following, $\mu(\partial\Omega)$ denotes the Hausdorff measure of $\partial\Omega$.

We suppose that Ω can be approximated by a sequence $\{\Omega_n\}$ of domains such that for every $n \in \mathbb{N}$:

$$(\mathcal{H}) \begin{cases} \Omega_n \text{ is bounded and Lipschitz;} \\ \Omega_n \subseteq \Omega_{n+1}; \\ \Omega = \displaystyle\bigcup_{n=1}^{\infty} \Omega_n. \end{cases}$$

The reader is referred to [15, 16] for examples of such domains.

We recall the definition of Besov space specialized to our case. For generalities on Besov spaces, we refer to [32].

Definition 1.3. Let $\mathcal{F}$ be a d-set with respect to a d-measure μ and $\alpha = s - \frac{N-d}{p}$. $B^{p,p}_\alpha(\mathcal{F})$ is the space of functions for which the following norm is finite:

$$\|u\|^p_{B^{p,p}_\alpha(\mathcal{F})} = \|u\|^p_{L^p(\mathcal{G})} + \iint_{|x-y|<1} \frac{|u(x) - u(y)|^p}{|x-y|^{d+\alpha p}} \, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y).$$

Let p' be the Hölder conjugate exponent of p. In the following, we will denote the dual of the Besov space $B^{p,p}_\alpha(\mathcal{F})$ with $(B^{p,p}_\alpha(\mathcal{F}))'$; we point out that this space coincides with the space $B^{p',p'}_{-\alpha}(\mathcal{F})$ (see [33]).

From now on, let

$$\alpha := s - \frac{N-d}{p} \in (0,1). \tag{1.3}$$

We now state a trace theorem for functions in $W^{s,p}(\Omega)$, where Ω is a bounded (ϵ, δ) domain with boundary $\partial\Omega$ a d-set. For the proof, we refer to [32, Theorem 1, Chapter VII].

Proposition 1.4. *Let* $\frac{N-d}{p} < s < 1$. $B^{p,p}_\alpha(\partial\Omega)$ *is the trace space of* $W^{s,p}(\Omega)$ *in the following sense:*

- *(i)* γ_0 *is a continuous linear operator from* $W^{s,p}(\Omega)$ *to* $B^{p,p}_\alpha(\partial\Omega)$*;*
- *(ii) there exists a continuous linear operator* Ext *from* $B^{p,p}_\alpha(\partial\Omega)$ *to* $W^{s,p}(\Omega)$ *such that* $\gamma_0 \circ \mathrm{Ext}$ *is the identity operator in* $B^{p,p}_\alpha(\partial\Omega)$.

We point out that, if $\Omega \subset \mathbb{R}^N$ is a Lipschitz domain, its boundary $\partial\Omega$ is a $(N-1)$-set. Hence, the trace space of $W^{s,p}(\Omega)$ is $B^{p,p}_{s-\frac{1}{p}}(\partial\Omega)$, and the latter space coincides with $W^{s-\frac{1}{p},p}(\partial\Omega)$.

The following result provides us with an equivalent norm on $W^{s,p}(\Omega)$. The proof can be achieved by adapting the proof of [50, Theorem 2.3].

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^N$ be a (ϵ, δ) domain having as boundary a d-set, with $p > 1$ and $\frac{N-d}{p} < s < 1$. Then there exists a positive constant $C = C(\Omega, N, s, p, d)$ such that for every $u \in W^{s,p}(\Omega)$*

$$\int_\Omega |u|^p \,\mathrm{d}\mathcal{L}_N \leq C \left(\frac{C_{N,p,s}}{2} \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) + \int_{\partial\Omega} |u|^p \,\mathrm{d}\mu \right). \tag{1.4}$$

Here, $C_{N,p,s}$ is the positive constant defined in Sect. 2.

Finally, we recall the following important extension property which holds for (ϵ, δ) domains having as boundary a d-set. For details, we refer to Theorem 1, page 103 and Theorem 3, page 155 in [32].

Theorem 1.6. *Let $0 < s < 1$. There exists a linear extension operator $\mathcal{E}\mathrm{xt}\colon W^{s,p}(\Omega) \to W^{s,p}(\mathbb{R}^N)$ such that*

$$\|\mathcal{E}\mathrm{xt}\, w\|^p_{W^{s,p}(\mathbb{R}^N)} \leq \bar{C}_s \|w\|^p_{W^{s,p}(\Omega)}, \tag{1.5}$$

with $\bar{C}_s$ depending on s.

Domains Ω satisfying property (1.5) are the so-called $W^{s,p}$-*extension domains.*

1.3. Sobolev embeddings

We now recall some important Sobolev-type embeddings for the fractional Sobolev space $W^{s,p}(\Omega)$ where Ω is a $W^{s,p}$-extension domain with boundary a d-set, see [19, Theorem 6.7] and [32, Lemma 1, p. 214] respectively.

We set

$$p^* := \frac{Np}{N-sp} \quad \text{and} \quad \bar{p} := \frac{dp}{N-sp}.$$

Theorem 1.7. *Let $s \in (0,1)$ and $p \geq 1$ be such that $sp < N$. Let $\Omega \subseteq \mathbb{R}^N$ be a $W^{s,p}$-extension domain. Then $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for every $q \in [1, p^*]$, i.e. there exists a positive constant $C = C(N, s, p, \Omega)$ such that, for every $u \in W^{s,p}(\Omega)$,*

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{s,p}(\Omega)}. \tag{1.6}$$

Theorem 1.8. *Let $p > 1$ and let $s \in (0,1)$ be such that $N - d < sp < N$. Let $\Omega \subseteq \mathbb{R}^N$ be a $W^{s,p}$-extension domain having as boundary $\partial\Omega$ a d-set, for $0 < d \leq N$. Then $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\partial\Omega)$ for every $q \in [1, \bar{p}]$, i.e. there exists a positive constant $C = C(N, s, p, d, \Omega)$ such that, for every $u \in W^{s,p}(\Omega)$,*

$$\|u\|_{L^q(\partial\Omega)} \leq C \|u\|_{W^{s,p}(\Omega)}. \tag{1.7}$$

We point out that $p^* \geq \bar{p} \geq p$.

We define the average of u on Ω and on $\partial\Omega$ respectively as

$$\bar{u}_\Omega := \frac{1}{|\Omega|} \int_\Omega u \,\mathrm{d}\mathcal{L}_N \quad \text{and} \quad \bar{u}_{\partial\Omega} := \frac{1}{\mu(\partial\Omega)} \int_{\partial\Omega} u \,\mathrm{d}\mu. \tag{1.8}$$

Using Theorems 1.7 and 1.8 and Hölder inequality, one can easily prove that

$$\|u - \bar{u}_\Omega\|_{L^{p^*}(\Omega)} \leq C|u|_{W^{s,p}(\Omega)} \tag{1.9}$$

and

$$\|u - \bar{u}_{\partial\Omega}\|_{L^{\bar{p}}(\partial\Omega)} \leq C|u|_{W^{s,p}(\Omega)}. \tag{1.10}$$

We point out that (1.9) and (1.10) imply that, for every $1 < q \leq p^*$, it holds

$$\|u\|_q^p \leq 2^{p-1} \left(C|u|^p_{W^{s,p}(\Omega)} + |\bar{u}_\Omega|^p |\Omega|^{\frac{p}{q}} + |\bar{u}_{\partial\Omega}|^p \mu(\partial\Omega)^{\frac{p}{q}} \right). \tag{1.11}$$

2. The regional fractional p-Laplacian and the Green formula

We recall the definition of the regional fractional p-Laplacian. We refer to [51] and the references listed in.

Let $s \in (0, 1)$ and $p > 1$. For $\mathcal{G} \subseteq \mathbb{R}^N$, we define the space

$$\mathcal{L}_s^{p-1}(\mathcal{G}) := \left\{ u \colon \mathcal{G} \to \mathbb{R} \text{ measurable } : \int_{\mathcal{G}} \frac{|u(x)|^{p-1}}{(1 + |x|)^{N+sp}} \, \mathrm{d}\mathcal{L}_N(x) < \infty \right\}.$$

The regional fractional p-Laplacian $(-\Delta_p)^s_{\mathcal{G}}$ is defined as follows, for $x \in \mathcal{G}$:

$$\begin{aligned} (-\Delta_p)^s_{\mathcal{G}} u(x) &= C_{N,p,s} \text{P.V.} \int_{\mathcal{G}} |u(x) - u(y)|^{p-2} \frac{u(x) - u(y)}{|x - y|^{N+sp}} \, \mathrm{d}\mathcal{L}_N(y) \\ &= C_{N,p,s} \lim_{\varepsilon \to 0+} \int_{\{y \in \mathcal{G} \,:\, |x-y| > \varepsilon\}} |u(x) - u(y)|^{p-2} \frac{u(x) - u(y)}{|x - y|^{N+sp}} \, \mathrm{d}\mathcal{L}_N(y), \end{aligned} \tag{2.1}$$

provided that the limit exists, for every function $u \in \mathcal{L}_s^{p-1}(\mathcal{G})$. The positive constant $C_{N,p,s}$ is defined as follows:

$$C_{N,p,s} = \frac{s 2^{2s} \Gamma(\frac{ps+p+N-2}{2})}{\pi^{\frac{N}{2}} \Gamma(1 - s)},$$

where Γ is the Euler function.

We now introduce the notion of *p-fractional normal derivative* on (ε, δ) domains having as boundary a d-set and satisfying hypotheses $(\mathcal{H})$ in Sect. 1.2 via a p-fractional Green formula by suitably generalizing the results in [16]. We recall its proof for the reader's convenience. For the case $p = 2$, we refer to [27,28] for the smooth case and to [15] for irregular sets.

We define the space

$$V((-\Delta_p)^s_\Omega, \Omega) := \{u \in W^{s,p}(\Omega) \,:\, (-\Delta_p)^s_\Omega u \in L^{p'}(\Omega) \text{ in the sense of distributions}\},$$

which is a Banach space equipped with the norm

$$\|u\|_{V((-\Delta_p)^s_\Omega, \Omega)} := \|u\|_{W^{s,p}(\Omega)} + \|(-\Delta_p)^s_\Omega u\|_{L^{p'}(\Omega)}.$$

We first define the p-fractional normal derivative on Lipschitz domains.

Definition 2.1. Let $\mathcal{T} \subset \mathbb{R}^N$ be a Lipschitz domain. Let $u \in V((-\Delta_p)^s_{\mathcal{T}}, \mathcal{T}) := \{u \in W^{s,p}(\mathcal{T}) \,:\, (-\Delta_p)^s_{\mathcal{T}} u \in L^{p'}(\mathcal{T})$ in the sense of distributions$\}$. We say that u has a weak p-fractional normal derivative in $(W^{s-\frac{1}{p},p}(\partial\mathcal{T}))'$ if there exists $g \in (W^{s-\frac{1}{p},p}(\partial\mathcal{T}))'$ such that

$$\begin{aligned} \langle g, v|_{\partial\Omega}\rangle_{(W^{s-\frac{1}{p},p}(\partial\mathcal{T}))', W^{s-\frac{1}{p},p}(\mathcal{T})} &= -\int_{\mathcal{T}} (-\Delta_p)^s_{\mathcal{T}} u\, v \,\mathrm{d}\mathcal{L}_N \\ &+ \frac{C_{N,p,s}}{2} \iint_{\mathcal{T}\times\mathcal{T}} |u(x)-u(y)|^{p-2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} \,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) \end{aligned} \tag{2.2}$$

for every $v \in W^{s,p}(\mathcal{T})$. In this case, g is uniquely determined and we call $C_{p,s}\mathcal{N}_p^{p'(1-s)} u := g$ the weak p-fractional normal derivative of u, where

$$C_{p,s} := \frac{(p-1)C_{1,p,s}}{(sp-(p-2))(sp-(p-2)-1)} \int_0^\infty \frac{|t-1|^{(p-2)+1-sp} - (t \vee 1)^{p-sp-1}}{t^{p-sp}} \,\mathrm{d}t.$$

We point out that, when $s \to 1^-$ in (2.2), we recover the quasi-linear Green formula for Lipschitz domains [8].

Theorem 2.2. (Fractional green formula) *There exists a bounded linear operator $\mathcal{N}_p^{p'(1-s)}$ from $V((-\Delta_p)^s_\Omega, \Omega)$ to $(B^{p,p}_\alpha(\partial\Omega))'$.*

The following generalized Green formula holds for every $u \in V((-\Delta_p)^s_\Omega, \Omega)$ and $v \in W^{s,p}(\Omega)$:

$$\begin{aligned} C_{p,s} \left\langle \mathcal{N}_p^{p'(1-s)} u, v|_{\partial\Omega} \right\rangle_{(B^{p,p}_\alpha(\partial\Omega))', B^{p,p}_\alpha(\partial\Omega)} &= -\int_\Omega (-\Delta_p)^s_\Omega u\, v \,\mathrm{d}\mathcal{L}_N \\ &+ \frac{C_{N,p,s}}{2} \iint_{\Omega\times\Omega} |u(x)-u(y)|^{p-2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} \,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y). \end{aligned} \tag{2.3}$$

Proof. For $u \in V((-\Delta_p)^s_\Omega, \Omega)$ and $v \in W^{s,p}(\Omega)$, we define

$$\langle l(u), v\rangle := -\int_\Omega (-\Delta_p)^s_\Omega u\, v \,\mathrm{d}\mathcal{L}_N + \frac{C_{N,p,s}}{2}(u,v)_{s,p}.$$

From Hölder inequality, we get

$$\begin{aligned} |\langle l(u), v\rangle| &\le \|(-\Delta_p)^s_\Omega u\|_{L^{p'}(\Omega)} \|v\|_{L^p(\Omega)} + \frac{C_{N,p,s}}{2} \|u\|_{W^{s,p}(\Omega)} \|v\|_{W^{s,p}(\Omega)} \\ &\le C\, \|u\|_{V((-\Delta_p)^s_\Omega,\Omega)} \|v\|_{W^{s,p}(\Omega)} \end{aligned} \tag{2.4}$$

We now prove that the operator $l(u)$ is independent from the choice of v and it is an element of $(B^{p,p}_\alpha(\partial\Omega))'$. From Proposition 1.4, for every $v \in B^{p,p}_\alpha(\partial\Omega)$ there exists a function $\tilde{w} := \operatorname{Ext} v \in W^{s,p}(\Omega)$ such that

$$\|\tilde{w}\|_{W^{s,p}(\Omega)} \le C \|v\|_{B^{p,p}_\alpha(\partial\Omega)} \tag{2.5}$$

and $\tilde{w}|_{\partial\Omega} = v$ μ-almost everywhere. From (2.3) we have that

$$C_{p,s} \left\langle \mathcal{N}_p^{p'(1-s)} u, v \right\rangle_{(B^{p,p}_\alpha(\partial\Omega))', B^{p,p}_\alpha(\partial\Omega)} = \langle l(u), \tilde{w}\rangle.$$

The thesis follows from (2.4) and (4.21).

We now recall that Ω is approximated by a sequence of Lipschitz domains Ω_n, for $n \in \mathbb{N}$, satisfying conditions $(\mathcal{H})$ in Sect. 1.2. From (2.2) we have that

$$C_{p,s}\left\langle \mathcal{N}_p^{p'(1-s)}u, v|_{\partial\Omega}\right\rangle_{(W^{s-\frac{1}{p},p}(\partial\Omega_n))',W^{s-\frac{1}{p},p}(\partial\Omega_n)} = -\int_\Omega \chi_{\Omega_n}(-\Delta_p)^s_\Omega u\, v\, \mathrm{d}\mathcal{L}_N$$
$$+\frac{C_{N,p,s}}{2}\iint_{\Omega\times\Omega}\chi_{\Omega_n}(x)\chi_{\Omega_n}(y)|u(x)-u(y)|^{p-2}$$
$$\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y).$$

From the dominated convergence theorem, we have

$$\lim_{n\to\infty} C_{p,s}\left\langle \mathcal{N}_p^{p'(1-s)}u, v|_{\partial\Omega}\right\rangle_{(W^{s-\frac{1}{p},p}(\partial\Omega_n))',W^{s-\frac{1}{p},p}(\partial\Omega_n)}$$
$$=\lim_{n\to\infty}\left(-\int_{\Omega_n}(-\Delta_p)^s_\Omega u\, v\,\mathrm{d}\mathcal{L}_N+\frac{C_{N,p,s}}{2}\iint_{\Omega_n\times\Omega_n}|u(x)-u(y)|^{p-2}\right.$$
$$\left.\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y)\right)$$
$$=-\int_\Omega(-\Delta_p)^s_\Omega u\, v\,\mathrm{d}\mathcal{L}_N$$
$$+\frac{C_{N,p,s}}{2}(u,v)_{s,p}=\langle l(u),v\rangle$$

for every $u \in V((-\Delta_p)^s_\Omega,\Omega)$ and $v \in W^{s,p}(\Omega)$. Hence, we define the p-fractional normal derivative on Ω as

$$\langle C_{p,s}\mathcal{N}_p^{p'(1-s)}u, v|_{\partial\Omega}\rangle_{(B^{p,p}_\alpha(\partial\Omega))',B^{p,p}_\alpha(\partial\Omega)} := -\int_\Omega(-\Delta_p)^s_\Omega u\, v\,\mathrm{d}\mathcal{L}_N+\frac{C_{N,p,s}}{2}(u,v)_{s,p}.$$

□

Remark 2.3. We note that $p'(1-s) = 2-\beta$, where $\beta = \frac{ps-1}{p-1}+1$, thus recovering the usual notation for the p-fractional normal derivative in (2.3).

Moreover, by proceeding as in [16, Remark 3.3], when $s \to 1^-$ in (2.3) we recover the Green formula proved in [37] for fractal domains.

3. The evolution problems

3.1. The energy functional

From now on, we suppose that $p \geq 2$, $sp < N$ and that $b \in C(\overline{\Omega})$ is strictly positive and continuous on $\overline{\Omega}$. We denote by $H := \mathbb{X}^2(\Omega,\partial\Omega)$ the Lebesgue space defined in Sect. 1.1.

We introduce the linear and continuous operator $\Theta_{p,\gamma}\colon B^{p,p}_\alpha(\partial\Omega) \to (B^{p,p}_\alpha(\partial\Omega))'$ defined as

$$\langle\Theta_{p,\gamma}(u),v\rangle := \tag{3.1}$$
$$\frac{1}{p}\iint_{\partial\Omega\times\partial\Omega}\zeta(x,y)\frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\gamma p}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y),$$

where $\zeta \in L^\infty(\partial\Omega \times \partial\Omega)$ is such that $\zeta \geq 0$, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(B^{p,p}_\alpha(\partial\Omega))'$ and $B^{p,p}_\alpha(\partial\Omega)$ and $\gamma \in (0, \alpha]$, where α is defined in (1.3).

For every $\mathbf{u} = (u, u|_{\partial\Omega}) \in H$, we introduce the following energy functional, with effective domain $D(\Phi_{p,s}) = W^{s,p}(\Omega)$:

$$\Phi_{p,s}[\mathbf{u}] := \begin{cases} \frac{C_{N,p,s}}{2p} \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) & \\ +\frac{1}{p}\int_{\partial\Omega} b|u|_{\partial\Omega}|^p \,\mathrm{d}\mu + \langle \Theta_{p,\gamma}(u|_{\partial\Omega}), u|_{\partial\Omega}\rangle & \text{if } u \in D(\Phi_{p,s}), \\ +\infty & \text{if } \mathbf{u} \in H \setminus D(\Phi_{p,s}), \end{cases} \tag{3.2}$$

We remark that, if $u \in W^{s,p}(\Omega)$, from Proposition 1.4 its trace $u|_{\partial\Omega}$, and hence the couple $\mathbf{u} = (u, u|_{\partial\Omega})$, is well-defined. In the following, with an abuse of notation, we will write $\mathbf{u} \in W^{s,p}(\Omega)$.

Proposition 3.1. *$\Phi_{p,s}$ is a weakly lower semicontinuous, proper and convex functional in H. Moreover, its subdifferential $\partial\Phi_{p,s}$ is single-valued.*

Proof. The functional $\Phi_{p,s}$ is clearly convex and proper. The weak lower semicontinuity follows from the weak lower semicontinuity of the $L^p(\partial\Omega)$-norm and by proceeding as in [34, Proposition 2.3]. Moreover, from Proposition 2.40 in [4], $\partial\Phi_{p,s}$ is single-valued. □

3.2. Abstract Cauchy problem

Let T be a fixed positive number. We consider the abstract Cauchy problem

$$(P) \begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \mathcal{A}\mathbf{u} = \tilde{F}, \ t \in [0, T] \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases}$$

where $\mathcal{A}$ is the subdifferential of $\Phi_{p,s}$ and $\tilde{F} = (f, g)$ and $\mathbf{u}_0$ are given data.

According to [3, Section 2.1, chapter II], we give the following definition.

Definition 3.2. A function $\mathbf{u}\colon [0, T] \to H$ is a strong solution of problem (P) if $\mathbf{u} \in C([0, T]; H)$, $\mathbf{u}(t)$ is differentiable a.e. in $(0, T)$, $\mathbf{u}(t) \in D(-\mathcal{A})$ a.e and $\frac{\partial \mathbf{u}}{\partial t} + \mathcal{A}\mathbf{u} = \tilde{F}$ for a.e. $t \in [0, T]$.

From [3, Theorem 2.1, chapter IV] the following existence and uniqueness result for the strong solution of problem (P) holds.

Theorem 3.3. *If $\mathbf{u}_0 \in \overline{D(-\mathcal{A})}$ and $(f, g) \in L^2([0, T]; H)$, then problem (P) has a unique strong solution $\mathbf{u} \in C([0, T]; H)$ such that $\mathbf{u} \in W^{1,2}((\delta, T); H)$ for every $\delta \in (0, T)$. Moreover $\mathbf{u} \in D(-\mathcal{A})$ a.e. for $t \in (0, T)$, $\sqrt{t}\frac{\partial \mathbf{u}}{\partial t} \in L^2(0, T; H)$ and $\Phi_{p,s}[\mathbf{u}] \in L^1(0, T)$.*

From Theorem 1 and Remark 2 in [7] (see also [3]) we have the following result.

Theorem 3.4. *Let $\varphi :\ H \to (-\infty, +\infty]$ be a proper, convex, lower semicontinuous functional on a real Hilbert space H, with effective domain $D(\varphi)$. Then the subdifferential $\partial\varphi$ is a maximal monotone m-accretive operator. Moreover,*

$\overline{D(\varphi)} = \overline{D(\partial\varphi)}$ *and* $-\partial\varphi$ *generates a nonlinear* C_0*-semigroup* $\{T(t)\}_{t\geq 0}$ *on* $\overline{D(\varphi)}$ *in the following sense: for each* $u_0 \in \overline{D(\varphi)}$*, the function* $u := T(\cdot)u_0$ *is the unique strong solution of the problem*

$$\begin{cases} u \in C(\mathbb{R}_+; H) \cap W^{1,\infty}_{loc}((0,\infty); H) \ \ \textit{and} \ u(t) \in D(\varphi) \ \ \textit{a.e.}, \\ \dfrac{\partial u}{\partial t} + \partial\varphi(u) \ni 0 \ \ \textit{a.e. on} \ \mathbb{R}_+, \\ u(0,x) = u_0(x). \end{cases}$$

In addition, $-\partial\varphi$ *generates a nonlinear semigroup* $\{\tilde{T}(t)\}_{t\geq 0}$ *on* H *where, for every* $t \geq 0$*,* $\tilde{T}(t)$ *is the composition of the semigroup* $T(t)$ *on* $\overline{D(\varphi)}$ *with the projection on the convex set* $\overline{D(\varphi)}$*.*

From Proposition 3.1 and Theorem 3.4 it follows that $\partial\Phi_{p,s}$ is a maximal, monotone and m-accretive operator on $H = \mathbb{X}^2(\Omega, \partial\Omega)$, with domain dense in H.

We denote by $T_{p,s}(t)$ the nonlinear semigroup generated by $-\partial\Phi_{p,s}$. From Proposition 3.2, page 176 in [45], the following result follows.

Proposition 3.5. $T_{p,s}(t)$ *is a strongly continuous and contractive semigroup on* H*.*

3.3. The strong problem

We give a characterization of $\partial\Phi_{p,s}$ in order to prove that the strong solution of the abstract Cauchy problem solves problem $(\tilde{P})$.

Theorem 3.6. *Let* $u \in W^{s,p}(\Omega)$ *for a.e.* $t \in (0, T]$*, and let* $\mathbf{f} = (f, f|_{\partial\Omega}) \in H$*. Then* $\mathbf{f} \in \partial\Phi_{p,s}[\mathbf{u}]$ *if and only if* $\mathbf{u} = (u, u|_{\partial\Omega})$ *solves the following problem:*

$$(\bar{P}) \begin{cases} (-\Delta_p)^s_\Omega u = f & \textit{in } L^{p'}(\Omega), \\ C_{p,s} \left\langle \mathcal{N}_p^{p'(1-s)} u, v \right\rangle_{(B^{p,p}_\alpha(\partial\Omega))', B^{p,p}_\alpha(\partial\Omega)} + \langle b|u|^{p-2}u, v\rangle_{L^{p'}(\partial\Omega), L^p(\partial\Omega)} & \\ +p \,\langle \Theta_{p,\gamma}(u), v\rangle_{(B^{p,p}_\alpha(\partial\Omega))', B^{p,p}_\alpha(\partial\Omega)} = \langle f, v\rangle_{L^2(\partial\Omega), L^2(\partial\Omega)} & \forall\, v \in B^{p,p}_\alpha(\partial\Omega). \end{cases}$$

Proof. Let $\mathbf{f} \in \partial\Phi_{p,s}$, i.e. $\Phi_{p,s}(\psi) - \Phi_{p,s}[\mathbf{u}] \geq (\mathbf{f}, \psi - \mathbf{u})_H$ for every $\psi \in W^{s,p}(\Omega)$. This means that

$$\begin{aligned} &\int_\Omega f(\psi - u)\,\mathrm{d}\mathcal{L}_N + \int_{\partial\Omega} f(\psi - u)\,\mathrm{d}\mu \\ &\quad \leq \frac{C_{N,p,s}}{2p} \iint_{\Omega\times\Omega} \frac{|\psi(x) - \psi(y)|^p - |u(x) - u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) \qquad (3.3) \\ &\qquad + \frac{1}{p}\int_{\partial\Omega} b(|\psi|^p - |u|^p)\,\mathrm{d}\mu + \langle\Theta_{p,\gamma}(\psi), \psi\rangle - \langle\Theta_{p,\gamma}(u), u\rangle. \end{aligned}$$

We choose $\psi = u + tv$, with $v \in W^{s,p}(\Omega)$ and $0 < t \leq 1$ in (3.3), thus obtaining

$$\begin{aligned} t\int_\Omega f\,v\,\mathrm{d}\mathcal{L}_N + t\int_{\partial\Omega} f\,v\,\mathrm{d}\mu \\ \leq \frac{C_{N,p,s}}{2p}\iint_{\Omega\times\Omega} \frac{|(u+tv)(x)-(u+tv)(y)|^p - |u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) \\ + \frac{1}{p}\int_{\partial\Omega} b(|u+tv|^p - |u|^p)\,\mathrm{d}\mu + \langle\Theta_{p,\gamma}(u+tv), u+tv\rangle - \langle\Theta_{p,\gamma}(u), u\rangle. \end{aligned} \tag{3.4}$$

We first take $v \in \mathcal{D}(\Omega)$ in (3.4) and, by passing to the limit for $t \to 0^+$, we get

$$\int_\Omega f\,v\,\mathrm{d}\mathcal{L}_N \leq \frac{C_{N,p,s}}{2}\iint_{\Omega\times\Omega} |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y).$$

If we take $-v$ in (3.4) we obtain the opposite inequality, thus getting the equality

$$\int_\Omega f\,v\,\mathrm{d}\mathcal{L}_N = \frac{C_{N,p,s}}{2}\iint_{\Omega\times\Omega} |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y).$$

Since $v \in \mathcal{D}(\Omega)$ and $p' \leq 2$, it turns out that in particular $f \in L^{p'}(\Omega)$. Hence, the p-fractional Green formula for irregular domains given by Theorem 2.2 yields that

$$(-\Delta_p)^s_\Omega u = f \quad \text{in } L^{p'}(\Omega) \tag{3.5}$$

(and in particular in $L^2(\Omega)$).

We go back to (3.4). Dividing by $t > 0$ and passing to the limit for $t \to 0^+$, we get

$$\begin{aligned} \int_\Omega f\,v\,\mathrm{d}\mathcal{L}_N + \int_{\partial\Omega} f\,v\,\mathrm{d}\mu \\ \leq \frac{C_{N,p,s}}{2}\iint_{\Omega\times\Omega} |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) \\ + \int_{\partial\Omega} b|u|^{p-2}u\,v\,\mathrm{d}\mu + p\langle\Theta_{p,\gamma}(u), v\rangle. \end{aligned}$$

As before, by taking $-v$ we obtain the opposite inequality, hence we get the equality. Then, from Theorem 2.2 and (3.5) we get

$$\begin{aligned} \int_{\partial\Omega} fv\,\mathrm{d}\mu = C_{p,s}\left\langle \mathcal{N}_p^{p'(1-s)}u, v\right\rangle_{(B^{p,p}_\alpha(\partial\Omega))', B^{p,p}_\alpha(\partial\Omega)} + \int_{\partial\Omega} b|u|^{p-2}u\,v\,\mathrm{d}\mu \\ + p\langle\Theta_{p,\gamma}(u), v\rangle_{(B^{p,p}_\alpha(\partial\Omega))', B^{p,p}_\alpha(\partial\Omega)}. \end{aligned} \tag{3.6}$$

Hence (3.6) holds in $(B^{p,p}_\alpha(\partial\Omega))'$ and we get the thesis.

We now prove the converse. Let then $\mathbf{u} \in W^{s,p}(\Omega)$ be the weak solution of problem $(\bar{P})$. We have to prove that $\Phi_{p,s}[\mathbf{v}] - \Phi_{p,s}[\mathbf{u}] \geq (\mathbf{f}, \mathbf{v}-\mathbf{u})_H$ for every $\mathbf{v} \in W^{s,p}(\Omega)$. By applying the inequality

$$\frac{1}{p}(|a|^p - |b|^p) \geq |b|^{p-2}b(a-b)$$

and taking into account that $\mathbf{u}$ is the weak solution of $(\bar{P})$, by using as test functions $\mathbf{v}$ and $\mathbf{u}$ respectively, we get

$$\Phi_{p,s}[\mathbf{v}] - \Phi_{p,s}[\mathbf{u}] \geq \int_\Omega f\, v \, \mathrm{d}\mathcal{L}_N + \int_{\partial\Omega} f|_{\partial\Omega}\, v|_{\partial\Omega} \, \mathrm{d}\mu - \int_{\partial\Omega} f\, u \, \mathrm{d}\mathcal{L}_N - \int_{\partial\Omega} f|_{\partial\Omega}\, u|_{\partial\Omega} \, \mathrm{d}\mu = (\mathbf{f}, \mathbf{v})_H - (\mathbf{f}, \mathbf{u})_H,$$

thus concluding the proof. □

From Theorem 3.6, we deduce that the unique strong solution $\mathbf{u}$ of the abstract Cauchy problem (P) solves the following Venttsel'-type problem $(\tilde{P})$ on Ω for a.e. $t \in (0, T]$ in the following weak sense:

$$(\tilde{P}) \begin{cases} \frac{\partial u}{\partial t}(t,x) + (-\Delta_p)^s_\Omega u(t,x) = f(t,x) & \text{for a.e. } x \in \Omega, \\ \left\langle \frac{\partial u|_{\partial\Omega}}{\partial t}, v|_{\partial\Omega} \right\rangle_{L^2(\partial\Omega), L^2(\partial\Omega)} + C_{p,s} \left\langle \mathcal{N}_p^{p'(1-s)} u, v|_{\partial\Omega} \right\rangle_{(B^{p,p}_\alpha(\partial\Omega))', B^{p,p}_\alpha(\partial\Omega)} & \\ + \langle b|u|_{\partial\Omega}|^{p-2} u|_{\partial\Omega}, v|_{\partial\Omega} \rangle_{L^{p'}(\partial\Omega), L^p(\partial\Omega)} + p\, \langle \Theta_{p,\gamma}(u|_{\partial\Omega}), v|_{\partial\Omega} \rangle_{(B^{p,p}_\alpha(\partial\Omega))', B^{p,p}_\alpha(\partial\Omega)} & \\ = \langle f|_{\partial\Omega}, v|_{\partial\Omega} \rangle_{L^2(\partial\Omega), L^2(\partial\Omega)} & \forall\, v \in B^{p,p}_\alpha(\partial\Omega), \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) & \text{in } H, \end{cases}$$

where we recall that $\alpha = s - \frac{N-d}{p}$.

3.4. Well-posedness of the (homogeneous) heat equation

In this subsection we prove that the homogeneous heat equation is well-posed. This will be achieved by investigating the order-preserving and Markovian properties for the semigroup $T_{p,s}(t)$ generated by $-\mathcal{A} = -\partial\Phi_{p,s}$ for every $p \geq 2$.

For the sake of completeness, we recall the following definitions. We refer to [11] for details.

Definition 3.7. Let X be a locally compact metric space and $\tilde{\mu}$ be a Radon measure on X. Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on $L^2(X, \tilde{\mu})$. The semigroup is *order-preserving* if, for every $u, v \in L^2(X, \tilde{\mu})$ such that $u \leq v$,

$$T(t)u \leq T(t)v \quad \forall\, t \geq 0.$$

The semigroup is *non-expansive* on $L^q(X, \tilde{\mu})$ if for every $t \geq 0$

$$\|T(t)u - T(t)v\|_{L^q(X,\tilde{\mu})} \leq \|u - v\|_{L^q(X,\tilde{\mu})} \quad \forall\, u \in L^2(X, \tilde{\mu}) \cap L^q(X, \tilde{\mu}).$$

The semigroup is *Markovian* if it is order-preserving and non-expansive on $L^\infty(X, \tilde{\mu})$.

We give two equivalent conditions for proving order-preserving and Markovian properties.

Proposition 3.8. *Let $\varphi : H \to (-\infty, +\infty]$ be a proper, convex, lower semicontinuous functional on a real Hilbert lattice H, with effective domain $D(\varphi)$. Let $\{T(t)\}_{t\geq 0}$ be the nonlinear semigroup on H generated by $-\partial\varphi$. Then the following assertions are equivalent:*

(i) The semigroup $\{T(t)\}_{t\geq 0}$ is order-preserving;

(ii) For every $u,\, v \in H$ one has

$$\varphi\left(\frac{1}{2}\left(u+u\wedge v\right)\right)+\varphi\left(\frac{1}{2}\left(v+u\vee v\right)\right)\,\leq\,\varphi(u)+\varphi(v),$$

where $u\wedge v:=\inf\{u,v\}$ and $u\vee v:=\sup\{u,v\}$.

Proposition 3.9. *Let $\varphi\colon L^2(X,\nu)\to(-\infty,+\infty]$ be a proper, convex, lower semicontinuous functional. Let $\{T(t)\}_{t\geq 0}$ be the nonlinear semigroup on $L^2(X,\nu)$ generated by $-\partial\varphi$. Assume that $\{T(t)\}_{t\geq 0}$ is order-preserving. Then, the following assertions are equivalent:*

(i) The semigroup $\{T(t)\}_{t\geq 0}$ is Markovian;

(ii) For every $u,v\in L^2(X,\nu)$ and $\tilde{\alpha}>0$

$$\varphi\left(v+g_{\tilde{\alpha}}(u,v)\right)+\varphi\left(u-g_{\tilde{\alpha}}(u,v)\right)\,\leq\,\varphi(u)+\varphi(v),$$

where

$$g_{\tilde{\alpha}}(u,v):=\frac{1}{2}\left[(u-v+\tilde{\alpha})^{+}-(u-v-\tilde{\alpha})^{-}\right],$$

with $u^{+}:=\sup\{u,0\}$ and $u^{-}:=\sup\{-u,0\}$.

For the proofs of Propositions 3.8 and 3.9 we refer to [18, p. 1] and [11, Theorem 3.6] respectively.

Theorem 3.10. *The semigroup $\{T_{p,s}(t)\}_{t\geq 0}$ generated by $-\mathcal{A}$ is Markovian on $\mathbb{X}^2(\Omega,\partial\Omega)$.*

Proof. We will apply Propositions 3.8 and 3.9.

Let $\mathbf{u},\mathbf{v}\in\mathbb{X}^2(\Omega,\partial\Omega)$. If $\mathbf{u}$ does not belong to $W^{s,p}(\Omega)$, then $\Phi_{p,s}[\mathbf{u}]=+\infty$ and the conclusion is obvious (and similarly if $\mathbf{v}\notin W^{s,p}(\Omega)$). Hence, we suppose that $\mathbf{u},\mathbf{v}\in W^{s,p}(\Omega)$.

We begin by proving the order-preserving property. We set, for $u,v\in W^{s,p}(\Omega)$,

$$g(u,v):=\frac{1}{2}\left(u+u\wedge v\right)\qquad\text{and}\qquad h(u,v):=\frac{1}{2}\left(v+u\vee v\right).$$

Since $W^{s,p}(\Omega)$ is a lattice, we have that both $g(u,v)$ and $h(u,v)$ belong to $W^{s,p}(\Omega)$. Proceeding as in [47, Theorem 3.1.4], we prove that

$$\int_{\partial\Omega}b|g(u,v)|^p\,\mathrm{d}\mu+\int_{\partial\Omega}b|h(u,v)|^p\,\mathrm{d}\mu\leq\int_{\partial\Omega}b|u|^p\,\mathrm{d}\mu+\int_{\partial\Omega}b|v|^p\,\mathrm{d}\mu. \tag{3.7}$$

Moreover, from the convexity of the functional and proceeding as in [48, Theorem 3.4] we have that

$$\begin{aligned}&|g(u,v)|^p_{W^{s,p}(\Omega)}+\langle\Theta_{p,\gamma}(g(u,v)),g(u,v)\rangle\\&\quad+|h(u,v)|^p_{W^{s,p}(\Omega)}+\langle\Theta_{p,\gamma}(h(u,v)),h(u,v)\rangle\leq|u|^p_{W^{s,p}(\Omega)}\\&\quad+\langle\Theta_{p,\gamma}(u),u\rangle+|v|^p_{W^{s,p}(\Omega)}+\langle\Theta_{p,\gamma}(v),v\rangle.\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$\Phi_{p,s}[\mathbf{g}(u,v)]+\Phi_{p,s}[\mathbf{h}(u,v)]\leq\Phi_{p,s}[\mathbf{u}]+\Phi_{p,s}[\mathbf{v}],$$

thus $\{T_{p_s}(t)\}_{t\geq 0}$ is order-preserving.

In order to complete the proof, we apply Proposition 3.9. In the notation of Proposition 3.9, given $u, v \in W^{s,p}(\Omega)$ and $\tilde{\alpha} > 0$, we set

$$g_{\tilde{\alpha}}(u, v) := \frac{1}{2}\left[(u - v + \tilde{\alpha})^+ - (u - v - \tilde{\alpha})^-\right].$$

We point out that $g_{\tilde{\alpha}}(u, v) \in W^{s,p}(\Omega)$. Proceeding again as in the proofs of [47, Theorem 3.1.4] and [48, Theorem 3.4], we get that

$$\Phi_{p,s}[\mathbf{v} + \mathbf{g}_{\tilde{\alpha}}(u, v)] + \Phi_{p,s}[\mathbf{u} - \mathbf{g}_{\tilde{\alpha}}(u, v)] \leq \Phi_{p,s}[\mathbf{u}] + \Phi_{p,s}[\mathbf{v}],$$

hence from Proposition 3.9 $\{T_{p,s}(t)\}_{t\geq 0}$ is non-expansive on $\mathbb{X}^\infty(\Omega, \partial\Omega)$, and thus $\{T_{p,s}(t)\}_{t\geq 0}$ is Markovian. $\square$

Remark 3.11. We remark that, from [9, Theorem 1] and [40, Corollary 3], $\{T_{p,s}(t)\}_{t\geq 0}$ can be extended to a non-expansive semigroup on $\mathbb{X}^q(\Omega, \partial\Omega)$ for every $q \in [2, \infty]$. Moreover, we can prove also the strong continuity of $\{T_{p,s}(t)\}_{t\geq 0}$ over $\mathbb{X}^q(\Omega, \partial\Omega)$ for every $q \in [2, \infty)$ by following the approach used in [47, Theorem 3.1.4].

4. Ultracontractivity of semigroups

We now focus on proving the ultracontractivity of the semigroup $T_{p,s}(t)$.

We first prove a logarithmic Sobolev inequality adapted to our case.

Proposition 4.1. *Let $p \geq 2$, $s > \frac{N-d}{p}$ and $sp < N$. Let $u \in W^{s,p}(\Omega)$ be non-negative on $\overline{\Omega}$ and such that $\|\mathbf{u}\|_p^p = \|u\|^p_{L^p(\Omega)} + \|u|_{\partial\Omega}\|^p_{L^p(\partial\Omega)} = 1$. We set*

$$\Lambda(\mathbf{u}) := \int_\Omega u \, \mathrm{d}\mathcal{L}_N + \int_{\partial\Omega} u|_{\partial\Omega} \, \mathrm{d}\mu. \tag{4.1}$$

Then there exists a positive constant $\bar{C} = \bar{C}(N, s, p, d, \Omega)$ such that, for every $\varepsilon > 0$,

$$\Lambda(\mathbf{u}^p \log \mathbf{u}) \leq \frac{d}{p(d-N+sp)}\left[\varepsilon\bar{C}\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \, \mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) - \log\varepsilon + \varepsilon\bar{C}\left(|\bar{u}_\Omega|^p + |\bar{u}_{\partial\Omega}|^p\right)\right], \tag{4.2}$$

where $\mathbf{u}^p \log \mathbf{u} := (u^p \log u, u|^p_{\partial\Omega} \log u|_{\partial\Omega})$ and $\bar{u}_\Omega$ and $\bar{u}_{\partial\Omega}$ are defined in (1.8).

Proof. Following [6, Proposition 2.1], we apply Jensen's inequality with $q = \bar{p} - p = \frac{p(d-N+sp)}{N-sp}$ and we obtain

$$\Lambda(\mathbf{u}^p \log \mathbf{u}) \leq \frac{1}{q}\log\Lambda(\mathbf{u}^{p+q}) = \frac{N - sp}{p(d - N + sp)}\log\|\mathbf{u}\|_{\bar{p}}^{\bar{p}} = \frac{d}{p(d - N + sp)}\log\|\mathbf{u}\|^p_{\bar{p}}. \tag{4.3}$$

Moreover, from the properties of the logarithmic function, for every $\varepsilon > 0$ we have that

$$\log\|\mathbf{u}\|^p_{\bar{p}} \leq \varepsilon\|\mathbf{u}\|^p_{\bar{p}} - \log\varepsilon. \tag{4.4}$$

From (1.11) with $q=\bar{p}$, we estimate $\|\mathbf{u}\|_{\bar{p}}^{p}$ in (4.4). Hence, there exists a positive constant $\bar{C}$ such that

$$\Lambda(\mathbf{u}^p \log \mathbf{u}) \leq \frac{d}{p(d-N+sp)}\left[\varepsilon\bar{C}\left(|u|^p_{W^{s,p}(\Omega)}+|\bar{u}_\Omega|^p+|\bar{u}_{\partial\Omega}|^p\right)-\log\varepsilon\right],$$

thus concluding the proof. □

In order to prove the ultracontractivity of $T_{p,s}(t)$, we now prove some preliminary lemmas. We adapt to the fractional framework the results of [34, Section 3.2], see also [49,51,52].

We first recall some known numerical inequalities. For more details we refer to [5].

Proposition 4.2. *Let $a, b \in \mathbb{R}^N$. If $r \in (1,\infty)$, it holds that*

$$\left(|a|^{r-2}a-|b|^{r-2}b\right)(a-b) \;\geq\; (r-1)\left(|a|+|b|\right)^{r-2}|a-b|^2. \tag{4.5}$$

If $r \in [2,\infty)$, then for $c_r^* := \min\{1/(r-1), 2^{-2-r}\,3^{-r/2}\} \in (0,1]$, it holds that

$$\left(|a|^{r-2}a-|b|^{r-2}b\right)(a-b) \;\geq\; c_r^*\,|a-b|^r. \tag{4.6}$$

We remark that (4.6) implies

$$\left(|a|^{r-2}a-|b|^{r-2}b\right)\operatorname{sgn}(a-b) \;\geq\; c_r^*\,|a-b|^{r-1}. \tag{4.7}$$

Lemma 4.3. *Let $\{T_{p,s}(t)\}_{t\geq 0}$ be the Markovian semigroup on $\mathbb{X}^2(\Omega,\partial\Omega)$ generated by $-\partial\Phi_{p,s}$. Given $t\geq 0$ and $\mathbf{u}_0$, $\mathbf{v}_0 \in \mathbb{X}^\infty(\Omega,\partial\Omega)$, let $\mathbf{u}(t,x):=T_{p,s}(t)\mathbf{u}_0(x)$ and $\mathbf{v}(t,x) := T_{p,s}(t)\mathbf{v}_0(x)$ be the solutions of the homogeneous problem associated to (P) with initial data $\mathbf{u}_0$ and $\mathbf{v}_0$ respectively. We set $U(t,x) := u(t,x)-v(t,x)$ and $\mathbf{U}=(U,U|_{\partial\Omega})$. Then, for every real number $r\geq 2$ and for a.e. $t\geq 0$, there exists a constant $\tilde{C}=\tilde{C}(N,s,p)$ such that*

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{U}(t)\|_r^r \leq\; & -r\tilde{C}\iint_{\Omega\times\Omega}\tfrac{|U(t,x)-U(t,y)|^{r+p-2}}{|x-y|^{N+sp}}\,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y)\\ & -c_p^* b_0 r\int_{\partial\Omega}|U|_{\partial\Omega}(t)|^{r+p-2}\,\mathrm{d}\mu,\end{aligned} \tag{4.8}$$

where $c_p^*>0$ is the constant given in Proposition 4.2 and $b_0=\min_{\overline{\Omega}} b$.

Proof. The proof can be obtained by suitably adapting the proof of Lemma 3.4 in [34]. □

We remark that, as a consequence of Lemma 4.3, we have that $G_r(t) := \|\mathbf{U}(t)\|_r^r = \|U(t)\|^r_{L^r(\Omega)}+\|U|_{\partial\Omega}(t)\|^r_{L^r(\partial\Omega)}$ is non-increasing w.r.t. t.

We now recall a useful estimate for the nonlocal term on Ω appearing in $\Phi_{p,s}[\mathbf{u}]$. We refer to [52, Lemma 4.1].

Lemma 4.4. *Let $p,r\geq 2$ and $s\in(0,1)$. Then, for every $u,v\in W^{s,p}(\Omega)$ it holds that*

$$\begin{aligned}C_{r,p}\left(|u|^{\frac{r+p-2}{p}},|u|^{\frac{r+p-2}{p}}\right)_{s,p} &\leq C_{r,p}(|u|^{\frac{r-2}{p}},|u|^{\frac{r-2}{p}})_{s,p}\\ &\leq (u,|u|^{r-2}u)_{s,p},\end{aligned} \tag{4.9}$$

where $C_{r,p} := (r-1)\left(\dfrac{p}{r+p-2}\right)^p$.

The next two lemmas follow by adapting to the fractional setting Lemmas 3.5 and 3.6 in [34] (see also [47]).

Lemma 4.5. *Under the same notations and assumptions of Lemma 4.3, if* $r : [0,\infty) \to [2,\infty)$ *is an increasing differentiable function, then for a.e.* $t \geq 0$ *and for every* $\varepsilon > 0$ *we have that*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \log \|\mathbf{U}(t)\|_{r(t)} \leq \frac{r'(t)}{r(t)} \Lambda\left(\frac{|\mathbf{U}(t)|^{r(t)}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}} \log \frac{|\mathbf{U}(t)|}{\|\mathbf{U}(t)\|_{r(t)}}\right) \\
&- \frac{\tilde{C}(r(t)-1)}{\varepsilon \bar{C}} \left(\frac{p}{r(t)+p-2}\right)^p \log \varepsilon \frac{\|\mathbf{U}(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}} \\
&+ C_{\overline{\Omega}} \tilde{C}(r(t)-1) \left(\frac{p}{r(t)+p-2}\right)^p \frac{\|\mathbf{U}(t)\|_{\frac{r(t)+p-2}{p}}^{r(t)+p-2}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}} \\
&- \frac{\tilde{C}(r(t)-1)}{\varepsilon \bar{C}} \left(\frac{p}{r(t)+p-2}\right)^{p-1} \frac{p(d-N+sp)}{d} \frac{\|\mathbf{U}(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}} \cdot \\
&\cdot \Lambda\left(\frac{|\mathbf{U}(t)|^{r(t)+p-2}}{\|\mathbf{U}(t)\|_{r(t)+p-2}^{r(t)+p-2}} \log \frac{|\mathbf{U}(t)|}{\|\mathbf{U}(t)\|_{r(t)+p-2}}\right),
\end{aligned} \tag{4.10}$$

where $\bar{C}$ *and* $\tilde{C}$ *are the constants appearing in* (4.2) *and* (4.8) *respectively,* Λ *is defined as in* (4.1) *and* $C_{\overline{\Omega}} := \max\left\{\frac{1}{|\Omega|^p}, \frac{1}{\mu(\partial\Omega)^p}\right\}$.

Proof. From the chain rule, we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{U}(t)\|_{r(t)}^{r(t)} &= r'(t) \frac{\partial}{\partial r} \|\mathbf{U}(t)\|_{r(t)}^{r(t)} + \frac{\partial}{\partial t} \|\mathbf{U}(t)\|_{r(t)}^{r(t)} \\
&= r'(t) \Lambda\left(|\mathbf{U}(t)|^{r(t)} \log |\mathbf{U}(t)|\right) + \frac{\partial}{\partial t} \|\mathbf{U}(t)\|_{r(t)}^{r(t)}.
\end{aligned}$$

Then, from Lemma 4.3 the following holds:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \log \|\mathbf{U}(t)\|_{r(t)} = -\frac{r'(t)}{r(t)} \log \|\mathbf{U}(t)\|_{r(t)} \\
&\quad + \frac{1}{r(t) \|\mathbf{U}(t)\|_{r(t)}^{r(t)}} \frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{U}(t)\|_{r(t)}^{r(t)} \leq -\frac{r'(t)}{r(t)} \log \|\mathbf{U}(t)\|_{r(t)} \\
&\quad + \frac{r'(t)}{r(t)} \frac{1}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}} \Lambda\left(|\mathbf{U}(t)|^{r(t)} \log |\mathbf{U}(t)|\right) \\
&\quad - \frac{c_p^* b_0}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}} \int_{\partial\Omega} |U|_{\partial\Omega}(t)|^{r+p-2} \,\mathrm{d}\mu - \frac{\tilde{C}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}} \cdot \\
&\cdot \iint_{\Omega\times\Omega} \frac{|U(t,x)-U(t,y)|^{p-2}(U(t,x)-U(t,y))(|U(t,x)|^{r(t)}U(t,x)-|U(t,y)|^{r(t)}U(t,y))}{|x-y|^{N+sp}} \,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y).
\end{aligned} \tag{4.11}$$

Recalling the definition of Λ, using Lemma 4.4 and estimating the boundary term with zero, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\log\|\mathbf{U}(t)\|_{r(t)} \le \frac{r'(t)}{r(t)}\Lambda\left(\frac{|\mathbf{U}(t)|^{r(t)}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}}\log\frac{|\mathbf{U}(t)|}{\|\mathbf{U}(t)\|_{r(t)}}\right)\\
&-\frac{\tilde{C}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}}(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^{p}\cdot\\
&\cdot\iint_{\Omega\times\Omega}\frac{\left||U(t,x)|^{\frac{r(t)+p-2}{p}}-|U(t,y)|^{\frac{r(t)+p-2}{p}}\right|^{p}}{|x-y|^{N+sp}}\,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) \qquad (4.12)\\
&=\frac{r'(t)}{r(t)}\Lambda\left(\frac{|\mathbf{U}(t)|^{r(t)}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}}\log\frac{|\mathbf{U}(t)|}{\|\mathbf{U}(t)\|_{r(t)}}\right)\\
&-\tilde{C}(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^{p}\frac{\|\mathbf{U}(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}}|F(t)|_{W^{s,p}(\Omega)}^{p},
\end{aligned}$$

where

$$F(t,x) := \frac{|U(t)|^{\frac{r(t)+p-2}{p}}}{\|\mathbf{U}(t)\|_{r(t)+p-2}^{\frac{r(t)+p-2}{p}}}$$

fulfills the hypotheses of Proposition 4.1. Thus we have that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\log\|\mathbf{U}(t)\|_{r(t)} \le \frac{r'(t)}{r(t)}\Lambda\left(\frac{|\mathbf{U}(t)|^{r(t)}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}}\log\frac{|\mathbf{U}(t)|}{\|\mathbf{U}(t)\|_{r(t)}}\right)\\
&+\tilde{C}(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^{p}\frac{\|\mathbf{U}(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}}\left(-\frac{p(d-N+sp)}{d\varepsilon\bar{C}}\Lambda(\mathbf{F}^p\log\mathbf{F})\right.\\
&\left.-\frac{\log\varepsilon}{\varepsilon\bar{C}}+|\bar{F}_\Omega|^p+|\bar{F}_{\partial\Omega}|^p\right).
\end{aligned} \qquad (4.13)$$

We now point out that

$$\Lambda(\mathbf{F}^p\log\mathbf{F}) = \frac{r(t)+p-2}{p}\Lambda\left(\frac{|\mathbf{U}(t)|^{r(t)+p-2}}{\|\mathbf{U}(t)\|_{r(t)+p-2}^{r(t)+p-2}}\log\frac{|\mathbf{U}(t)|}{\|\mathbf{U}(t)\|_{r(t)+p-2}}\right). \qquad (4.14)$$

Moreover, we have that

$$|\bar{F}_\Omega|^p = \frac{1}{|\Omega|^p\|\mathbf{U}(t)\|_{r(t)+p-2}^{r(t)+p-2}}\left|\int_\Omega |U(t)|^{\frac{r(t)+p-2}{p}}\,\mathrm{d}\mathcal{L}_N\right|^p = \frac{\|U(t)\|_{L^{\frac{r(t)+p-2}{p}}(\Omega)}^{r(t)+p-2}}{|\Omega|^p\|\mathbf{U}(t)\|_{r(t)+p-2}^{r(t)+p-2}}; \qquad (4.15)$$

an analogous equality holds for $|\bar{F}_{\partial\Omega}|^p$. Hence, from (4.13), (4.14) and (4.15) we deduce

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\log\|\mathbf{U}(t)\|_{r(t)} \le \frac{r'(t)}{r(t)}\Lambda\left(\frac{|\mathbf{U}(t)|^{r(t)}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}}\log\frac{|\mathbf{U}(t)|}{\|\mathbf{U}(t)\|_{r(t)}}\right)\\
&-\tilde{C}(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^p\frac{\log\varepsilon}{\varepsilon\bar{C}}\frac{\|\mathbf{U}(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}}\\
&-\frac{\tilde{C}(r(t)-1)p(d-N+sp)}{d\varepsilon\bar{C}}\left(\frac{p}{r(t)+p-2}\right)^{p-1}\frac{\|\mathbf{U}(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}}\cdot\\
&\cdot\Lambda\left(\frac{|\mathbf{U}(t)|^{r(t)+p-2}}{\|\mathbf{U}(t)\|_{r(t)+p-2}^{r(t)+p-2}}\log\frac{|\mathbf{U}(t)|}{\|\mathbf{U}(t)\|_{r(t)+p-2}}\right)\\
&+\tilde{C}(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^p\frac{1}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}}\cdot\\
&\cdot\left(\frac{\|U(t)\|_{L^{\frac{r(t)+p-2}{p}}(\Omega)}^{r(t)+p-2}}{|\Omega|^p}+\frac{\|U(t)\|_{L^{\frac{r(t)+p-2}{p}}(\partial\Omega)}^{r(t)+p-2}}{\mu(\partial\Omega)^p}\right),
\end{aligned}
\tag{4.16}
$$

and, defining

$$C_{\overline{\Omega}} := \max\left\{\frac{1}{|\Omega|^p}, \frac{1}{\mu(\partial\Omega)^p}\right\},$$

we get the thesis. □

Lemma 4.6. *Under the assumptions of Lemma 4.5, for a.e. $t \ge 0$ we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\|\mathbf{U}(t)\|_{r(t)} \le -A(t)\log\|\mathbf{U}(t)\|_{r(t)} - B(t), \tag{4.17}$$

where

$$A(t) := \frac{r'(t)(p-2)d}{r(t)(r(t)+p-2)(d-N+sp)}, \tag{4.18}$$

$$
\begin{aligned}
B(t) :=& -\frac{r'(t)(N-sp)(p-2)}{r(t)(r(t)+p-2)(d-N+sp)}\log\omega - \hat{C}p\|\mathbf{U}_0\|_2^{p-2}\\
&+\frac{r'(t)d}{r(t)(r(t)+p-2)(d-N+sp)}\cdot\\
&\cdot\log\left[\frac{r(t)}{r'(t)}\frac{p(d-N+sp)}{d}\frac{\tilde{C}(r(t)-1)}{\bar{C}}\left(\frac{p}{r(t)+p-2}\right)^{p-1}\right],
\end{aligned}
\tag{4.19}
$$

$\hat{C} = \hat{C}(N,s,p,\Omega)$ is a positive constant, $\mathbf{U}_0 := \mathbf{U}(0) = \mathbf{u}_0 - \mathbf{v}_0$ and $\omega =$ $\max\{|\Omega|, \mu(\partial\Omega)\}$.

Proof. We choose $\varepsilon > 0$ as follows:

$$\varepsilon := \frac{r(t)}{r'(t)} \frac{p(d-N+sp)}{d} \frac{\tilde{C}}{\bar{\bar{C}}}(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^{p-1} \frac{\|\mathbf{U}(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}}.$$

Hence from (4.10) we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\log\|\mathbf{U}(t)\|_{r(t)} \leq{}& \frac{r'(t)}{r(t)}\Lambda\left(\frac{|\mathbf{U}(t)|^{r(t)}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}}\log\frac{|\mathbf{U}(t)|}{\|\mathbf{U}(t)\|_{r(t)}}\right)\\
&-\frac{r'(t)}{r(t)}\Lambda\left(\frac{|\mathbf{U}(t)|^{r(t)+p-2}}{\|\mathbf{U}(t)\|_{r(t)+p-2}^{r(t)+p-2}}\log\frac{|\mathbf{U}(t)|}{\|\mathbf{U}(t)\|_{r(t)+p-2}}\right)\\
&-\frac{r'(t)}{r(t)(r(t)+p-2)}\frac{d}{d-N+sp}\cdot\\
&\cdot\log\left[\frac{r(t)}{r'(t)}\frac{p(d-N+sp)}{d}\frac{\tilde{C}}{\bar{\bar{C}}}(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^{p-1}\frac{\|\mathbf{U}(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}}\right]\\
&+C_{\overline{\Omega}}\tilde{C}(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^{p}\frac{\|\mathbf{U}(t)\|_{\frac{r(t)+p-2}{p}}^{r(t)+p-2}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}}.
\end{aligned}\tag{4.20}$$

We point out that, since $r(t) \geq 2$ and $p \geq 2$, it holds that

$$(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^{p} \leq p.$$

From Hölder and interpolation inequalities, we get that

$$\|U(t)\|_{L^{\frac{r(t)+p-2}{p}}(\Omega)} \leq |\Omega|^{\frac{1}{r(t)+p-2}}\|U(t)\|_{L^1(\Omega)}^{\frac{p-2}{r(t)+p-2}}\|U(t)\|_{L^{r(t)}(\Omega)}^{\frac{r(t)}{r(t)+p-2}};\tag{4.21}$$

an analogous inequality holds for the $L^{\frac{r(t)+p-2}{p}}(\partial\Omega)$-norm of $U(t)$.

We now set

$$K(q,\mathbf{U}) := \Lambda\left(\frac{|\mathbf{U}|^q}{\|\mathbf{U}\|_q^q}\log\frac{|\mathbf{U}|}{\|\mathbf{U}\|_q}\right) = \int_\Omega \frac{|U|^q}{\|\mathbf{U}\|_q^q}\log\frac{|U|}{\|\mathbf{U}\|_q}\,\mathrm{d}\mathcal{L}_N + \int_{\partial\Omega}\frac{|U|_{\partial\Omega}|^q}{\|\mathbf{U}\|_q^q}\log\frac{|U|_{\partial\Omega}|}{\|\mathbf{U}\|_q}\,\mathrm{d}\mu.$$

The functional $K(q,\mathbf{U})$ satisfies the following property: for every $q_2 \geq q_1 \geq 1$ and for every $\mathbf{U} \in \mathbb{X}^\infty(\Omega,\partial\Omega)$

$$K(q_2,\mathbf{U}) - K(q_1,\mathbf{U}) \geq \log\frac{\|\mathbf{U}\|_{q_1}}{\|\mathbf{U}\|_{q_2}}.\tag{4.22}$$

Applying (4.21) and (4.22) with $q_1 = r(t)$ and $q_2 = (r(t)+p-2)$ and defining $C^* := C_{\overline{\Omega}}\omega\tilde{C}$, from (4.20) we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\log \|\mathbf{U}(t)\|_{r(t)} \leq \frac{r'(t)}{r(t)} \log \frac{\|\mathbf{U}(t)\|_{r(t)+p-2}}{\|\mathbf{U}(t)\|_{r(t)}} \\
&\quad + pC^*\|\mathbf{U}(t)\|_1^{p-2} - \frac{r'(t)}{r(t)(r(t)+p-2)}\frac{d}{d-N+sp} \cdot \\
&\quad \cdot \log\left[\frac{r(t)}{r'(t)}\frac{p(d-N+sp)}{d}\frac{\tilde{C}}{\bar{C}}(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^{p-1}\frac{\|\mathbf{U}(t)\|_{r(t)+p-2}^{r(t)+p-2}}{\|\mathbf{U}(t)\|_{r(t)}^{r(t)}}\right].
\end{aligned} \tag{4.23}$$

Now, using the properties of the logarithmic function, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\log \|\mathbf{U}(t)\|_{r(t)} \leq \frac{r'(t)}{r(t)}\left(1-\frac{d}{d-N+sp}\right)\log \|\mathbf{U}(t)\|_{r(t)+p-2} + pC^*\|\mathbf{U}(t)\|_1^{p-2} \\
&\quad - \frac{r'(t)}{r(t)}\left(1-\frac{r(t)d}{(r(t)+p-2)(d-N+sp)}\right)\log \|\mathbf{U}(t)\|_{r(t)} \\
&\quad - \frac{r'(t)}{r(t)(r(t)+p-2)}\frac{d}{d-N+sp}\log\left[\frac{r(t)}{r'(t)}\frac{p(d-N+sp)}{d}\frac{\tilde{C}}{\bar{C}}(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^{p-1}\right].
\end{aligned} \tag{4.24}$$

We remark that, since $sp<N$, $1-\frac{d}{d-N+sp}=\frac{sp-N}{d-N+sp}<0$. Hence, from Hölder inequality we have that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\log \|\mathbf{U}(t)\|_{r(t)} \leq \frac{r'(t)}{r(t)}\frac{d(2-p)}{(r(t)+p-2)(d-N+sp)}\log \|\mathbf{U}(t)\|_{r(t)} \\
&\quad + \frac{r'(t)}{r(t)}\frac{N-sp}{d-N+sp}\frac{p-2}{r(t)+p-2}\log\omega + pC^*\|\mathbf{U}(t)\|_1^{p-2} \\
&\quad - \frac{r'(t)}{r(t)(r(t)+p-2)}\frac{d}{d-N+sp}\log\left[\frac{r(t)}{r'(t)}\frac{p(d-N+sp)}{d}\frac{\tilde{C}}{\bar{C}}(r(t)-1)\left(\frac{p}{r(t)+p-2}\right)^{p-1}\right].
\end{aligned} \tag{4.25}$$

From Hölder inequality and the Markovian property of $T_{p,s}(t)$, we deduce

$$\|U(t)\|_{L^1(\Omega)} \leq |\Omega|^{\frac{1}{2}}\|U(t)\|_{L^2(\Omega)} \leq |\Omega|^{\frac{1}{2}}\|U(0)\|_{L^2(\Omega)} = |\Omega|^{\frac{1}{2}}\|U_0\|_{L^2(\Omega)};$$

an analogous inequality holds on $\partial\Omega$. Hence, for a suitable constant $\hat{C}$ depending on N, s, p and Ω, taking into account the definitions of $A(t)$ and $B(t)$ in (4.18) and (4.19) respectively, estimate (4.17) follows. □

We now prove the ultracontractivity of $T_{p,s}(t)$, the main result of this section.

Theorem 4.7. *Let $p>2$ and $sp<N$. In the notations of the above lemmas, if $q\in[2,\infty]$, then there exist two positive constants C_1, C_2 depending on N, s, p, q, d and Ω such that*

$$\|T_{p,s}(t)\mathbf{u}_0 - T_{p,s}(t)\mathbf{v}_0\|_\infty \leq C_1(\max\{|\Omega|,\mu(\partial\Omega)\})^{\lambda_1(s)} e^{C_2 t\|\mathbf{u}_0-\mathbf{v}_0\|_2^{p-2}} t^{-\lambda_2(s)}\|\mathbf{u}_0-\mathbf{v}_0\|_q^{\lambda_3(s)}, \tag{4.26}$$

for every $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{X}^q(\Omega,\partial\Omega)$ and for every $t>0$, where

$$\lambda_1(s) = \frac{N-sp}{d}\left[1-\left(\frac{q}{q+p-2}\right)^{\frac{d}{d-N+sp}}\right], \quad \lambda_2(s) = \frac{1}{p-2}\left[1-\left(\frac{q}{q+p-2}\right)^{\frac{d}{d-N+sp}}\right],$$
$$\lambda_3(s) = \left(\frac{q}{q+p-2}\right)^{\frac{d}{d-N+sp}}. \tag{4.27}$$

Proof. We first take $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{X}^\infty(\Omega, \partial\Omega)$ and we use the same assumptions and notations of Lemma 4.6. In particular, we consider an increasing differentiable function $r\colon [0,\infty) \to [2,\infty)$ and we define $A(t)$ and $B(t)$ as in (4.18) and (4.19) respectively.

We set

$$y(t) := \log \|\mathbf{U}(t)\|_{r(t)};$$

then, from (4.17), $y(t)$ satisfies the following ordinary differential inequality:

$$y'(t) + A(t)y(t) + B(t) \le 0. \tag{4.28}$$

We now consider the following ODE:

$$\begin{cases} x'(t) + A(t)x(t) + B(t) = 0, \\ x(0) = y(0). \end{cases} \tag{4.29}$$

The unique solution $x(t)$ of (4.29) can be written in the following way:

$$x(t) = \exp\left(-\int_0^t A(\tau)\,\mathrm{d}\tau\right)\left[y(0) - \int_0^t B(\tau)\exp\left(\int_0^\tau A(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\tau\right]; \tag{4.30}$$

hence, the solution $y(t)$ of the ordinary differential inequality (4.28) is such that $y(t) \le x(t)$ for every $t \in [0,\infty)$.

We now fix $t > 0$, for any given $q \ge 2$ and for $\tau \in [0,t)$ we set

$$r(\tau) := \frac{qt}{t-\tau}. \tag{4.31}$$

The function $r(\cdot)$ satisfies the hypotheses of Lemma 4.5, i.e. it is increasing and differentiable on $[0,t)$ and $r(\tau) \ge 2$ for every $\tau \in [0,t)$.

Using (4.31), we obtain that

$$A(\tau) = \frac{d(p-2)}{d-N+sp}\;\frac{1}{t(q+p-2)-\tau(p-2)}$$

and

$$B(\tau) = -\frac{(N-sp)(p-2)}{d-N+sp}\;\frac{1}{t(q+p-2)-\tau(p-2)}\log\omega - \hat{C}p\|\mathbf{U}_0\|_2^{p-2} + \frac{d}{d-N+sp}\cdot$$
$$\cdot\frac{1}{t(q+p-2)-\tau(p-2)}\log\left[\frac{p(d-N+sp)}{d}\frac{\tilde{C}}{\bar{C}}(t(q-1)+\tau)\left(\frac{p(t-\tau)}{t(q+p-2)-\tau(p-2)}\right)^{p-1}\right],$$

where $\bar{C}, \tilde{C}$ and $\hat{C}$ are the constants in Proposition 4.1, Lemma 4.3 and Lemma 4.6 respectively.

Our aim is now to write $x(t)$ in a more explicit way. From standard calculations, we have that

$$\int_0^\tau A(\sigma)\,\mathrm{d}\sigma = \frac{d}{d-N+sp}\log\frac{t(q+p-2)}{t(q+p-2)-\tau(p-2)},$$

hence

$$\lim_{\tau\to t^-}\exp\left(-\int_0^\tau A(\sigma)\,\mathrm{d}\sigma\right) = \left(\frac{q}{q+p-2}\right)^{\frac{d}{d-N+sp}}. \tag{4.32}$$

Moreover, again from standard cumbersome calculations, we can prove that

$$\begin{aligned}
&\lim_{\tau\to t^-}\int_0^\tau B(\sigma)\exp\left(\int_0^\sigma A(\xi)\,\mathrm{d}\xi\right)\mathrm{d}\sigma\\
&= -\frac{N-sp}{d}\log\omega\left[\left(\frac{q+p-2}{q}\right)^{\frac{d}{d-N+sp}}-1\right]-\check{C}t\|\mathbf{U}_0\|_2^{p-2}\\
&+\frac{1}{p-2}\left[\left(\frac{q+p-2}{q}\right)^{\frac{d}{d-N+sp}}-1\right]\\
&\left[\log\left(p^p\frac{d-N+sp}{d}\frac{\tilde{C}}{\bar{\bar{C}}}\right)+\log t\right]+I^{(1)}+I^{(2)}-I^{(3)},
\end{aligned} \tag{4.33}$$

where $\check{C}$ is a suitable positive constant depending on N, s, p, Ω, d and q and $I^{(1)}$, $I^{(2)}$ and $I^{(3)}$ are integral terms which do not depend on t and can be explicitly computed as in [10, proof of Lemma 3.9].

From (4.32) and (4.33) it follows that

$$\begin{aligned}
\lim_{\tau\to t^-}x(\tau) &= \left(\frac{q}{q+p-2}\right)^{\frac{d}{d-N+sp}}y(0)+\frac{N-sp}{d}\log\omega\\
&\left[1-\left(\frac{q}{q+p-2}\right)^{\frac{d}{d-N+sp}}\right]+C_2t\|\mathbf{U}_0\|_2^{p-2}\\
&-\frac{1}{p-2}\left[1-\left(\frac{q}{q+p-2}\right)^{\frac{d}{d-N+sp}}\right]\\
&\left[\log\left(p^p\frac{d-N+sp}{d}\frac{\tilde{C}}{\bar{\bar{C}}}\right)+\log t\right]+C_I,
\end{aligned} \tag{4.34}$$

where $C_2 = \left(\frac{q}{q+p-2}\right)^{\frac{d}{d-N+sp}}\check{C}$ and $C_I = \left(\frac{q}{q+p-2}\right)^{\frac{d}{d-N+sp}}(I^{(3)}-I^{(1)}-I^{(2)})$.

We now point out that, as a consequence of Lemma 4.3, for every $0\le\tau<t$ it holds

$$\begin{aligned}
\|\mathbf{U}(t)\|_{r(\tau)} &= \|\mathbf{u}(t)-\mathbf{v}(t)\|_{r(\tau)} \le \|\mathbf{u}(\tau)-\mathbf{v}(\tau)\|_{r(\tau)}\\
&= \|\mathbf{U}(\tau)\|_{r(\tau)} = e^{y(\tau)} \le e^{x(\tau)}.
\end{aligned} \tag{4.35}$$

Since $y(0) = \log \|\mathbf{U}(0)\|_{r(0)} = \log \|\mathbf{u}_0 - \mathbf{v}_0\|_q$, from (4.34) and (4.35) we obtain

$$\begin{aligned}\lim_{\tau\to t^-} \|\mathbf{U}(t)\|_{r(\tau)} &\leq \lim_{\tau\to t^-} e^{x(\tau)} \\ &= \|\mathbf{u}_0 - \mathbf{v}_0\|_q^{\lambda_3(s)}\, \omega^{\lambda_1(s)}\, e^{C_2 t \|\mathbf{U}_0\|_2^{p-2}}\, t^{-\lambda_2(s)} \left(p^p \frac{d-N+sp}{d} \frac{\tilde{C}}{\bar{C}}\right)^{-\lambda_2(s)} \end{aligned} \tag{4.36}$$

where the constants $\lambda_1(s)$, $\lambda_2(s)$ and $\lambda_3(s)$ are as defined in (4.27).

Finally, we remark that

$$\lim_{\tau\to t^-} r(\tau) = +\infty;$$

hence, from the definition of ω, we have that, for a suitable constant C_1 depending on N, s, p, Ω, d and q,

$$\begin{aligned}\|\mathbf{U}(t)\|_\infty &= \|T_{p,s}(t)\mathbf{u}_0 - T_{p,s}(t)\mathbf{v}_0\|_\infty \\ &\leq C_1(\max\{|\Omega|, \mu(\partial\Omega)\})^{\lambda_1(s)} e^{C_2 t\|\mathbf{u}_0-\mathbf{v}_0\|_2^{p-2}} t^{-\lambda_2(s)} \|\mathbf{u}_0 - \mathbf{v}_0\|_q^{\lambda_3(s)},\end{aligned}$$

thus the thesis follows in the case $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{X}^\infty(\Omega, \partial\Omega)$. The proof in the case $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{X}^q(\Omega, \partial\Omega)$ is then achieved by a density argument as in the proof of [47, Theorem 3.2.7]. □

We remark that also in the linear case, i.e. $p = 2$, the semigroup $T_{2,s}(t)$ is ultracontractive. The proof follows by adapting the techniques of [25, Theorem 2.16].

5. The elliptic problem

In this section we investigate the elliptic Venttsel' problem, under the same assumptions and notations of the previous sections. In particular, we prove a priori estimates for its (unique) weak solution.

Let $(f, g) \in \mathbb{X}^{q,r}(\Omega, \partial\Omega)$. The elliptic Venttsel' problem is formally given by

$$(P_e) \begin{cases} (-\Delta_p)^s_\Omega u = f & \text{in } \Omega, \\ C_{p,s}\mathcal{N}_p^{p'(1-s)} u + b|u|_{\partial\Omega}|^{p-2} u|_{\partial\Omega} + p\Theta_{p,\gamma}(u|_{\partial\Omega}) = g & \text{on } \partial\Omega. \end{cases}$$

We observe that, from Theorems 1.7 and 1.8, the space $W^{s,p}(\Omega)$ is continuously embedded in $\mathbb{X}^{q,r}(\Omega, \partial\Omega)$ for every $q \in [1, p^*]$ and $r \in [1, \bar{p}]$; hence, there exists a positive constant C such that, for every $u \in W^{s,p}(\Omega)$,

$$\|\mathbf{u}\|_{q,r} \leq C\|u\|_{W^{s,p}(\Omega)}. \tag{5.1}$$

We first aim to prove the existence and uniqueness of a weak solution of the elliptic problem (P_e).

We say that $\mathbf{u} \in W^{s,p}(\Omega)$ is a weak solution of problem (P_e) if

$$
\begin{aligned}
&\frac{C_{N,p,s}}{2} \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} \, \mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) \\
&\quad + \int_{\partial\Omega} b|u|_{\partial\Omega}|^{p-2} u|_{\partial\Omega} \, v|_{\partial\Omega} \, \mathrm{d}\mu \\
&\quad + p\langle \Theta_{p,\gamma}(u|_{\partial\Omega}), v|_{\partial\Omega}\rangle = \int_\Omega f v \, \mathrm{d}\mathcal{L}_N + \int_{\partial\Omega} g v|_{\partial\Omega} \, \mathrm{d}\mu \qquad \text{for every } \mathbf{v} \in W^{s,p}(\Omega).
\end{aligned}
$$

Theorem 5.1. *If $(f, g) \in \mathbb{X}^{q,r}(\Omega, \partial\Omega)$ with $q \in [1, p^*]$ and $r \in [1, \bar{p}]$, problem (P_e) admits a unique weak solution $\mathbf{u} \in W^{s,p}(\Omega)$.*

Proof. We introduce the following form, for $\mathbf{u}, \mathbf{v} \in W^{s,p}(\Omega)$:

$$
\begin{aligned}
\Psi_{p,s}(\mathbf{u}, \mathbf{v}) := &\frac{C_{N,p,s}}{2} \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} \, \mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) \\
&+ \int_{\partial\Omega} b|u|_{\partial\Omega}|^{p-2} u|_{\partial\Omega} \, v|_{\partial\Omega} \, \mathrm{d}\mu + p\langle \Theta_{p,\gamma}(u|_{\partial\Omega}), v|_{\partial\Omega}\rangle.
\end{aligned} \tag{5.2}
$$

Firstly, we prove that $\Psi_{p,s}(\mathbf{u}, \cdot) \in (W^{s,p}(\Omega))'$ for every $\mathbf{u} \in W^{s,p}(\Omega)$. From Hölder and trace inequalities, the hypotheses on b and ζ and the definition of α in (1.3), for $\mathbf{u} \in W^{s,p}(\Omega)$ it holds that

$$
\begin{aligned}
|\Psi_{p,s}(\mathbf{u}, \mathbf{v})| &\leq C\|u\|^{p-1}_{W^{s,p}(\Omega)} \|v\|_{W^{s,p}(\Omega)} + (\max_{\overline{\Omega}} b) \|u|_{\partial\Omega}\|^{p-1}_{L^p(\partial\Omega)} \|v|_{\partial\Omega}\|_{L^p(\partial\Omega)} \\
&\quad + p\|\zeta\|_{L^\infty(\partial\Omega\times\partial\Omega)} \|u|_{\partial\Omega}\|^{p-1}_{B^{p,p}_\alpha(\partial\Omega)} \|v|_{\partial\Omega}\|_{B^{p,p}_\alpha(\partial\Omega)} \\
&\leq C \max\{1, \max_{\overline{\Omega}} b, p\|\zeta\|_{L^\infty(\partial\Omega\times\partial\Omega)}\} \|u\|^{p-1}_{W^{s,p}(\Omega)} \|v\|_{W^{s,p}(\Omega)}
\end{aligned}
$$

for a suitable constant $C > 0$ and for every $\mathbf{v} \in W^{s,p}(\Omega)$. Hence $\Psi_{p,s}(\mathbf{u}, \cdot) \in (W^{s,p}(\Omega))'$ for every $\mathbf{u} \in W^{s,p}(\Omega)$.

Next, we claim that $\Psi_{p,s}$ is hemicontinuous, strictly monotone, and coercive. The hemicontinuity follows from the continuity of the norm function in any Banach space, while the strict monotonicity follows from (4.5).

As to the coercivity, from Theorem 1.5 and the hypotheses on b and ζ we deduce that

$$
\|u\|^p_{W^{s,p}(\Omega)} \leq C\left(\frac{C_{N,p,s}}{2}|u|^p_{W^{s,p}(\Omega)} + \|u\|^p_{L^p(\partial\Omega)}\right) \leq C \max\left\{1, \frac{1}{b_0}\right\} \Psi_{p,s}(\mathbf{u}, \mathbf{u});
$$

this implies that

$$
\frac{\Psi_{p,s}(\mathbf{u}, \mathbf{u})}{\|u\|_{W^{s,p}(\Omega)}} \to +\infty \qquad \text{when} \qquad \|u\|_{W^{s,p}(\Omega)} \to +\infty,
$$

thus yielding the coercivity of $\Psi_{p,s}$.

The above claim implies that for every $\mathbf{u} \in W^{s,p}(\Omega)$ there exists an operator $\mathcal{S}(\mathbf{u}) \in (W^{s,p}(\Omega))'$ such that

$$
\Psi_{p,s}(\mathbf{u}, \mathbf{v}) = \langle \mathcal{S}(\mathbf{u}), \mathbf{v}\rangle_{s,p} \qquad \text{for every } \mathbf{v} \in W^{s,p}(\Omega), \tag{5.3}
$$

where $\langle \cdot, \cdot \rangle_{s,p}$ denotes the duality pairing between $(W^{s,p}(\Omega))'$ and $W^{s,p}(\Omega)$. Hence, from the properties of $\Psi_{p,s}$, equality (5.3) defines an hemicontinuous, bounded, strictly monotone and coercive operator $\mathcal{S} \colon W^{s,p}(\Omega) \to (W^{s,p}(\Omega))'$.

From [45, Corollary 2.2, page 39], $\mathcal{S}$ is surjective; hence, from Browder theorem (see e.g. [20, Theorem 5.3.22]), for every $\mathbf{w} \in (W^{s,p}(\Omega))'$, there exists a unique $\mathbf{u} \in W^{s,p}(\Omega)$ solution of

$$\langle \mathcal{S}(\mathbf{u}), \mathbf{v}\rangle_{s,p} = \langle \mathbf{w}, \mathbf{v}\rangle_{s,p} \qquad \text{for every } \mathbf{v} \in W^{s,p}(\Omega).$$

Now, for $q \in [1, p^*]$ and $r \in [1, \bar{p}]$, we take $(f, g) \in \mathbb{X}^{q,r}(\Omega, \partial\Omega)$ and we define the operator $L\colon W^{s,p}(\Omega) \to \mathbb{R}$ by

$$L(\mathbf{v}) := \int_\Omega f v \,\mathrm{d}\mathcal{L}_N + \int_{\partial\Omega} g v|_{\partial\Omega} \,\mathrm{d}\mu.$$

From (5.1), it follows that $L \in (W^{s,p}(\Omega))'$. Therefore, for every $\mathbf{v} \in W^{s,p}(\Omega)$ there exists a unique weak solution $\mathbf{u} \in W^{s,p}(\Omega)$ of $\langle \mathcal{S}(\mathbf{u}), v\rangle_{s,p} = L(\mathbf{v})$, hence the thesis follows. $\square$

We now recall a technical lemma, see [43].

Lemma 5.2. *Let $\varphi = \varphi(t)$ be a non-negative, non-increasing function on a half line $\{t \geq k_0 \geq 0\}$, such that there exist $c, \lambda > 0$ and $\tilde{\delta} > 1$ with*

$$\varphi(h) \leq c(h-k)^{-\lambda} \varphi(k)^{\tilde{\delta}},$$

for $h > k \geq k_0$. Then

$$\varphi(k_0 + \eta) = 0,$$

with

$$\eta^\lambda = c\varphi(k_0)^{\tilde{\delta}-1} 2^{\lambda\tilde{\delta}/(\tilde{\delta}-1)}.$$

The following result follows from [22, Lemma 5.2 a)].

Lemma 5.3. *Let $u \in W^{s,p}(\Omega)$ and $k \geq 0$ be a real number. We set $u_k := (|u| - k)^+ \mathrm{sgn}(u)$. Then for every $k \geq 0$ we have that $u_k \in W^{s,p}(\Omega)$ and*

$$\Psi_{p,s}(u_k, u_k) \leq \Psi_{p,s}(u, u_k). \tag{5.4}$$

We now prove the main result of this section, in which we prove a priori estimates for the weak solution of problem (P_e).

Theorem 5.4. *Let $q \in [1, p^*]$ and $r \in [1, \bar{p}]$.*

(a) *If $f \in (W^{s,p}(\Omega))'$ and $g \in (B^{p,p}_\alpha(\partial\Omega))'$, problem (P_e) admits a unique weak solution $\mathbf{u} \in W^{s,p}(\Omega)$. Moreover, there exists a positive constant C such that*

$$\|u\|^{p-1}_{W^{s,p}(\Omega)} \leq C \left(\|f\|_{(W^{s,p}(\Omega))'} + \|g\|_{(B^{p,p}_\alpha(\partial\Omega))'} \right). \tag{5.5}$$

(b) *If $(f, g) \in \mathbb{X}^{q,r}(\Omega, \partial\Omega)$, then the unique weak solution $\mathbf{u}$ of problem (P_e) belongs in particular to $\mathbb{X}^\infty(\Omega, \partial\Omega)$. Moreover, there exists a positive constant C such that*

$$\|\mathbf{u}\|^{p-1}_\infty \leq C \|(f, g)\|_{q,r}. \tag{5.6}$$

(c) *If* $(f,g)\in\mathbb{X}^{q,r}(\Omega,\partial\Omega)$ *and* $\mathbf{u},\mathbf{v}\in W^{s,p}(\Omega)$ *satisfy*

$$\Psi_{p,s}(\mathbf{u},\psi)-\Psi_{p,s}(\mathbf{v},\psi)=\int_\Omega f\psi\,\mathrm{d}\mathcal{L}_N+\int_{\partial\Omega} g\psi|_{\partial\Omega}\,\mathrm{d}\mu \tag{5.7}$$

for every $\psi\in W^{s,p}(\Omega)$, *then there exists a positive constant* $\tilde{C}=\tilde{C}(N,s,p,q,r,d,\Omega)$ *such that*

$$\|\mathbf{u}-\mathbf{v}\|_\infty^{p-1}\leq\tilde{C}\|(f,g)\|_{q,r}. \tag{5.8}$$

Proof. We begin by proving a). The existence and uniqueness with "irregular" data (f,g) can be achieved as in Theorem 5.1. As to (5.5), we take $v=u$ as test function in the weak formulation of problem (P_e); then, from Theorem 1.5, Hölder inequality and the trace theorem we get

$$\begin{aligned}\|u\|^p_{W^{s,p}(\Omega)}&\leq C\left(\frac{C_{N,p,s}}{2}\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y)\right.\\&\qquad\left.+\int_{\partial\Omega}|u|_{\partial\Omega}|^p\,\mathrm{d}\mu\right)\leq C\Psi_{p,s}(\mathbf{u},\mathbf{u})\\&=C\left(\langle f,u\rangle_{(W^{s,p}(\Omega))',W^{s,p}(\Omega)}+\langle g,u|_{\partial\Omega}\rangle_{(B^{p,p}_\alpha(\partial\Omega))',B^{p,p}_\alpha(\partial\Omega)}\right)\\&\leq C\left(\|f\|_{(W^{s,p}(\Omega))'}+\|g\|_{(B^{p,p}_\alpha(\partial\Omega))'}\right)\|u\|_{W^{s,p}(\Omega)},\end{aligned}$$

and this implies (5.5).

We now prove c); part b) can be proven in a similar way.

Let $\mathbf{u},\mathbf{v}\in W^{s,p}(\Omega)$ satisfy (5.7) and let $k\geq k_0\geq 0$ be real numbers. We set

$$w:=u-v,\quad w_k:=(|w|-k)^+\mathrm{sgn}(w),\quad \mathbf{w}_k:=(w_k,w_k|_{\partial\Omega}).$$

We remark that from Lemma 5.3 in particular it follows that $w_k\in W^{s,p}(\Omega)$, hence its trace $w_k|_{\partial\Omega}$ is well-defined. We also introduce the sets

$$A_k:=\{x\in\overline{\Omega}\,:\,|\mathbf{w}(x)|>k\},\quad B_k:=\overline{\Omega}\setminus A_k=\{x\in\overline{\Omega}\,:\,|\mathbf{w}(x)|\leq k\},$$

where

$$|\mathbf{w}(x)|=\begin{cases}|w(x)| & \text{if } x\in\Omega,\\ |w|_{\partial\Omega}(x)| & \text{if } x\in\partial\Omega.\end{cases}$$

We now take $\psi=w_k$ in (5.7) and from (4.6) and (5.4) we obtain

$$\Psi_{p,s}(\mathbf{u},\mathbf{w}_k)-\Psi_{p,s}(\mathbf{v},\mathbf{w}_k)\geq c_p^*\Psi_{p,s}(\mathbf{w},\mathbf{w}_k)\geq c_p^*\Psi_{p,s}(\mathbf{w}_k,\mathbf{w}_k).$$

Now, from the coercivity of $\Psi_{p,s}$ and (5.7) we have that there exists a positive constant $\tilde{C}_1$ such that

$$\tilde{C}_1\|w_k\|^p_{W^{s,p}(\Omega)}\leq\Psi_{p,s}(\mathbf{u},\mathbf{w}_k)-\Psi_{p,s}(\mathbf{v},\mathbf{w}_k)=\int_\Omega fw_k\,\mathrm{d}\mathcal{L}_N+\int_{\partial\Omega}gw_k|_{\partial\Omega}\,\mathrm{d}\mu. \tag{5.9}$$

Let now $q_1,r_1\in[1,\infty)$ be such that

$$\frac{1}{q_1}+\frac{1}{p^*}+\frac{1}{q}=1\quad\text{and}\quad\frac{1}{r_1}+\frac{1}{\bar{p}}+\frac{1}{r}=1.$$

We point out that from the bounds on q and r we have respectively that

$$q_1 < \frac{p^*}{p-1} \quad \text{and} \quad r_1 < \frac{\bar{p}}{p-1}. \tag{5.10}$$

From Hölder inequality, Theorems 1.7 and 1.8 we have that, for a suitable positive constant $\tilde{C}_2$,

$$\begin{aligned} \int_\Omega f w_k \, \mathrm{d}\mathcal{L}_N + \int_{\partial\Omega} g w_k|_{\partial\Omega} \, \mathrm{d}\mu &\leq \|f\|_{L^q(\Omega)} \|w_k\|_{L^{p^*}(\Omega)} \|\chi_{A_k}\|_{L^{q_1}(\Omega)} \\ &\quad + \|g\|_{L^r(\partial\Omega)} \|w_k|_{\partial\Omega}\|_{L^{\bar{p}}(\partial\Omega)} \|\chi_{A_k}\|_{L^{r_1}(\partial\Omega)} \\ &\leq \tilde{C}_2 \|(f,g)\|_{q,r} \|w_k\|_{W^{s,p}(\Omega)} \|\chi_{A_k}\|_{q_1,r_1}. \end{aligned} \tag{5.11}$$

Inequalities (5.9) and (5.11) together with Theorems 1.7 and 1.8 yield that, for a suitable constant $\tilde{C}_3 > 0$,

$$\|\mathbf{w}_k\|_{p^*,\bar{p}}^{p-1} \leq \tilde{C}_3 \|(f,g)\|_{q,r} \|\chi_{A_k}\|_{q_1,r_1}. \tag{5.12}$$

Let now $h > k$. Therefore, we point out that $A_h \subset A_k$ and for every $x \in A_h$ it holds that $|w_k(x)| \geq h-k$. Hence

$$\|\mathbf{w}_k\|_{p^*,\bar{p}}^{p-1} \geq \|(h-k)\chi_{A_h}\|_{p^*,\bar{p}}^{p-1},$$

and from (5.12) for every $h > k \geq 0$ we have

$$\|\chi_{A_h}\|_{p^*,\bar{p}}^{p-1} \leq \tilde{C}_3 (h-k)^{-(p-1)} \|(f,g)\|_{q,r} \|\chi_{A_k}\|_{q_1,r_1}. \tag{5.13}$$

We now set

$$\delta_0 := \min\left\{\frac{p^*}{q_1}, \frac{\bar{p}}{r_1}\right\} > p-1 \quad \text{and} \quad \tilde{\delta} := \frac{\delta_0}{p-1} > 1.$$

By using the definition of A_k and of the $\|\cdot\|_{q,r}$-norm, for every $k \geq 0$ we have that there exists a positive constant $\tilde{C}_4$ such that

$$\|\chi_{A_k}\|_{q_1,r_1} \leq \tilde{C}_4 \|\chi_{A_k}\|_{p^*,\bar{p}}^{\delta_0}.$$

Hence, there exists a positive constant $\tilde{C}_5$ such that

$$\|\chi_{A_h}\|_{p^*,\bar{p}}^{p-1} \leq \tilde{C}_5 (h-k)^{-(p-1)} \|(f,g)\|_{q,r} \left(\|\chi_{A_k}\|_{p^*,\bar{p}}^{p-1}\right)^{\tilde{\delta}}. \tag{5.14}$$

We now set

$$\varphi(h) := \|\chi_{A_h}\|_{p^*,\bar{p}}^{p-1}$$

for every $h \in [0,\infty)$. We apply Lemma 5.2 with $k_0 = 0$ and $\lambda = p-1$, thus obtaining that

$$\varphi(\eta) = 0 \quad \text{with} \quad \eta^{p-1} = \tilde{C}_5 \|(f,g)\|_{q,r} \varphi(0)^{\tilde{\delta}-1} 2^{(p-1)\tilde{\delta}/(\tilde{\delta}-1)} \tag{5.15}$$

Finally, we point out that (5.15) in particular implies that the set $\{x \in \overline{\Omega} : |\mathbf{w}(x)| > \tilde{C}\|(f,g)\|_{q,r}^{\frac{1}{p-1}}\}$, where $\tilde{C}$ can be computed explicitly, has zero measure. Hence, we have that

$$|\mathbf{w}(x)| = |\mathbf{u}(x) - \mathbf{v}(x)| \leq \tilde{C}\|(f,g)\|_{q,r}^{\frac{1}{p-1}} \qquad \text{a.e. on} \quad \overline{\Omega},$$

and this leads us to (5.8), thus concluding the proof. □

Remark 5.5. We point out that most of the results of this paper can be adapted to more general operators. In particular, one can replace the regional fractional p-Laplacian and the nonlocal term on the boundary with more general operators satisfying suitable growth hypotheses. This will be object of a forthcoming paper.

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References

[1] Abe, S., Thurner, S.: Anomalous diffusion in view of Einstein's 1905 theory of Brownian motion. Phys. A **356**, 403–407 (2005)

[2] Adams, D.R., Hedberg, L.I.: Function Spaces and Potential Theory. Springer, Berlin (1996)

[3] Barbu, V.: Nonlinear semigroups and differential equations in Banach spaces. Noordhoff International Publishing, Leiden (1976).. (**Translated from the Romanian**)

[4] Barbu, V., Precupanu, T.: Convexity and Optimization in Banach Spaces, 4th edn. Springer Monographs in Mathematics, Springer (2012)

[5] Biegert, M.: A priori estimate for the difference of solutions to quasi-linear elliptic equations. Manuscripta Math. **133**, 273–306 (2010)

[6] Bonforte, M., Grillo, G.: Ultracontractive bounds for nonlinear evolution equations governed by the subcritical p-Laplacian, Progr. Nonlinear Differential Equations Appl., 61, Trends in partial differential equations of mathematical physics, 15–26, Birkhäuser, Basel (2005)

[7] Brézis, H.: Propriétés régularisantes de certains semi-groupes non linéaires. Israel J. Math. **9**, 513–534 (1971)

[8] Brezzi, F., Gilardi, G.: Fundamentals of PDE for numerical analysis. In: Kardestuncer, H., Norrie, D.H. (eds.) Finite Element Handbook. McGraw-Hill Book Co., New York (1987)

[9] Browder, F.E.: Remarks on nonlinear interpolation in Banach spaces. J. Funct. Anal. **4**, 390–403 (1969)

[10] Cipriani, F., Grillo, G.: Uniform bounds for solutions to quasilinear parabolic equations. J. Differ. Equ. **177**, 209–234 (2001)

[11] Cipriani, F., Grillo, G.: Nonlinear Markov semigroups, nonlinear Dirichlet forms and applications to minimal surfaces. J. Reine. Angew. Math. **562**, 201–235 (2003)

[12] Claus, B., Warma, M.: Realization of the fractional Laplacian with nonlocal exterior conditions via forms method. J. Evol. Equ. (2020). https://doi.org/10.1007/s00028-020-00567-0

[13] Creo, S., Lancia, M.R., Nazarov, A.I.: Regularity results for nonlocal evolution Venttsel' problems. Fract. Calc. Appl. Anal. **23**(5), 1416–1430 (2020)

[14] Creo, S., Lancia, M.R., Vélez-Santiago, A., Vernole, P.: Approximation of a nonlinear fractal energy functional on varying Hilbert spaces. Commun. Pure Appl. Anal. **17**, 647–669 (2018)

[15] Creo, S., Lancia, M.R., Vernole, P.: Convergence of fractional diffusion processes in extension domains. J. Evol. Equ. **20**, 109–139 (2020)

[16] Creo, S., Lancia, M.R., Vernole, P.: M-convergence of p-fractional energies in irregular domains. J. Convex Anal. **28**, 2 (2021). (**in press**)

[17] Creo, S., Durante, V.R.: Convergence and density results for parabolic quasilinear Venttsel' problems in fractal domains. Discrete Contin. Dyn. Syst. Ser. S **12**, 65–90 (2019)

[18] Davies, E.B.: Heat Kernels and Spectral Theory. Cambridge University Press, Cambridge (1989)

[19] Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**, 521–573 (2012)

[20] Drábek, P., Milota, J.: Methods of nonlinear analysis. Birkhäuser Adv. Texts, Birkhäuser, Basel, Applications to Differential Equations (2007)

[21] Dubkov, A.A., Spagnolo, B., Uchaikin, V.V.: Lèvy flight superdiffusion: An introduction. Int. J. Bifurc. Chaos **18**, 2649–2672 (2008)

[22] Gal, C.G., Warma, M.: Bounded solutions for nonlocal boundary value problems on Lipschitz manifolds with boundary. Adv. Nonlinear Stud. **16**, 529–550 (2016)

[23] Gal, C.G., Warma, M.: Elliptic and parabolic equations with fractional diffusion and dynamic boundary conditions. Evol. Equat. Control Theory **5**, 61–103 (2016)

[24] Gal, C.G., Warma, M.: Reaction-diffusion equations with fractional diffusion on non-smooth domains with various boundary conditions. Discrete Contin. Dyn. Syst. **36**, 1279–1319 (2016)

[25] Gal, C.G., Warma, M.: Nonlocal transmission problems with fractional diffusion and boundary conditions on non-smooth interfaces. Commun. Partial Differ. Equ. **42**, 579–625 (2017)

[26] Gorenflo, R., Mainardi, F., Vivoli, A.: Continuous-time random walk and parametric subordination in fractional diffusion. Chaos, Solitons Fractals **34**, 87–103 (2007)

[27] Guan, Q.Y.: Integration by parts formula for regional fractional Laplacian. Commun. Math. Phys. **266**, 289–329 (2006)

[28] Guan, Q.Y., Ma, Z.M.: Boundary problems for fractional Laplacians. Stoch. Dyn. **5**, 385–424 (2005)

[29] Jara, M.: Nonequilibrium scaling limit for a tagged particle in the simple exclusion process with long jumps. Commun. Pure Appl. Math. **62**, 198–214 (2009)

[30] Jones, P.W.: Quasiconformal mapping and extendability of functions in Sobolev spaces. Acta Math. **147**, 71–88 (1981)

[31] Jonsson, A.: Besov spaces on closed subsets of $\mathbb{R}^n$. Trans. Am. Math. Soc. **341**, 355–370 (1994)

[32] Jonsson, A., Wallin, H.: Function spaces on subsets of R^n, Part 1, Math. Reports, vol. 2, Harwood Acad. Publ., London (1984)

[33] Jonsson, A., Wallin, H.: The dual of Besov spaces on fractals. Stud. Math. **112**, 285–300 (1995)

[34] Lancia, M.R., Vélez-Santiago, A., Vernole, P.: Quasi-linear Venttsel' problems with nonlocal boundary conditions on fractal domains. Nonlinear Anal. Real World Appl. **35**, 265–291 (2017)

[35] Lancia, M.R., Vélez-Santiago, A., Vernole, P.: A quasi-linear nonlocal Venttsel' problem of Ambrosetti-Prodi type on fractal domains. Discrete Contin. Dyn. Syst. **39**, 4487–4518 (2019)

[36] Lancia, M.R., Vernole, P.: Irregular heat flow problems. SIAM J. Math. Anal. **42**, 1539–1567 (2010)

[37] Lancia, M.R., Vernole, P.: Semilinear fractal problems: approximation and regularity results. Nonlinear Anal. **80**, 216–232 (2013)

[38] Lancia, M.R., Vernole, P.: Venttsel' problems in fractal domains. J. Evol. Equ. **14**, 681–712 (2014)

[39] Lancia, M.R., Vernole, P.: Semilinear Venttsel' problems in fractal domains. Appl. Math. **5**(12), 1820–1833 (2014)

[40] Maligranda, L.: Some remarks on Orlicz's interpolation theorem. Stud. Math. **95**, 43–58 (1989)

[41] Mandelbrot, B.B., Van Ness, J.W.: Fractional Brownian motions, fractional noises and applications. SIAM Rev. **10**, 422–437 (1969)

[42] Mellet, A., Mischler, S., Mouhot, C.: Fractional diffusion limit for collisional kinetic equations. Arch. Ration. Mech. Anal. **199**, 493–525 (2011)

[43] Murthy, M.R.V., Stampacchia, G.: Boundary value problems for some degenerate-elliptic operators. Ann. Mat. Pura Appl. **80**, 1–122 (1968)

[44] Schneider, W.R.: Grey noise. In: Albeverio, S., Casati, G., Cattaneo, U., Merlini, D., Moresi, R. (eds.) Stochastic Processes, pp. 676–681. Physics and Geometry, Teaneck, NJ, USA, World Scientific (1990)

[45] Showalter, R.E.: Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, Mathematical Surveys and Monographs 49. American Mathematical Society, Providence, RI (1997)

[46] Valdinoci, E.: From the long jump random walk to the fractional Laplacian. Bol. Soc. Esp. Mat. Apl. SeMA **49**, 33–44 (2009)

[47] Vélez-Santiago, A.: Quasi-linear variable exponent boundary value problems with Wentzell-Robin and Wentzell boundary conditions. J. Funct. Anal. **266**, 560–615 (2014)

[48] Vélez-Santiago, A., Warma, M.: A class of quasi-linear parabolic and elliptic equations with nonlocal Robin boundary conditions. J. Math. Anal. Appl. **372**, 120–139 (2010)

[49] Warma, M.: The p-Laplace operator with the nonlocal Robin boundary conditions on arbitrary open sets. Ann. Mat. Pura Appl. **193**, 771–800 (2014)

[50] Warma, M.: A fractional Dirichlet-to-Neumann operator on bounded Lipschitz domains. Commun. Pure Appl. Anal. **14**, 2043–2067 (2015)

[51] Warma, M.: The fractional Neumann and Robin type boundary conditions for the regional fractional p-Laplacian. NoDEA Nonlinear Differential Equations Appl., 23 (2016), Art. 1, 46 pp

[52] Warma, M.: On a fractional (s, p)-Dirichlet-to-Neumann operator on bounded Lipschitz domains. J. Elliptic Parabol. Equ. **4**, 223–269 (2018)

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