Supplemental Material for
Full spectrum of open dissipative quantum systems in the Zeno limit

Vladislav Popkov and Carlo Presilla

Lamb shift Hamiltonian and effective dissipator of Eq. (2)

Assume that the kernel of $\mathcal{D}$ is one-dimensional, i.e., its 0 eigenvalue, $\mathcal{D}[\psi_0] = 0$, is nondegenerate and $\mathcal{D}$ is diagonalizable, i.e., there exists a basis $\{\psi_k\}$ (not necessarily orthogonal) such that $\mathcal{D}[\psi_k] = c_k \psi_k$. Let $\{\varphi_k\}$ be a complementary basis, trace-orthonormal to the basis $\{\psi_k\}$, $\mathrm{tr}(\varphi_k \psi_j) = \delta_{k,j}$. In Ref. [18] it has been shown that

$$\tilde{H}_a = \sum_{m>0} \sum_{n>0} \beta_{m,n} g_m^\dagger g_n, \tag{S1}$$

$$\tilde{\mathcal{D}}[\cdot] = \sum_{m>0} \sum_{n>0} \gamma_{m,n} \left(g_n \cdot g_m^\dagger - \frac{1}{2} g_m^\dagger g_n \cdot - \frac{1}{2} \cdot g_m^\dagger g_n \right), \tag{S2}$$

where $g_k = \mathrm{tr}_{\mathcal{H}_0}((\psi_k \otimes I_{\mathcal{H}_1})H)$ are the operators in Eq. (5) and $\gamma_{m,n} = Y_{m,n} + Y_{n,m}^*$ and $\beta_{m,n} = (Y_{m,n} - Y_{n,m}^*)/(2i)$ with $Y_{m,n} = -\,\mathrm{tr}\left(\varphi_m^\dagger \varphi_n \psi_0\right)/c_m^*$ are the elements of two matrices which are, respectively, positive and Hermitian. Note that the dissipation-projected Hamiltonian of Eq. (2) is $h_D = g_0$.

For the dissipator $\mathcal{D} = ((1+\mu)/2)\mathcal{D}_1 + ((1-\mu)/2)\mathcal{D}_2$ with $\mathcal{D}_1$ and $\mathcal{D}_2$ given by Eq. (19), we have $\beta_{m,n} = 0$ and $\gamma_{m,n} = \gamma_m \delta_{m,n}$, where $\gamma_1 = (1+\mu)/2$, $\gamma_2 = (1-\mu)/2$ and $\gamma_3 = (1-\mu^2)/4$.

Proof of Statement: nondegenerate eigenvalues

We start introducing the spectral projection P_k according to

$$P_k X = \psi_k \otimes X_k, \qquad X_k = \mathrm{tr}_{\mathcal{H}_0}\left((\varphi_k \otimes I_{\mathcal{H}_1})X\right). \tag{S3}$$

We have $P_k P_m = \delta_{k,m} P_m$ and $P_k \rho(\tau) = \psi_k \otimes R_k(\tau)$. From Eq. (1), scaling the time by $\tau = t/\Gamma$, we find $d\rho(t)/dt = \mathcal{L}_0[\rho(t)] + K[\rho(t)]$ with $\mathcal{L}_0[\cdot] = \mathcal{D}[\cdot]$ and $K[\cdot] = -(i/\Gamma)[H, \cdot]$. If we now apply the Liouvillian propagator $\epsilon_t = \exp \mathcal{L}t$ on P_k with $k > 0$, we can use the Dyson expansion with respect to the small perturbation K and obtain

$$\begin{aligned}\epsilon_t P_k = {} & e^{c_k t}\left(P_k + P_k K P_k t\right) + \frac{1}{c_k} P_0 K P_k (e^{c_k t} - 1) \\ & + \sum_{m>0,\ m\neq k} \frac{e^{c_m t}}{c_k - c_m} P_m K P_k (e^{(c_k - c_m)t} - 1) + O(K^2).\end{aligned} \tag{S4}$$

The term $P_0 K P_k$ describes the flow towards the dissipation-free subspace; as expected, its norm is of order $1/\Gamma$ due to presence of K. The term in (S4) containing $P_m K P_k$ describes the intra-sector flow $\psi_k \otimes R_k(0) \to \psi_m \otimes R_m(t)$, and is at most of order $1/\Gamma$ at any time. Finally, the inter-sector flow $\psi_k \otimes R_k(0) \to \psi_k \otimes R_k(t)$ is given by the first two terms, namely,

$$P_k \epsilon_t P_k = e^{c_k t}\left(P_k + P_k K P_k t\right) + O(K^2). \tag{S5}$$

The evolution $R_k(0) \to R_k(t)$ resulting from Eq. (S5) can be cast in differential form by using $dR_k(t)/dt = \lim_{t\to 0}(R_k(t) - R_k(0))/t$. Applying $P_k \epsilon_t P_k$ on $\rho(t)$ we find $\psi_k \otimes dR_k(t)/dt = c_k \psi_k \otimes R_k + P_k K P_k \rho(t)$. Scaling back the time by $t = \Gamma\tau$, after some algebra we get

$$\begin{aligned}\frac{dR_k(\tau)}{d\tau} = {} & \Gamma c_k R_k(\tau) + i[g_0, R_k(\tau)] \\ & + i \sum_{n>0} \left(\mathrm{tr}(\varphi_k \varphi_n^\dagger \psi_k) g_n^\dagger R_k(\tau) - R_k(\tau)\, \mathrm{tr}(\varphi_k \psi_n \varphi_k^\dagger) g_n^\dagger \right) + O(1/\Gamma),\end{aligned} \tag{S6}$$

which, by virtue of Eq. (16), is Eq. (8) up to terms $O(1)$. The $O(1/\Gamma)$ corrections can be obtained by accounting for the next, second order term of the Dyson expansion, see later.

Proof of Statement: degenerate eigenvalues

Suppose that there exists a degenerate dissipator eigenvalue with degeneracy $\deg$, let's say, $c_k = c_{k+1} = \cdots = c_{k+\deg-1}$. Equation (S4) is not applicable directly, since there would be a pole singularity in the terms $1/(c_k - c_m)$. In order to eliminate this singularity, we group together the respective spectral projections P_k, defining $\mathbf{P} = P_k + P_{k+1} + \cdots + P_{k+\deg-1}$. One can check that Eq. (S4) with the substitution $(P_k, P_{k+1}, \dots, P_{k+\deg-1}) \to \mathbf{P}$ remains valid provided the sum over m has the constraint $m \neq k, k+1, \dots, k+\deg-1$, and we obtain $\mathbf{P}\epsilon_t\mathbf{P} = e^{c_k t}(\mathbf{P} + \mathbf{P}K\mathbf{P}t) + O(K^2)$. For the equation of motion of the components $R_k(\tau), R_{k+1}(\tau), \dots, R_{k+\deg-1}(\tau)$, we get

$$\begin{aligned}\frac{dR_k(\tau)}{d\tau} &= \Gamma c_k R_k(\tau) + i[g_0, R_k(\tau)] \\ &\quad + i\sum_{n>0}\sum_{s:c_s=c_k}\left(\mathrm{tr}(\varphi_k\varphi_n^\dagger\psi_s)g_n^\dagger R_s(\tau) - R_s(\tau)\,\mathrm{tr}(\varphi_k\psi_n\varphi_s^\dagger)g_n^\dagger\right) + O(1/\Gamma),\end{aligned} \tag{S7}$$

which, by virtue of Eq. (9), is Eq. (8) up to terms $O(1)$.

Proof of statement: Dyson expansion at second order

To obtain the $O(1/\Gamma)$ terms in the equation of motion for $R_k(t)$ we need to include in the Dyson expansion the terms of order 2 in the perturbation K. The $O(K^2)$ term for the evolution projected onto the R_k subspace is given by the operator $P_k\exp(\mathcal{L}t) = P_k\epsilon_t$. Recalling that $P_k\rho = \psi_k \otimes R_k$, we have

$$\psi_k \otimes R_k(t) = P_k\rho(t) = P_k\epsilon_t\rho(0) = \sum_j P_k\epsilon_t P_j\rho(0). \tag{S8}$$

In differential form we have $dR_k(\tau)/d\tau = \Gamma dR_k(t)/dt$, i.e.,

$$\psi_k \otimes \frac{dR_k(\tau)}{d\tau} = \Gamma\lim_{t\to 0}\frac{\sum_j P_k\epsilon_t P_j\rho(0) - P_k\rho(0)}{t}. \tag{S9}$$

It turns out that the $O(1/\Gamma)$ contribution to the equation of motion (S9) for $R_k(t)$ are given only by the terms $P_k\epsilon_t P_s\rho(0)$, with $c_s = c_k$, while the terms $P_k\epsilon_t P_n\rho(0)$ with $c_n \neq c_k$ give no $O(1/\Gamma)$ contribution. The Dyson expansion for $P_k\epsilon_t P_s$ with $c_s = c_k$ yields

$$P_k\epsilon_t P_s = \delta_{s,k}P_k + O(K) + te^{c_k t}\sum_{n:c_n\neq c_k}\frac{1}{c_k - c_n}P_k K P_n K P_s, \tag{S10}$$

where the $O(K)$ terms are those calculated before. At the leading order in time, $te^{c_k t} = t + O(t^2)$. In differential form, the respective terms for $R_k(\tau)$ are given by

$$\psi_k \otimes \frac{dR_k(\tau)}{d\tau} = O(1) + \Gamma\sum_{n:c_n\neq c_k}\sum_{s:c_s=c_k}\frac{1}{c_k - c_n}P_k K P_n K P_s\rho(\tau). \tag{S11}$$

Using the following formulas

$$\rho = \sum_k \psi_k \otimes R_k, \tag{S12}$$

$$\mathrm{tr}(\varphi_k\psi_n) = \delta_{k,n}, \tag{S13}$$

$$P_k A = \psi_k \otimes \mathrm{tr}(A\varphi_k), \qquad P_k\rho = \psi_k \otimes R_k, \tag{S14}$$

$$H = \sum_m \varphi_m \otimes g_m = \sum_m \varphi_m^\dagger \otimes g_m^\dagger, \tag{S15}$$

$$KA = -\frac{i}{\Gamma}[H, A], \tag{S16}$$

we calculate the term $P_k K P_n K P_s\rho$, step by step, as follows (summation over repeated indices m is implied)

$$P_n K P_s\rho = -\frac{i}{\Gamma}P_n[H, \psi_s \otimes R_s]$$

$$
\begin{aligned}
&= -\frac{i}{\Gamma} P_n (H(\psi_s \otimes R_s) - (\psi_s \otimes R_s) H) \\
&= -\frac{i}{\Gamma} P_n \left((\varphi_m \otimes g_m)(\psi_s \otimes R_s) - (\psi_s \otimes R_s)(\varphi_m^\dagger \otimes g_m^\dagger) \right) \\
&= -\frac{i}{\Gamma} P_n \left((\varphi_m \psi_s \otimes g_m R_s) - (\psi_s \varphi_m^\dagger \otimes R_s g_m^\dagger) \right) \\
&= -\frac{i}{\Gamma} \psi_n \otimes \left(\mathrm{tr}(\varphi_n \varphi_m \psi_s) g_m R_s - \mathrm{tr}(\varphi_n \psi_s \varphi_m^\dagger) R_s g_m^\dagger \right) \\
&= -\frac{i}{\Gamma} \psi_n \otimes \left(C_{m,s,n} g_m R_s - A_{m,s,n} R_s g_m^\dagger \right),
\end{aligned}
$$

and then (now, summation over repeated indices m and z is implied)

$$
\begin{aligned}
&P_k K(P_n K P_s \rho) \\
&= -\frac{1}{\Gamma^2} P_k \left(C_{m,s,n} [H, \psi_n \otimes g_m R_s] - A_{m,s,n} [H, \psi_n \otimes R_s g_m^\dagger] \right) \\
&= -\frac{1}{\Gamma^2} P_k (C_{m,s,n} [\varphi_z^\dagger \otimes g_z^\dagger), (\psi_n \otimes g_m R_s] - A_{m,s,n} [\varphi_z \otimes g_z), (\psi_n \otimes R_s g_m^\dagger]) \\
&= -\frac{1}{\Gamma^2} P_k (C_{m,s,n} (\varphi_z^\dagger \psi_n \otimes g_z^\dagger g_m R_s - \psi_n \varphi_z^\dagger \otimes g_m R_s g_z^\dagger) - A_{m,s,n} (\varphi_z \psi_n \otimes g_z R_s g_m^\dagger - \psi_n \varphi_z \otimes R_s g_m^\dagger g_z) \\
&= -\frac{1}{\Gamma^2} \psi_k \otimes (C_{m,s,n} B_{z,n,k} g_z^\dagger g_m R_s - C_{m,s,n} A_{z,n,k} g_m R_s g_z^\dagger - A_{m,s,n} C_{z,n,k} g_z R_s g_m^\dagger + A_{m,s,n} C_{k,n,z} R_s g_m^\dagger g_z) \\
&= -\frac{1}{\Gamma^2} \psi_k \otimes (-(C_{m,s,n} A_{z,n,k} + A_{z,s,n} C_{m,n,k}) g_m R_s g_z^\dagger + C_{m,s,n} B_{z,n,k} g_z^\dagger g_m R_s + A_{z,s,n} C_{k,n,m} R_s g_z^\dagger g_m).
\end{aligned}
$$

In passing from the second-last line to the last one, we exchanged the summation indices $m \leftrightarrow z$ in half of the terms. Finally, denoting

$$\gamma_{m,z}^{n,s,k} = C_{m,s,n} A_{z,n,k} + A_{z,s,n} C_{m,n,k}, \tag{S17}$$

$$\epsilon_{z,m}^{n,s,k} = C_{m,s,n} B_{z,n,k}, \tag{S18}$$

$$\delta_{z,m}^{n,s,k} = A_{z,s,n} C_{k,n,m}, \tag{S19}$$

and multiplying by Γ, we obtain the $O(1/\Gamma)$ terms of Eq. (8).

Equivalence of two open spin chains with flipped boundary fields

Suppose that we have two operators $f_\pm$ of the form

$$f_\pm = \sum_{j=1}^{N-1} \sum_{\alpha=x,y,z} J_\alpha \sigma_j^\alpha \sigma_{j+1}^\alpha \pm \sum_{\alpha=x,y,z} n_\alpha \sigma_1^\alpha, \tag{S20}$$

where J_α, n_α are some constants. Let us choose a representation in which the boundary term becomes diagonal, by an appropriate rotation of the basis, $\sum_{\alpha=x,y,z} n_\alpha \sigma_1^\alpha = A\tilde{\sigma}_1^z$. Under this transformation the operators $f_\pm$ take the form

$$f_\pm = \sum_{j=1}^{N-1} \sum_{\alpha,\beta=x,y,z} K_{\alpha\beta} \tilde{\sigma}_j^\alpha \tilde{\sigma}_{j+1}^\beta \pm A\tilde{\sigma}_1^z, \tag{S21}$$

where $K_{\alpha\beta}$ and A are constants. Then, the unitary operator

$$U = \bigotimes_{j=1}^{N} \tilde{\sigma}_j^x, \quad U^2 = I, \tag{S22}$$

transforms f_+ into f_- and vice versa,

$$f_\pm = U f_\mp U, \tag{S23}$$

which follows from $\tilde{\sigma}_j^x \tilde{\sigma}_j^z \tilde{\sigma}_j^{x,y} = -\tilde{\sigma}_j^z$ and $\tilde{\sigma}_j^x \tilde{\sigma}_j^x \tilde{\sigma}_j^x = \tilde{\sigma}_j^x$.

The XYZ spin chain: spectrum associated to the dissipator eigenvalue $c_0 = 0$.

This is the stripe closest to the origin in Fig. 1. The equation for R_0 was obtained in [18]. It has the Lindblad form (2) with $\tilde{H}_a = 0$,

$$h_D = \sum_{j=1}^{N-1} \vec{\sigma}_j \cdot (\hat{J}\vec{\sigma}_{j+1}) + \mu(\hat{J}\vec{n}_0) \cdot \vec{\sigma}_1, \tag{S24}$$

where $\hat{J} = \mathrm{diag}(J_x, J_y, J_z)$, and effective dissipator $\tilde{\mathcal{D}}[R_0] = \sum_{p=1}^{3}(\tilde{L}_p R_0 \tilde{L}_p^\dagger - \frac{1}{2}\tilde{L}_p^\dagger \tilde{L}_p R_0 - \frac{1}{2} R_0 \tilde{L}_p^\dagger \tilde{L}_p)$ with

$$\begin{aligned}
\tilde{L}_1 &= \sqrt{2(1+\mu)}\left(\hat{J}(\vec{n}_0' - i\vec{n}_0)\right) \cdot \vec{\sigma}_1, \\
\tilde{L}_2 &= \tilde{L}_1^\dagger \sqrt{(1-\mu)}/\sqrt{(1+\mu)}, \\
\tilde{L}_3 &= \sqrt{(1-\mu^2)/2}\left(\hat{J}\vec{n}_0\right) \cdot \vec{\sigma}_1,
\end{aligned}$$

where $\vec{n}_0 = \vec{n}(\theta, \varphi) \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, and $\vec{n}_0' = \vec{n}(\frac{\pi}{2} - \theta, \varphi + \pi)$, $\vec{n}_0 = \vec{n}(\frac{\pi}{2}, \varphi + \frac{\pi}{2})$.

Neglecting $O(1/\Gamma)$ corrections, eigencomponents of the matrix R_0 have form $|\psi_0\rangle \otimes |\alpha\rangle\langle\beta|$, with respective eigenvalues $\lambda_{0,\alpha,\beta} = i(\epsilon_\beta - \epsilon_\alpha) + O(1/\Gamma)$, where $h_D|\alpha\rangle = \epsilon_\alpha|\alpha\rangle$. Note that the eigenvalues ϵ_α are real because h_D is Hermitian. Including the $O(1/\Gamma)$ corrections, the eigenvalues $\lambda_{0,\alpha,\beta}$ are given by the perturbative formula

$$\lambda_{0,\alpha,\beta} = i(\epsilon_\beta - \epsilon_\alpha) + \frac{1}{\Gamma}\sum_{p=1}^{3}\left(\langle\alpha|\tilde{L}_p|\alpha\rangle\langle\beta|\tilde{L}_p^\dagger|\beta\rangle - \frac{1}{2}\langle\alpha|\tilde{L}_p^\dagger\tilde{L}_p|\alpha\rangle - \frac{1}{2}\langle\beta|\tilde{L}_p^\dagger\tilde{L}_p|\beta\rangle\right). \tag{S25}$$

The above $O(1/\Gamma)$ corrections are valid only for eigenvalues nondegenerate at the zeroth order, i.e., for $\alpha \neq \beta$. For degenerate eigenvalues $\lambda_{0,\alpha,\alpha}$, to resolve the degeneracy we write down equations for $\nu_\alpha(\tau) = \langle\alpha|R_0(\tau)|\alpha\rangle$ using Eq. (2). We obtain (see also [25]) a classical Markov process $d\nu_\alpha(\tau)/d\tau = \Gamma^{-1}\sum_\beta M_{\alpha\beta}\,\nu_\beta(\tau)$, where M is the stochastic matrix with elements $M_{\alpha\beta} = \sum_p |\langle\alpha|\tilde{L}_p|\beta\rangle|^2$, for $\alpha \neq \beta$, and $M_{\alpha\alpha} = -\sum_{\beta\neq\alpha} M_{\beta\alpha}$. The eigenvalues of M, namely, $M|P_\alpha\rangle = \mu_\alpha|P_\alpha\rangle$, determine the $O(1/\Gamma)$ corrections to the 2^{N-1} degenerate eigenvalues $\lambda_{0,\alpha,\alpha}$

$$\lambda_{0,\alpha,\alpha} = \frac{1}{\Gamma}\mu_\alpha + O(1/\Gamma^2). \tag{S26}$$

According to the Perron-Frobenius theorem, all eigenvalues μ_α have a strictly negative real part, except for $\alpha = 0$ which is $\mu_0 = 0$. This zero eigenvalue corresponds to an eigenvector $|P_0\rangle$ with real nonnegative entries ν_α. In the original quantum problem, the ν_α have the meaning of eigenvalues of the reduced density matrix in the Zeno limit [18]. We remark that the $O(1/\Gamma)$ corrections in Eqs. (S25) and (S26) have strictly negative real part and, in addition, all μ_α from Eq. (S26) are real, which is a highly nontrivial property.

In the top right panel of Fig. S1, we compare the Liouvillian eigenvalues of this stripe evaluated numerically with those obtained by the above perturbative formulas. As expected according to Fig. 2, for the chosen value $\Gamma = 8000$ we have an excellent agreement between the two sets of data.

The XYZ spin chain: corrections $O(1/\Gamma)$ for the spectrum associated to the nondegenerate dissipator eigenvalue $c_3 = -1$

First of all, we note that for the XYZ spin chain with dissipation at site 0 the operators $g_k = \mathrm{tr}_{\mathcal{H}_0}((\psi_k \otimes I_{\mathcal{H}_1})H)$ are given by [**?**], Eqs. (41) and (42),

$$\begin{aligned}
g_1 &= \left(\hat{J}(\vec{n}_0' - i\vec{n}_0)\right) \cdot \vec{\sigma}_1, \\
g_2 &= g_1^\dagger, \\
g_3 &= \left(\hat{J}\vec{n}_0\right) \cdot \vec{\sigma}_1.
\end{aligned}$$

The $O(1/\Gamma)$ corrections $\delta\lambda_{3,\alpha,\beta}$ to the Liouvillian eigenvalues $\lambda_{3,\alpha,\beta} = c_3\Gamma + i(\epsilon_\alpha - \epsilon_\beta)$ are obtained from the second order of Dyson expansion and correspond to the terms $O(1/\Gamma)$ of Eq. (8). By explicitly calculating the coefficients γ_{mz}^{nsk}, ϵ_{zm}^{nsk} and δ_{zm}^{nsk} with $s = k = 3$ and $n = 0, 1, 2$, we find

$$\frac{dR_3}{d\tau} = \Gamma c_3 R_3 + i(U_3 R_3 - R_3 W_3) + \frac{1}{\Gamma}\left((1+\mu)E_{2,1}[R_3] + (1-\mu)E_{1,2}[R_3] - \frac{1-\mu^2}{2}\mathcal{D}_{g_3}[R_3]\right), \tag{S27}$$

where

$$U_3 = V_3 = g_0 - \mu\, g_3^\dagger = \sum_{k=1}^{N-1} h_{k,k+1} - \mu(J\vec{n}_0)\cdot\vec{\sigma}_1, \tag{S28}$$

$$E_{n,m}[X] = g_n^\dagger g_n X + X g_n^\dagger g_n + 2 g_m X g_m^\dagger, \tag{S29}$$

$$\mathcal{D}_g[X] = g X g^\dagger - \frac{1}{2} g^\dagger g X - \frac{1}{2} X g^\dagger g. \tag{S30}$$

For Γ large, the last term in Eq. (S27) can be treated as a perturbation $V_3[R_3]$ of order $1/\Gamma$. The $O(1/\Gamma)$ corrections to the Liouvillian eigenvalues are then obtained via the standard perturbative formula $\delta\lambda_{3,\alpha,\beta} = \langle\alpha\beta|\hat{V}_3|\alpha\beta\rangle$, where $\hat{V}_3$ is the vectorized superoperator acting on the vectorized reduced density matrix $|R_3\rangle = |\alpha,\beta\rangle = |\alpha\rangle\otimes|\beta\rangle^*$ defined by $\hat{V}_3|R_3\rangle = V_3[R_3]$. We recall that $|\alpha\rangle$ and $|\beta\rangle$ are the eigenvectors of $V_3 = V_3$,

$$U_3|\alpha\rangle = \epsilon_\alpha|\alpha\rangle. \tag{S31}$$

Note that U_3 is Hermitian and its eigenvalues ϵ_α are real.

To explicitly illustrate the evaluation of $\delta\lambda_{3,\alpha,\beta}$, let's start considering the simplest case $\mu = 1$. By making the substitution $R_3(\tau) = e^{c_3\Gamma\tau} r_3(\tau)$, we obtain

$$\begin{aligned} \frac{dr_3}{d\tau} &= i(U_3 r_3 - r_3 U_3) + \frac{2}{\Gamma}\left(g_2^\dagger g_2 r_3 + r_3 g_2^\dagger g_2 + 2 g_1 r_3 g_1^\dagger\right) \\ &= \mathcal{L}_3^{(0)}[r_3] + V_3[r_3]. \end{aligned} \tag{S32}$$

In the Zeno limit $\Gamma\to\infty$, Eq. (S32) for r_3 is linearized in terms of modes $|\alpha\rangle\langle\beta|$. In fact, U_3 can be obtained from h_D by flipping the boundary term, therefore h_D and U_3 are equivalent and have the same set of eigenvalues ϵ_α. It follows that, in an equivalent representation, the solution of the eigenvalue problem for the Liouvillian $\mathcal{L}_3^{(0)}[\cdot]$, namely, $\mathcal{L}_3^{(0)}[\psi_j] = \Lambda_j\psi_j$, is given by $\psi_j = |\alpha\rangle\langle\beta|$ and $\Lambda_j = i(\epsilon_\alpha - \epsilon_\beta)$.

The expectation of an arbitrary superoperator of the form $V[r_3] = Q r_3 W$ on the state $\psi_j = |\alpha\rangle\langle\beta|$ can be calculated in a vectorized form as

$$\langle\psi_j|\hat{V}|\psi_j\rangle = \langle\alpha|\otimes\langle\beta|^*(Q\otimes W^t)|\alpha\rangle\otimes|\beta\rangle^* = \langle\alpha|Q|\alpha\rangle\langle\beta|^* W^t|\beta\rangle^* = \langle\alpha|Q|\alpha\rangle\langle\beta|W|\beta\rangle. \tag{S33}$$

It follows that, accounting for the corrections $O(1/\Gamma)$, for $\mu = 1$ we obtain

$$\begin{aligned} \lambda_{3,\alpha,\beta} &= c_3\Gamma + i(\epsilon_\alpha - \epsilon_\beta) + \langle\psi_j|\hat{V}_3|\psi_j\rangle \\ &= -\Gamma + i(\epsilon_\alpha - \epsilon_\beta) + \frac{2}{\Gamma}\left(\langle\alpha|g_2^\dagger g_2|\alpha\rangle + \langle\beta|g_2^\dagger g_2|\beta\rangle + 2\langle\alpha|g_1|\alpha\rangle\langle\beta|g_1^\dagger|\beta\rangle\right). \end{aligned} \tag{S34}$$

This result is immediately generalised to arbitrary μ

$$\begin{aligned} \lambda_{3,\alpha,\beta} = &-\Gamma + i(\epsilon_\alpha - \epsilon_\beta) \\ &+ \frac{1}{\Gamma}\Big((1+\mu)\left(\langle\alpha|g_2^\dagger g_2|\alpha\rangle + \langle\beta|g_2^\dagger g_2|\beta\rangle + 2\langle\alpha|g_1|\alpha\rangle\langle\beta|g_1^\dagger|\beta\rangle\right) \\ &\quad + (1-\mu)\left(\langle\alpha|g_1^\dagger g_1|\alpha\rangle + \langle\beta|g_1^\dagger g_1|\beta\rangle + 2\langle\alpha|g_2|\alpha\rangle\langle\beta|g_2^\dagger|\beta\rangle\right) \\ &\quad + \frac{1-\mu^2}{4}\left(\langle\alpha|g_3^\dagger g_3|\alpha\rangle + \langle\beta|g_3^\dagger g_3|\beta\rangle - 2\langle\alpha|g_3|\alpha\rangle\langle\beta|g_3^\dagger|\beta\rangle\right)\Big). \end{aligned} \tag{S35}$$

The above perturbative formula can be applied only if the unperturbed eigenvalue is nondegenerate. For $O(1)$ degenerate eigenvalues, $\lambda_{3,\alpha,\alpha} = -\Gamma$, the $O(1/\Gamma)$ corrections must be found in a different way. In the Zeno limit, we have a stationary solution $r_3(\infty) = \sum_\alpha \nu_\alpha|\alpha\rangle\langle\alpha|$. Taking into account the $O(1/\Gamma)$ terms, we can assume the finite-time $r_3(\tau)$ to have the same form but with coefficients ν_α which depend on time, $r_3(\tau) = \sum_\alpha \nu_\alpha(\tau)|\alpha\rangle\langle\alpha|$. Then, from Eq. (S32) we have, for $\mu = 1$,

$$\frac{d\nu_\alpha}{d\tau} = \frac{2}{\Gamma}\sum_\beta T_{\alpha,\beta}\nu_\beta, \tag{S36}$$

where

$$T_{\alpha,\beta} = 2 w_{1,\alpha,\beta}, \quad \beta\neq\alpha, \qquad T_{\alpha,\alpha} = 2\sum_\beta w_{2,\beta,\alpha} + 2 w_{1,\alpha,\alpha}, \tag{S37}$$

with

$$w_{n,\alpha,\beta} = |\langle\alpha|g_n|\beta\rangle|^2 . \tag{S38}$$

For arbitrary values of μ we have, instead,

$$T_{\alpha,\beta} = w_{\alpha,\beta}(\mu), \quad \beta \neq \alpha, \qquad T_{\alpha,\alpha} = \sum_{\beta} f_{\beta,\alpha}(\mu) + w_{\alpha,\alpha}(\mu), \tag{S39}$$

with

$$w_{\alpha,\beta}(\mu) = (1+\mu)w_{1,\alpha,\beta} + (1-\mu)w_{2,\alpha,\beta} - \frac{1-\mu^2}{4}w_{3,\alpha,\beta}, \tag{S40}$$

$$f_{\beta,\alpha}(\mu) = (1+\mu)w_{2,\beta,\alpha} + (1-\mu)w_{1,\beta,\alpha} + \frac{1-\mu^2}{4}w_{3,\beta,\alpha}. \tag{S41}$$

By finding the eigenvalues μ_α of the matrix T, we resolve the degeneracy problem. In fact, in terms of the corresponding eigenvectors $\tilde{\nu}_\alpha$ of T, we have

$$\frac{d\tilde{\nu}_\alpha}{d\tau} = \frac{2}{\Gamma}\mu_\alpha\tilde{\nu}_\alpha, \tag{S42}$$

the set of the values $(2/\Gamma)\mu_\alpha$ being the $1/\Gamma$ correction to the set of the degenerate eigenvalues $\lambda_{3,\alpha,\alpha}$,

$$\lambda_{3,\alpha,\alpha} = -\Gamma + \frac{2}{\Gamma}\mu_\alpha, \qquad \alpha = 1, 2, \dots, 2^N. \tag{S43}$$

Numerically, for the integrable XYZ model, we find the matrix T to be equivalent to a symmetric real matrix, so that all its eigenvalues μ_α are real. Since $c_3 = -1$ is real too, the eigenvalues (S43) lie on the real axis.

The XYZ spin chain: spectrum associated to the degenerate dissipator eigenvalue $c_1 = c_2 = -1/2$

Equation (8) for $k = 1, 2$ has the form

$$\begin{aligned}\frac{dR_k}{d\tau} &= \Gamma c_1 R_k + i\sum_{s=1}^{2}(U_{k,s}R_s - R_s W_{k,s}) \\ &\quad + \frac{1}{\Gamma}\sum_{z>0}\sum_{m>0}\sum_{n=0,3}\sum_{s=1}^{2}\frac{1}{c_n - c_1}\left(-\gamma^{n,s,k}_{m,z}g_m R_s g_z^\dagger + \epsilon^{n,s,k}_{z,m}g_z^\dagger g_m R_s + \delta^{n,s,k}_{z,m}R_s g_z^\dagger g_m\right) \\ &= \Gamma c_1 R_k + i\sum_{s=1}^{2}(U_{k,s}R_s - R_s W_{k,s}) \\ &\quad + \frac{2}{\Gamma}\sum_{z>0}\sum_{m>0}\sum_{s=1}^{2}\left(-\tilde{\gamma}^{s,k}_{m,z}g_m R_s g_z^\dagger + \tilde{\epsilon}^{s,k}_{z,m}g_z^\dagger g_m R_s + \tilde{\delta}^{s,k}_{z,m}R_s g_z^\dagger g_m\right),\end{aligned} \tag{S44}$$

where

$$\tilde{\gamma}^{s,k}_{m,z} = \gamma^{0,s,k}_{m,z} - \gamma^{3,s,k}_{m,z}, \tag{S45}$$

$$\tilde{\epsilon}^{s,k}_{m,z} = \epsilon^{0,s,k}_{m,z} - \epsilon^{3,s,k}_{m,z}, \tag{S46}$$

$$\tilde{\delta}^{s,k}_{m,z} = \delta^{0,s,k}_{m,z} - \delta^{3,s,k}_{m,z}, \tag{S47}$$

with

$$\gamma^{n,s,k}_{m,z} = C_{m,s,n}A_{z,n,k} + A_{z,s,n}C_{m,n,k}, \tag{S48}$$

$$\epsilon^{n,s,k}_{z,m} = C_{m,s,n}B_{z,n,k}, \tag{S49}$$

$$\delta^{n,s,k}_{z,m} = A_{z,s,n}C_{k,n,m}. \tag{S50}$$

The only nonzero coefficients $\tilde{\gamma}^{s,k}_{m,z}$, $\tilde{\epsilon}^{s,k}_{m,z}$ and $\tilde{\delta}^{s,k}_{m,z}$ are

$$\begin{aligned}
\tilde{\gamma}^{1,1}_{1,1} &= \tilde{\gamma}^{2,2}_{1,1} = 1+\mu,\\
\tilde{\gamma}^{1,1}_{2,2} &= \tilde{\gamma}^{2,2}_{2,2} = 1-\mu,\\
\tilde{\epsilon}^{2,2}_{2,2} &= -\tilde{\epsilon}^{1,1}_{1,1} = \mu,\\
\tilde{\epsilon}^{2,1}_{1,2} &= 1-\mu, \qquad \tilde{\epsilon}^{1,2}_{2,1} = 1+\mu,\\
\tilde{\delta}^{1,1}_{2,2} &= -\tilde{\epsilon}^{2,2}_{1,1} = \mu,\\
\tilde{\delta}^{2,1}_{1,2} &= 1+\mu, \qquad \tilde{\delta}^{1,2}_{2,1} = 1-\mu.
\end{aligned}$$

After the substitution $R_1(\tau) = e^{c_1\Gamma\tau}r_1(\tau)$ and $R_2(\tau) = e^{c_1\Gamma\tau}r_2(\tau)$, we obtain the following equations of motion for $r_1(\tau)$ and $r_2(\tau)$.

$$\begin{aligned}
\frac{dr_1}{d\tau} &= i\left(f_+r_1 - r_1f_-\right)\\
&+\frac{2}{\Gamma}\Big(-(1+\mu)g_1r_1g_1^\dagger - (1-\mu)g_2r_1g_2^\dagger - \mu g_1^\dagger g_1 r_1 + \mu r_1 g_2^\dagger g_2 +\\
&\quad +(1-\mu)g_1^\dagger g_2 r_2 + (1+\mu)r_2 g_1^\dagger g_2\Big), \qquad \text{(S51)}\\
\frac{dr_2}{d\tau} &= i\left(f_-r_2 - r_2f_+\right)\\
&+\frac{2}{\Gamma}\Big(-(1+\mu)g_1r_2g_1^\dagger - (1-\mu)g_2r_2g_2^\dagger + \mu g_2^\dagger g_2 r_2 - \mu r_2 g_1^\dagger g_1 +\\
&\quad +(1+\mu)g_2^\dagger g_1 r_1 + (1-\mu)r_1 g_2^\dagger g_1\Big), \qquad \text{(S52)}
\end{aligned}$$

where, we recall that $g_0 = h_D$,

$$f_\pm = g_0 \pm \frac{1\mp\mu}{2}g_3^\dagger = \sum_{j=1}^{N-1} h_{j,j+1} \pm (J\vec{n}_0)\cdot\vec{\sigma}_1. \qquad \text{(S53)}$$

At zeroth order in $1/\Gamma$, the eigenmodes of Eqs. (S51) and (S52) are, respectively, $|\alpha\rangle\langle\tilde{\beta}|$ and $|\tilde{\alpha}\rangle\langle\beta|$, where $|\alpha\rangle$ and $|\tilde{\alpha}\rangle$ are the eigenvectors of f_+ and f_-, namely, $f_+|\alpha\rangle = \epsilon_\alpha|\alpha\rangle$ and $f_-|\tilde{\alpha}\rangle = \epsilon_\alpha|\tilde{\alpha}\rangle$. Note that f_+ and f_-, being related by a unitary transformation, have the same eigenvalues. It follows that the zeroth order eigenvalues of the Liouvillian are twice degenerate,

$$\lambda_{1,\alpha,\beta} = -\frac{\Gamma}{2} + i(\epsilon_\alpha - \epsilon_\beta) + O(1/\Gamma), \qquad \text{(S54)}$$

$$\lambda_{2,\alpha,\beta} = \lambda_{1,\alpha,\beta} + O(1/\Gamma), \qquad \text{(S55)}$$

the respective eigenvectors being $r_1^{(0)} = |\alpha\rangle\langle\tilde{\beta}|$ and $r_2^{(0)} = |\tilde{\alpha}\rangle\langle\beta|$. Note that the zeroth-order eigenvalues $\lambda_{1,\alpha,\beta}$ and $\lambda_{2,\alpha,\beta}$ have a double degeneracy for $\alpha \neq \beta$ and a degeneracy 2^{N+1} for $\alpha = \beta$.

To obtain the $O(1/\Gamma)$ corrections to the degenerate eigenvalues $\lambda_{1,\alpha,\beta} = \lambda_{2,\alpha,\beta} = -\Gamma/2 + i(\epsilon_\alpha - \epsilon_\beta) \equiv \Lambda_{\alpha\beta}$, we substitute the Ansatz $r_1(\tau) = x_1(\tau)|\alpha\rangle\langle\tilde{\beta}|$ and $r_2(\tau) = x_2(\tau)|\tilde{\alpha}\rangle\langle\beta|$ into Eqs. (S51) and (S52), obtaining the following equations for $x_1(\tau)$ and $x_2(\tau)$

$$\begin{aligned}
\frac{dx_1}{d\tau} &= \Lambda_{\alpha\beta}x_1 + \frac{2}{\Gamma}(V_{11}x_1 + V_{12}x_2),\\
\frac{dx_2}{d\tau} &= \Lambda_{\alpha\beta}x_2 + \frac{2}{\Gamma}(V_{21}x_1 + V_{22}x_2),
\end{aligned}$$

where

$$\begin{aligned}
V_{11} &= -(1+\mu)\langle\alpha|g_1|\alpha\rangle\langle\tilde{\beta}|g_1^\dagger|\tilde{\beta}\rangle - (1-\mu)\langle\alpha|g_2|\alpha\rangle\langle\tilde{\beta}|g_2^\dagger|\tilde{\beta}\rangle - \mu\langle\alpha|g_1^\dagger g_1|\alpha\rangle + \mu\langle\tilde{\beta}|g_2^\dagger g_2|\tilde{\beta}\rangle,\\
V_{22} &= -(1+\mu)\langle\tilde{\alpha}|g_1|\tilde{\alpha}\rangle\langle\beta|g_1^\dagger|\beta\rangle - (1-\mu)\langle\tilde{\alpha}|g_2|\tilde{\alpha}\rangle\langle\beta|g_2^\dagger|\beta\rangle + \mu\langle\tilde{\alpha}|g_2^\dagger g_2|\tilde{\alpha}\rangle - \mu\langle\beta|g_1^\dagger g_1|\beta\rangle,\\
V_{12} &= (1+\mu)\langle\alpha|\tilde{\alpha}\rangle\langle\beta|g_1^\dagger g_2|\tilde{\beta}\rangle + (1-\mu)\langle\alpha|g_1^\dagger g_2|\tilde{\alpha}\rangle\langle\beta|\tilde{\beta}\rangle,
\end{aligned}$$

$$V_{21} = (1+\mu)\langle\tilde{\alpha}|g_2^\dagger g_1|\alpha\rangle\langle\tilde{\beta}|\beta\rangle + (1-\mu)\langle\tilde{\alpha}|\alpha\rangle\langle\tilde{\beta}|g_2^\dagger g_1|\beta\rangle.$$

The eigenvalues v_1, v_2 of the matrix V with elements V_{ij} give the corrections to the eigenvalues $-\Gamma/2 + i(\epsilon_\alpha - \epsilon_\beta)$,

$$\lambda_{1\alpha\beta} = -\Gamma/2 + i(\epsilon_\alpha - \epsilon_\beta) + \frac{2}{\Gamma}v_1, \tag{S56}$$

$$\lambda_{2\alpha\beta} = -\Gamma/2 + i(\epsilon_\alpha - \epsilon_\beta) + \frac{2}{\Gamma}v_2. \tag{S57}$$

For degenerate eigenvalues $\lambda_{1,\alpha,\alpha} = \lambda_{2,\alpha,\alpha} = -\Gamma/2$, the $O(1/\Gamma)$ corrections have to be calculated in the following way. In the Zeno limit, the stationary solutions of Eqs. (S51) and (S52) are, respectively, $r_1(\infty) = \sum_\alpha \nu_\alpha|\alpha\rangle\langle\tilde{\alpha}|$ and $r_2(\infty) = \sum_\alpha \mu_\alpha|\tilde{\alpha}\rangle\langle\alpha|$. Therefore, for $r_1(\tau)$ and $r_2(\tau)$ we may assume the form $r_1(\tau) = \sum_\alpha \nu_\alpha(\tau)|\alpha\rangle\langle\tilde{\alpha}|$ and $r_2(\tau) = \sum_\alpha \mu_\alpha(\tau)|\tilde{\alpha}\rangle\langle\alpha|$ with coefficients ν_α and μ_α depending on time. Inserting these expressions into Eqs. (S51) and (S52) and writing down the equations for the components $\langle\alpha|r_1(\tau)|\tilde{\alpha}\rangle = \nu_\alpha(\tau)$ and $\langle\tilde{\alpha}|r_2(\tau)|\alpha\rangle = \mu_\alpha(\tau)$, we have

$$\frac{d\nu_\alpha}{d\tau} = \frac{2}{\Gamma}\sum_\beta \left(T^{11}_{\alpha\beta}\nu_\beta + T^{12}_{\alpha\beta}\mu_\beta\right), \tag{S58}$$

$$\frac{\partial\mu_\alpha}{\partial\tau} = \frac{2}{\Gamma}\sum_\beta \left(T^{21}_{\alpha\beta}\nu_\beta + T^{22}_{\alpha\beta}\mu_\beta\right), \tag{S59}$$

where

$$T^{11}_{\alpha\beta} = w_1(\alpha,\beta), \qquad \beta \neq \alpha, \tag{S60}$$

$$T^{22}_{\alpha\beta} = w_2(\alpha,\beta), \qquad \beta \neq \alpha, \tag{S61}$$

$$T^{12}_{\alpha\beta} = w_{12}(\alpha,\beta), \tag{S62}$$

$$T^{21}_{\alpha\beta} = w_{21}(\alpha,\beta), \tag{S63}$$

$$T^{11}_{\alpha\alpha} = w_1(\alpha,\alpha) + \sum_\beta f(\alpha,\beta), \tag{S64}$$

$$T^{22}_{\alpha\alpha} = w_2(\alpha,\alpha) + \sum_\beta f(\alpha,\beta), \tag{S65}$$

and

$$w_1(\alpha,\beta) = -(1+\mu)\langle\alpha|g_1|\beta\rangle\langle\tilde{\beta}|g_1^\dagger|\tilde{\alpha}\rangle - (1-\mu)\langle\alpha|g_2|\beta\rangle\langle\tilde{\beta}|g_2^\dagger|\tilde{\alpha}\rangle, \tag{S66}$$

$$w_2(\alpha,\beta) = -(1+\mu)\langle\tilde{\alpha}|g_1|\tilde{\beta}\rangle\langle\beta|g_1^\dagger|\alpha\rangle - (1-\mu)\langle\tilde{\alpha}|g_2|\tilde{\beta}\rangle\langle\beta|g_2^\dagger|\alpha\rangle, \tag{S67}$$

$$f(\alpha,\beta) = \mu|\langle\tilde{\beta}|g_2|\tilde{\alpha}\rangle|^2 - \mu|\langle\beta|g_1|\alpha\rangle|^2, \tag{S68}$$

$$w_{12}(\alpha,\beta) = (1-\mu)\langle\alpha|\tilde{\beta}\rangle\langle\beta|g_1^\dagger g_2|\tilde{\alpha}\rangle + (1+\mu)\langle\beta|\tilde{\alpha}\rangle\langle\alpha|g_1^\dagger g_2|\tilde{\beta}\rangle, \tag{S69}$$

$$w_{21}(\alpha,\beta) = (1-\mu)\langle\tilde{\alpha}|\beta\rangle\langle\tilde{\beta}|g_2^\dagger g_1|\alpha\rangle + (1+\mu)\langle\tilde{\beta}|\alpha\rangle\langle\tilde{\alpha}|g_2^\dagger g_1|\beta\rangle. \tag{S70}$$

By finding the eigenvalues q_α of the block matrix

$$T = \begin{pmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{pmatrix}, \tag{S71}$$

we resolve the degeneracy problem. The real eigenvalues with $O(1/\Gamma)$ corrections, belonging to the degenerate eigenvalue $c_1 = c_2$ of the dissipator, are given by

$$\lambda_{1\&2,\alpha,\alpha} = \frac{\Gamma}{2} + \frac{2}{\Gamma}q_\alpha, \qquad \alpha = 1, 2, \ldots, 2\times 2^N. \tag{S72}$$

Numerically, we find that all the coefficients of the matrix T, as the operators $f_\pm$, are μ independent and, therefore, the corrections q_α in Eq. (S72) are μ independent. This property is exceptional and probably connected with the integrability of the XYZ model.

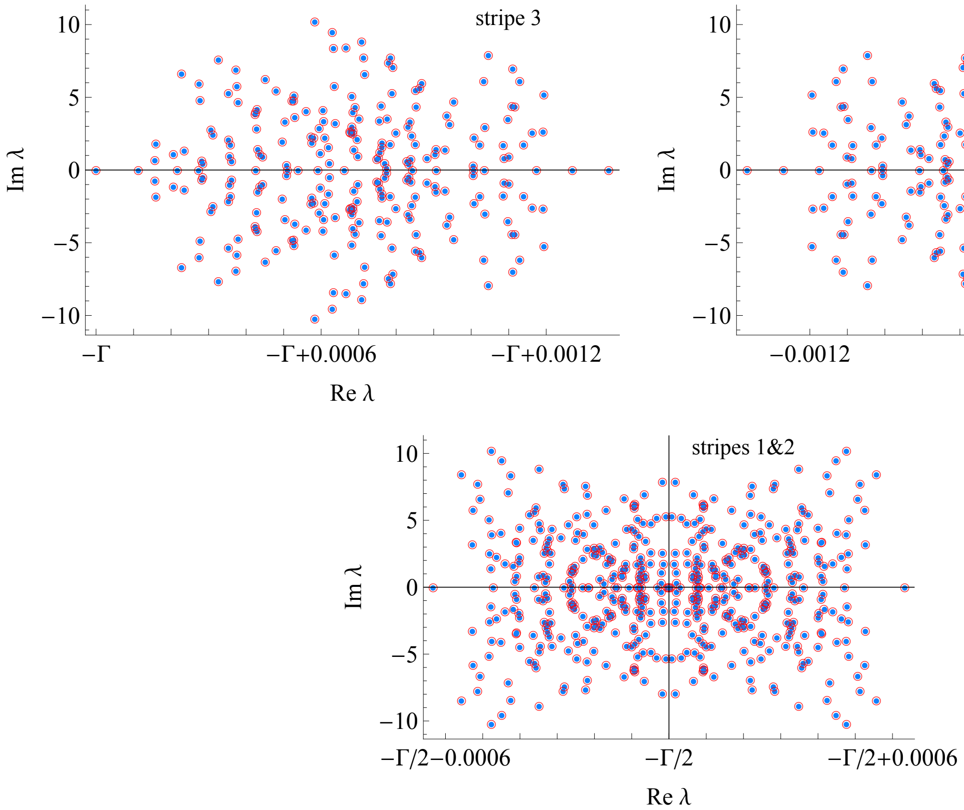

Figure S1. Complex eigenvalues of the Liouvillian belonging to the stripes 0, 3 and 1&2 for $\Gamma = 8000$. Approximated eigenvalues (open red circles) are computed at order $1/\Gamma$ by Eqs. (S25) and (S26) for stripe 0, Eqs. (S35) and (S43) for stripe 3 and Eqs. (S56), (S57) and (S72) for stripes 1&2, and compare very well with the exact numerical results (blue dots). Parameters as in Fig. 1.

Properties of the auxiliary Markov Matrix M_{ab}

It is well known that the eigenvalues of a generic stochastic matrix are complex. Nevertheless, for our case example – the XYZ model with Zeno boundary dissipation – all the eigenvalues happen to be real.

Here we prove this exceptional property, namely, that the eigenvalues μ_a of the Markov matrix M_{ab} in Eq (S26) are all real, for pure state boundary driving $\mu = 1$. We observe (numerically) that the elements M_{ab} of the Matrix Markov process,

$$\frac{d\nu_\alpha(\tau)}{d\tau} = \frac{1}{\Gamma}\sum_\beta M_{\alpha\beta}\, \nu_\beta(\tau), \tag{S73}$$

satisfy the so-called Kolmogorov condition

$$M_{ab}M_{bc}M_{ca} = M_{ac}M_{cb}M_{ba}, \tag{S74}$$

with a, b, c arbitrary and all different, if the targeted state at the boundary is pure, i.e., for $\mu = 1$. The Kolmogorov condition and the positivity of the non-diagonal elements M_{ab} entail

$$M_{ab} = s(a,b)\pi_b,$$
$$s(a,b) = s(b,a),$$

with $s(a,b)$ and π_b real and positive. Introducing the diagonal matrix $\hat{\pi}$ with elements π_a, we can write the Markov matrix M as

$$M = \hat{\pi} S,$$

where S is the matrix with non-diagonal elements $S_{ab} = s(a,b)$ and $S_{aa} = M_{aa}/\pi_a$. The above relation can be rewritten as

$$\hat{\pi}^{-1/2} M \hat{\pi}^{1/2} = \hat{\pi}^{1/2} S \hat{\pi}^{1/2}.$$

Obviously, the RHS of the above equation is a real symmetric matrix, since S is a real symmetric matrix. Consequently, $\hat{\pi}^{-1/2} M \hat{\pi}^{1/2}$ is also a real symmetric matrix, i.e., the Markov matrix M is equivalent to a real symmetric matrix. Therefore, the eigenvalues μ_a of M are all real. It follows that the 2^N Liouvillian eigenvalues belonging to the first stripe (S26) lie, in the Zeno limit, on the real axis.

The same argument can be repeated for all stripes, and consequently, all the Liouvillian eigenvalues of type $\lambda_{k,\alpha,\alpha}$ are, near the Zeno limit, real. In total, for our XYZ spin chain, there are $4 \times 2^N = 2^{N+2}$ real Liouvillian eigenvalues, while all the remaining Liouvillian eigenvalues $\lambda_{k,\alpha,\beta}$, with $\alpha \neq \beta$ generically, i.e., in the absence of extra degeneracies, have a nonzero imaginary part.

Finally, for $\mu \neq 1$ we observe numerically the same situation, i.e., the eigenvalues of the Markov matrix M (and its analogs for the other stripes) are all real, so that the Zeno-limit Liouvillian spectrum contains 2^{N+2} real entries. Clearly, also in this case M must be equivalent to a Hermitian matrix. However, this fact can no longer be explained by the Kolmogorov property (S74), (equivalent to a detailed balance condition for the Markov rates $w_{ab} = M_{ba}$) since this property is violated for $\mu \neq \pm 1$, and the detailed balance condition $\pi_a w_{ab} = \pi_b w_{ba}$ is consequently not satisfied. Further studies are required to clarify this subtle issue.

Zeno limit for a problem with two qubits

Consider a problem (1) with $H = \vec{\sigma}_0 \cdot (\hat{J} \vec{\sigma}_1)$, where $\hat{J} = \mathrm{diag}(J_x, J_y, J_z) \equiv \mathrm{diag}(1, \gamma, \Delta)$, and

$$\mathcal{D}[\rho] = \sigma_0^+ \rho \sigma_0^- - \frac{1}{2} \sigma_0^- \sigma_0^+ \rho - \frac{1}{2} \rho \sigma_0^- \sigma_0^+. \tag{S75}$$

According to our general theory, the stripe closest to the imaginary axis, in the Zeno limit contains 4 eigenvalues. They are governed by the effective Hamiltonian (S24)

$$h_D = \Delta\, \sigma^z \tag{S76}$$

and by the effective Lindblad operator

$$\tilde{L}_1 = -\begin{pmatrix} 0 & 1+\gamma \\ 1-\gamma & 0 \end{pmatrix}. \tag{S77}$$

The near-Zeno limit eigenvalues for the first stripe are given by Eq. (S25),

$$\begin{aligned} \lambda_{0,1,1} &= 0, \\ \lambda_{0,1,2} &= -4\frac{1+\gamma^2}{\Gamma} - 2\Delta i, \\ \lambda_{0,2,1} &= \lambda_{0,1,2}^*, \\ \lambda_{0,2,2} &= -8\frac{1+\gamma^2}{\Gamma}. \end{aligned}$$

Analogously, we obtain the other Liouvillian eigenvalues. The full set of 16 Liouvillian eigenvalues λ up to order $1/\Gamma$ is given by

$$\begin{aligned} \lambda_{0,\alpha,\beta} &= \left\{0, -2\frac{\gamma_+}{\Gamma}, -\frac{\gamma_+}{\Gamma} \pm 2\Delta i\right\}, \\ \lambda_{1\&2,\alpha,\beta} &= \left\{-\frac{\Gamma}{2}, -\frac{\Gamma}{2}, -\frac{\Gamma}{2} \pm \frac{2\gamma_-}{\Gamma}, -\frac{\Gamma}{2} \pm \frac{8\gamma}{\Gamma} \pm 2\Delta i\right\}, \\ \lambda_{3,\alpha,\beta} &= \left\{-\Gamma, -\Gamma + 2\frac{\gamma_+}{\Gamma}, -\Gamma + \frac{\gamma_+}{\Gamma} \pm 2\Delta i, \right\}, \end{aligned} \tag{S78}$$

where $\gamma_\pm = 4(1 \pm \gamma^2)$. The respective eigenfunctions are fully analytic functions of Γ in the Zeno regime ($\Gamma > \Gamma_{\mathrm{cr}}$, see later for its definition) so the Liouvillian is diagonalizable in any point. In the following considerations, the free fermion point $\Delta = 0$ must be excluded, since it corresponds to zero h_D and multiple degeneracies even in the Zeno limit (S78).

As discussed in the main text, the analyticity of Liouvillian eigenvalues breaks down at the branch points, which can be located by finding the eigenvalues of the Liouvillian for arbitrary Γ, γ, Δ. An inspection shows that among the 16 eigenvalues for $\Delta \neq 0$, apart from $\lambda = 0$ there is a double degenerate real eigenvalue $\lambda = -\Gamma/2$, the eigenvalue $\lambda = -\Gamma$ and all the other eigenvalues contain branch points. Depending on the parameters, there can be up to 8 values of $\Gamma = \Gamma_i$ where branchings occur.

Two points are $\Gamma_1 = 8$ and $\Gamma_2 = 8|\gamma|$, while the location of the other branch points $\Gamma_3, \ldots, \Gamma_8$ involves radicals of a quartic equation. In particular, for small Δ we find a singularity, for $\max(\Gamma_3, \ldots \Gamma_8) = O(1/|\Delta|)$, which has a probable origin in the repulsion of the eigenvalues, which, for $|\Delta| \ll 1$, become too close each other. The onset of the fully analytic Zeno regime sets in beyond the rightmost branching points, i.e., for $\Gamma > \Gamma_{\mathrm{cr}} \equiv \max_i \Gamma_i$. The value of Γ_{cr} is easily estimated numerically for a generic choice of the model parameters, see Fig. 3 for an example.