Available online at www.easdprocedia.org
EASD Procedia EURODYN (2020) 1445-1458

EASD Procedia

EURODYN 2020
XI International Conference on Structural Dynamics
M. Papadrakakis, M. Fragiadakis, C. Papadimitriou (eds.)
Athens, Greece, 23–26 November 2020

OPTIMAL FEEDBACK CONTROL LAW FOR VISCOELASTIC MATERIALS WITH MEMORY EFFECTS

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Keywords: Viscoelasticity, Memory effects, Optimal control, Feedback control, Fractional derivatives, Integral-differential equations

Abstract. *Viscoelastic materials have excellent properties of absorbing vibrational energy which makes their use very attractive in structural, aerospace and biomechanics engineering applications. The macroscopic dynamical behaviour of such materials depends on the time history, or memory, of the strain. The stress-strain viscoelastic relation can be described by a convolution integral with a memory kernel, according to Boltzmann's formulation of hereditary elasticity, or by using Caputo or Riemann-Liouville fractional derivatives. In order to emphasize the vibrations damping attitude of these materials, by actively controlling their stress-strain behaviour, novel optimal control logics are required which involve memory effects. This paper deals with a feedback control strategy applied to a structural-dynamic problem described by integral-differential equations. It is shown how to obtain a feedback control, called $PD^{(N)}$, i.e. Proportional-Nth-order-Derivatives control, by using a variational approach. Numerical simulations show how the $PD^{(N)}$ controller is an effective tool to improve the viscoelastic materials performance.*

ISSN:2311-9020 © 2020 The Authors. Published by EASD Procedia.
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doi: 10.47964/1120.9117.19567

1 INTRODUCTION

The use of viscoelastic materials is crucially important in many engineering disciplines and there are applications in a wide variety of areas such as aerospace, transport components, biomechanics and civil infrastructures [1-5]. These materials have excellent properties of absorbing vibrational energy which makes their use in engineering applications very attractive, such as the dynamic instabilities and the fluid-structures interaction control.

Often, the macroscopic dynamical behaviour of such materials depends on the time history or memory effects, of the strain response. According to Boltzmann's formulation of hereditary elasticity [6], the stress-strain relation in linear viscoelasticity can be described by a convolution integral with a memory kernel. However, the mathematical form of this kernel is not easily predictable by theoretical tools, but it rather relies on experimental identification.

In the past, combinations of elemental rheological components such as the spring and dashpot, were used to generate a whole family of rheological models, such as Maxwell and Kelvin–Voigt, or the four-parameters Burger's model, generalizing the viscoelastic response, but without including memory effects [7]. Moreover, experimental tests have shown some inconsistencies of viscoelastic materials, especially when the effects of relaxation and creep would be kept into the model [8]. For these reasons in the second part of the last century, many researches have been carried out a more realistic description of creep and/or relaxation given by the use of power law functions with real order exponent [9]. Such formulation captures the creep behaviour and generates the fractional hereditary model. In fact, the constitutive law of viscoelastic materials makes use of the Riemann-Liouville and Caputo's fractional derivative [10, 11]. Due to the high mathematical complexity of such models, which imply integral-differential equations (IDEs), difficulties arise in the formulation of optimal control algorithm.

In recent years the interest in model-based control of mechanical systems incorporating active controlled viscoelastic dampers has been increased in order to improve the mechanical system efficiency. Systems with viscoelastic dampers has been investigated by [12] where the viscoelastic term is approximated by damping terms which are converted into stochastic differential equations solved by a dynamic programming method; meanwhile multi-input and multi-output mechanical systems incorporating viscoelastic dampers are controlled with discrete-time sliding mode control algorithm [13]. Fractal-fractional model are used by [14] where the nonlinear algebraic system is solving by the Gauss-Legendre quadrature rule. Moreover, active control for standard Kelvin-Maxwell viscoelastic structures can be designed accurately with poles and zeros assignment defining an ad-hoc receptance transfer function [15]. Numerical scheme for solving fractional differential equations are proposed by [16] using the approximation of Laguerre integral formula.

In general, the optimal control strategies, applied to viscoelastic model, make use of direct methods. This paper introduces an indirect optimal feedback control algorithm, which is still missing in the literature panorama. Starting from the prototype integral-differential beam equation, the authors show how it is possible to obtain a Proportional-Nth-order-Derivatives control, called PD(N), by using the variational approach. PD(N) algorithm depends on the structure of the kernel and is a sort of hyper-derivative proportional control. The PD(N) control belongs to the category of Variational Feedback Controls-VFC, in the context of which the authors are developing different control architectures [17-19].

The paper consists of three main sections. The first presents the viscoelastic mathematical beam model, using the Caputo's fractional derivatives. The PD(N) feedback optimal control algorithm is derived in the second section, and in the third, numerical results show the effective performance of the proposed control for a bridge deflection control when it is subject to earthquake excitation and impulsive disturbances, still maintaining low the cost function in comparison with the LQR method and the implicit Pontryagin solution for IDEs already formulated by authors [20-22].

2 VISCOELASTIC BEAM MODEL

A brief description of the viscoelasticity fractional Kelvin–Voigt model is the premise to obtain the partial differential equations (PDE) of an Euler–Bernoulli viscoelastic beam, of length L and subjected to a basement seismic motion z_b (see Figure 1).

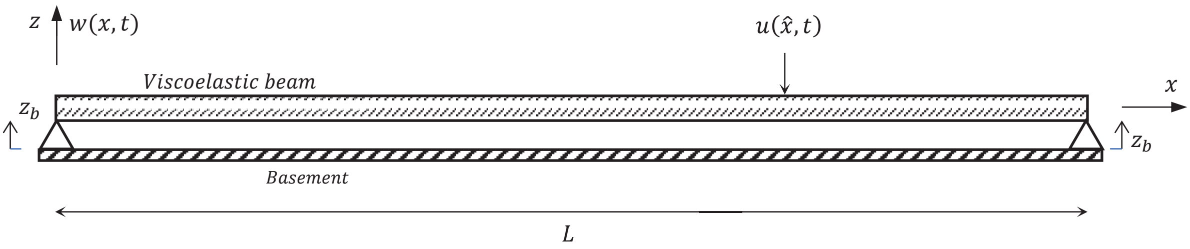

Figure 1: Viscoelastic beam.

The classical Kelvin–Voigt model, using one spring and one viscous pot in parallel, presents a time-varying linear stress-strain relationship $\sigma(\varepsilon)$ which can be expressed by the Caputo's fractional derivative D^{β}:

$$\sigma = E\varepsilon + E_{\beta} D^{\beta}[\varepsilon] \tag{1}$$

with E the Young's modulus, E_{β} the damping modulus and the time varying operator $D^{\beta}[\ \ ]$, with $0 \leqslant \beta \leqslant 1$ depending on material proprieties, defined as:

$$D^{\beta}[\varepsilon] = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\dot{\varepsilon}(\tau)}{(t-\tau)^{\beta}} d\tau \tag{2}$$

with $\Gamma(\cdot)$ the Gamma function.

Considering constant ρ density, A cross-section area and I inertial moment the PDE is derived by defining the conservation of momentum and of moment of momentum:

$$\begin{gathered} \frac{\partial T_z(x,t)}{\partial x} = \rho A \frac{\partial^2 w_b(x,t)}{\partial t^2} - u(\hat{x},t) - \rho A \ddot{z}_b(t) \\ \frac{\partial M_y(x,t)}{\partial x} = T_z(x,t) \end{gathered} \tag{3}$$

where $w_b(x,t)$, $T_z(x,t)$, $M_y(x,t)$ and $u(\hat{x},t)$ are the vertical displacement along the z-axis, the shear, the bending moment and the punctual actuation acting on $\hat{x}$, respectively.

The introduction of the kinematic Euler-Bernoulli and the σ stress and bending moment M_y relations

$$\begin{gathered} \varepsilon(x,t) = -z \frac{\partial^2 w_b(x,t)}{\partial x^2} \\ \sigma(x,t) = z \frac{M_y(x,t)}{I} \end{gathered} \tag{4}$$

leads to the definition of the viscoelastic beam equation as formulated by [23]:

$$\rho A \frac{\partial^2 w_b(x,t)}{\partial t^2} + EI \frac{\partial^4 w_b(x,t)}{\partial x^4} + E_{\beta} I D^{\beta}\left[\frac{\partial^4 w_b(x,t)}{\partial x^4}\right] = u(\hat{x},t) + \rho A \ddot{z}_b(t) \tag{5}$$

Finally, equation (5) can be expressed in a compact form with the convolution integral term $k * \dot{w}_b^{IV}$ instead of the fractional operator $E_{\beta} D^{\beta}[w_b^{IV}]$ as formulated by Nutting [24]. By using the derivative convolution properties and considering the supported beam case, the final equation with the associated boundary conditions is:

$$\rho A\ddot{w}_b(x,t) + EIw_b^{IV}(x,t) + I\dot{k} * w_b^{IV}(x,t) = \delta(x-\hat{x}\,)u(t) + \rho A\ddot{z}_b(t)$$
$$w_b''|_0^L = 0 \tag{6}$$
$$w_b|_0^L = 0$$

where $\delta(\quad)$ is the Dirac function. In the case of translational base motion, the final transverse displacement w is given by the summation of two contributions:

$$w(x,t) = w_b(x,t) + z_b(t) \tag{7}$$

Decoupling equation (6) by the modal decomposition:

$$w_b(x,t) = \sum_r^{\infty} q_r(t)\phi_r(x) \tag{8}$$

with ϕ_r the eigenfunction of the r-*th* vibration mode, one obtains:

$$\rho A\phi_r\ddot{q}_r + EI\phi_r{}^{IV}q_r + I\phi_r{}^{IV}\,\dot{k} * q_r = \delta(x-\hat{x}\,)u(t) + \rho A\ddot{z}_b(t) \tag{9}$$

With the following normalization:

$$\int_0^L \phi_m\rho A\phi_r = \delta_{mr} \quad , \qquad \int_0^L \phi_m EI\phi_r^{IV} = \omega_r^2\delta_{mr} \qquad , \quad \widehat{\omega}_r^2 = \frac{\omega_r^2}{E}$$
$$\gamma_r = \int_0^L \phi_r\,\rho A \tag{10}$$

equation (9) becomes:

$$\ddot{q}_r + \omega_r^2 q_r + \widehat{\omega}_r^2\,\dot{k} * q_r = \phi_r(\hat{x})u(t) + \gamma_r\ddot{z}_b(t) \tag{11}$$

Finally, equation (11) is a second order integral-differential equation which represents the decoupled dynamic of the controlled viscoelastic Euler-Bernoulli beam when an external basement disturbance is acting on it. In the following section, is investigated a novel optimal control approach to equation (11).

3 THE VISCOELASTIC OPTIMAL CONTROL

For equation (11), it is possible to introduce an optimal feedback control law by using the variational calculus. The control solution is a sort of hyper-derivative proportional control called Proportional-Nth-order-Derivatives, i.e. PD$^{(N)}$ controller, depending on the structure of the kernel [25]. Introducing the new state variable $\boldsymbol{x} = [\boldsymbol{q}, \dot{\boldsymbol{q}}]^T$ and $\boldsymbol{q} = [q_1, \dots, q_r, \dots q_N]^T$ equation (11) takes the space state form:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{K} * \boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{d} \tag{12}$$

where $\boldsymbol{A} \in \mathbb{R}^{N,N}$ and $\boldsymbol{B} \in \mathbb{R}^{N,1}$ are constant matrixes, $\boldsymbol{u}$ in this example is a scalar control variable, $\boldsymbol{d} \in \mathbb{R}^{N,1}$ the external disturbances vector and $\boldsymbol{K} \in \mathbb{R}^{N,N}$ include the Caputo's fractional derivative.

The optimal control law is based on the minimization of a generic quadratic cost function J which depends on the state and the control variables together with the Lagrange multiplier vector $\boldsymbol{\lambda}$, introduced to account for the system's dynamics expressed by equation (12):

$$J(\boldsymbol{x},\boldsymbol{u},\boldsymbol{\lambda}) = \int_0^T \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \frac{1}{2}\boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u} + \boldsymbol{\lambda}^T[\dot{\boldsymbol{x}} - \boldsymbol{A}\boldsymbol{x} - \boldsymbol{K} * \boldsymbol{x} - \boldsymbol{B}\boldsymbol{u} - \boldsymbol{d}]dt \tag{13}$$

The variational calculus finds a solution to the stated problem by using the stationary condition $\delta J(\boldsymbol{x},\boldsymbol{u},\boldsymbol{\lambda}) = 0$. The variations of the term $\boldsymbol{K} * \boldsymbol{x}$ present some nontrivial problems, and a technique to deal with this term is considered in details in [21, 25]. Considering a scalar term

$k * x$, the kernel causality proprieties $k(t-\tau)=0$ for $t<0$ and $\tau>t$, produces its variation in the form:

$$\delta\int_0^T x(t)\int_0^t k(t-\tau)x(\tau)d\tau\,dt=\int_0^T\int_t^T k(\tau-t)x(\tau)d\tau\,\delta f dt=\int_0^T k\diamond x\,\delta x\,dt \tag{14}$$

Proceeding with standard variational calculus rules, and using equation (14), the associated extremal conditions are found:

$$\begin{aligned}&\dot{\boldsymbol{\lambda}}=\boldsymbol{Q}\boldsymbol{x}-\boldsymbol{A}^T\boldsymbol{\lambda}-\boldsymbol{K}^T\diamond\boldsymbol{\lambda}\\&\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{u}+\boldsymbol{K}*\boldsymbol{x}+\boldsymbol{d}\\&\boldsymbol{u}=\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{\lambda}\\&\boldsymbol{\lambda}(T)=\boldsymbol{0}\,,\boldsymbol{x}(0)=\boldsymbol{x_0}\end{aligned} \tag{15}$$

Eq. (15) gives the optimal open loop solution to problem (13) and it can be solved with several direct control methods [26]. However, our goal is to obtain an explicit feedback formulation of the control $\boldsymbol{u}(\boldsymbol{x})$, that has tremendous advantages under the engineering application point of view. To this aim, we replace the fractional operator (2) by its approximation by a truncated exponential series. Therefore, the Nutting's kernel $k(t)$ in (11), can be reformulated as:

$$k*x=\frac{E_\beta}{\Gamma(1-\beta)}\int_0^t\frac{x(\tau)}{(t-\tau)^\beta}d\tau\approx\frac{E_\beta}{\Gamma(1-\beta)}\int_0^t\sum_i^N\alpha_i e^{-c_i(t-\tau)}x(\tau)d\tau=\hat{k}*x \tag{16}$$

where α_i and c_i are the N-constants of the exponential series approximation and $\hat{k}$ is introduced to indicate the approximated kernel. In this way, eq. (16) leads to transform the integral-differential equations (15) into the Laplace domain. In fact, both the terms, $\hat{k}*x$ and $\hat{k}\diamond x$ have the corresponding Laplace transform $\mathcal{L}\{\quad\}$ in s as sum of polynomials:

$$\begin{aligned}\mathcal{L}\{\hat{k}*x\}&=X(s)\sum\frac{\alpha_i}{s+c_i}=X(s)\frac{P(N-1)(s)}{D(N)(s)}\\\mathcal{L}\{\hat{k}\diamond x\}&=X(s)\sum\frac{\alpha_i}{s-c_i}=-X(s)\frac{P(N-1)(-s)}{D(N)(-s)}\end{aligned} \tag{17}$$

where P and D are used to indicate the N-order polynomials.

Then, using the equations (16) and (17) to express the integral terms in the first two of equations (15), setting $\boldsymbol{d}=\boldsymbol{0}$ and eliminating the control $\boldsymbol{u}$ in the equation $\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{u}+\boldsymbol{K}*\boldsymbol{x}$, using $\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{\lambda}+\boldsymbol{K}*\boldsymbol{x}$, we can produce the Laplace transform of the obtained equations in terms of $\boldsymbol{X}(s)$ and $\boldsymbol{\Lambda}(s)$ only. Rearranging these equations, and transforming back to time domain, we finally obtain:

$$\begin{aligned}&\sum_{i=0}^{N+1}\boldsymbol{C}_i\boldsymbol{x}^{(i)}+\sum_{i=0}^{N}\boldsymbol{D}_i\boldsymbol{\lambda}^{(i)}=\boldsymbol{0}\\&\sum_{i=0}^{N+1}\boldsymbol{F}_i\boldsymbol{\lambda}^{(i)}+\sum_{i=0}^{N}\boldsymbol{E}_i\boldsymbol{x}^{(i)}=\boldsymbol{0}\end{aligned} \tag{18}$$

where $\boldsymbol{x}^{(i)}=\frac{d^i x}{dt^i}$, and the problem is reduced to a pure differential equation set, linear and time-invariant. Collecting the variables into $\boldsymbol{\xi}=[\boldsymbol{x}^{(N)}\ldots\boldsymbol{x}^{(i)}\ldots\boldsymbol{x}]$ and $\boldsymbol{\eta}=[\boldsymbol{\lambda}^{(N)}\ldots\boldsymbol{\lambda}^{(i)}\ldots\boldsymbol{\lambda}]$, equations (18) can be reduced to a first order normal form differential problem:

$$\begin{cases}\dot{\boldsymbol{\xi}}=\boldsymbol{H}_{\xi\xi}\boldsymbol{\xi}+\boldsymbol{H}_{\xi\eta}\boldsymbol{\eta}\\\dot{\boldsymbol{\eta}}=\boldsymbol{H}_{\eta\xi}\boldsymbol{\xi}+\boldsymbol{H}_{\eta\eta}\boldsymbol{\eta}\end{cases} \tag{19}$$

Equation (19) is characterized by the matrix $\boldsymbol{H} = [\boldsymbol{H}_{\xi\xi}, \boldsymbol{H}_{\xi\eta}; \boldsymbol{H}_{\eta\xi}, \boldsymbol{H}_{\eta\eta}] \in \mathbb{R}^{2N,2N}$ that is not in general of Hamiltonian type. For this reason, it is not possible to solve the optimal problem by satisfying the Riccati's stationary equation, but a direct solution through the *2N*-eigenvalues and *2N*-eigenvectors is used. By assuring at least N negative real part eigenvalues of matrix $\boldsymbol{H}$, one obtains:

$$\begin{aligned} &\boldsymbol{\eta} = \boldsymbol{G}\boldsymbol{\xi} \qquad with \qquad \boldsymbol{G} = \boldsymbol{\Theta}\boldsymbol{\Psi}^{-1} \in \mathbb{R}^{N,N} \\ &\boldsymbol{\Psi} = [\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_N] \\ &\boldsymbol{\Theta} = [\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_N] \end{aligned} \tag{20}$$

where $\boldsymbol{\psi}_i$ and $\boldsymbol{\theta}_i$ are half of the *2N* eigenvectors of $\boldsymbol{H}$, those associated to negative real part eigenvalues. Finally, since $= \boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{\lambda}$, extracting from $\boldsymbol{\eta}$ the last sub-vector $\boldsymbol{\eta}_N = \boldsymbol{\lambda}$, the optimal feedback control variable $\boldsymbol{u}$ is:

$$\boldsymbol{u} = \boldsymbol{R}^{-1}\boldsymbol{B}^T \sum_{i=1}^{N+1} \boldsymbol{G}_{ji}\, \boldsymbol{x}^{(i-1)} = \boldsymbol{K}_{\boldsymbol{pdn}}\boldsymbol{\xi} \tag{21}$$

The structure of the solution (21) shows that the optimal control of an integral-differential system is strongly related to the structure of the kernel $\hat{k}(t)$ and contains a combination of state derivatives of order equal to the number of exponential terms of $\hat{k}(t)$. For this reason, (21) is called hyper-derivative proportional control, PD(N). The (21) can also be expressed through a $\boldsymbol{K}_{\boldsymbol{pdn}}$ gain matrix multiplied by the vector $\boldsymbol{\xi}$. The initial state derivatives $\frac{d^i\boldsymbol{x}(0)}{dt^i} = \boldsymbol{x}_0^{(i)}$ for $i > 2$ can be evaluated from the Laplace transformation of (12) in the case of uncontrolled and undisturbed system and replacing the approximated $\hat{k}$ exponential kernel function.

4 NUMERICAL RESULTS

In the following section the numerical results are shown for the deflection control of a bridge when external excitations are acting on it. The bridge has been considered as a three-mode decomposition model. Two different simulations have been performed, considering an impulsive and an earthquake disturbance on the bridge basement. For both cases, we assume the actuator is placed at $\hat{x} = L/2 + 0.2L/4$ in order to control not only the first and third vibrational modes but also the 20% of the second mode. To measure the vibrations of the bridge, three sensors $s1$, $s2$, $s3$, for example accelerometers, have been placed on the structure at $x_{s1} = L/2$, $x_{s2} = L/4$ and $x_{s3} = L/3$, in such a way that it can be easily reconstruct the modal displacements $\boldsymbol{q}$. Table 1 shows the geometrical parameters describing the simulated bridge which has been chosen with a H-beam cross-section.

Description	Parameters	Value
Density [kg/m³]	ρ	7500
Young modulus [N/m²]	E	$200 \cdot 10^9$
Length [m]	L	24
Thickness [m]	b	0.18
Height [m]	h	0.4
Momentum of inertia [m⁴]	I	$23130 \cdot 10^{-8}$
Cross-sectional area [m²]	A	$84.46 \cdot 10^{-4}$

Table 1**:** H-beam parameters.

In Table 2 are listed the main dynamic parameters and control settings used to develop numerical simulations in which a kernel function composed by two exponential terms has been considered.

Description	Parameters	Value
Beam eigenfunctions in x_{s1}	$[\phi_1(L/2); \phi_2(L/2); \phi_3(L/2)]$	[0.036; 0; -0.0363]
Beam eigenfunctions in x_{s2}	$[\phi_1(L/4); \phi_2(L/4); \phi_3(L/4)]$	[0.0256; 0.0363; 0.0256]
Beam eigenfunctions in x_{s3}	$[\phi_1(L/3); \phi_2(L/3); \phi_3(L/3)]$	[0.0314; 0.0314; 0]
Beam eigenfunctions in $\hat{x}$	$[\phi_1(\hat{x}); \phi_2(\hat{x}); \phi_3(\hat{x})]$	[0.0358; -0.0112; -0.0323]
Natural frequencies [rad/s]	$[\omega_1; \omega_2; \omega_3]$	[14.6434; 58.6032; 131.8943]
Modal kernel frequencies	$[\widehat{\omega}_1; \widehat{\omega}_2; \widehat{\omega}_3]$	$[0.0327; 0.13; 0.29]\cdot 10^{-3}$
Kernel coefficients	$\left[\frac{\alpha_1 E_\beta}{\Gamma(1-\beta)}; c_1; \frac{\alpha_2 E_\beta}{\Gamma(1-\beta)}; c_2\right]$	$[8\cdot 10^{10}; 100; 2\cdot 10^{10}; 10]$
Control gain	$\mathbf{Q}\,;\mathbf{R}$	$\begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 80 & 0 \\ 0 & 0 & 0 & 80 \end{bmatrix}; 0.01$
Control gain PDN	$\boldsymbol{K_{pdn}}$	$[-102.6;\ 63.7;\ 4.1; -3;\ 1.2; \dots$ $-0.0117; -0.0833;\ 0.0009]$
Initial displacement [m]	$\boldsymbol{q_0}$	$[0;\ 0;\ 0]$
Initial velocity [m/s]	$\dot{\boldsymbol{q}}_{\boldsymbol{0}}$	$[0.1;\ 0.1;\ 0.1]$

Table 2: Parameters and control settings used for the simulations.

Regarding the first simulation, the impulsive basement disturbance is simulated by considering an initial condition on the modal velocity $\dot{\boldsymbol{q}}_{\boldsymbol{0}}$ different from zero taking into account the first three vibrational modes $r = 1,2,3$ of eq. (11). The performances of the PD(N) controller are compared with the Pontryagin implicit solution, recently proposed by the authors [20, 21], and the benchmarking LQR method by solving eq. (12) neglecting the convolution term.
From Figure 2 to Figure 6 the time evolution of the beam displacement $w(x,t)$, cost function J, power and force actuation, are shown respectively. The bridge deflection has been computed in two different positions x_{s1} and x_{s2} in order to be able to catch the contributions of all the three vibrational modes.

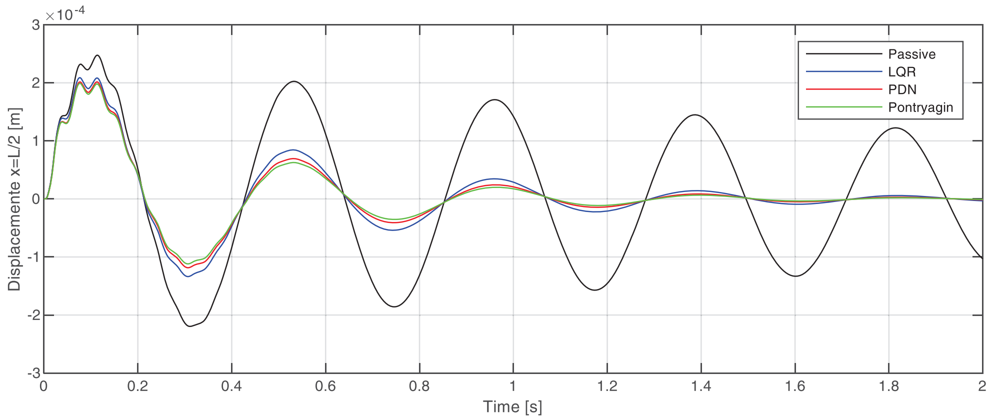

Figure 2: Beam vertical displacement at x_{s1} under impulsive load.

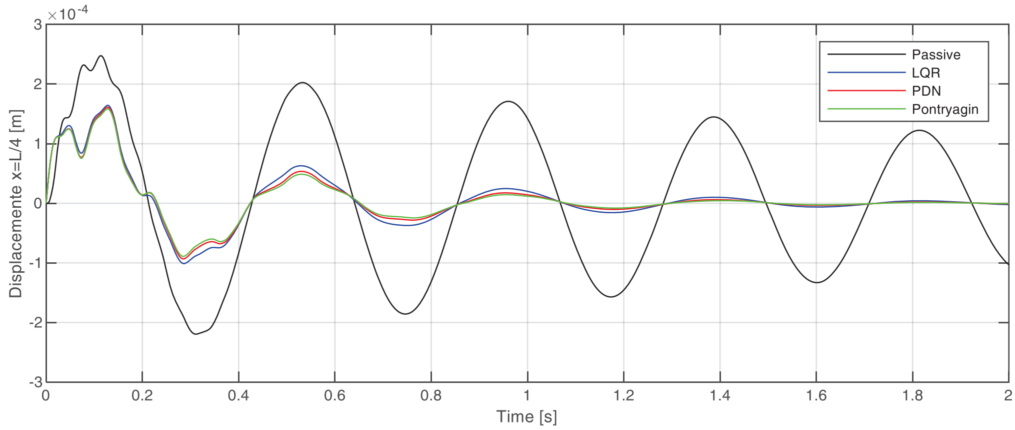

Figure 3: Beam vertical displacement at x_{s2} under impulsive load.

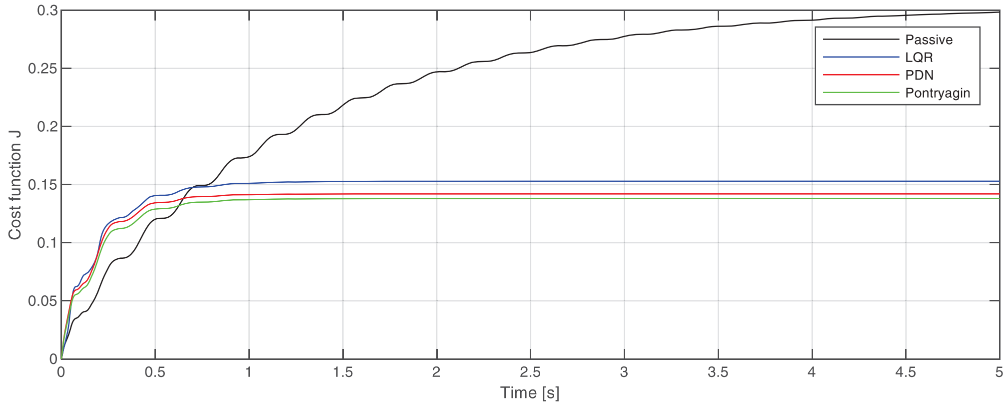

Figure 4: Comparisons of cost functions under impulsive load.

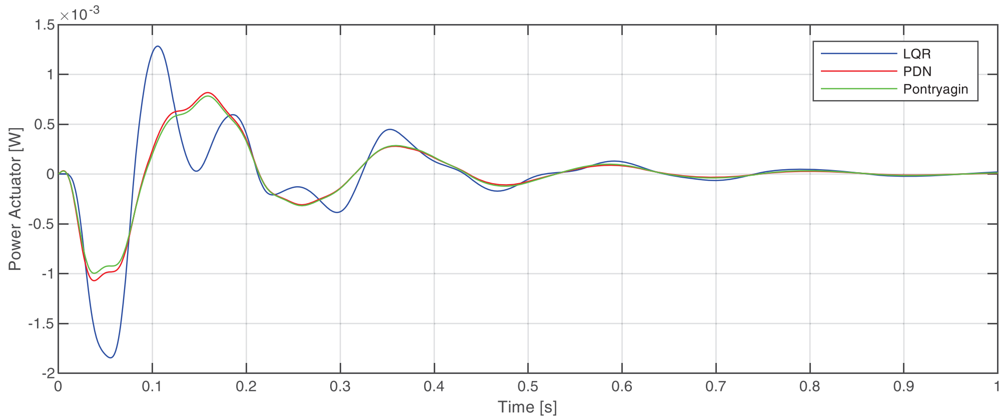

Figure 5: Comparisons of power actuation under impulsive load.

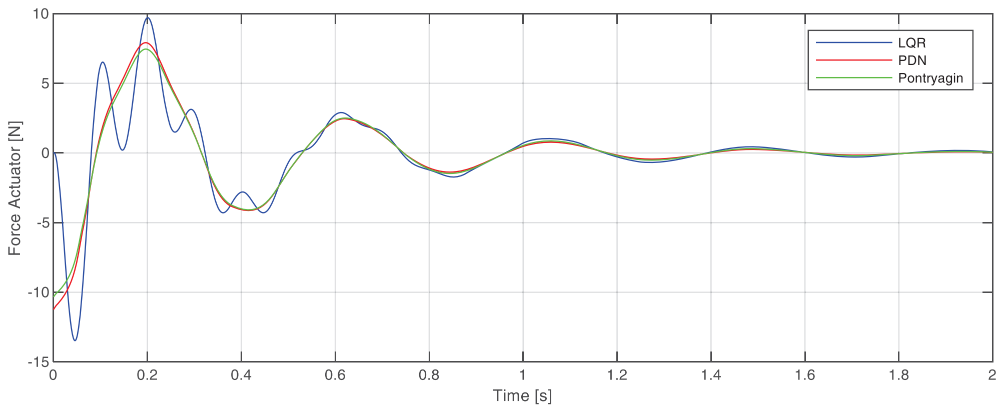

Figure 6: Comparisons of force actuation under impulsive load.

The numerical results show very good performances of the PD(N) algorithm in comparison with the standard LQR method in terms of vibrations minimization. In fact, the LQR, which cannot take into account the viscoelastic memory effects, shows higher values of the actuation force and power in comparison with the PD(N) controller. In fact, the PD(N) algorithm reach half of the actuation power and the 80% of the maximum force reached by LQR, respectively. Moreover, observing Figure 2 and Figure 3 the proposed controller presents lower beam deflection (higher in x_{s2}) and a lower cost function value (Figure 4), stretching its convenience in comparison to the LQR method. The PD(N) performance are at the same level of the Pontryagin implicit solution. Of course, the implicit solution, which is the best optimal solution, presents the lower values of cost function and arrival time.

In the second tested case, the bridge is excited by an earthquake disturb. The Class A earthquake load has been formulated by [27] according to the Eurocode 8 which introduce the possibility to use the stochastic analysis in the design of structures in seismic zone. Figure 7 and Figure 8 describe the earthquake disturbance in term of PSD and acceleration, respectively.

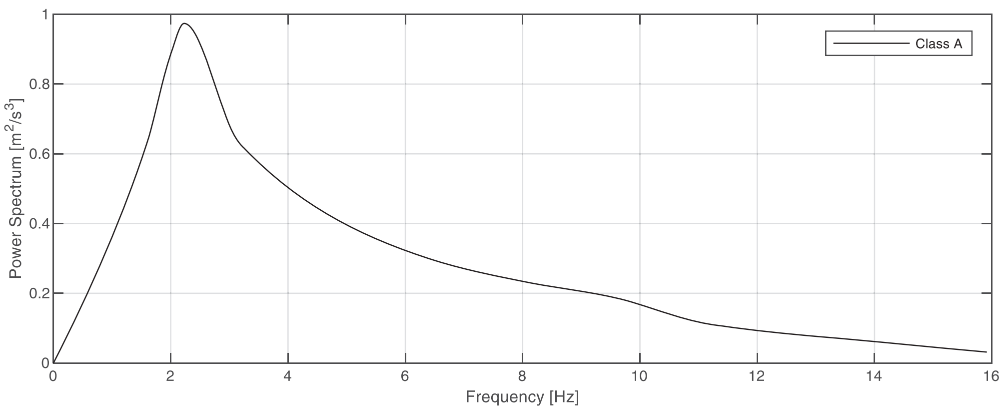

Figure 7: Earthquake PSD

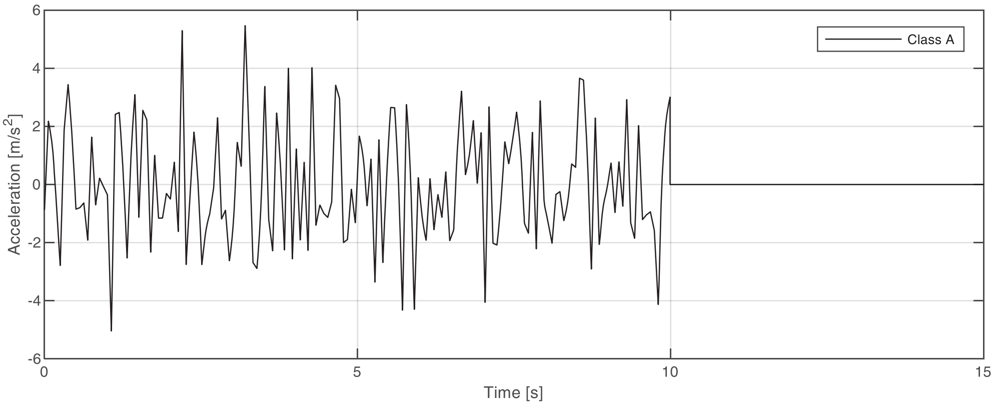

Figure 8: Earthquake acceleration

The class A earthquake PSD amplified the first vibrational mode of the bridge showing a maximum value corresponding to the bridge first natural frequency ($\sim 2Hz$) and on the contrary it does not excite the third mode, because is very far from the spectrum. Moreover, given the symmetry of the noise, even modes are not excited. For all these reasons, the results will be shown only for the position x_{s1}. The earthquake acceleration, $\ddot{z}_b(t)$, in Figure 8 has been generated from the PSD in Figure 7, acting for 10s and in the final 5s the bridge system is free to extinguish its residual vibrations.

In this simulation the performance of the proposed PD(N) controller are only compared with the benchmarking LQR method. In fact, the external disturbance considered in this case is stochastic one, which is impossible to predict and to include in the Pontryagin solution.

From Figure 9 to Figure 12 the time evolution of the bridge deflection in x_{s1}, the cost function J, the power and the force actuation, are illustrated respectively. It is possible to notice that, even when an external random disturbance is acting on the bridge, the proposed PD(N) controller presents better performance respect to the LQR method. The PD(N) algorithm shows a minimization of the bridge deflection requiring a lower force and power of actuation and for these reasons, it is also presenting a lower cost function J.

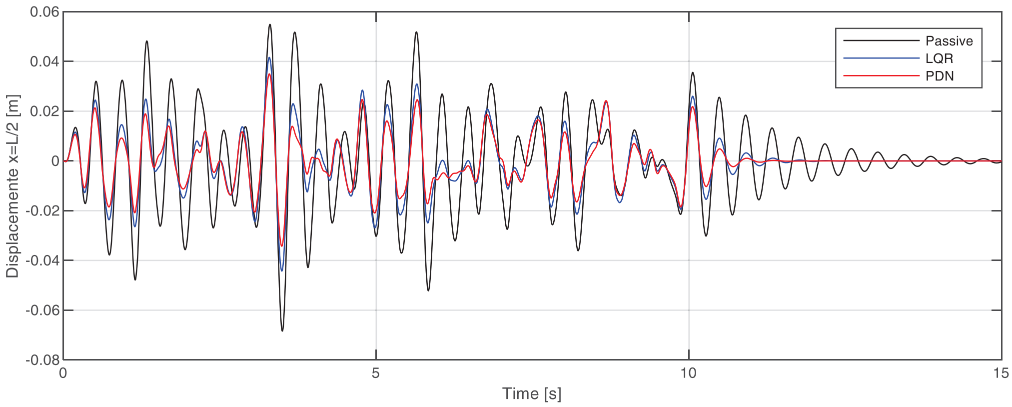

Figure 9: Comparisons of deflection under earthquake load.

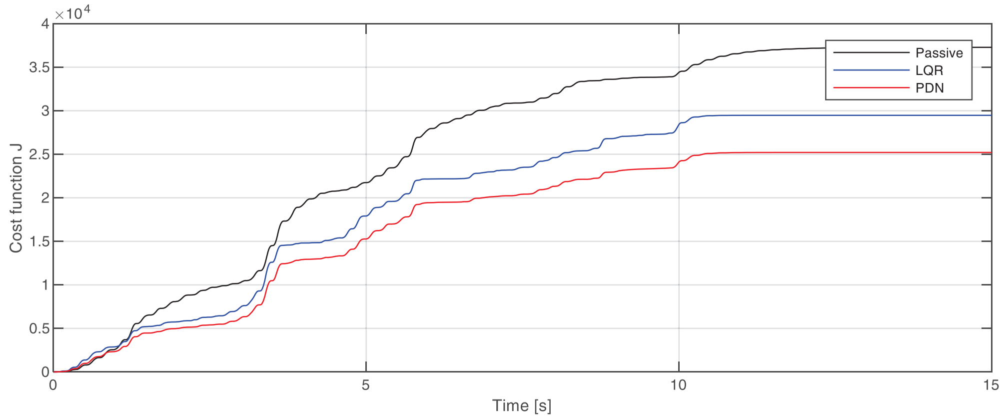

Figure 10: Comparisons of cost function under earthquake load.

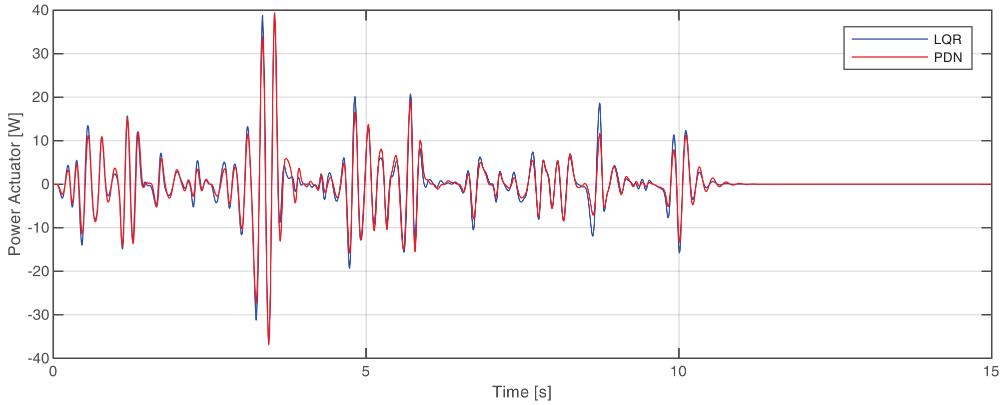

Figure 11: Comparisons of power under earthquake load.

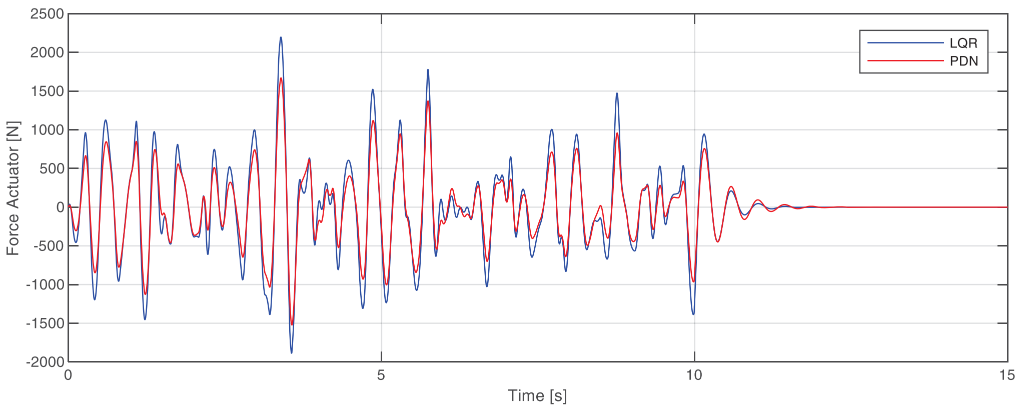

Figure 12: Comparisons of force actuation under earthquake load.

5 CONCLUSIONS

In this paper the authors propose an optimal control algorithm to actively control viscoelastic materials for the structural vibration's minimization.

The proposed algorithm applies to a viscoelastic Euler-Bernoulli beam including Boltzmann hereditary effects by using a Nutting's memory kernel function. The prototype equation is of integro-differential type and requires a non-conventional optimal control strategy, which considers convolution terms (memory effects). In general, the optimal control strategies, applied to this viscoelastic model, make use of direct methods, while in this paper the authors offer an indirect optimal feedback control algorithm, which is still missing in the literature panorama. The authors show how it is possible to obtain a Proportional-Nth-order-Derivatives control, called PD(N), by using the variational approach. Numerical simulations show very good result of the PD(N) solution compared to the benchmarking LQR and the implicit Pontryagin solution for IDEs, already formulated by the authors in recent works.

Thanks to the hyper derivative properties, the PD(N) controller shows a capability of minimization of beam deflection, in term of force and power of actuation and cost both for different external disturbances acting on the model and also for multiple kernel function compared to the benchmarking optimal control algorithm.

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