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Rational operators based on $q-$integers

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Dedicated to Prof. Ioan Raşa on the occasion of his 65$^{\text{th}}$ *birthday*

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Abstract Shepard-type rational operators based on $q-$integers are studied and convergence results and pointwise approximation error estimates improving previous statements are obtained. Influence of choice of q on the error estimates is also discussed. Techniques in CAGD for shape modeling of rational curves based on above operators are also presented and numerical examples are given.

Keywords Shepard operators · q-integers · Pointwise approximation error estimates · Shepard-type curves · Shape modeling

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1 Introduction

In the pioneered work [15] A. Lupaş introduced the q-analogue of the Bernstein operator and numerous generalizations based on q-integers have been developed (see, e.g., [17,18]). q-extensions of positive operators find application also in CAGD (see, e.g. [13,16]).

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On the other hand in [8] the authors proved that there exist simple rational functions r_n and q_n s.t. $\forall f \in C([0,1])$ and $\forall x \in [0,1]$

$$|f(x) - r_n(x)| \le C\omega\left(f; \frac{(x(1-x))^\gamma}{n}\right), \tag{1}$$

$$|f(x) - q_n(x)| \le C\omega\left(f; \frac{|x-1/2|^\gamma}{n}\right), \tag{2}$$

with $0 < \gamma < 1$, $\omega(f)$ the usual modulus of continuity of f and C a positive constant assuming in this paper different values even in the same formula. Estimates of type (1) with $1/2 < \gamma < 1$ and of type (2) with $0 < \gamma < 1$ are not achievable by polynomials (cfr. [12]). Moreover in [8] the authors proved that the exponent γ in (1) and (2) cannot be equal to 1. For example if $f \in C([0,1])$, the estimate

$$|f(x) - R_n(x)| \le C\omega\left(f; \frac{x}{n}\right), \; x \in [0,1],$$

is not possible by rational approximant R_n (see the remark to Corollary 2.5 in [8] suggested to authors by Totik).

Here extending the idea of Lupaş, we construct simple rational operators based on q-integers and prove they are a good tool to approximate functions from $C([0,1])$, achieving pointwise approximation error estimates improving (1)-(2). In Section 2 the $q-$analogue of Shepard operators are considered and uniform convergence results and pointwise approximation error estimates are given in Theorems 1–10. Influence of choice of q on the error estimates is also discussed. In Section 3 simple procedures in CAGD for shape modeling of rational curves based on above operators are presented. Numerical examples are given in Section 4. Finally the proofs are in Section 5 and are based on direct estimates of our operators and careful analysis of nodes mesh distributions.

2 Main results

First of all we recall the definition of Shepard operators. For $n \in \mathbb{N}$ introduce the nodes matrix

$$\overline{X} = (\overline{x}_{n,k} = \overline{x}_k, \; k = 0, \dots, n, \; n \in \mathbb{N}) \subseteq [0,1], \tag{3}$$

with $\overline{x}_0 = 0$ and $\overline{x}_n = 1$. Then for any function $f \in C([0,1])$ we consider the Shepard operator S_n defined by

$$S_n(\overline{X}; f; x) = \frac{\sum_{k=0}^{n} \frac{f(\overline{x}_k)}{(x-\overline{x}_k)^s}}{\sum_{k=0}^{n} \frac{1}{(x-\overline{x}_k)^s}}, \tag{4}$$

with $x \in [0,1]$, $\overline{x}_k$ as in (3) and s even > 2. From (4) we deduce that S_n is a linear, positive operator of interpolatory type, $\min_{0 \le x \le 1} |f(x)| \le |S_n(f;x)| \le \max_{0 \le x \le 1} |f(x)|, \forall x \in [0,1]$, $S_n(f)$ is a rational function of degree (sn, sn), with $S_n'(f;\overline{x}_k) = 0$, $k = 0,\ldots,n$. Shepard operators are widely studied in classical approximation theory and in scattered data interpolation problems (see, e.g., [1,2,7,8,10,22,23]). If the nodes mesh in (4) is equispaced, then direct and converse results are well-known for S_n (see, e.g. [19,20]). If the nodes mesh in (4) is of algebraic type, then pointwise approximation error estimates, direct and converse results, bridge theorems, saturation statements and simultaneous approximation error estimates not possible by polynomials were achieved in [7–9,21].

The case of geometric progression nodes mesh for S_n was an open problem. Aim of this paper is to give a positive answer to this question, proving that Shepard operators on $q-$integers uniformly approximate functions from $C([0,1])$ and allow pointwise estimates improving (1)–(2).

Indeed if $q > 0$, for any $n = 0,1,2,\ldots$, the q-integer $[n]_q$ is defined by

$$[n]_q := \frac{1-q^n}{1-q} = 1+q+\ldots+q^{n-1},\ n = 1,2,\ldots,\ [0]_q = 0.$$

Consider the nodes matrix

$$X = \left(x_{n,k} = x_k = \frac{[k]_q}{[n]_q},\ k = 0,\ldots,n,\ n \in \mathbb{N}\right). \tag{5}$$

Note that if $q = 1$, then $x_k = k/n$, $k = 0,\ldots,n$, which corresponds to the equispaced mesh case.

Now let $q = q(n) > 1$, with $\lim_n q(n) = 1$. Define for $f \in C([0,1])$ the operator

$$S_n(X;f;x) = \frac{\sum_{k=0}^{n} \frac{f(x_k)}{(x-x_k)^s}}{\sum_{k=0}^{n} \frac{1}{(x-x_k)^s}}, \tag{6}$$

with $x \in [0,1]$, x_k as in (5) and s even > 2. We call $S_n(X)$ the q-analogue of Shepard operator. Indeed if $q = 1$, then (6) gives back the Shepard operator (4) on equispaced mesh, whose approximation behaviour has been completely studied in [1,19,20]. In such case pointwise estimates of type (1) and (2) are against nature; indeed it is well-known that the approximation behaviour of the S_n operator is strongly influenced by the mesh distribution and for uniformly spaced mesh the endpoints 0 and 1 or the inner point 1/2 do not play a special role (cfr. [8]).

If $||\ ||$ denotes the usual supremum norm on $[a,b]$, for $f \in C([a,b])$, for any fixed $a < b$, we have

Theorem 1 *Let $q = q(n) = (1 + \log_2 n/n)$. Then for any $f \in C([0,1])$ and $n \in \mathbb{N}, n > 1$,*

$$\lim_n ||S_n(X;f) - f|| = 0$$

and

$$|f(x) - S_n(X;f;x)| \le C\omega\left(f;\left(x+\frac{1}{n}\right)\frac{\log_2 n}{n}\right),\quad \forall x \in [0,1]. \tag{7}$$

Remark 1 Under the assumptions of Theorem 1 we deduce the uniform convergence of $S_n(X;f)$ to f, $\forall f \in C([0,1])$. From (29) it follows that the mesh (5) is thicker near 0 and this affects the approximation error (see the presence of x at the r.h.s. in (7)). For example if $f \in C^1([0,1])$ and $x \le C/n$, then by (7) the approximation error is $O\left(\log_2 n/n^2\right)$, which is better than $O\left(1/n^{1+\gamma}\right)$ coming from (1) for $f \in C^1([0,1])$ and $x \le C/n$. In other words the slight worsening of the uniform approximation error rate in (7) (which is $O\left(\log_2 n/n\right)$) with respect to (1) (which is $O(1/n)$) is compensated by a better approximation rate near 0.

In general we can prove

Theorem 2 *Let $q = q(n) > 1$, $\lim_n q(n) = 1$ and $\lim_n q(n)^n = \infty$. Then for any $f \in C([0,1])$ and $n \in \mathbb{N}, n > 1$,*

$$\lim_n ||S_n(X;f) - f|| = 0$$

and

$$|f(x) - S_n(X;f;x)| \le C\omega\left(f; \left(x + \frac{1}{q^n - 1}\right)(q-1)\right), \quad \forall x \in [0,1].$$

Moreover we can get analogous results as Theorems 1–2 for the endpoint 1, for both the endpoints ± 1 or for any interior point. For example, let

$$y_k = 1 - \frac{q^k - 1}{q^n - 1}, \; k = 0, \ldots, n, \tag{8}$$

and consider the matrix $Y = (y_k, \; k = 0, \ldots, n, \; n \in \mathbb{N})$. Note that $y_0 = 1, y_n = 0$ and this mesh is finer near 1. Then for any $f \in C([0,1])$ define the operator

$$S_n(Y;f;x) = \frac{\displaystyle\sum_{k=0}^{n} \frac{f(y_k)}{(x - y_k)^s}}{\displaystyle\sum_{k=0}^{n} \frac{1}{(x - y_k)^s}},$$

with $x \in [0,1]$ and s even > 2. We have

Theorem 3 *Let $q = q(n) = (1 + \log_2 n/n)$. Then for any $f \in C([0,1])$ and $n \in \mathbb{N}, n > 1$,*

$$\lim_n ||S_n(Y;f) - f|| = 0$$

and

$$|f(x) - S_n(Y;f;x)| \le C\omega\left(f; \left(1 - x + \frac{1}{n}\right)\frac{\log_2 n}{n}\right), \quad \forall x \in [0,1]. \tag{9}$$

Remark 2 From Theorem 3 we get the uniform convergence of $S_n(Y;f)$ to f, $\forall f \in C([0,1])$. The mesh thickness near 1 influences the error estimate (see the presence of $1-x$ at the r.h.s. in (16)). For example if $f \in C^1([0,1])$ and $1-x \le C/n$, then by (16) the approximation error is $O(\log_2 n/n^2)$, which is better than $O(1/n^{1+\gamma})$, coming from (1) for $f \in C^1([0,1])$ and $(1-x) \le C/n$. This means the uniform approximation error rate in (9) (which is $O(\log_2 n/n)$) is slightly slower than in (1) (which is $O(1/n)$), but is balanced by a faster approximation rate near 1.

Analogously if n is even, let

$$z_k = \begin{cases} \frac{1}{2}\frac{q^k-1}{q^{n/2}-1}, & k=0,\dots,\frac{n}{2}, \\ 1-z_{n-k}, & k=\frac{n}{2}+1,\dots,n. \end{cases}$$

This mesh is thicker near 0 and 1. Then denote by $S_n(Z)$ the S_n operator based on the matrix $Z = (z_k,\ \ k=0,1,\dots,n,\ n\in\mathbb{N})$. If n is odd, then replace $S_n(Z)$ by $S_{n+1}(Z)$. We have

Theorem 4 *Let* $q(n) = (1+\log_2 n/n)$. *Then for any* $f \in C([0,1])$ *and* $n \in \mathbb{N}, n>1$,

$$\lim_n ||S_n(Z;f)-f|| = 0$$

and

$$|f(x)-S_n(Z;f;x)| \le C\omega\left(f;\left(x(1-x)+\frac{1}{n}\right)\frac{\log_2 n}{n}\right), \quad \forall x\in[0,1]. \quad (10)$$

Remark 3 From Theorem 4 we obtain the uniform convergence of $S_n(Z;f)$ to f, $\forall f \in C([0,1])$. Error estimate (10) is strongly influenced by the mesh thickness near both the endpoints (see the presence of $x(1-x)$ at the r.h.s. in (10)).

Also if n is even, letting

$$v_k = \begin{cases} \frac{q^k-1}{q^{n/2}-1}, & k=0,\dots,\frac{n}{2}, \\ -v_{-k}, & k=-\frac{n}{2},\dots,-1, \end{cases}$$

put

$$t_k = \frac{1}{2}\left(v_{k-n/2}+1\right), \quad k=0,1,\dots,n.$$

This mesh is thicker near the inner point $1/2$, with $t_0=0$ and $t_n=1$. Then denote by $S_n(T)$ the S_n operator based on the matrix $T=(t_k,\ \ k=0,\dots,n,$ $n\in\mathbb{N})$. If n is odd, then replace $S_n(T)$ by $S_{n+1}(T)$. We prove

Theorem 5 *Let* $q(n) = (1 + \log_2 n/n)$. *Then for any function* $f \in C([0,1])$ *and* $n \in \mathbb{N}, n > 1$,

$$\lim_n ||S_n(T; f) - f|| = 0$$

and

$$|f(x) - S_n(T; f; x)| \leq C\omega \left(f; \left(\left| x - \frac{1}{2} \right| + \frac{1}{n} \right) \frac{\log_2 n}{n} \right), \quad \forall x \in [0,1]. \qquad (11)$$

Remark 4 From Theorem 5 we deduce the uniform convergence of $S_n(T; f)$ to f, $\forall f \in C([0,1])$. The mesh thickness near 1/2 influences the approximation error (see the presence of $|x - 1/2|$ at the r.h.s. in (11)). For example if $f \in C^1([0,1])$ and $|x - 1/2| \leq C/n$, then by (11) the approximation error is $O(\log_2 n/n^2)$, which is better than $O(1/n^{1+\gamma})$, coming from (2). Equivalently the uniform approximation error rate in (11) (which is $O(\log_2 n/n)$) is slightly slower than in (2) (which is $O(1/n)$), on the other hand is made up for by a quicker decay rate near 1/2.

Finally consider the case $q = q(n) < 1$, with $\lim_n q(n) = 1$ and $\lim_n q(n)^n = 0$. We can prove convergence results in this case too. For example if

$$q = q(n) = 1 - \log_2 n/n, \qquad (12)$$

consider the matrix

$$U = \left(u_{n,k} = u_k = \frac{[k]_q}{[n]_q},\ k = 0, \ldots, n,\ n \in \mathbb{N} \right). \qquad (13)$$

Note that this mesh is thicker near 1 (compare with the nodes mesh X in Theorem 1). Then denote by $S_n(U)$ operator the S_n operator based on the matrix U. We have

Theorem 6 *Then for any function* $f \in C([0,1])$ *and* $n \in \mathbb{N}, n > 1$,

$$\lim_n ||S_n(U; f) - f|| = 0$$

and

$$|f(x) - S_n(U; f; x)| \leq C\omega \left(f; \left(1 - x + \frac{1}{n} \right) \frac{\log_2 n}{n} \right), \quad \forall x \in [0,1]. \qquad (14)$$

Remark 5 From Theorem 6 we deduce the uniform convergence of $S_n(U; f)$ to f, $\forall f \in C([0,1])$. The mesh thickness near 1 influences the approximation error (see the presence of $1-x$ at the r.h.s. in (14)). Comparing Theorem 1 with Theorem 6, we can see that the choice (12) for q reversed the approximation behaviour of S_n operator near the endpoints, in the sense that now the error estimate is better near the right-end point.

In general we can prove

Theorem 7 *Let $q = q(n) < 1$, $\lim_n q(n) = 1$ and $\lim_n q(n)^n = 0$ in (13). If $S_n(U)$ denotes the S_n operator on the matrix U, then for any function $f \in C([0,1])$ and $n \in \mathbb{N}, n > 1$,*

$$\lim_n ||S_n(U; f) - f|| = 0$$

and

$$|f(x) - S_n(U; f; x)| \leq C\omega\left(f; \left(\frac{1}{1-q^n} - x\right)(1-q)\right), \quad \forall x \in [0,1]. \tag{15}$$

Remark 6 The choice $q < 1$ in Theorem 7 made the nodes mesh in (13) thicker near 1 and this influenced the pointwise error estimate near 1, as one can see comparing Theorem 2 with Theorem 7.

Moreover consider the matrix $\overline{Y} = (\overline{y}_k = 1 - u_k,\ k = 0, \ldots, n,\ n \in \mathbb{N})$, with u_k given in (13) and q given by (12). Note that $\overline{y}_0 = 1, \overline{y}_n = 0$ and this mesh is finer near 0 (compare with the nodes mesh Y in Theorem 3). Then denote by $S_n(\overline{Y})$ the S_n operator based on the matrix $\overline{Y}$. We have

Theorem 8 *Let $q = q(n) = (1 - \log_2 n/n)$. Then for any $f \in C([0,1])$ and $n \in \mathbb{N}, n > 1$,*

$$\lim_n ||S_n\left(\overline{Y}; f\right) - f|| = 0$$

and

$$\left|f(x) - S_n(\overline{Y}; f; x)\right| \leq C\omega\left(f; \left(x + \frac{1}{n}\right)\frac{\log_2 n}{n}\right), \quad \forall x \in [0,1]. \tag{16}$$

Remark 7 From Theorem 8 we get the uniform convergence of $S_n(\overline{Y}; f)$ to f, $\forall f \in C([0,1])$. The mesh thickness near 0 influences the error estimate (see the presence of x at the r.h.s. in (16)). Hence the choice (12) for q here reversed the approximation behaviour near the endpoints with respect to Theorem 3, in the sense that now the error rate is better near 0.

Analogously if n is even, let

$$\overline{z}_k = \begin{cases} \dfrac{1}{2}\dfrac{1-q^k}{1-q^{n/2}}, & k = 0, \ldots, \dfrac{n}{2}, \\ 1 - \overline{z}_{n-k}, & k = \dfrac{n}{2} + 1, \ldots, n. \end{cases}$$

This mesh is thicker near 1/2 (compare with nodes matrix Z). Then denote by $S_n(\overline{Z})$ the S_n operator based on the matrix $\overline{Z} = (\overline{z}_k,\quad k = 0, 1, \ldots, n,\ n \in \mathbb{N})$. If n is odd, then replace $S_n(\overline{Z})$ by $S_{n+1}(\overline{Z})$. We have

Theorem 9 *Let $q(n) = (1 - \log_2 n/n)$. Then for any $f \in C([0,1])$ and $n \in \mathbb{N}, n > 1$,*

$$\lim_n ||S_n(\overline{Z}; f) - f|| = 0$$

and

$$|f(x) - S_n(\overline{Z}; f; x)| \leq C\omega\left(f; \left(\left|x - \frac{1}{2}\right| + \frac{1}{n}\right)\frac{\log_2 n}{n}\right), \quad \forall x \in [0,1]. \tag{17}$$

Remark 8 From Theorem 9 we obtain the uniform convergence of $S_n(\overline{Z}; f)$ to f, $\forall f \in C([0,1])$. Error estimate (17) is strongly influenced by the mesh thickness near 1/2 (see the presence of $|x-1/2|$ at the r.h.s. in (17)). Comparing Theorem 9 with Theorem 4, we can see the choice (12) for q modified the approximation behaviour of S_n operator, in the sense that now pointwise error is smaller near the inner point 1/2.

Also if n is even, letting

$$\overline{v}_k = \begin{cases} \dfrac{1-q^k}{1-q^{n/2}}, & k = 0, \ldots, \dfrac{n}{2}, \\ -\overline{v}_{-k}, & k = -\dfrac{n}{2}, \ldots, -1, \end{cases}$$

put

$$\bar{t}_k = \frac{1}{2}\left(\overline{v}_{k-n/2} + 1\right), \quad k = 0, 1, \ldots, n.$$

This mesh is thicker near the endpoints 0 and 1, with $\bar{t}_0 = 0$ and $\bar{t}_n = 1$. Then denote by $S_n(\overline{T})$ the S_n operator based on the matrix $\overline{T} = (\bar{t}_k, \ k = 0, \ldots, n, n \in \mathbb{N})$. If n is odd, then replace $S_n(\overline{T})$ by $S_{n+1}(\overline{T})$. We prove

Theorem 10 *Let $q(n) = (1 - \log_2 n/n)$. Then for any function $f \in C([0,1])$ and $n \in \mathbb{N}, n > 1$,*

$$\lim_n ||S_n(\overline{T}; f) - f|| = 0$$

and

$$|f(x) - S_n(\overline{T}; f; x)| \leq C\omega\left(f; \left(x(1-x) + \frac{1}{n}\right)\frac{\log_2 n}{n}\right), \quad \forall x \in [0,1]. \tag{18}$$

Remark 9 From Theorem 10 we deduce the uniform convergence of $S_n(T; f)$ to f, $\forall f \in C([0,1])$. The mesh thickness near 0 and 1 influences the approximation error (see the presence of $x(1-x)$ at the r.h.s. in (18)). Hence the choice (12) for q changed the approximation behaviour of S_n operator with respect to Theorem 5, since here the error rate is better near the endpoints 0 and 1.

3 Modelling by Shepard-type curves

In this Section we consider modeling techniques in CAGD by Shepard-type operators. Following the notation in [4], let $A_n(t) = [A_{n,0}(t), A_{n,1}(t), \ldots, A_{n,n}(t)]^T$, where

$$A_{n,k}(t) = \frac{\dfrac{1}{(t-\overline{x}_k)^s + \lambda}}{\displaystyle\sum_{k=0}^{n} \frac{1}{(t-\overline{x}_k)^s + \lambda}}, \tag{19}$$

for $0 \leq k \leq n$, $n \in \mathbb{N}$, $t \in [0,1]$, $\overline{x}_k$ given by (3) and $0 < \lambda$. The presence of λ at the denominators overcomes the flat spots drawback affecting original Shepard operator and preserves the fundamental properties in CAGD

$$0 \leq A_{n,i}(t) \leq 1, \quad i = 0, \ldots, n, \quad \sum_{i=0}^{n} A_{n,i}(t) = 1.$$

Given the functions $A_{n,i}(t)$ defined by (19) and a control polygon $P = [P_0, P_1, \ldots, P_n]^T$, $P_i \in \mathbb{R}^d$, $i = 0, \ldots, n$, $d \geq 2$, consider the Shepard-type curve defined by

$$\mathcal{S}_n[P,t] = \sum_{i=0}^{n} A_{n,i}(t) P_i = A_n(t) P. \tag{20}$$

From the definition we deduce that $\mathcal{S}_n[P,t]$ is a rational curve of degree (sn, sn), it reproduces points, it is symmetric, it is smooth, it is nondegenerate, it lies in the convex hull of the control points P_i, it satisfies the pseudo-local control property (indeed each function $A_{n,j}(t)$, $0 \leq j \leq n$, attains its maximum value close to 1 at $t = t_j$ and is very small for $|t - t_j| > \frac{1}{2n}$, in other words the point P_j influences strongly the shape of the curve in a neighborhood of $t = t_j$).

In the nonparametric case, i.e. $P_i = (x_i, f(x_i)), f \in C([0,1]), 0 \leq i \leq n$, if x_i are given from (3) and $\lambda = 0$, then we find back the interpolating function (6). If $\overline{x}_k = k/n$, $k = 0, 1, \ldots, n$, then $\mathcal{S}_n[P,t]$ curves were studied in [4]. If $\overline{x}_k$ are given by (5), the corresponding Shepard-type curves are called $q-$analogue of Shepard-type curves. Obviously if $q = 1$ we find back Shepard-type curves studied in [4].

The parameter $q \geq 1$ can be interpreted as a shape handle, allowing to model the form of a curve. Indeed playing on the mesh thickness regulated by $q-$integers, the designer has the freedom to change the contour of Shepard-type curve (20) with respect to the case $q = 1$ (corresponding to the equispaced mesh). Obviously when the nodes mesh is very thick, the parameter λ at denominators in (19) plays a significative role, hence the curve is approximating the corresponding control points; on the other hand, when the mesh is less dense, the parameter λ has less weight at denominators in (19), which means the interpolatory character of the curve takes over with respect to the case $q = 1$. For example if the nodes mesh is given by (5), it is thicker near 0 (see Remark to Theorem 1), consequently the corresponding $q-$analogue of Shepard-type curve approximates the control polygon and is closer to the last part of it than the case $q = 1$ (see Example 4.1 in Section 4). Analogously if nodes mesh is given by (8), the nodes mesh is thicker near 1 (see e.g. Remark to Theorem 3), hence the corresponding $q-$analogue of Shepard-type curve is nearer to the first part of the control polygon than the equispaced case (see Example 4.2 in Section 4).

Finally, if the knots are equispaced, we present a modeling technique by two shape parameters. Recently in [5] the authors introduced and studied the parametric curves

$$
\begin{aligned}
\mathcal{S}_{n,1}[P,t] &= \mathcal{S}_n[P,t],\\
\mathcal{S}_{n,m+1}[P,t] &= \sum_{i=0}^{n} A_{n,i}(t)P_i^m,\ m\in\mathbb{N},\\
P_i^0 &= P_i\\
P_i^{m+1} &= C_m P_i^0\\
C_m &= C_{m-1}(2I - KC_{m-1})\\
C_0 &= I,
\end{aligned}
\tag{21}
$$

with K the collocation matrix of the basis $A_{n,i}(t), i = 0, \dots, n$, i.e.

$$
K = \begin{pmatrix} A_{n,0}(\overline{x}_0) & A_{n,1}(\overline{x}_0) & \cdots & A_{n,n}(\overline{x}_0)\\ A_{n,0}(\overline{x}_1) & A_{n,1}(\overline{x}_1) & \cdots & A_{n,n}(\overline{x}_1)\\ \vdots & \vdots & \ddots & \vdots\\ A_{n,0}(\overline{x}_n) & A_{n,1}(\overline{x}_n) & \cdots & A_{n,n}(\overline{x}_n) \end{pmatrix},
$$

with $\overline{x}_i = i/n$, $0 \le i \le n$. Such procedure called progressive iterative approximation (PIA in short) technique is based on a suitable combination of iterates of operator introduced in (20) (properties of iterates of positive operators can be seen for ex. in [11]).

In Theorems 5–7 in [5] it was proved that

$$
\lim_m \mathcal{S}_{n,m}[P,t_i] = P_i,\ 0 \le i \le n, \tag{22}
$$

in other words (see [5]) the control points of the active curve are changed to deform it towards the target shape represented by sampled points from the given curve. Equivalently (see [5]) PIA process makes possible to construct a sequence of control points converging to the control polygon of an interpolating curve of Shepard-type. Hence the parameter m can be considered as a shape handle in order to model different shapes, obtaining as an extreme case original $\mathcal{S}_n$ curve and global interpolating Shepard-type curve. Convergence theorems and approximation error estimates for process (21) can be found in [5] . Here we stop iterations for the smallest m such that $\|P_i^m - P_i^{m-1}\| < 10^{-10}$, where $\|\ \|$ denotes the usual supremum norm in $\mathbb{R}^d$. We remark the convergence rate in (22) is exponential, while in [4] an analogous process had algebraic rate. A similar technique bridging Bernstein and Lagrange polynomials was studied in [3]. A further generalization of PIA format for shape modeling was examined in [6].

Now we introduce the function

$$
\sigma(c;t) = \begin{cases} \dfrac{t^2}{t^2+(1-t)^2}, & c=0\\ \dfrac{(1-t)^2}{t^2+(1-t)^2}, & c=1\\ \dfrac{t^2(1-t)^2}{t^2(1-t)^2+(t-c)^2(1-t)^2+t^2(t-c)^2}, & 0<c<1, \end{cases} \tag{23}
$$

with $c, t \in [0,1]$. From the definition it follows that $\sigma(c;t)$ is a rational function of degree $(2,2)$ for $c = 0,1$ and $(4,4)$ for $0 < c < 1$; moreover $\sigma(0;0) = 0$, $\sigma(0;1) = 1$, $\sigma(1;1) = 0$, $\sigma(1;0) = 1$, $\sigma(c;c) = 1$, $\sigma(c;0) = \sigma(c;1) = 0$, in other words the function $\sigma(0;t)$ is a rational approximation of signum function, $\sigma(1;t)$ is its complement (i.e. $\sigma(1;t) = 1 - \sigma(0;t)$) and $\sigma(c;t)$ is a composition of two signum-type functions, that is vanishes at the endpoints and is equal to 1 at $t = c$. Then we construct the curve

$$\tilde{\mathcal{S}}_{n,m}[P,c,t] = \sigma(c;t)\mathcal{S}_n[P,t] + (1 - \sigma(c;t))\mathcal{S}_{n,m}[P,t]. \tag{24}$$

Such curve is a rational curve of degree $(sn+2, sn+2)$ for $c = 0,1$, or $(sn+4, sn+4)$ for $0 < c < 1$, generally not positive, preserving constants. Obviously $\tilde{\mathcal{S}}_{n,1}[P,c,t] = \mathcal{S}_n[P,t], \forall c,t \in [0,1]$. Acting on the parameter c in (24) we can give more or less weight to $\mathcal{S}_n[P]$ with respect to $\mathcal{S}_{n,m}[P]$ on certain parts of the control polygon, consequently from (22) playing with the parameter m, we can approximate or go closer and closer to some control points or to other ones. In other words the two parameters c and m make the curve more flexible than in [4,5]. For example if $c = 0$, from (23) and (24) $\tilde{\mathcal{S}}_{n,m}[P]$ curve approximates the first parts of the control polygon and is very close to the last parts of it (see Example 4.3 in Section 4), while if $c = 1$ from (23) and (24) $\tilde{\mathcal{S}}_{n,m}[P]$ is very narrow to the first parts of the control polygon and approximates the last parts of it (see Example 4.3 in Section 4). Similarly if $c = 1/2$, from (23) and (24) $\tilde{\mathcal{S}}_{n,m}[P]$ is very narrow to the middle parts of the control polygon and is approximating the remaining parts of it (see Example 4.3 in Section 4).

4 Examples

4.1 Example 1

Consider the curve

$$(x(t), y(t), z(t)) = \left(\frac{1}{8}\log_q\left(t\left(q^8-1\right)-1\right), \cos 2\pi t, t\right),\ t \in [0,1]. \tag{25}$$

A sequence of 9 control points denoted by circle symbol in Fig. 1 is sampled from curve (25) as

$$(x(u_i), y(u_i), z(u_i)),\quad u_i = \frac{q^i - 1}{q^8 - 1},\quad i = 0,\dots,8,\ q = 1 + \frac{1}{8^{0.1}} \approx 1.8027. \tag{26}$$

Note that the mesh in (26) is thicker near 0. Starting with these control points, we fit curve (25) by $q-$analogue of Shepard-type curve on nodes mesh $u_i, i = 0,1,\dots,8$, given in (26) with $s = 4$ and $\lambda = 5\ 10^{-4}$ in Fig. 1. We also consider the Shepard-type curve on equispaced nodes $\overline{x}_i = i/8, i = 0,1,\dots,8$, for $s = 4$ and $\lambda = 5\ 10^{-4}$ in Fig. 1. Comparing the two curves we can see the modeling effect of $q-$integers nodes mesh, pushing the corresponding curve closer to the last parts of control polygon than to equispaced case and approximating the remaining parts of it.

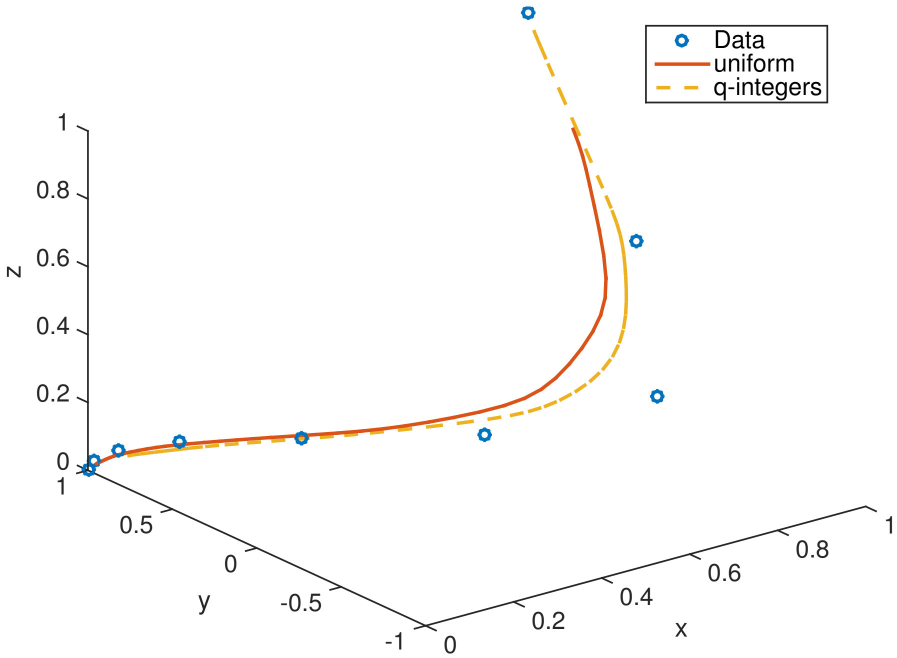

Fig. 1 Modeling by Shepard-type curves (20) on uniform and $q-$integers mesh

4.2 Example 2

Consider the curve

$$(x(t), y(t), z(t)) = (\sin 2\pi t, \cos 2\pi t, t), \quad t \in [0, 1]. \tag{27}$$

Then we sample a sequence of 9 control points from curve (27) as

$$(x(\overline{u}_i), y(\overline{u}_i), z(\overline{u}_i)), \ \overline{u}_i = 1 - \frac{q^{8-i} - 1}{q^8 - 1}, \ i = 0, \ldots, 8, \ q = 1 + \frac{1}{8^{0.1}} \approx 1.8027. \tag{28}$$

Note that $\overline{u}_0 = 0, \overline{u}_n = 1$ and the mesh in (28) is denser near 1. Starting with these control points, in Fig. 2 we fit curve (27) by $q-$analogue of Shepard-type curve on nodes mesh $\overline{u}_i$ given in (28) and by Shepard-type curve on equispaced nodes $\overline{x}_i = i/8$, $i = 0, 1, \ldots, 8$, for $s = 4$ and $\lambda = 4\ 10^{-6}$. From Fig. 2 we can see the shaping power of $q-$integers-type nodes, moving the corresponding curve nearer to the first parts of control polygon than to equispaced case and approximating the remaining parts of it.

4.3 Example 3

Consider a helix of radius 5 given by (cfr. [14])

$$(x(t), y(t), z(t)) = (5\cos 6\pi t, 5\sin 6\pi t, t), \ t \in [0, 1].$$

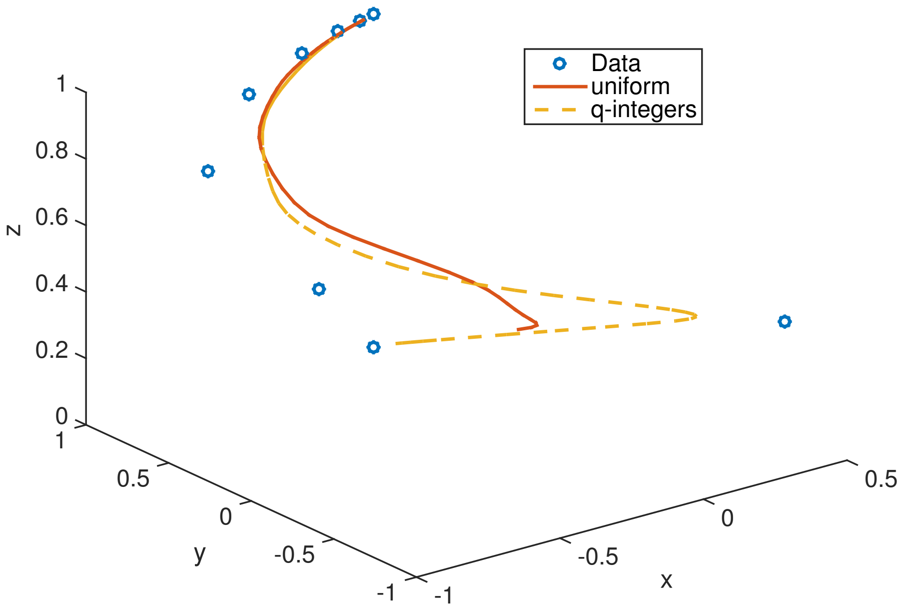

Fig. 2 Modeling by Shepard-type curves (20) on uniform and $q-$integers mesh

A sequence of 19 control points denoted by circle symbol in Fig. 3 is sampled from the helix as

$$(x(s_i), y(s_i), z(s_i)), \quad s_i = \frac{\pi}{3} i, \quad i = 0, \ldots, 18.$$

Starting with these control points we fit the helix by a sequence of five curves generated by processes (20), (21) with $m = 2$, (24) with $m = 2$ and $c = 0$, (24) with $m = 2$ and $c = 1$, (24) with $m = 2$ and $c = 1/2$, respectively, for $s = 4$ and $\lambda = 4\ 10^{-6}$. Comparing the five curves from Fig. 3 we can see the modeling power of technique (24), pushing the corresponding curves very close to the last, first, middle parts of control polygon respectively and approximating the remaining parts of it.

Analogous curves for $m = 3$ are given in Fig. 4, emphasizing the shape control behavior showed in Fig. 3. We stopped iterations at $m = 3$, since we met the stop criterion. So the designer can choose among different shapes modeling the helix.

5 Proofs

Proof of Theorem 1. Because of the interpolatory behaviour of S_n, we may assume $x \neq x_k$, $k = 0, \ldots, n$. Denote by x_j the closest knot to x, with $x_j \leq$

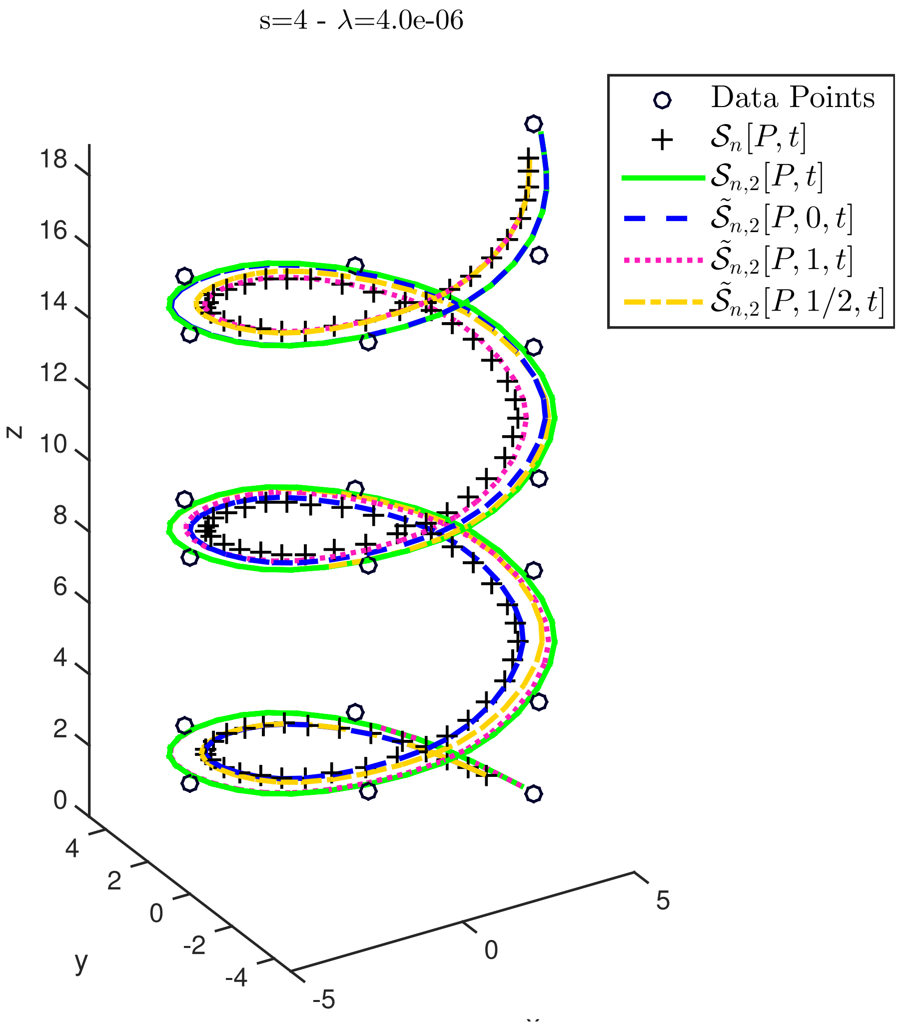

Fig. 3 Modeling the helix by curves (20), (21) and (24)

$x \le x_{j+1}$ (we can work analogously if the closest knot to x is x_{j+1}). Then

$$x_j \le x, \text{ i.e., } q^j < x(q^n - 1) + 1,$$

therefore

$$\begin{aligned}(x - x_j) \le x_{j+1} - x_j &= \frac{q^{j+1}-1}{q^n-1} - \frac{q^j-1}{q^n-1} = \frac{q^j(q-1)}{q^n-1} \\ &\le \left(x + \frac{1}{q^n-1}\right)\frac{\log_2 n}{n}.\end{aligned} \tag{29}$$

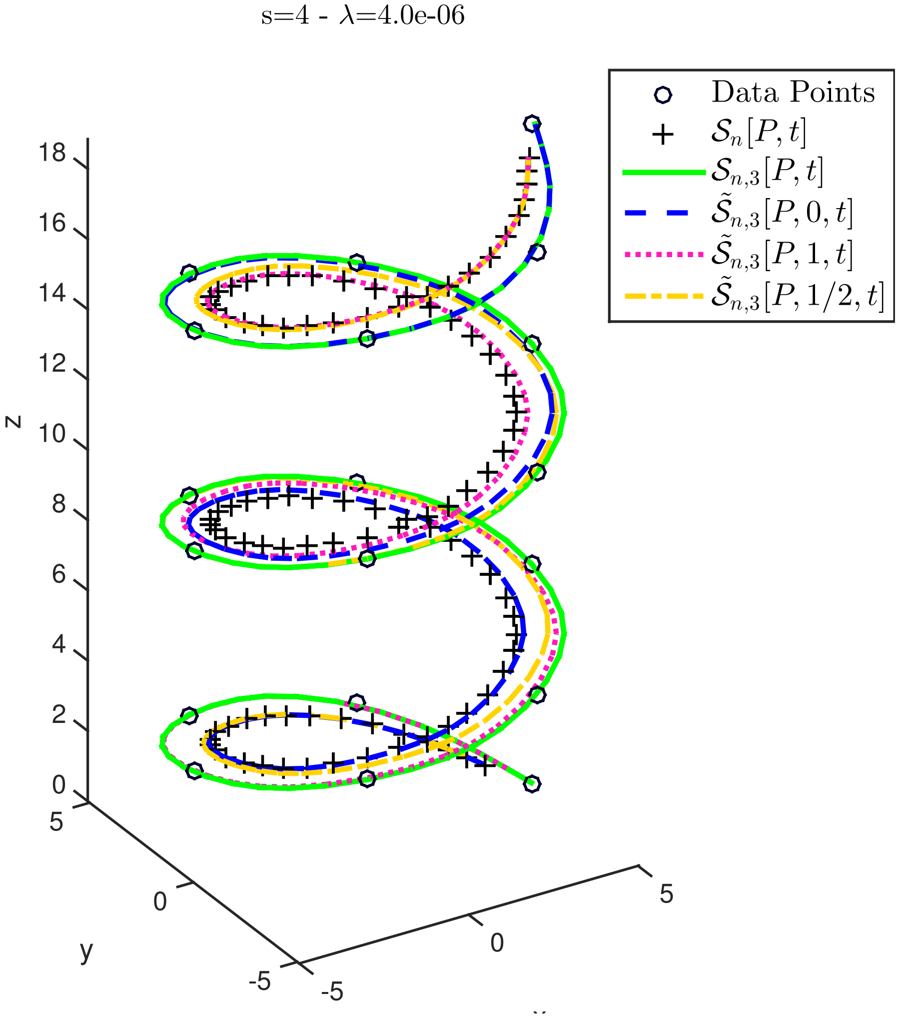

Fig. 4 Modeling the helix by curves (20), (21) and (24)

We have

$$\begin{aligned}|f(x)-S_n(f;x)| &\le \frac{\sum_{k=0}^{n}\dfrac{|f(x)-f(x_k)|}{(x-x_k)^s}}{\sum_{k=0}^{n}1/(x-x_k)^s}\\ &= \frac{\dfrac{|f(x)-f(x_j)|}{(x-x_j)^s}+\dfrac{|f(x)-f(x_{j+1})|}{(x-x_{j+1})^s}+\sum_{k\neq j,j+1}\dfrac{|f(x)-f(x_k)|}{(x-x_k)^s}}{\sum_{k=0}^{n}\dfrac{1}{(x-x_k)^s}}\\ &:= \Sigma_1+\Sigma_2+\Sigma_3.\end{aligned}$$

Working as usual (see, e.g., [8])

$$\Sigma_1 \le |f(x) - f(x_j)| \le \omega(f; |x - x_j|)$$

and

$$\begin{aligned}\Sigma_2 &\le |f(x) - f(x_{j+1})| \frac{|x - x_j|^s}{|x - x_{j+1}|^s} \le \omega\left(f; \frac{|x - x_{j+1}|}{|x - x_j|}|x - x_j|\right) \frac{|x - x_j|^s}{|x - x_{j+1}|^s} \\ &\le \left(1 + \frac{|x - x_{j+1}|}{|x - x_j|}\right) \frac{|x - x_j|^s}{|x - x_{j+1}|^s} \omega(f; |x - x_j|) \\ &\le 2\omega(f; |x - x_j|).\end{aligned}$$

Moreover

$$\begin{aligned}\Sigma_3 &\le \sum_{k \ne j, j+1} \frac{\omega(f; |x - x_k|)}{|x - x_k|^s}(x - x_j)^s \\ &\le \omega(f; |x - x_j|) \sum_{k \ne j, j+1} \frac{1 + |x - x_k|/|x - x_j|}{(x - x_k)^s}|x - x_j|^s \\ &= \omega(f; |x - x_j|) \sum_{k \ne j, j+1} \left(\frac{|x - x_j|^s}{|x - x_k|^s} + \frac{|x - x_j|^{s-1}}{|x - x_k|^{s-1}}\right) \\ &:= \Sigma_4 + \Sigma_5.\end{aligned}$$

First we estimate Σ_4 (we can work similarly for Σ_5). If $k < j$, then $x - x_k > x_j - x_k$, hence

$$\sum_{k<j} \frac{|x - x_j|^s}{|x - x_k|^s} \le \sum_{k<j} \frac{\left(q^{j+1} - q^j\right)^s}{(q^n - 1)^s} \frac{(q^n - 1)^s}{(q^j - q^k)^s}. \tag{30}$$

It results

$$q^j - q^k > q^k (j - k) \log_e q$$

and

$$\frac{q^j}{q^j - q^k} < \frac{q^j - q^k + q^k}{q^j - q^k} = 1 + \frac{q^k}{q^j - q^k} \le 1 + \frac{1}{(j - k) \log_e q},$$

from which

$$\frac{q^{js}(q - 1)^s}{(q^j - q^k)^s} \le \left(\frac{\log_2 n}{n}\right)^s \left(1 + \frac{1}{(j - k) \log_e q}\right)^s.$$

If $k > j + 1$, then $x_k - x > x_k - x_{j+1}$, consequently

$$\sum_{k>j+1} \frac{|x - x_j|^s}{|x - x_k|^s} \le \sum_{k>j+1} \frac{\left(q^{j+1} - q^j\right)^s}{(q^n - 1)^s} \frac{(q^n - 1)^s}{(q^k - q^{j+1})^s}.$$

If $k > j+1$, then $q^k - q^{j+1} > q^{j+1}(k-j-1)\log_e q$ and

$$\frac{q^{js}(q-1)^s}{(q^k-q^j)^s} \leq \frac{\log_2^s n}{q^s n^s (k-j-1)^s \log_e^s q} \leq \frac{\log_2^s n}{n^s (k-j-1)^s \log_e^s q}. \tag{31}$$

Hence working as usual (see, e.g., [8]) from (30)–(31)

$$\Sigma_4 \leq C\omega(f; |x-x_j|). \tag{32}$$

Now if $[x]$ denotes the integer part of x, from the well-known inequality $(1+1/m) \geq 2, \forall m \in \mathbb{N}$, we obtain for $n > 1$

$$\begin{aligned}(1+\log_2 n/n)^n &= \left(1+\frac{1}{n/\log_2 n}\right)^n \geq \left(1+\frac{1}{[n/\log_2 n]+1}\right)^n \\ &= \left(1+\frac{1}{[n/\log_2 n]+1}\right)^{\frac{n([n/\log_2 n]+1)}{[n/\log_2 n]+1}} \geq \\ &\geq 2^{\frac{n}{[n/\log_2 n]+1}} \geq 2^{\frac{n}{n/\log_2 n+1}} \geq 2^{\log_2 n - \frac{\log_2^2 n}{n+\log_2 n}}.\end{aligned}$$

So

$$\frac{1}{(1+\log_2 n/n)^n - 1} \leq \frac{2^{\log_2^2 n/(n+\log_2 n)}}{n - 2^{\log_2^2 n/(n+\log_2 n)}} \leq \frac{C}{n}.$$

Therefore from (29) and (32), (7) follows. □

Proof of Theorem 2. Following the proof of Theorem 1, one gets

$$|x-x_j| \leq \left(x + \frac{1}{q^n-1}\right)(q-1).$$

Working as in the proof of Theorem 1, the assertion follows. □

Proof of Theorem 3. From the proof of Theorem 1, we deduce

$$|y_{j+1}-y_j| = \frac{q^j(q-1)}{q^{n/2}-1} \quad \text{and } x - y_j \leq \left(1-x+\frac{1}{q^{n/2}-1}\right)(q-1)$$

and working as in the proof of Theorem 1, we obtain (16). □

Proof of Theorem 4. We can follow the proof of Theorems 1 and 3 and get

$$x - z_j \leq \left(x(1-x) + \frac{1}{q^{n/2}-1}\right)(q-1).$$

If $x, z_k > 1/2$, we can work as in the proof of Theorem 1. If $x > 1/2$ and $z_k < 1/2$, then $|x-z_k| > |x-1/2+z_k|$ and we can follow the proof of Theorem 1. The case $x < 1/2$ can be treated similarly.

□

Proof of Theorem 5. We can follow the proof of Theorems 1 and 2 and conclude

$$x - t_j \le \left(\left| x - \frac{1}{2} \right| + \frac{1}{q^{n/2} - 1} \right) (q-1).$$

Working as in the proof of Theorem 4 we deduce (11). □

Proof of Theorem 6. Working as in the proof of Theorem 1, denote by u_{j+1} the closest knot to x, with $u_j \le x \le u_{j+1}$ (we can work analogously if the closest knot to x is u_j). Then

$$x \le u_{j+1}, \text{ i.e., } q^{j+1} < 1 - x(1-q^n),$$

therefore

$$\begin{aligned} (u_{j+1} - x) &\le u_{j+1} - u_j = \frac{1-q^{j+1}}{1-q^n} - \frac{1-q^j}{1-q^n} = \frac{q^j(1-q)}{1-q^n} \\ &\le C \left(\frac{1}{1-q^n} - x \right) \frac{\log_2 n}{n}. \end{aligned} \tag{33}$$

We have

$$\begin{aligned} |f(x) - S_n(U;f;x)| &\le \frac{\sum_{k=0}^{n} \frac{|f(x)-f(u_k)|}{(x-u_k)^s}}{\sum_{k=0}^{n} 1/(x-u_k)^s} \\ &= \frac{\frac{|f(x)-f(u_j)|}{(x-u_j)^s} + \frac{|f(x)-f(u_{j+1})|}{(x-u_{j+1})^s} + \sum_{k \ne j,j+1} \frac{|f(x)-f(u_k)|}{(x-u_k)^s}}{\sum_{k=0}^{n} \frac{1}{(x-u_k)^s}} \\ &:= \overline{\Sigma}_1 + \overline{\Sigma}_2 + \overline{\Sigma}_3. \end{aligned}$$

Working as before

$$\overline{\Sigma}_1 \le \frac{|f(x)-f(u_j)|(x-u_{j+1})^s}{(x-u_j)^s} \le C\omega(f; u_{j+1} - x)$$

and

$$\overline{\Sigma}_2 \le \omega(f; u_{j+1} - x).$$

Moreover

$$\begin{aligned} \overline{\Sigma}_3 &\le \sum_{k \ne j,j+1} \frac{\omega(f; |x-u_k|)}{|x-u_k|^s} (u_{j+1}-x)^s \\ &\le \omega(f; u_{j+1}-x) \sum_{k \ne j,j+1} \frac{1 + |x-u_k|/(u_{j+1}-x)}{(x-u_k)^s} (u_{j+1}-x)^s \\ &= \omega(f; u_{j+1}-x) \sum_{k \ne j,j+1} \left(\frac{(u_{j+1}-x)^s}{|x-u_k|^s} + \frac{(u_{j+1}-x)^{s-1}}{|x-u_k|^{s-1}} \right) \\ &:= \overline{\Sigma}_4 + \overline{\Sigma}_5. \end{aligned}$$

Following the proof of Theorem 1 we obtain

$$\overline{\Sigma}_4 \leq C\omega(f; u_{j+1} - x).$$

Analogously we can work for $\overline{\Sigma}_5$.

Working as in the proof of Theorem 1, we can verify that $q^n < \frac{1}{n}$, hence $1/(1-q^n) < 1 + \frac{1}{n-1}$. From (33) collecting above estimates, the assertion follows.

□

Proof of Theorem 7. Following the proof of Theorem 6, one gets

$$u_{j+1} - x \leq C\left(\frac{1}{1-q^n} - x\right)(1-q).$$

Working as in the proof of Theorem 6, we deduce Theorem 7. □

Proof of Theorem 8. Working as in the proof of Theorem 1 and Theorem 6, denote by $\overline{y}_{j+1}$ the closest knot to x, with $\overline{y}_{j+1} < x < \overline{y}_j$ (we can work analogously if the closest knot to x is $\overline{y}_j$). Then

$$q^{j+1} < 1 - (1-x)(1-q^n),$$

therefore

$$\begin{aligned} x - \overline{y}_{j+1} \leq \overline{y}_j - \overline{y}_{j+1} &= -\frac{1-q^j}{1-q^n} + \frac{1-q^{j+1}}{1-q^n} \\ &= \frac{q^j}{1-q^n}\frac{\log_2 n}{n} \leq C\left(\frac{1}{1-q^n} - (1-x)\right)\frac{\log_2 n}{n}. \end{aligned}$$

Working as in the proof of Theorem 6, since $1/(1-q^n) - 1 + x \leq x + 1/(n-1)$, the statement follows. □

Proof of Theorem 9. We can follow the proof of Theorems 1 and 5 and get

$$x - \overline{z}_{j+1} \leq C\left(\left|x - \frac{1}{2}\right| + \frac{1}{n}\right)\frac{\log_2 n}{n}.$$

Working as in the proofs above, we obtain Theorem 9. □

Proof of Theorem 10. We can follow the proof of Theorems 1 and 6 and conclude

$$x - \overline{t}_j \leq C\left(x(1-x) + \frac{1}{n}\right)\frac{\log_2 n}{n}.$$

Working as in the proof of Theorem 4, the assertion follows. □

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