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Constrained Control of Linear Discrete-Time Systems under Quartic Performance Criterion

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***Abstract*—In several applications, a performance criterion that is *quartic* in the state can be a desired alternative to the classic quadratic control. This paper proposes a receding horizon controller for linear time-invariant systems subject to linear state and input constraints, which makes use of a running cost that is quadratic in the input and quartic in the state. Stability and recursive feasibility of the proposed receding horizon scheme are proven. Numerical simulations are presented, considering the problem of controlling a single-link inverted pendulum on a cart.**

***Index Terms*—Linear systems, Lyapunov methods, optimal control, predictive control for linear systems.**

I. INTRODUCTION

THIS paper focuses on the constrained control of discrete-time linear time invariant systems under a performance criterion that is quadratic in the control and *quartic* in the state variables. This problem can be seen as a natural extension of the classic model predictive control (MPC) with quadratic cost. The quadratic cost is arguably the most used performance criterion in control theory, mostly because of the linear quadratic (LQ) regulator, which has ubiquitous applications due to its simplicity, stability properties, the energy minimization concept that it naturally embeds, and, most of all, because it admits, under mild conditions and even in the infinite-horizon case, a closed expression of the optimal control as a function of the current state [1], [2].

In some applications it is however preferable to consider performance criteria with higher-order terms in the state variables. This choice is desirable in practice when a more "gentle" control action is desired close to the operating point, and a more aggressive one farther from it.

Some examples of applications of quartic optimal control, and higher order control more in general, are discussed in the following. Among the pioneering works, [3] provides the solution for the unconstrained problem in continuous-time, with performance criteria that are the integral of sums of positive semi-definite homogeneous polynomials of positive degrees equal or greater than 2 (see (4) in [3]). The motivation for their work is to derive nonlinear control laws to keep the evolution of the state variables within allowed bounds. A closed-form solution is achieved by introducing a constraint to limit "the mean amplitude of the square of the linear and the square of the nonlinear terms in the optimal control law" (see (30) in [3]). It is shown that the optimal control corresponding to quartic terms in the objective function is cubic in the state. In [4], the main paper inspiring the present work, the authors address the problem of human postural regulation and balancing about the standing upright position. As the authors explain, empirical evidence shows that balance control is characterised by small amplitude motion (called sway motion) close to the upright posture, and more aggressive control actions when farther from the equilibrium. The authors in [4] therefore propose a *finite-time* quartic optimal control problem for balance maintenance, and perform simulations to compare it with LQ control. In particular, [4] discusses both finite-time open loop control, in which the finite-time optimal control problem is solved once, and the corresponding closed-loop, MPC-like implementation, considering the same performance criterion and re-optimizing at each time. No terminal constraints and no terminal costs are considered in [4]. Simulations in [4] show that the quartic controller results in motion compatible with experimental evidence on human balancing and consume less energy that the LQ controller.

In [5] the authors propose, among the others, a quartic performance criterion to tune a proportional integral derivative (PID) controller and an approximate method to solve the deriving optimization problem. In [6], higher order performance criteria are explored for the problem of optimal control in continuous-time, with the motivation of deriving a feedback control law $u(x)$ "which is super-linear, i.e., $u(x)$ is progressive for larger values of the state x" [6] (i.e. to implement soft box constraints, as also done in [3]). A method based on results in [7] is presented to derive an approximated solution of the problem.

In [8] the authors use a quartic optimal control law to design a "suspension controller that reacts in a soft way to small disturbances and in a hard way to large ones", which improves suspension performance by reducing wheel hop. The control problem is set in continuous time and a procedure is presented to derive a suboptimal control law u as an expansion of a desired number of higher order terms.

In [9] quartic terms are considered in the receding horizon control of satellites motion, with the objective of improving trajectory tracking. The stability of the resulting control is not proved but only shown via simulations.

Finally, [10] presents an application in the field of macroeconomy, and compares the results derived by choosing different objective functions, including a quartic one [10].

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In this context, and motivated by the above works, in this paper we present a receding horizon strategy with guaranteed recursive feasibility and convergence properties and where the performance criterion is quartic in the state and quadratic in the control. As customary in the MPC literature [2], we do this via the selection of proper "terminal ingredients" (terminal set and terminal cost term), which allow to derive suitable upper bounds on the infinite-time optimal cost.

The reminder of the paper is organized as follows. Section II presents the formulation of the problem statement. Section III presents the proposed quartic control strategy approximating the control problem in Section II. Section IV presents simulations to validate the proposed approach and compare it with classic LQ-based MPC and with the quartic MPC controller proposed in [4]. Finally, Section V presents conclusions and discusses future works.

II. PROBLEM STATEMENT

The nomenclature in this paper is consistent with the one adopted in reference MPC works (see e.g. [1]).

Consider a linear time invariant (LTI) system described by the state-space model

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \tag{1}$$

where $x_k \in \mathrm{R}^{\mathrm{n_x}}$, $u_k \in \mathrm{R}^{\mathrm{n_u}}$ and $y_k \in \mathrm{R}^{\mathrm{n_y}}$. We assume the system is subject to affine constraints in the state and the inputs, which can be written without loss of generality as

$$Fx_k + Gu_k \le \mathbf{1} \tag{2}$$

where $\mathbf{1}$ is a column vector of ones of appropriate dimension.

In line of principle in this paper we want to develop an MPC scheme able to satisfy constraints (2) and where at each time instant we minimize a cost index that is quartic in the state variables and quadratic in the control variables

$$J(x_k, \{u_k, u_{k+1}, ...\}) = \sum_{i=k}^{\infty} \left[(x_i^T Q x_i)^2 + u_i^T R u_i \right], \tag{3}$$

with $\{u_k, u_{k+1}, ...\}$ the infinite sequence of control variables.

In the following, we denote with J^*, u_k^* and x_k^*, respectively, the optimal value of the objective function over the admissible control sequences, the optimal control at time k, and the optimal controlled state at time k.

We make the following standard assumption.

Assumption 1. $Q \ge 0$ and $R > 0$, (A, B) is stabilizable and (A, G) is observable, with G the Cholesky factorization of Q (i.e., $Q = G^T G$).

The condition "(A, B) stabilizable" is needed as otherwise the uncontrollable and unstable state components would make (3) go to infinity.

Observation 1. Given Assumption 1, the optimal cost

$$J^*(x_k) := \min_{\{u_k, u_{k+1}, \dots\}} J(x_k, \{u_k, u_{k+1}, ...\}) \tag{4}$$

is a positive definite function of x_k, i.e., $J^*(x_k) = 0$ if and only if $x_k = 0$.

Proof. It is clear that $J^*(x_k) \ge 0 \; \forall x_k$. It is also clear that if $x_k = 0$ then $J^*(x_k) = 0$. Vice versa, if $J^*(x_k) = 0$, it must be first of all $u_i^* = 0 \; \forall i \ge k$, since $R > 0$ ($u_i^* \ne 0$ for some i would otherwise result in $J^*(x_k) > 0$). Therefore from (1) we have $x_i^* = A^{i-k} x_k$, $\forall i \ge k$ and hence $J^*(x_k) = \sum_{i=k}^{\infty} (x_k^T {A^{i-k}}^T G^T G A^{i-k} x_k)^2$. Now $J^*(x_k) = 0$ implies that $GA^{i-k} x_k = 0 \; \forall i \ge k$ which, given the assumed condition "(A, G) observable", implies $x_k = 0$. □

Observation 2. Given Assumption 1, the optimal cost $J^*(x_k)$ is finite, and

$$\lim_{k \to \infty} x_k^* = 0 \;\; \text{and} \;\; \lim_{k \to \infty} u_k^* = 0. \tag{5}$$

Proof. The optimal cost is finite because the series (3), evaluated at the optimum, is monotonically increasing and upper bounded. An upper bound can be found indeed by considering the corresponding LQ problem[1]. It is known (see e.g. [1]) that in the LQ case $J^{*LQ}(x_k)$ is finite, and that $\lim_{k\to\infty} x_k^{*LQ} = 0$ and $\lim_{k\to\infty} u_k^{*LQ} = 0$. The optimal LQ control sequence $\{u_k^{*LQ}, u_{k+1}^{*LQ}, \dots\}$ is a feasible control sequence for the quartic problem, hence $J^*(x_k) \le J(x_k, \{u_k^{*LQ}, u_{k+1}^{*LQ}, \dots\})$. Finally, $J(x_k, \{u_k^{*LQ}, u_{k+1}^{*LQ}, \dots\})$ is finite because it is $({x_i^{*LQ}}^T Q x_i^{*LQ})^2 \le {x_i^{*LQ}}^T Q x_i^{*LQ}$ for ${x_i^{*LQ}}^T Q x_i^{*LQ} \le 1$, which is eventually the case since $\lim_{k\to\infty} x_k^{*LQ} = 0$. □

Remark 1. Like in the case of classic LQ infinite-time control, finding the value of $J^*(x_k)$ in the *unconstrained* quartic case, as a function of the initial state x_k, would allow to approximate the optimal *constrained* linear quartic control via MPC schemes (potentially exactly for large control horizons). In this paper we propose a receding horizon approximation of the infinite time quartic control through a particular selection of a terminal set and terminal cost function. These two are, together with the stage cost function, the standard "ingredients" that are used in the literature to establish the stability of MPC algorithms (see e.g. [1], [2]). The novel contribution of the paper in particular is in the inclusion of the condition $x^T Q x \le 1$, already emerged in Observation 2, for the design of the terminal set, under which the stability of the proposed controller can be proven.

III. PROPOSED RECEDING HORIZON APPROXIMATION

The main idea of the proposed method is that of considering an MPC scheme with an "hybrid" objective function, which is quartic for the explicitly computed predictions up to the horizon N and quadratic afterward. To do this, we approximate the "pure" quartic objective function (3) with

$$\begin{aligned} \tilde{J}(x_k, \{u_k, u_{k+1}, ...\}) := \sum_{i=k}^{k+N-1} & \left[(x_i^T Q x_i)^2 + u_i^T R u_i \right] + \\ & \sum_{i=k+N}^{\infty} \left[x_i^T Q x_i + u_i^T R u_i \right]. \end{aligned} \tag{6}$$

[1]We denote in the following with the superscript LQ all the quantities related to the LQ problem corresponding to the quartic one (i.e. the LQ problem obtained by dropping exponent 2 in (3)).

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As well known [11], the optimal control u_i^* for (6) for $i \geq k+N$ in the unconstrained case is the optimal LQ regulator:

$$u_i^* = Kx_i^* \tag{7}$$

with $K = -(B^TWB+R)^{-1}B^TWA$, where $W > 0$ is the solution of the algebraic Riccati equation $W = A^TWA + Q - A^TWB(B^TWB+R)^{-1}B^TWA$ (see e.g. [1], Theorem 2.1). From the theory of optimal LQ control it is known as well that $\sum_{i=k+N}^{\infty}\left[x_i^{*T}Qx_i^* + u_i^{*T}Ru_i^*\right] = x_{k+N}^{*}{}^TWx_{k+N}^*$. Hence the proposed receding horizon approximation of the optimal constrained quartic control considers, at the generic time k, the following target function:

$$\begin{aligned}\tilde{J}(x_k,\{\hat{u}_{0|k},\hat{u}_{1|k},\ldots,\hat{u}_{N-1|k}\}) = \sum_{i=0}^{N-1}\Big[(\hat{x}_{i|k}^TQ\hat{x}_{i|k})^2 + \\ + \hat{u}_{i|k}^TR\hat{u}_{i|k}\Big] + \hat{x}_{N|k}^TW\hat{x}_{N|k}\end{aligned} \tag{8}$$

where, as customary in MPC studies, notation $\hat{x}_{i|k}$ and $\hat{u}_{i|k}$ is introduced to refer to the predicted value of the state and control at time $k+i$, referred to the optimization problem taking place at time k, with the initial condition $\hat{x}_{0|k} = x_k$.

The proposed MPC strategy is presented in Algorithm 1.

Algorithm 1 (Hybrid Quartic-Quadratic MPC). At each time $k = 1, 2, ...$ do:

1) Solve the optimization problem:

$$\underset{\{\hat{u}_{0|k},\hat{u}_{1|k},\ldots,\hat{u}_{N-1|k}\}}{\text{argmin}} \tilde{J}(x_k,\{\hat{u}_{0|k},\hat{u}_{1|k},\ldots,\hat{u}_{N-1|k}\}) \tag{9}$$

subject to:

$$F\hat{x}_{i|k} + G\hat{u}_{i|k} \leq \mathbf{1} \quad \text{for} \quad i = 0,1,...,N-1 \tag{10}$$

$$\hat{x}_{0|k} = x_k \tag{11}$$

$$\hat{x}_{i+1|k} = A\hat{x}_{i|k} + B\hat{u}_{i|k} \quad \text{for} \quad i = 0,1,...,N-1 \tag{12}$$

$$\hat{x}_{N|k} \in \mathbb{X}_f \tag{13}$$

where $\mathbb{X}_f \subseteq \mathrm{R}^{\mathrm{n_x}}$ is a (terminal) set that under the optimal LQ control (7) is: i) positive invariant for system (1); and ii) for any $x \in \mathbb{X}_f$, the constraints $Fx + GKx \leq \mathbf{1}$ and the fictitious quadratic constraint $x^TQx \leq 1$ hold true;

2) Apply to the system the first sample of the optimal control sequence, $u_k = \hat{u}_{0|k}^*$.

The following theorem gives a possible way to compute the terminal set $\mathbb{X}_f$. [2]

Theorem 1 (Computation of $\mathbb{X}_f$). The set $\mathbb{X}_f = \{x : x^TWx \leq \gamma^2\}$, such that γ is solution of:

$$\begin{aligned}&\underset{\gamma}{\text{argmax}}\,\gamma \\ &\text{subject to:} \\ &\gamma \leq \frac{1}{\max_i\{||[(F+GK)W^{-\frac{1}{2}}]_i||\}} \\ &Q - \gamma^{-2}W \leq 0,\end{aligned} \tag{14}$$

where $[\cdot]_i$ denotes the i-th row of a matrix, satisfies the following properties:

i) is positive invariant for the system (1) under control law $u = Kx$ (i.e., if $x_k \in \mathbb{X}_f$ then $(A+BK)x_k \in \mathbb{X}_f$);

ii) ensures that $(F+GK)x \leq \mathbf{1}\ \forall x \in \mathbb{X}_f$;

iii) ensures that $x^TQx \leq 1\ \forall x \in \mathbb{X}_f$.

Proof. i) follows from the fact that $\mathbb{X}_f$ is a level set of the Lyapunov function x^TWx (so that trajectories starting in $\mathbb{X}_f$ will remain in $\mathbb{X}_f$); ii) the ellipsoid $\mathbb{X}_f$ can be equivalently defined as $\mathbb{X}_f = \{x : x = \gamma W^{-\frac{1}{2}}y, \forall ||y|| \leq 1\}$ (see e.g. (5)-(7) in [12]). Hence it is $(F+GK)x \leq \mathbf{1}\ \forall x \in \mathbb{X}_f$ (i.e. $\mathbb{X}_f$ lies within the constraints polyhedron $(F+GK)x \leq \mathbf{1}$) if and only if $(F+GK)\gamma W^{-\frac{1}{2}}y \leq \mathbf{1}$, $\forall ||y|| \leq 1$. This last constraint can be equivalently written row by row as $\gamma[(F+GK)W^{-\frac{1}{2}}]_iy \leq 1$, $\forall i$, $||y|| \leq 1$ (recall that $[\cdot]_i$ denotes the i-th row of a matrix), which is equivalent to $\gamma\sup_{||y||\leq 1}[(F+GK)W^{-\frac{1}{2}}]_iy \leq 1$, $\forall i$, which in turn is equivalent to $\gamma||[(F+GK)W^{-\frac{1}{2}}]_i|| \leq 1\ \forall i$ (this last equivalence following from the fact that $\sup_{||y||\leq 1}[(F+GK)W^{-\frac{1}{2}}]_iy = ||[(F+GK)W^{-\frac{1}{2}}]_i||$, because of the definition of the scalar product of two vectors). Hence, it is $(F+GK)x \leq \mathbf{1}\ \forall x \in \mathbb{X}_f$ if and only if $\gamma||[(F+GK)W^{-\frac{1}{2}}]_i|| \leq 1\ \forall i$, i.e., equivalently, $\gamma \leq 1/\max_i\{||[(F+GK)W^{-\frac{1}{2}}]_i||\}$; iii) the linear matrix inequality $Q-\gamma^{-2}W \leq 0$ implies $x^TQx \leq \gamma^{-2}x^TWx \leq 1$ in $\mathbb{X}_f$ (since $x^TWx \leq \gamma^2$ in $\mathbb{X}_f$). Note that choosing γ as in (14) ensures that $\mathbb{X}_f$ has the maximum possible volume. □

The following two results prove, respectively, the feasibility and the stability of the proposed algorithm.

Theorem 2 (Recursive feasibility of Algorithm 1). Algorithm 1 is recursively feasible, i.e., if it is feasible at a given time k, then it feasible for all $i \geq k$.

Proof. This can be proven with standards arguments (see e.g. [1], Section 2.5) by showing that if $\hat{\mathbf{u}}_{\mathbf{k}}^* = \{\hat{u}_{0|k}^*, \hat{u}_{1|k}^*, ..., \hat{u}_{N-1|k}^*\}$ is a solution for (9)-(13) at time k, then since under the state feedback $u = Kx$ every state in the terminal set $\mathbb{X}_f$ satisfies the constraints (10)-(13) and at the next step remains in the same set (positive invariance), the sequence $\tilde{\mathbf{u}}_{\mathbf{k+1}} = \{\hat{u}_{1|k}^*, ..., \hat{u}_{N-1|k}^*, K\hat{x}_{N|k}\}$ is feasible for (9)-(13) at time $k+1$. □

Theorem 3 (Stability). Under Algorithm 1, $x = 0$ is an asymptotically stable equilibrium point for system (1).

Proof. Consider the solution of the optimization step at time k, $\hat{\mathbf{u}}_{\mathbf{k}}^* = \{\hat{u}_{0|k}^*, \hat{u}_{1|k}^*, ..., \hat{u}_{N-1|k}^*\}$, and derive from it the *feasible* (but in general not optimal) control sequence at time $k+1$ given by $\tilde{\mathbf{u}}_{\mathbf{k+1}} = \{\hat{u}_{1|k}^*, ..., \hat{u}_{N-1|k}^*, K\hat{x}_{N|k}\}$. From this choice of $\tilde{\mathbf{u}}_{\mathbf{k+1}}$ it follows that

$$\begin{aligned}&\tilde{J}^*(x_{k+1}^*) \leq \tilde{J}(x_{k+1}^*, \tilde{\mathbf{u}}_{\mathbf{k+1}}) = \tilde{J}^*(x_k^*) + \\ &- \left[\left(\hat{x}_{0|k}^{*T}Q\hat{x}_{0|k}^*\right)^2 + \hat{u}_{0|k}^{*T}R\hat{u}_{0|k}^*\right] - \hat{x}_{N|k}^{*T}W\hat{x}_{N|k}^* + \\ &\left(\hat{x}_{N|k}^{*T}Q\hat{x}_{N|k}^*\right)^2 + \hat{x}_{N|k}^{*T}K^TRK\hat{x}_{N|k}^* + \\ &\hat{x}_{N|k}^{*T}(A+BK)^TW(A+BK)\hat{x}_{N|k}^* \leq \\ &\leq \tilde{J}^*(x_k) - \left[\left(\hat{x}_{0|k}^{*T}Q\hat{x}_{0|k}^*\right)^2 + \hat{u}_{0|k}^{*T}R\hat{u}_{0|k}^*\right].\end{aligned} \tag{15}$$

[2]In the theorem and in the rest of the paper, we use the Euclidean norm for vectors (i.e. $||x|| = \sqrt{x^Tx}$, $x \in \mathrm{R}^{n_x}$).

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Where we have used the two facts: i) $(\hat{x}^{*T}_{N|k}Q\hat{x}^{*}_{N|k})^2 \leq \hat{x}^{*T}_{N|k}Q\hat{x}^{*}_{N|k}$ (since $\hat{x}^{*}_{N|k} \in \mathbb{X}_f$ and by the definition of $\mathbb{X}_f$ $\hat{x}^{T}_{N|k}Q\hat{x}_{N|k} \leq 1$) and, ii) $\hat{x}^{*T}_{N|k}Q\hat{x}^{*}_{N|k} + \hat{x}^{*T}_{N|k}K^TRK\hat{x}^{*}_{N|k} + \hat{x}^{*T}_{N|k}(A+BK)^TW(A+BK)\hat{x}^{*}_{N|k} = \hat{x}^{*T}_{N|k}W\hat{x}^{*}_{N|k}$. Hence we have

$$\tilde{J}^*(x^*_{k+1}) - \tilde{J}^*(x^*_k) \leq -\left[\left(\hat{x}^{*T}_{0|k}Q\hat{x}^{*}_{0|k}\right)^2 + \hat{u}^{*T}_{0|k}R\hat{u}^{*}_{0|k}\right] \quad (16)$$

which shows that $\tilde{J}^*$ is a Lyapunov function [3] strictly decreasing along the trajectories of the controlled system, which proves the asymptotic stability. □

Remark 2. The result could be alternatively proven using Theorem 2.24 in [2], which in turn would require to prove that some needed assumptions are valid for the case at study, which is not trivial, in particular for Assumptions 2.23 (a) and (b) in [2]. The proof presented is based instead on the standard argument of using a feasible control sequence to derive the needed bounds on the optimal cost [1].

Remark 3. The fictitious constraint $\hat{x}^{*T}_{N|k}Q\hat{x}^{*}_{N|k} \leq 1$ enforced on the terminal set is the key tool used to prove the stability of the proposed MPC quartic controller. This condition ensures than in the terminal set $\mathbb{X}_f$ the quadratic cost in the state is an upper bound of the quartic one, which in turns allows us in (15) to fall back via bounding to quadratic terms that cancel out. As done in Theorem 1, $\mathbb{X}_f$ should be designed as the maximal one satisfying the required properties (see [1] Section 2.7.1 for other approaches to the design of $\mathbb{X}_f$).

Observation 3 (Bounds on the optimal costs). From (16), by summing both members for $k \in \{0, 1, 2, \dots\}$, a bound for the closed loop performance achieved by the proposed MPC algorithm, which we call $\tilde{J}^*_{MPC}(x_0)$, is found as[4]:

$$\begin{aligned}\tilde{J}^*_{MPC}(x_0) &:= \sum_{k=0}^{\infty}\left[\left(\hat{x}^{*T}_{0|k}Q\hat{x}^{*}_{0|k}\right)^2 + \hat{u}^{*T}_{0|k}R\hat{u}^{*}_{0|k}\right] \leq \\ &\leq \tilde{J}^*(x_0) - \lim_{k\to\infty}\tilde{J}^*(x^*_k) = \tilde{J}^*(x_0),\end{aligned} \quad (17)$$

since $\lim_{k\to\infty}\tilde{J}^*(x^*_k) = 0$. Also, we have $\tilde{J}^*(x_0) \geq \tilde{J}^*_{MPC}(x_0) \geq J^*(x_0)$ where $\tilde{J}^*(x_0) \geq \tilde{J}^*_{MPC}(x_0)$ derives from (17), while $\tilde{J}^*_{MPC}(x_0) \geq J^*(x_0)$ derives from the fact that the proposed algorithm provides a sub optimal solution to the original quartic infinite time problem. Future work is devoted to characterize and reduce the gap $\tilde{J}^*_{MPC}(x_0) - J^*(x_0)$, by devising new approximations of the quartic objective function (3), as outlined in Section V.

IV. NUMERICAL TESTS ON A CART-INVERTED PENDULUM

We test the proposed methodology on the control of an inverted pendulum on a cart (Fig. 1), to provide an illustrative and instructive comparison with classic LQ MPC and quartic MPC in [4]. As well known, the nonlinear differential equations characterizing the system are (see e.g. [13]):

$$(M+m)\ddot{s} + b\dot{s} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = u \quad (18)$$

$$(I+ml^2)\ddot{\theta} + mgl\sin\theta + ml\ddot{s}\cos\theta = 0 \quad (19)$$

where $s(t)$ is the horizontal displacement of the cart and $\theta(t)$ the angular displacement of the pendulum about the vertical. The above equations can be rewritten in the state space form, with the state given by $x = [s, \dot{s}, \theta, \dot{\theta}]^T$ and $u \in \mathrm{R}$.

Simulations have been performed using Julia 0.7 [14]. Two models of the system are considered in the simulations:

1) An LTI discrete-time *control model*, obtained from the discretization of the linearization of (18) and (19), about the unstable equilibrium ($\bar{x}^T = [0, 0, \pi, 0]$, $\bar{u} = 0$). The system is discretized with a zero-order hold, and sampling rate of $50Hz$, which is compatible with the solving time of the optimization problem in Algorithm 1. This model is used in Algorithm 1 to compute, through (2), the (approximated) evolution of the systems within the control window $[k, k+N-1]$ (k being the current time);
2) A *simulation model*, i.e., the nonlinear representation (18) and (19), used to accurately simulate the evolution of the system when controlled with Algorithm 1.

The optimization problem in Algorithm 1 is convex and nonlinear (quartic polynomial). It has been modeled in Julia with the JuMP library [15] and solved with IPOPT [16] (for tailored algorithms for solving quartic optimization see e.g. [17], [18]). Equations (18) and (19) have been numerically integrated in Julia using the package DifferentialEquations.jl [19]. All the above tools are open-source.

Parameters in (18) and (19) are as in [20]: $M = 0.5Kg$, $m = 0.2$, $b = 0.1$, $I = 0.006$, $g = 9.8$, $l = 0.3$ with the appropriate units in the international system of units. We consider the following constraints:

$$\begin{aligned}-2 \leq x_1 \leq 2 \\ -2.5 \leq u(t) \leq 2.5\end{aligned} \quad (20)$$

The main control design parameters are Q, R and N. The reader is referred to e.g. [5], [21] for a discussion on tuning of the parameters, in the LQ case.

In the following we explore different parameters selections with the objective of illustrating the peculiar features of the proposed controller and the comparison with classic quadratic MPC control and the quartic MPC control in [4].

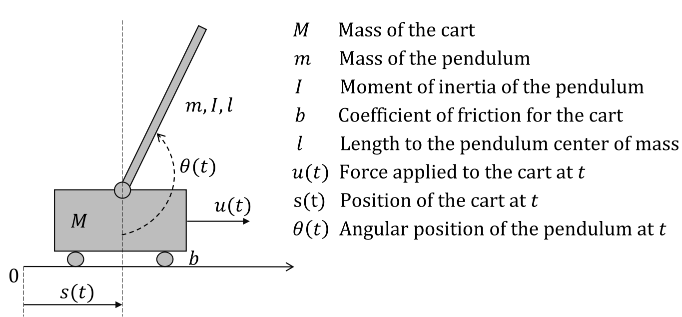

Fig. 1. Single-link inverted pendulum on a cart.

[3] $\tilde{J}^*$ (see (8)), can be shown to be positive definite following similar arguments in Observation 1 and considering that $W > 0$ and A is non-singular.

[4] For clarity, $\tilde{J}^*(x_0)$ is the minimum of (8) subject to (10)-(13), while $J^*(x_0)$ is the minimum of (3) subject to $Fx_i + Gu_i \leq \mathbf{1}$, $\forall i$.

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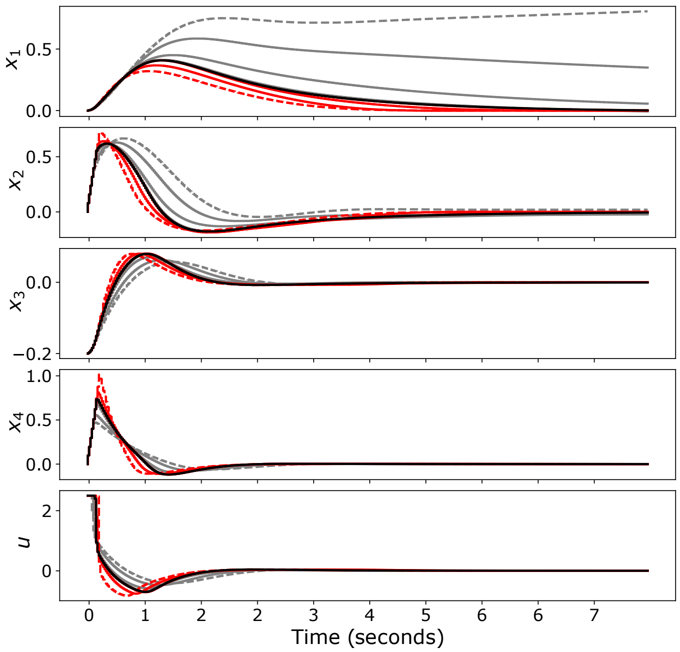

Fig. 2. Simulation 1: Closed-loop trajectories about the equilibrium, for the proposed controller (red lines) and the one in [4] (gray lines).

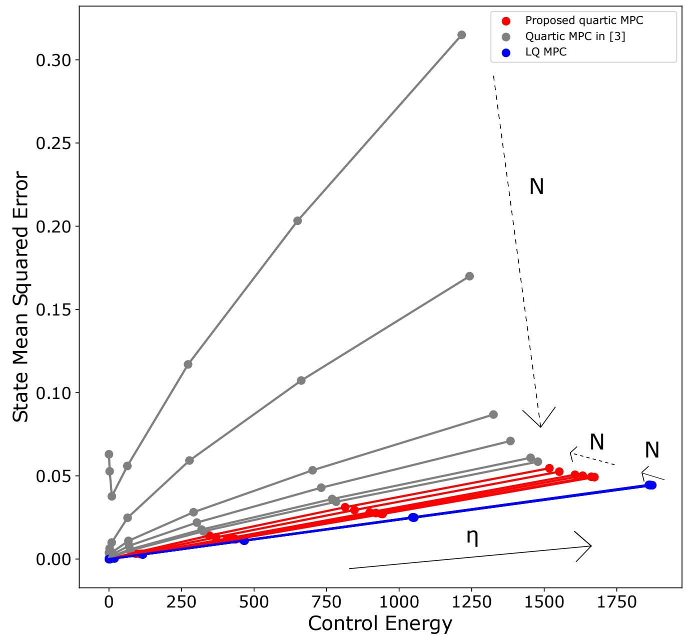

Fig. 3. Simulation 2: Balancing mean squared error against energy consumption.

1) Simulation 1: Comparison with Quartic MPC in [4] - Balance Recovery Task: We set $R = 1$, $Q = 10I_4$ (I_4 the identity matrix) and progressively increase N. The system is initialised in the state $x_0^T = [0, 0, \pi - 0.2, 0]$ (i.e. with the pendulum 0.2 radiants on the right of the equilibrium). Hence we consider here the task of restoring the upright balance position of the pendulum. We consider here the nominal case, with no disturbances or noise affecting the system.

Figure 2 reports the state trajectories and control sequences for the proposed controller and the one in [4]. The proposed controller is recursive-feasible from the given initial state for $N \geq 23$. The quartic MPC control in [4] is not recursive-feasible for $N \leq 26$ for the given initial state. The curves in grey refer to the quartic MPC controller in [4], for values of N of 26 (dashed gray line - notice this trajectory leads to infeasibility), 30, 50, 100, 200 and 300 (also marked with a solid black line). The curves in red refer to the proposed controller, for values of N of 24 (dashed red line), 50, 100, 200 and 300 (again also marked with a solid black line). Classic LQ MPC has been simulated as well: trajectories do not significantly vary with N and are very close to the ones resulting from the proposed controller with $N = 24$ (dashed red line). This is not surprising considering that the two strategies have the same *terminal* cost, and therefore tend to coincide for low values of N. As expected, notice that the responses of the two controllers (the proposed one and the one in [4]) tend to coincide as N grows, and are practically the same for $N \geq 300$ (solid black line). The response of the proposed controller appears less variable with N, compared to the strategy in [4].

2) Simulation 2: Performance in the Balancing Maintenance Task: We now test the behaviour of the system about the equilibrium, in presence of additive disturbance on the state feedback. In this configuration, the control task of the system is that of maintaining the balance position. Specifically, Gaussian noise is added to the state feedback of the angular position and angular velocity at each sampling time. To setup the experiments, we first extract a matrix $\mathcal{N} \in \mathrm{R}^{2\times \mathrm{T}}$ of i.i.d. Gaussian variables with zero mean and unitary standard deviation, where $T = 10\,000$ sampling intervals is the duration of each experiment (500 s). At the generic sampling time k, the state feedback x_k is then corrupted as $\hat{x}_k = x_k + \eta \mathcal{N}_k$, where $\mathcal{N}_k$ is the k-th column of $\mathcal{N}$ and η is a real number used to tune the variance of the additive noise. We compare in the following the performance of the three controllers in terms of mean squared error of the state trajectory with respect to the equilibrium, and the corresponding "control energy" (defined, with little abuse of terminology, as the integral of the control effort, $\sum_i u_i^2$, in coherence with the definition of the objective function and the choice $R = 1$).

Figure 3 plots the squared mean error achieved during the balancing task, against the spent control energy, for $N \in [27, 30, 40, 50, 75, 100, 200, 300]$, $\eta \in [0.0001, 0.0005, 0.001, 0.0025, 0.005, 0.0075, 0.01]$ and for the three controllers we aim to compare (the one proposed in this paper in red lines, the quartic MPC one in [4] in gray lines, and classic LQ MPC in blue lines). The above choice of exploration for N and η has been made in order to have a comprehensive picture of the behaviour of the system in the parameters' space (for $N \leq 26$ the controller in [4] is not feasible). It is interesting to see from the figure that, for $\eta \geq 0.0025$, the performance points $\{\text{control energy}, \text{state squared error}\}$ across the 3 controllers lie on a Pareto-like curve. For the quartic MPC controller proposed in [4], the curve is descended as N increases. For the other two controllers the opposite happens. For $N \to \infty$, the performance of the two quartic

MPC controllers tends to converge towards the same point, the one characterizing the performance of the infinite-time "pure" quartic controller.

A gap instead is found between the performance of the quartic MPC controllers and the quadratic MPC one: the quadratic controller achieves a lower balancing error, at the expense of higher energy consumption. It is then confirmed the property highlighted in [4], that quartic controllers perform posture balancing with reduced control effort, while allowing larger sway motion around the equilibium. It appears also that, at least for small N, the "performance points" of the proposed quartic controller are closer (smaller distance) to the performance points of the ideal infinite-time quartic control than the quartic MPC control in [4]. On the other hand, the controller in [4] is the one ensuring the least control effort for every given configuration of the parameters (at the expense of a higher motion of the state about the equilibrium, much higher in case of small N).

V. Conclusions and Future Works

This paper has presented a receding-horizon controller for constrained, discrete-time, linear, time-invariant systems under a performance criterion which is quartic in the state variable and quadratic in the control variables. The recursive feasibility and the stability of the proposed controller have been proven, which is of fundamental importance for practical implementation, and constitutes a relevant extension of the quartic MPC presented in [4]. Numerical simulations have been discussed, testing the proposed method on a single link, cart-inverted pendulum, and comparing its performances with those achieved by classic LQ-based MPC and the MPC quartic controller proposed in [4]. The controller in [4] is the one spending the least control energy in the proposed balancing tests, which is a highly desirable feature in practical applications. Nevertheless, the controller proposed in this paper has been proven to be stable and recursive feasible and, in the provided numeric simulations on the balancing task, achieves a significant energy consumption reduction compared with standard LQ-based MPC, with only a minimal increase of the sway motion about the equilibrium, especially for low values of N (which is relevant for real time applications). Future works will focus on the derivation of tighter bounds of the quartic infinite-horizon optimal cost, which is relevant in order to improve the performance of MPC (both in terms of optimality of the solution and decrease of computational effort). Research lines currently investigated involve the application of sum of square [22], interpolation techniques [23] and the transposition of results derived in continuous time [8].

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