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Topology of the set of univoque bases

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ABSTRACT

Given a positive integer M, a number $q > 1$ is called a *univoque base* if there is exactly one sequence $(c_i) = c_1c_2\cdots$ with integer digits c_i belonging to the set $\{0, 1, \dots, M\}$, such that $1 = \sum_{i=1}^{\infty} c_i q^{-i}$. The topological and combinatorial properties of the set of univoque bases $\mathcal{U}$ and their corresponding sequences (c_i) have been investigated in many papers since a pioneering work of Erdős, Horváth and Joó 25 years ago. While in most studies the attention was restricted to univoque bases belonging to $(M, M+1]$, a recent work of Kong and Li on the Hausdorff dimension of unique expansions demonstrated the necessity to extend the earlier results to all univoque bases. This is the object of this paper. Although the general research strategy remains the same, a number of new arguments are needed, several new properties are uncovered, and some formerly known results become simpler and more natural in the present framework.

1. Introduction

Fix a positive integer M and an *alphabet* $\{0, 1, \dots, M\}$. By a *sequence* we mean an element $c = (c_i) = c_1c_2\cdots$ of $\{0, 1, \dots, M\}^\infty$. We will frequently use the lexicographic order between sequences and blocks (i.e., elements of $\{0, 1, \dots, M\}^n$ for some $n \ge 1$). Furthermore, we give each coordinate $\{0, 1, \dots, M\}$ the discrete topology and endow the set $\{0, 1, \dots, M\}^\infty$ with the Tychonoff product topology. The corresponding convergence is the component-wise convergence of sequences.

Given a real *base* $q > 1$, by an *expansion* of a real number x we mean a sequence $c = (c_i)$ satisfying the equality

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$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} = x.$$

Expansions of this type in *non-integer* bases have been extensively investigated since a pioneering paper of Rényi [22].

One of the striking features of non-integer bases is that if $q \in (M, M+1)$, almost every number in $[0, M/(q-1)]$ has a continuum of expansions, see, e.g., [23]. In fact, using the theory of *random β-expansions* (see [4]) it was shown by [3] that almost every $x \in [0, M/(q-1)]$ has a continuum of so called *universal* expansions, i.e., expansions containing all possible blocks in $\{0, 1, \ldots, M\}^n$, $n = 1, 2, \ldots$. On the other hand, [10] constructed numbers $q \in (M, M+1)$ such that $x = 1$ has a unique expansion in base q, a discovery that stimulated many works during the past 25 years. We refer to the surveys [24,13,9] for more information.

In this paper we investigate only expansions of $x = 1$. (The only exception is Theorem 2.1 and its applications in the proof of several lemmas.) Hence by a *q-expansion* or an *expansion* we mean a sequence $c = (c_i)$ satisfying the equality

$$\pi_q(c) := \sum_{i=1}^{\infty} \frac{c_i}{q^i} = 1. \tag{1.1}$$

Since

$$0 \le \sum_{i=1}^{\infty} \frac{c_i}{q^i} \le \frac{M}{q-1}$$

for all sequences (c_i), such expansions may exist only if $q \in (1, M+1]$. Conversely, the greedy algorithm of [22] provides a q-expansion for each $q \in (1, M+1]$, which is defined recursively as follows: if for some positive integer n the digits $\beta_1, \ldots, \beta_{n-1}$ are already defined (no condition if $n = 1$), then β_n is the largest digit in $\{0, \ldots, M\}$ such that the inequality $\sum_{i=1}^{n} \beta_i q^{-i} \le 1$ holds. The resulting *greedy* or *β-expansion* $\beta(q)$ or $(\beta_i(q))$ of $x = 1$ is clearly the lexicographically largest q-expansion.

Examples 1.1. Let $M = 1$.

(i) If $q \in (1, \varphi)$, where $\varphi \approx 1.618$ denotes the Golden Ratio, then there is a continuum of q-expansions [11].

(ii) If $q = \varphi$, then there are countably many q-expansions [10]: symbolically

$$(10)^\infty, \quad \text{and} \quad (10)^k 110^\infty, \quad (10)^k 01^\infty, \quad k = 0, 1, \ldots.$$

(iii) If $q = \varphi_n$ is a Multinacci number, i.e., the positive solution of $q^n = q^{n-1} + \cdots + q + 1$ for some $n = 3, 4, \ldots$, then there are countably many q-expansions [18]:

$$(1^{n-1}0)^\infty, \quad \text{and} \quad (1^{n-1}0)^k 1^n 0^\infty, \quad k = 0, 1, \ldots.$$

(iv) If $q = 2$, then 1^∞ is the only q-expansion.

(v) Let $c = 1(1^n 0^k)^\infty$ for some integers $n \ge k \ge 1$, and define $q \in (1, 2)$ by the equation (1.1). Then c is the only q-expansion [11,10].

See also [13] for short elementary proofs of the above statements.

A sequence is called *finite* if it has a last non-zero digit, and *infinite* otherwise. Thus 0^∞ is considered to be an infinite sequence: this unusual terminology simplifies many statements in the sequel.

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A slight modification of the greedy algorithm yields the lexicographically largest *infinite* q-expansion: if for some integer $n \geq 1$, $\alpha_1, \ldots, \alpha_{n-1}$ is already defined, then α_n is the largest digit in $\{0, \ldots, M\}$ such that the strict inequality $\sum_{i=1}^{n} \alpha_i q^{-i} < 1$ holds. The resulting sequence $\alpha(q)$ or $(\alpha_i(q))$ is called the *quasi-greedy q-expansion*. We recall the basic properties of greedy and quasi-greedy expansions in the next section.

As a direct consequence of Proposition 2.3 below, we will see that the quasi-greedy q-expansions are even *doubly infinite*, i.e., their *conjugates* $(M - \alpha_i(q)) := (M - \alpha_1(q))(M - \alpha_2(q)) \cdots$ are also infinite. Hence there exists a doubly infinite q-expansion for every $q \in (1, M+1]$.

Let us observe that, although there are infinitely many expansions for the Multinacci numbers and for the Golden Ratio, in the first case there is only one infinite expansion, and in the second case there is only one doubly infinite expansion. One of the purposes of this paper is to characterize the bases having these properties. This leads to a surprisingly rich combinatorial and topological picture.

This investigation was carried out in [18,8,14] under the assumption $M < q \leq M+1$. We extend these results to all $1 < q \leq M+1$. This requires some new arguments, and the results explain some differences between the case of odd and even values of M.

Our results also explain, from a different point of view, some important recent results of [19] on the Hausdorff dimension of univoque sets.

In order to state the main results of this paper we denote by $\mathcal{U}$ the set of *univoque* bases $q \in (1, M+1]$ which are by definition the bases $q \in (1, M+1]$ for which there is exactly one q-expansion.

Our first result describes the topology of $\mathcal{U}$ and of its closure $\overline{\mathcal{U}}$. Furthermore, it characterizes the bases in which there is a unique infinite q-expansion:

Theorem 1.2.

(i) *The set $\mathcal{U}$ is closed from above but not from below.*[1]

(ii) *Its closure $\overline{\mathcal{U}}$ is a Cantor set. Moreover, $\overline{\mathcal{U}} \setminus \mathcal{U}$ is a countable dense set in $\overline{\mathcal{U}}$.*

(iii) *We have a disjoint union*

$$(1, M+1] \setminus \overline{\mathcal{U}} = \cup^* (q_0, q_0^*)$$

where q_0 runs over $\{1\} \cup (\overline{\mathcal{U}} \setminus \mathcal{U})$ and q_0^* runs over a proper subset $\mathcal{U}^*$ of $\mathcal{U}$.

(iv) *If $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$, then $(\alpha_i) := \alpha(q)$ is periodic. Furthermore, all q-expansions are given by (α_i) and the sequences*

$$(\alpha_1 \cdots \alpha_m)^N \alpha_1 \cdots \alpha_{m-1}(\alpha_m + 1)0^\infty, \quad N = 0, 1, \ldots,$$

where m is the smallest period of (α_i).

(v) $q \in \overline{\mathcal{U}} \Longleftrightarrow \alpha(q)$ *is the only infinite q-expansion.*

We recall that a Cantor set is a non-empty closed set having neither isolated, nor interior points.

Note that $\mathcal{U}^*$ is dense in $\overline{\mathcal{U}}$. Indeed, if $q \in \mathcal{U} \setminus \mathcal{U}^*$ and $p \in (1, q)$, then $(p, q) \cap \mathcal{U} \neq \varnothing$; hence by parts (ii) and (iii) of the above theorem we have $[q_0, q_0^*] \subset (p, q)$ for some connected component (q_0, q_0^*) of $(1, M+1] \setminus \overline{\mathcal{U}}$. Since p can be arbitrarily close to q, this implies that $\mathcal{U}^*$ is dense in $\overline{\mathcal{U}}$. Kong and Li [19] proved that all elements of $\mathcal{U}^*$ (called by them *De Vries–Komornik constants*) are transcendental. This extended some earlier results in [17]. It follows that the transcendental univoque bases are dense in $\overline{\mathcal{U}}$.

[1] We call a set $A \subseteq (1, M+1]$ *closed from above* (closed from below) if the limit of every decreasing (increasing) sequence of elements in A belongs to A.

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De Vries [7] proved that the algebraic univoque bases in $(M, M+1]$ are also dense in $\overline{\mathcal{U}} \cap (M, M+1]$. Repeating that proof for all bases $q \in (1, M+1]$ (one only has to redefine the conjugate of a digit by $\overline{c} := M - c$), we may conclude that the set of algebraic univoque bases is dense in $\overline{\mathcal{U}}$ as well.

Let us denote by $\mathcal{V}$ the set of bases $q \in (1, M+1]$ for which there is a unique *doubly infinite* q-expansion. Since $\alpha(q)$ is always doubly infinite, the last part of the preceding theorem implies that $\mathcal{U} \subseteq \overline{\mathcal{U}} \subseteq \mathcal{V}$.

For example, in case $M = 1$ we have $2 \in \mathcal{U}$, $\varphi \in \mathcal{V} \setminus \overline{\mathcal{U}}$, while the Multinacci numbers belong to $\overline{\mathcal{U}} \setminus \mathcal{U}$.

Theorem 1.3.

(i) *$\mathcal{V}$ is compact, and $\mathcal{V} \setminus \overline{\mathcal{U}}$ is a countable dense subset of $\mathcal{V}$.*

(ii) *$\mathcal{V} \setminus \overline{\mathcal{U}}$ is a discrete set. Moreover, for each connected component (q_0, q_0^*) of $(1, M+1] \setminus \overline{\mathcal{U}}$, we have*

$$\mathcal{V} \cap (q_0, q_0^*) = \{q_n \ : \ n = 1, 2, \ldots\}, \tag{1.2}$$

where (q_n) is a strictly increasing sequence converging to q_0^*.

(iii) *If $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, then $(\alpha_i) := \alpha(q)$ is periodic.*
If $M = 2m$ is even and $q = m+1$, then all q-expansions are given by (α_i) and the sequences

$$m^N (m+1) 0^\infty \quad \text{and} \quad m^N (m-1) M^\infty, \quad N = 0, 1, \ldots.$$

Otherwise all q-expansions are given by (α_i) and the sequences

$$(\alpha_1 \cdots \alpha_{2n})^N \alpha_1 \cdots \alpha_{2n-1} (\alpha_{2n} + 1) 0^\infty, \quad N = 0, 1, \ldots$$

and

$$(\alpha_1 \cdots \alpha_{2n})^N \alpha_1 \cdots \alpha_{n-1} (\alpha_n - 1) M^\infty, \quad N = 0, 1, \ldots,$$

where $2n$ is the smallest even period of (α_i).

The proof of Theorems 1.2(iii) and 1.3(ii) will also provide a construction of (q_n) and q_0^*.

The proof of Theorems 1.2 and 1.3 is based on lexicographic characterizations of $\mathcal{U}$, $\overline{\mathcal{U}}$ and $\mathcal{V}$: see Theorem 2.5, Definition 3.2 and Theorem 3.9 below. In the next section we provide a short review of similar well-known characterizations, which we shall apply frequently in the remainder of this paper.

Theorems 1.2 and 1.3 are proved in Section 3.

2. Review of lexicographic characterizations

Most results of this section are known; in the sequel we will apply them frequently without explicit citation.

An obvious modification of the greedy algorithm from the previous paragraph shows that each real number $x \in [0, M/(q-1)]$ has a lexicographically largest expansion. It is called the *greedy* or *β-expansion* of x, and it is denoted by (b_i) or $(b_i(x))$ (the value of q will always be understood). Parry [20] gave a lexicographic characterization of these expansions. Daróczy and Kátai [5,6] gave a more elegant form of his result by introducing the quasi-greedy expansion $(\alpha_i) := \alpha(q)$ of $x = 1$. The following result is essentially equivalent to their result:

Theorem 2.1. *A sequence (b_i) is the greedy expansion of some real number x if and only if the following condition is satisfied:*

$$b_{n+1}b_{n+2}\cdots < \alpha_1\alpha_2\cdots \quad \textit{whenever} \quad b_n < M. \tag{2.1}$$

Furthermore, if a sequence $(b_i) \neq M^\infty$ *satisfies* (2.1), *then there exists a sequence* $1 \leq n_1 < n_2 < \cdots$ *such that for each* $i \geq 1$,

$$b_{n_i} < M, \quad \textit{and} \quad b_{m+1}\cdots b_{n_i} < \alpha_1\cdots\alpha_{n_i-m} \quad \textit{if} \quad 1 \leq m < n_i \quad \textit{and} \quad b_m < M. \tag{2.2}$$

Proof. For the first part we refer to [17].

For the second we define a sequence $(n_i)_{i\geq 1}$ satisfying the requirements by induction.

Let r be the least positive integer for which $b_r < M$. Then, (2.2) with r in place of n_i holds clearly. Set $n_1 := r$ and let ℓ be a positive integer.

Suppose we have already defined $n_1 < \cdots < n_\ell$ such that (2.2) holds for each i with $1 \leq i \leq \ell$. Since (b_i) is greedy and $b_{n_\ell} < M$, there exists by (2.1) a *smallest* integer $n_{\ell+1} > n_\ell$ such that

$$b_{n_\ell+1}\cdots b_{n_{\ell+1}} < \alpha_1\cdots\alpha_{n_{\ell+1}-n_\ell}. \tag{2.3}$$

Note that $b_{n_{\ell+1}} < \alpha_{n_{\ell+1}-n_\ell}$, hence $b_{n_{\ell+1}} < M$. It remains to verify that

$$b_{m+1}\cdots b_{n_{\ell+1}} < \alpha_1\cdots\alpha_{n_{\ell+1}-m} \tag{2.4}$$

if $1 \leq m < n_{\ell+1}$ and $b_m < M$. If $m < n_\ell$, then (2.4) follows from the induction hypothesis. If $m = n_\ell$, then (2.4) reduces to (2.3). If $n_\ell < m < n_{\ell+1}$, then

$$b_{n_\ell+1}\cdots b_m = \alpha_1\cdots\alpha_{m-n_\ell},$$

by minimality of $n_{\ell+1}$, and thus by (2.3) and (2.6) below,

$$b_{m+1}\cdots b_{n_{\ell+1}} < \alpha_{m-n_\ell+1}\cdots\alpha_{n_{\ell+1}-n_\ell} \leq \alpha_1\cdots\alpha_{n_{\ell+1}-m}. \qquad \square$$

Setting $\beta(1) := 10^\infty$ for commodity, we have the following characterization of greedy q-expansions.

Proposition 2.2. *The map* $q \mapsto \beta(q)$ *is an increasing bijection of the interval* $[1, M+1]$ *onto the set of all sequences* (β_i) *satisfying*

$$\beta_{n+1}\beta_{n+2}\cdots < \beta_1\beta_2\cdots \quad \textit{whenever} \quad \beta_n < M. \tag{2.5}$$

For $q \neq M+1$ *the inequalities* (2.5) *are satisfied in fact for all* $n \geq 1$. *Hence* $\beta_1 > 0$, *and* $\beta_i \leq \beta_1$ *for all* i.

Furthermore, the map $q \mapsto \beta(q)$ *is continuous from the right.*

Proof. For the first part we refer to [1]. For the second, let $(\beta_i) := \beta(q)$ with $q \in (1, M+1)$. Furthermore, let N be an arbitrarily large positive integer. The following inequalities follow from the definition of the greedy algorithm:

$$\sum_{i=1}^{n} \frac{\beta_i}{q^i} > 1 - \frac{1}{q^n} \text{ whenever } \beta_n < M \text{ and } n \in \{1, \ldots, N\}.$$

The same set of inequalities holds with q' in place of q if $q' \in (q, M+1]$ is close enough to q, whence $\beta_1(q')\cdots\beta_N(q') \leq \beta_1(q)\cdots\beta_N(q)$. The reverse inequality is clear. $\square$

Next we investigate the quasi-greedy q-expansions. Setting $\alpha(1)=0^\infty$ for commodity,[2] we have the

Proposition 2.3. *The map $q\mapsto\alpha(q)$ is an increasing bijection of the interval $[1,M+1]$ onto the set of all* infinite *sequences (α_i) satisfying the inequality*

$$\alpha_{n+1}\alpha_{n+2}\cdots\le\alpha_1\alpha_2\cdots\quad\textit{whenever}\quad\alpha_n<M.\tag{2.6}$$

The inequalities (2.6) are satisfied in fact for all $n\ge 0$. Hence $\alpha_i\le\alpha_1$ for all i, and $\alpha_1>0$ unless $(\alpha_i)=0^\infty$.

Furthermore, the map $q\mapsto\alpha(q)$ is continuous from the left.

Proof. For the first part we refer to [1]. The proof of the second part is analogous (but even simpler) to that of the right-continuity of the map $q\mapsto\beta(q)$. □

Remark 2.4. Let us clarify the relations between the greedy and quasi-greedy q-expansions.

For $q=M+1$ we have simply $\alpha(M+1)=\beta(M+1)=M^\infty$.

For $q\in[1,M+1)$ the sequence (β_i) is finite if and only if (α_i) is periodic. In this case, denoting by β_m the last non-zero element of (β_i), the length of the smallest period of (α_i) is equal to m, and

$$\alpha_i=\beta_i\quad\text{for}\quad i=1,\dots,m-1,\qquad\alpha_m=\beta_m-1.$$

Otherwise, the two sequences coincide: $\alpha_i=\beta_i$ for all i.

For example,

$$\begin{aligned}&\beta(1)=10^\infty\quad\text{and}\quad\alpha(1)=0^\infty;\\&\beta(\varphi)=110^\infty\quad\text{and}\quad\alpha(\varphi)=(10)^\infty;\\&\beta(2)=\alpha(2)=1^\infty\quad\text{if}\quad M=1;\\&\beta(2)=20^\infty\quad\text{and}\quad\alpha(2)=1^\infty\quad\text{if}\quad M\ge 2.\end{aligned}$$

We recall from [11] and [17] a characterization of univoque bases. Henceforth we denote by $\mathcal{U}'$ the set of q-expansions when q runs over $\mathcal{U}$. We recall that the conjugate of a digit c_i is defined by $\overline{c_i}:=M-c_i$. We also set $\overline{c_1\cdots c_n}:=\overline{c_1}\cdots\overline{c_n}$ $(n\ge 1)$ and $\overline{c_1c_2\cdots}:=\overline{c_1}\,\overline{c_2}\cdots$.

Theorem 2.5. *A sequence $c=(c_i)$ belongs to $\mathcal{U}'$ if, and only if, the following two conditions are satisfied:*

$$c_{n+1}c_{n+2}\cdots<c_1c_2\cdots\quad\textit{whenever}\quad c_n<M\tag{2.7}$$

and

$$\overline{c_{n+1}c_{n+2}\cdots}<c_1c_2\cdots\quad\textit{whenever}\quad c_n>0.\tag{2.8}$$

For $c\in\mathcal{U}'$ and $c\ne M^\infty$ the inequalities (2.7) are satisfied for all $n\ge 1$, and the inequalities (2.8) are satisfied for all $n\ge 0$.

Finally, the map $c\mapsto q$, where $q\in(1,M+1]$ is defined by (1.1), defines an increasing homeomorphism between $\mathcal{U}'$ and $\mathcal{U}$.

[2] Despite the fact that 0^∞ is not an expansion of $x=1$, this convention simplifies several statements.

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3. Proof of Theorems 1.2 and 1.3

The proof is divided into a series of lemmas. The first one is a bit surprising because $\mathcal{U}$ is characterized by strict lexicographic inequalities:

Lemma 3.1. *$\mathcal{U}$ is closed from above.*

Proof. Fix a number $q \notin \mathcal{U}$ and set $(\beta_i) := \beta(q)$. We have to show that $q' \notin \mathcal{U}$ for all $q' > q$, sufficiently close to q.

Since $q \notin \mathcal{U}$, there exists by Proposition 2.2 and Theorem 2.5 an index k such that $\beta_k > 0$ and

$$\overline{\beta_{k+1}\beta_{k+2}\cdots} \geq \beta_1\beta_2\cdots .$$

Since we cannot have equality here (this would imply an equality in (2.5) for $n = 2k$), there exists an index m such that

$$\overline{\beta_{k+1}\cdots\beta_{k+m}} > \beta_1\cdots\beta_m. \tag{3.1}$$

If $q' > q$ is close enough to q, then $\beta(q)$ and $\beta(q')$ start with the same word of length $k+m$, and (3.1) implies that $q' \notin \mathcal{U}$. $\square$

In order to characterize the closure $\overline{\mathcal{U}}$ of $\mathcal{U}$, it is convenient to change the definition of $\mathcal{V}$, given in the introduction:

Definition 3.2. A number $q \in (1, M+1]$ belongs to $\mathcal{V}$ if $(\alpha_i) := \alpha(q)$ satisfies the following conditions:

$$\overline{\alpha_{n+1}\alpha_{n+2}\cdots} \leq \alpha_1\alpha_2\cdots \quad \text{whenever} \quad \alpha_n > 0. \tag{3.2}$$

We will show later (in Lemma 3.15) that this definition is equivalent to the former one.

We will also show (see Remarks 3.6(ii)) that for $q \in \mathcal{V}$ the inequality (3.2) holds for all $n \geq 0$.

It follows from Theorem 2.5 that $\mathcal{U} \subseteq \mathcal{V}$.

Lemma 3.3. *$\mathcal{V}$ is compact.*

Proof. Since $M+1 \in \mathcal{V}$, it suffices to prove that the complement of $\mathcal{V}$ in $(1, M+1]$ is open. Let $q \in (1, M+1] \setminus \mathcal{V}$ and set $(\alpha_i) := \alpha(q)$. Choose two integers $k, m \geq 1$ such that

$$\alpha_k > 0 \quad \text{and} \quad \overline{\alpha_{k+1}\cdots\alpha_{k+m}} > \alpha_1\cdots\alpha_m. \tag{3.3}$$

If $q' < q$ is close enough to q, then $\alpha(q)$ and $\alpha(q')$ start with the same word of length $k+m$, and (3.3) implies that $q' \notin \mathcal{V}$.

If $q' > q$ is close enough to q, then $\beta(q)$ and $\alpha(q')$ start with the same word of length $k+m$. In case of $\alpha(q) = \beta(q)$ the inequality (3.3) implies $q' \notin \mathcal{V}$ again.

If $\alpha(q) \neq \beta(q)$, then $\beta(q)$ has a last non-zero digit β_k. If $q' > q$ is close enough to q, then $(\alpha_i') := \alpha(q')$ starts with $\beta_1\cdots\beta_k 0^{k+1}$. Hence

$$\alpha_k' > 0 \quad \text{and} \quad \overline{\alpha_{k+1}'\cdots\alpha_{2k+1}'} = M^{k+1} > \beta_1\cdots\beta_k 0 = \alpha_1'\cdots\alpha_{k+1}',$$

and therefore $q' \notin \mathcal{V}$. $\square$

Lemma 3.4. *Fix $q \in \mathcal{V}$ and set $(\alpha_i) := \alpha(q)$.*

(i) *If for some $k \geq 1$,*

$$\alpha_k > 0 \quad \textit{and} \quad \alpha_{k+1} \cdots \alpha_{2k} = \overline{\alpha_1 \cdots \alpha_k},$$

then

$$(\alpha_i) = (\alpha_1 \cdots \alpha_k \overline{\alpha_1 \cdots \alpha_k})^\infty.$$

(ii) *Let, moreover, n be the smallest index k in (i). Then $2n$ is the smallest period of (α_i), except if $M = 2m$ is even and $q = m + 1$.*

Proof.

(i) Let $r := \alpha_1 \cdots \alpha_k$. We must show that if (α_i) starts with $(r\overline{r})^N$ for some positive integer N, then (α_i) also starts with $(r\overline{r})^{N+1}$. Let

$$s := \alpha_{2kN+1} \cdots \alpha_{2kN+k} \quad \text{and} \quad t := \alpha_{2kN+k+1} \cdots \alpha_{2k(N+1)}.$$

Applying (2.6) for $n = 2kN$ and (3.2) for $n = 2kN - k$, we obtain that

$$s \leq r \quad \text{and} \quad \overline{\overline{r}s} \leq r\overline{r};$$

they imply that $s = r$. Applying (2.6) and (3.2) again with the same choices of n, we obtain that

$$rt \leq r\overline{r} \quad \text{and} \quad \overline{\overline{r}rt} \leq r\overline{r}r;$$

hence $t = \overline{r}$.

(ii) If

$$\overline{\alpha_1 \cdots \alpha_n} < \alpha_1 \cdots \alpha_n,$$

then the smallest period of (α_i) is a divisor of $2n$, but not of n, so that it is an even number, say $2k$, and $2n = 2mk$ with an *odd* integer m. Therefore $n - k$ is a multiple of $2k$, so that

$$\alpha_k = \alpha_n > 0 \text{ and } \alpha_{k+1} \cdots \alpha_{2k} = \alpha_{n+1} \cdots \alpha_{n+k} = \overline{\alpha_1 \cdots \alpha_k}.$$

This contradicts the minimality of n, unless $k = n$.
If $\overline{\alpha_1 \cdots \alpha_n} \geq \alpha_1 \cdots \alpha_n$, then we have equality by Proposition 2.3. Hence $M = 2m$ is even, and $\alpha_1 = \cdots = \alpha_n = m$. This implies that $\alpha(q) = m^\infty$. Hence $\beta(q) = (m + 1)0^\infty$ and therefore $q = m + 1$. □

The following result shows that the above exceptional value $q = m+1$ is the least element of $\mathcal{V}$ if $M = 2m$:

Lemma 3.5. *$\mathcal{V}$ has a smallest element $\tilde{q}$, given by the formulas*

$$\tilde{q} = \begin{cases} m + 1 & \textit{if } M = 2m, \\ (m + \sqrt{m^2 + 4m})/2 & \textit{if } M = 2m - 1 \end{cases}$$

for $m = 1, 2, \ldots$. Furthermore,

$$\begin{cases} \beta(\tilde{q}) = (m+1)0^\infty \quad \text{and} \quad \alpha(\tilde{q}) = m^\infty & \text{if } M = 2m, \\ \beta(\tilde{q}) = mm0^\infty \quad \text{and} \quad \alpha(\tilde{q}) = (m(m-1))^\infty & \text{if } M = 2m-1. \end{cases}$$

For $M = 1$, $\tilde{q} = \varphi$ is the Golden Ratio. For $M > 1$ we recover the "generalized golden ratios", introduced earlier for other purposes by [2]. He was motivated by some earlier theorems of [12] and [15].

Proof. Let $M = 2m$ or $M = 2m-1$ for some $m = 1, 2, \ldots$. It suffices to establish the formula $\alpha(\tilde{q}) = (m\overline{m})^\infty$. Since $(m\overline{m})^\infty$ clearly satisfies (2.6) and (3.2), it remains to show that $\alpha(q) \geq (m\overline{m})^\infty$ for each $q \in \mathcal{V}$.

Let $q \in \mathcal{V}$ and $(\alpha_i) := \alpha(q)$. Since $q > 1$, we have $\alpha_1 > 0$. Furthermore, we have $\alpha_2 \leq \alpha_1$ by Proposition 2.3 and $\overline{\alpha_2} \leq \alpha_1$ by (3.2). It follows that $M \leq \alpha_1 + \alpha_2 \leq 2\alpha_1$, i.e., $\alpha_1 \geq M/2$. Hence, $\alpha_1 \geq m$ and $\alpha_2 \geq \overline{\alpha_1}$. If we have equalities here, then applying Lemma 3.4 we conclude that $\alpha(q) = (m\overline{m})^\infty$. □

Remarks 3.6.

(i) Let $q \in \mathcal{V}$. It follows from the preceding lemma that $\beta_1(q) > M/2$ and therefore $\overline{\beta_1(q)} < \beta_1(q)$. Moreover, we even have $\alpha_1(q) > M/2$ and $\overline{\alpha_1(q)} < \alpha_1(q)$, except for $q = \tilde{q}$ when M is even. For $M = 2m - 1$ this follows from the relation $\alpha_1(q) \geq m$. If $M = 2m$ and $q > \tilde{q}$, then $\beta(q) > \beta(\tilde{q}) = (m+1)0^\infty$ and therefore $\alpha_1(q) \geq m + 1$ by Remark 2.4.

(ii) Now we show that for $q \in \mathcal{V}$ the inequalities (3.2) are satisfied for all $n \geq 0$. For $n = 0$ this follows from the previous observations.
If $n \geq 1$ and $\alpha_n = 0$, then (since $\alpha_1 > 0$), there exists a largest index $1 \leq k < n$ such that $\alpha_k > 0$. Since $\overline{\alpha_{k+1} \cdots \alpha_n} = M^{n-k}$, we deduce from the inequality

$$\overline{\alpha_{k+1}\alpha_{k+2}\cdots} \leq \alpha_1\alpha_2\cdots$$

and from Proposition 2.3 that

$$\overline{\alpha_{n+1}\alpha_{n+2}\cdots} \leq \alpha_{n-k+1}\alpha_{n-k+2}\cdots \leq \alpha_1\alpha_2\cdots.$$

Since $\mathcal{V}$ is closed, we have $\mathcal{U} \subseteq \overline{\mathcal{U}} \subseteq \mathcal{V}$, and a characterization of $\overline{\mathcal{U}}$ may be expected by strengthening somewhat the condition (3.2). Let us therefore investigate the case of equality in (3.2).

Lemma 3.7. *Fix $q \in \mathcal{V}$ and set $(\alpha_i) := \alpha(q)$. If not all inequalities (3.2) are strict, then $q \notin \mathcal{U}$, and q is an isolated point of $\mathcal{V}$. Hence $q \notin \overline{\mathcal{U}}$.*

Proof. By Theorem 2.5, the number q does not belong to $\mathcal{U}$. By assumption, there exists a smallest index n such that

$$\alpha(q) = (\alpha_1 \cdots \alpha_n \overline{\alpha_1 \cdots \alpha_n})^\infty \quad \text{with} \quad \alpha_n > 0.$$

If $(\alpha_i) = m^\infty$ with $m = M/2$, then $\beta(q) = (m+1)0^\infty$. If $q' > q$ is close enough to q, then $\alpha(q')$ begins with $(m+1)00$ by Proposition 2.2, and this sequence does not satisfy (3.2) for $\alpha_1(q') > 0$.

If $q' \leq q$ is close enough to q, then $\alpha(q')$ begins with mm by Proposition 2.3. If $q' \in \mathcal{V}$, then $q' = q$ by Lemma 3.4.

The proof is similar in the other cases. By Lemma 3.4(ii) we have

$$\beta(q) = \alpha_1 \cdots \alpha_n \overline{\alpha_1 \cdots \alpha_{n-1}(\alpha_n - 1)}0^\infty.$$

If $q' > q$ is close enough to q, then $\alpha(q')$ begins with

$$\alpha_1 \cdots \alpha_n \overline{\alpha_1 \cdots \alpha_{n-1}(\alpha_n - 1)} 0^{2n+1}$$

by Proposition 2.2, and this sequence does not satisfy (3.2) for $\alpha_{2n}(q') > 0$.

If $q' \le q$ is close enough to q, then $\alpha(q')$ begins with $\alpha_1 \cdots \alpha_n \overline{\alpha_1 \cdots \alpha_n}$ by Proposition 2.3. If $q' \in \mathcal{V}$, then $q' = q$ by Lemma 3.4. $\square$

It follows from the above property that if $q \in \overline{\mathcal{U}}$, then the inequalities (3.2) are strict. The converse also holds true:

Lemma 3.8. *Fix $q \in \mathcal{V}$ and let $(\alpha_i) := \alpha(q)$. Suppose that all inequalities (3.2) are strict:*

$$\overline{\alpha_{n+1}\alpha_{n+2}\cdots} < \alpha_1\alpha_2\cdots \quad \textit{whenever} \quad \alpha_n > 0. \tag{3.4}$$

Then

(i) *there exist arbitrarily large positive integers m such that*

$$\overline{\alpha_{k+1}\cdots\alpha_m} < \alpha_1\cdots\alpha_{m-k} \quad \textit{whenever} \quad 0 \le k < m; \tag{3.5}$$

(ii) *$q \in \overline{\mathcal{U}}$.*

Note that (3.5) for $k = m - 1$ implies that $\alpha_m > 0$.

Proof.

(i) By Theorem 2.1 there exist arbitrarily large indices m with $\alpha_m > 0$ such that

$$\overline{\alpha_{k+1}\cdots\alpha_m} < \alpha_1\cdots\alpha_{m-k} \quad \text{whenever} \quad 1 \le k < m \quad \text{and} \quad \alpha_k > 0.$$

If $1 \le k < m$ and $\alpha_k = 0$ and $\ell := \max\{i \in \{1, \ldots, k-1\} \mid \alpha_i > 0\}$ (recall that $\alpha_1 > 0$), then

$$\overline{\alpha_{k+1}\cdots\alpha_m} < \overline{\alpha_{\ell+1}\cdots\alpha_{m-k+\ell}} \le \alpha_1\cdots\alpha_{m-k}.$$

If $k = 0$, then (3.5) follows from $\overline{\alpha_1} < \alpha_1$ (see Remarks 3.6(i)).

(ii) We shall construct a sequence (p_m^*) of numbers belonging to $\mathcal{U}$ that converges to q. For each m satisfying (3.5) we define a sequence (c_i) by recursion as follows. First set

$$c_1 \cdots c_m := \alpha_1 \cdots \alpha_m.$$

Then, if $c_1 \cdots c_{2^N m}$ is already defined for some nonnegative integer N, set

$$c_{2^N m+1} \cdots c_{2^{N+1} m} := \overline{c_1 \cdots c_{2^N m-1}(c_{2^N m} - 1)}.$$

Note that $c_m = \alpha_m > 0$, and $c_{2^N m} > 0$ implies $c_{2^{N+1} m} = \overline{c_{2^N m} - 1} > 0$ for $N = 0, 1, \ldots$; hence $c_i \in \{0, \ldots, M\}$ for each $i \ge 1$.

Assume for the moment that the sequence (c_i) satisfies the conditions (2.7) and (2.8). Then it is the unique p_m^*-expansion for some base p_m^*. Since $\alpha(q)$ and $\alpha(p_m^*)$ start with the same block of length m, letting $m \to \infty$ we conclude that $p_m^* \to q$. It remains to prove (2.7) and (2.8).

To this end we show by induction on $N = 0, 1, \ldots$ the following inequalities:

$$\overline{c_{j+1} \cdots c_{2^N m}} < c_1 \cdots c_{2^N m-j} \quad \text{for all} \quad 0 \le j < 2^N m \tag{3.6}$$

and

$$c_{j+1} \cdots c_{j+2^N m} < c_1 \cdots c_{2^N m} \quad \text{for all} \quad 1 \le j < 2^N m. \tag{3.7}$$

Consider first the case $N = 0$. If $0 \le j < m$, then (3.6) holds because

$$\overline{c_{j+1} \cdots c_m} = \overline{\alpha_{j+1} \cdots \alpha_m} < \alpha_1 \cdots \alpha_{m-j} = c_1 \cdots c_{m-j}$$

by (i).
Next, for $1 \le j < m$ we have

$$\alpha_{j+1} \cdots \alpha_m \le \alpha_1 \cdots \alpha_{m-j}$$

by Proposition 2.3, and

$$\overline{\alpha_{m-j+1} \cdots \alpha_m} < \alpha_1 \cdots \alpha_j$$

by applying (i) for $m - j$ instead of j. Taking the conjugate of the last inequality, we obtain (3.7):

$$\begin{aligned} c_{j+1} \cdots c_{j+m} &= \alpha_{j+1} \cdots \alpha_m \overline{\alpha_1 \cdots \alpha_j} \\ &< \alpha_1 \cdots \alpha_{m-j} \alpha_{m-j+1} \cdots \alpha_m \\ &= c_1 \cdots c_m. \end{aligned}$$

Now assume by induction that (3.6) and (3.7) hold for some $N \ge 0$. In order to establish them for $N + 1$, it suffices to show that

$$\overline{c_{k+1} \cdots c_{2^{N+1} m}} < c_1 \cdots c_{2^{N+1} m-k} \quad \text{for all} \quad 2^N m \le k < 2^{N+1} m \tag{3.8}$$

and

$$c_{k+1} \cdots c_{k+2^N m} < c_1 \cdots c_{2^N m} \quad \text{for all} \quad 2^N m \le k < 2^{N+1} m. \tag{3.9}$$

Writing $k = 2^N m + j$, so that $0 \le j < 2^N m$, (3.8) follows from the induction hypothesis and from the definition of (c_i):

$$\begin{aligned} \overline{c_{k+1} \cdots c_{2^{N+1} m}} &= c_{j+1} \cdots c_{2^N m-1} (c_{2^N m} - 1) \\ &< c_{j+1} \cdots c_{2^N m-1} c_{2^N m} \\ &\le c_1 \cdots c_{2^N m-j} \\ &= c_1 \cdots c_{2^{N+1} m-k}. \end{aligned}$$

For $k = 2^N m$ the inequality (3.9) follows from $\overline{\alpha_1} < \alpha_1$ which in turn follows from Remarks 3.6(i), and therefore

$$c_{2^N m+1} = \overline{c_1} = \overline{\alpha_1} < \alpha_1 = c_1.$$

In the remaining cases we may write $k = 2^N m + j$ with $1 \le j < 2^N m$. Using again the definition of (c_i) and applying (3.6) we have

$$\begin{aligned} c_{k+1} \cdots c_{k+2^N m} &= c_{k+1} \cdots c_{2^{N+1}m-1} c_{2^{N+1}m} c_{2^{N+1}m+1} \cdots c_{2^{N+1}m+j} \\ &= \overline{c_{j+1} \cdots c_{2^N m-1}(c_{2^N m} - 1) c_1 \cdots c_j} \\ &\le c_1 \cdots c_{2^N m - j} \overline{c_1 \cdots c_j}. \end{aligned}$$

In order to get (3.9) we have to show that

$$\overline{c_1 \cdots c_j} < c_{2^N m-j+1} \cdots c_{2^N m},$$

or equivalently

$$\overline{c_{2^N m-j+1} \cdots c_{2^N m}} < c_1 \cdots c_j.$$

This follows by applying (3.6) for $2^N m - j$ instead of j (this is possible because $j \neq 0$). $\square$

The last two lemmas imply the following lexicographic characterization of $\overline{\mathcal{U}}$:

Theorem 3.9. *A number $q \in (1, M+1]$ belongs to $\overline{\mathcal{U}}$ if, and only if, the sequence $(\alpha_i) := \alpha(q)$ satisfies (3.4).*

Adapting Remarks 3.6(ii) in the obvious way, we see that for $q \in \overline{\mathcal{U}}$ the inequality (3.4) holds for all $n \ge 0$.

We proceed to prove our main theorems.

Lemma 3.10. *$\overline{\mathcal{U}} \setminus \mathcal{U}$ is a countable dense set in $\overline{\mathcal{U}}$.*

Proof. Each element $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$ is algebraic because $\alpha(q)$ is periodic by Proposition 2.3, Theorems 2.5 and 3.9. Hence $\overline{\mathcal{U}} \setminus \mathcal{U}$ is countable. On the other hand, $\overline{\mathcal{U}} \setminus \mathcal{U}$ is an infinite set because by the preceding theorem it contains the bases q_n with $\alpha(q_n) = (M^n 0)^\infty$, $n = 2, 3, \ldots$.

It remains to show that each fixed $q \in \mathcal{U}$ can be approximated arbitrarily closely by numbers belonging to $\overline{\mathcal{U}} \setminus \mathcal{U}$. By Lemma 3.8(i) there exist arbitrarily large positive integers m satisfying (3.5) with $(\alpha_i) := \alpha(q)$. Assume for the moment that the corresponding periodic sequences

$$(\eta_i) := \Big(\alpha_1 \cdots \alpha_m \, \overline{\alpha_1 \cdots \alpha_{m-1}(\alpha_m - 1)\alpha_1 \cdots \alpha_m}\Big)^\infty$$

are quasi-greedy r_m-expansions for some bases $r_m \in \overline{\mathcal{U}} \setminus \mathcal{U}$. Since $\alpha(r_m)$ and $\alpha(q)$ start with the same block of length m, letting $m \to \infty$ we conclude that $r_m \to q$.

It remains to prove that

$$\eta_{k+1}\eta_{k+2} \cdots \le \eta_1 \eta_2 \cdots \tag{3.10}$$

and

$$\overline{\eta_{k+1}\eta_{k+2} \cdots} < \eta_1 \eta_2 \cdots \tag{3.11}$$

for all $k \ge 0$. Thanks to the periodicity of (η_i) it suffices to consider the cases $0 \le k < 3m$.

The relation (3.10) is obvious for $k = 0$, while for $k = m$ and $k = 2m$ it follows from the inequality $\overline{\alpha_1} < \alpha_1$ (see Remarks 3.6(i)). The relation (3.11) for $k = 0$ also follows from the inequality $\overline{\alpha_1} < \alpha_1$. For $k = m$ and $k = 2m$ the relation (3.11) follows from the inequalities

$$\alpha_1 \cdots \alpha_{m-1}(\alpha_m - 1) < \alpha_1 \cdots \alpha_{m-1}\alpha_m$$

and

$$\alpha_1 \cdots \alpha_m \overline{\alpha_1 \cdots \alpha_m} < \alpha_1 \cdots \alpha_m \overline{\alpha_1 \cdots \alpha_{m-1}(\alpha_m - 1)},$$

respectively.

For the rest we distinguish the three cases

$$0 < k < m, \quad m < k < 2m \quad \text{and} \quad 2m < k < 3m.$$

If $0 < k < m$, then using (3.5) we have

$$\begin{aligned} \overline{\eta_{k+1} \cdots \eta_m} &= \overline{\alpha_{k+1} \cdots \alpha_m} \\ &< \alpha_1 \cdots \alpha_{m-k} \\ &= \eta_1 \cdots \eta_{m-k} \end{aligned}$$

which implies (3.11), while using Proposition 2.3 and (3.5) for $m - k$ instead of k we obtain

$$\begin{aligned} \eta_{k+1} \cdots \eta_{k+m} &= \alpha_{k+1} \cdots \alpha_m \overline{\alpha_1 \cdots \alpha_k} \\ &\le \alpha_1 \cdots \alpha_{m-k} \overline{\alpha_1 \cdots \alpha_k} \\ &< \alpha_1 \cdots \alpha_{m-k}\alpha_{m-k+1} \cdots \alpha_m \\ &= \eta_1 \cdots \eta_m, \end{aligned}$$

proving (3.10).

If $m < k < 2m$, then writing $k = m + j$ and using Proposition 2.3 we have

$$\begin{aligned} \overline{\eta_{k+1} \cdots \eta_{2m}} &= \alpha_{j+1} \cdots \alpha_{m-1}(\alpha_m - 1) \\ &< \alpha_{j+1} \cdots \alpha_{m-1}\alpha_m \\ &\le \alpha_1 \cdots \alpha_{m-j} \\ &= \eta_1 \cdots \eta_{m-j} \\ &= \eta_1 \cdots \eta_{2m-k}, \end{aligned}$$

while applying (3.5) for j and $m - j$, we obtain that

$$\begin{aligned} \eta_{k+1} \cdots \eta_{k+m} &= \overline{\alpha_{j+1} \cdots \alpha_{m-1}(\alpha_m - 1)\alpha_1 \cdots \alpha_j} \\ &\le \alpha_1 \cdots \alpha_{m-j}\overline{\alpha_1 \cdots \alpha_j} \\ &< \alpha_1 \cdots \alpha_{m-j}\alpha_{m-j+1} \cdots \alpha_m \\ &= \eta_1 \cdots \eta_m. \end{aligned}$$

Finally, if $2m < k < 3m$, then writing $k = 2m + j$ and using (3.5) we obtain that

$$\begin{aligned}\eta_{k+1}\cdots\eta_{3m} &= \overline{\alpha_{j+1}\cdots\alpha_m}\\ &< \alpha_1\cdots\alpha_{m-j}\\ &= \eta_1\cdots\eta_{m-j}\\ &= \eta_1\cdots\eta_{3m-k}.\end{aligned}$$

Furthermore, applying Proposition 2.3 and then (3.5) for $m - j$, we obtain that

$$\begin{aligned}\overline{\eta_{k+1}\cdots\eta_{k+m}} &= \alpha_{j+1}\cdots\alpha_m\overline{\alpha_1\cdots\alpha_j}\\ &\le \alpha_1\cdots\alpha_{m-j}\overline{\alpha_1\cdots\alpha_j}\\ &< \alpha_1\cdots\alpha_{m-j}\alpha_{m-j+1}\cdots\alpha_m\\ &= \eta_1\cdots\eta_m.\end{aligned}$$

These two relations imply (3.10) and (3.11) again. □

Lemma 3.11. *If $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$, then $(\alpha_i) := \alpha(q)$ is periodic. Furthermore, all q-expansions are given by (α_i) and the sequences*

$$(\alpha_1\cdots\alpha_m)^N\alpha_1\cdots\alpha_{m-1}(\alpha_m+1)0^\infty, \quad N = 0, 1, \ldots, \tag{3.12}$$

where m is the smallest period of (α_i).

Proof. Fix $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$. Since $M + 1 \in \mathcal{U}$, we have $(\alpha_i) < M^\infty$. Furthermore, it follows from Theorems 2.1 and 3.9 that $(\overline{\alpha_i})$ is the greedy expansion of $M/(q-1) - 1$, and hence (α_i) is the lexicographically smallest q-expansion. Theorem 2.5 implies that at least one of the inequalities (2.6) must be an equality, so that (α_i) is periodic. If m is the smallest period of (α_i), then we have

$$(\beta_i) := \beta(q) = \alpha_1\cdots\alpha_{m-1}(\alpha_m+1)0^\infty.$$

It follows by induction on N that the sequences (3.12) are also q-expansions.

Now let $c = (c_i)$ be an arbitrary q-expansion. If

$$(c_i) \ne (\alpha_i) = (\alpha_1\cdots\alpha_m)^\infty,$$

then let N be the biggest nonnegative integer such that (c_i) begins with $(\alpha_1\cdots\alpha_m)^N$. We complete the proof by showing that c has the form (3.12).

Since

$$\pi_q\left((c_i)_{i=mN+1}^\infty\right) = 1, \tag{3.13}$$

we have

$$(\alpha_i) \le (c_i)_{i=mN+1}^\infty \le (\beta_i)$$

and hence

$$\alpha_1 \cdots \alpha_m \le c_{mN+1} \cdots c_{mN+m} \le \alpha_1 \cdots \alpha_{m-1}(\alpha_m + 1).$$

In view of the choice of N we conclude that

$$c_{mN+1} \cdots c_{mN+m} = \alpha_1 \cdots \alpha_{m-1}(\alpha_m + 1) = \beta_1 \cdots \beta_m.$$

Since $\beta_i = 0$ for all $i > m$, it follows from (3.13) that c has the form (3.12) indeed. □

Lemma 3.12. *$\mathcal{V} \setminus \overline{\mathcal{U}}$ is a discrete set, dense in $\mathcal{V}$.*

Proof. We already know that $\overline{\mathcal{U}}$ is characterized by Theorem 3.9. Therefore Lemma 3.7 shows that $\mathcal{V} \setminus \overline{\mathcal{U}}$ is discrete.

Next we show that each fixed $q \in \overline{\mathcal{U}}$ can be approximated arbitrarily closely by numbers belonging to $\mathcal{V} \setminus \overline{\mathcal{U}}$. By Lemma 3.8(i) there exist arbitrarily large positive integers m satisfying (3.5) with $(\alpha_i) := \alpha(q)$. Assume for the moment that the sequences $(\alpha_1 \cdots \alpha_m \overline{\alpha_1 \cdots \alpha_m})^\infty$ are quasi-greedy t_m-expansions for some bases $t_m \in \mathcal{V} \setminus \overline{\mathcal{U}}$. Since $\alpha(t_m)$ and $\alpha(q)$ start with the same block of length m, letting $m \to \infty$ we conclude that $t_m \to q$.

It remains to prove for each fixed m that the sequence

$$\eta := (\alpha_1 \cdots \alpha_m \overline{\alpha_1 \cdots \alpha_m})^\infty$$

satisfies the inequalities

$$\eta_{k+1}\eta_{k+2} \ldots \le \eta_1\eta_2 \cdots \tag{3.14}$$

and

$$\overline{\eta_{k+1}\eta_{k+2} \cdots} \le \eta_1\eta_2 \cdots \tag{3.15}$$

for all $k \ge 0$. Since (3.14) for $k+m$ is equivalent to (3.15) for k and (3.15) for $k+m$ is equivalent to (3.14) for k, it suffices to verify (3.14) and (3.15) for $0 \le k < m$.

The relation (3.15) follows from (3.5):

$$\overline{\eta_{k+1} \cdots \eta_m} = \overline{\alpha_{k+1} \cdots \alpha_m} < \alpha_1 \cdots \alpha_{m-k} = \eta_1 \cdots \eta_{m-k}.$$

The relation (3.14) for $k = 0$ is obvious. If $0 < k < m$, then

$$\alpha_{k+1} \cdots \alpha_m \le \alpha_1 \cdots \alpha_{m-k}.$$

Furthermore, applying (3.5) again we get

$$\overline{\alpha_{m-k+1} \cdots \alpha_m} < \alpha_1 \cdots \alpha_k,$$

which is equivalent to

$$\overline{\alpha_1 \cdots \alpha_k} < \alpha_{m-k+1} \cdots \alpha_m.$$

Consequently, we have

$$\begin{aligned}\eta_{k+1}\cdots\eta_{m+k} &= \alpha_{k+1}\cdots\alpha_m\overline{\alpha_1\cdots\alpha_k}\\ &\le \alpha_1\cdots\alpha_{m-k}\overline{\alpha_1\cdots\alpha_k}\\ &< \alpha_1\cdots\alpha_{m-k}\alpha_{m-k+1}\cdots\alpha_m\\ &= \eta_1\cdots\eta_m,\end{aligned}$$

and (3.14) follows again. □

Remark 3.13. We show that the numbers t_m which appear in the proof of Lemma 3.12 are all less than q.

Since $q \in \overline{\mathcal{U}}$, we have $\overline{\alpha_{m+1}\alpha_{m+2}\cdots} < \alpha_1\alpha_2\cdots$, and hence

$$\overline{\alpha_1\cdots\alpha_m} \le \alpha_{m+1}\cdots\alpha_{2m}.$$

In case of equality we would have $\alpha = \eta$ and thus $t_m = q$ by Lemma 3.4. This is impossible because $q \in \overline{\mathcal{U}}$ and $t_m \notin \overline{\mathcal{U}}$.

Thus we have

$$\overline{\alpha_1\cdots\alpha_m} < \alpha_{m+1}\cdots\alpha_{2m};$$

this is equivalent to

$$\eta_1\cdots\eta_{2m} < \alpha_1\cdots\alpha_{2m},$$

implying $\eta < \alpha$ and thus $t_m < q$.

Lemma 3.14. *If $q \in \mathcal{V}\backslash\overline{\mathcal{U}}$, then $(\alpha_i) := \alpha(q)$ is periodic.*

If $M = 2m$ is even and $q = m + 1$, then all q-expansions are given by (α_i) and the sequences

$$m^N(m+1)0^\infty \quad \textit{and} \quad m^N(m-1)M^\infty, \quad N = 0, 1, \ldots. \tag{3.16}$$

Otherwise all q-expansions are given by (α_i) and the sequences

$$(\alpha_1\cdots\alpha_{2n})^N\alpha_1\cdots\alpha_{2n-1}(\alpha_{2n}+1)0^\infty, \quad N = 0, 1, \ldots \tag{3.17}$$

and

$$(\alpha_1\cdots\alpha_{2n})^N\alpha_1\cdots\alpha_{n-1}(\alpha_n-1)M^\infty, \quad N = 0, 1, \ldots, \tag{3.18}$$

where $2n$ is the smallest even period of (α_i).

Proof. Fix $q \in \mathcal{V}\setminus\overline{\mathcal{U}}$. Since $M + 1 \in \mathcal{U}$, we have $(\alpha_i) < M^\infty$. It follows from the definition of $\mathcal{V}$ (see (3.2)) and from Theorem 3.9 that at least one of the inequalities (3.2) must be an equality. Let n be the smallest positive integer for which equality holds in (3.2), then $\alpha_n > 0$ and (α_i) is periodic:

$$(\alpha_i) = (\alpha_1\cdots\alpha_n\overline{\alpha_1\cdots\alpha_n})^\infty.$$

Applying Lemma 3.4(ii) we distinguish two cases.

If $(\alpha_i) = m^\infty$ with $m = M/2$, then $\beta(q) = (m+1)0^\infty$, and the sequences (3.16) are easily seen to be q-expansions. Since $\pi_q(M^\infty) = M/(q-1) = 2$, the conjugate $(m-1)M^\infty$ of $\beta(q)$ is also a q-expansion, hence it is the lexicographically smallest q-expansion.

Conversely, let $c = (c_i)$ be an arbitrary q-expansion. If $(c_i) \neq m^\infty$, then let N be the biggest nonnegative integer such that (c_i) begins with m^N. Then

$$\pi_q\left((c_i)_{i=N+1}^\infty\right) = 1,$$

so that

$$(m-1)M^\infty \leq (c_i)_{i=N+1}^\infty \leq (m+1)0^\infty.$$

Hence either $c_{N+1} = m-1$ or $c_{N+1} = m+1$. In the first case we have $c_i = M$ for all $i > N+1$; in the second case we have $c_i = 0$ for all $i > N+1$. Thus c is one of the sequences in (3.16).

Next we consider the case where $2n$ is the smallest period of (α_i). Then

$$\beta(q) = \alpha_1 \cdots \alpha_n \overline{\alpha_1 \cdots \alpha_{n-1}(\alpha_n - 1)} 0^\infty,$$

and the sequences (3.17)–(3.18) are easily seen to be q-expansions again. Using Theorem 2.1 we see also that $\alpha_1 \cdots \alpha_{n-1}(\alpha_n - 1)M^\infty$ is the lexicographically smallest q-expansion.

Conversely, let $c = (c_i)$ be an arbitrary q-expansion. If $(c_i) \neq (\alpha_i)$, then let N be the biggest nonnegative integer such that (c_i) begins with $(\alpha_1 \cdots \alpha_{2n})^N$. Then

$$\pi_q\left((c_i)_{i=2nN+1}^\infty\right) = 1,$$

so that

$$\alpha_1 \cdots \alpha_{n-1}(\alpha_n - 1)M^\infty \leq (c_i)_{i=2nN+1}^\infty \leq \alpha_1 \cdots \alpha_n \overline{\alpha_1 \cdots \alpha_{n-1}(\alpha_n - 1)} 0^\infty. \tag{3.19}$$

Hence we have either

$$c_{2nN+1} \cdots c_{2nN+n} = \alpha_1 \cdots \alpha_{n-1}(\alpha_n - 1)$$

or

$$c_{2nN+1} \cdots c_{2nN+n} = \alpha_1 \cdots \alpha_{n-1}\alpha_n.$$

In the first case we deduce from the first inequality of (3.19) that

$$(c_i)_{i=2nN+1}^\infty = \alpha_1 \cdots \alpha_{n-1}(\alpha_n - 1)M^\infty,$$

so that c is one of the sequences (3.18).

In the second case, we have

$$\pi_q\left((\overline{c_i})_{i=2nN+n+1}^\infty\right) = 1,$$

so that

$$\overline{c_{2nN+n+1} \cdots c_{2n(N+1)}} \leq \beta_1(q) \cdots \beta_n(q) = \alpha_1 \cdots \alpha_n$$

or

$$c_{2nN+n+1}\cdots c_{2n(N+1)} \ge \overline{\alpha_1\cdots\alpha_n}.$$

We cannot have equality here by our choice of N. Therefore we deduce from the second inequality of (3.19) that

$$c_{2nN+n+1}\cdots c_{2n(N+1)} = \overline{\alpha_1\cdots\alpha_{n-1}(\alpha_n-1)},$$

whence c is one of the sequences (3.17). □

Now we are able to prove the promised characterizations of $\overline{\mathcal{U}}$ and $\mathcal{V}$:

Lemma 3.15.

(i) $q \in \overline{\mathcal{U}} \iff$ *there is a unique infinite q-expansion.*
(ii) $q \in \mathcal{V} \iff$ *there is a unique doubly infinite q-expansion.*

Proof. The direct implications $\Longrightarrow$ of (i) and (ii) follow from Lemmas 3.11 and 3.14 together with the fact that $\alpha(q)$ is doubly infinite for each $q \in (1, M+1]$. The proofs of the inverse implications, given in [14, Theorem 1.4(c)] for $M < q < M+1$ remain valid for $1 < q < M+1$. The case $q = M+1$ is obvious: the unique q-expansion M^∞ is doubly infinite by definition. □

The just proven lemma establishes the equivalence of the two definitions of $\mathcal{V}$, given in the introduction and at the beginning of this section.

Lemma 3.16. *Let $q_2 \in \mathcal{V}\setminus\overline{\mathcal{U}}$, $q_2 > \tilde{q}$, and write*

$$\alpha(q_2) = (\alpha_1\cdots\alpha_k\overline{\alpha_1\cdots\alpha_k})^\infty$$

where k is chosen to be minimal. Then

$$\overline{\alpha_{i+1}\cdots\alpha_k} < \alpha_1\cdots\alpha_{k-i}, \quad i = 0,\dots,k-1.$$

Proof. For $i = 0$ the inequality follows from the relation $\overline{\alpha_1} < \alpha_1$ (see Remarks 3.6(i)).
For $i > 0$ we may repeat the proof of Lemma 6.2 in [8]. □

Lemma 3.17. *Let q_2 be as in the preceding lemma and let $(\alpha_i) := \alpha(q_2)$. There is a greatest $q_1 \in \mathcal{V}$ that is smaller than q_2, and*

$$\alpha(q_1) = (\alpha_1\cdots\alpha_{k-1}(\alpha_k-1))^\infty. \tag{3.20}$$

Furthermore, $q_1 \in \mathcal{V}\setminus\mathcal{U}$.

Proof. The relation $q_1 \in \mathcal{V}\setminus\mathcal{U}$ will follow from the first part because $\alpha(q_1)$ is periodic and not equal to M^∞, whence $q_1 \notin \mathcal{U}$.

For the proof of the first part, in case $k \ge 2$ we may repeat the proof of Lemma 6.3 in [8], by defining in that proof the conjugate of a digit c by $\overline{c} := M - c$, and by applying Lemmas 3.4 and 3.16 above instead of Lemmas 4.2 and 6.2 there. Suppose that $k = 1$. It follows from Lemma 3.4 that q_2 is the smallest element q of $\mathcal{V}$ with $\alpha_1(q) = \alpha_1$. It remains to show that the sequence $(\alpha_1 - 1)^\infty$ satisfies (3.2). For this we have to show that $\alpha_1 - 1 \ge M/2$.

If $M = 2m$ or $M = 2m-1$, then $\alpha(\tilde{q}) = (m\overline{m})^\infty$ and $\alpha(q_2) = (\alpha_1\overline{\alpha_1})^\infty$. Since $q_2 > \tilde{q}$, we conclude that $\alpha_1 \ge m+1 \ge M/2+1$ as required. □

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Lemma 3.18. *Let $q_1 \in \mathcal{V} \setminus \mathcal{U}$, and let β_m be the last non-zero element of $(\beta_i) := \beta(q_1)$.*

There is a smallest $q_2 \in \mathcal{V}$ that is greater than q_1. Moreover, we have

$$\alpha(q_2) = (\beta_1 \cdots \beta_m \overline{\beta_1 \cdots \beta_m})^\infty$$

and

$$\beta(q_2) = \beta_1 \cdots \beta_m \overline{\beta_1 \cdots \beta_{m-1}(\beta_m - 1)} 0^\infty.$$

Furthermore, $q_2 \in \mathcal{V} \setminus \overline{\mathcal{U}}$.

Proof. We may repeat the proof of Lemma 6.5 in [8] with the following minor changes.

Instead of Lemmas 4.2 and 6.4 there we apply Lemma 3.4 and Proposition 2.2 here.

For the proof of (6.3) for $k = 0$ and of (6.4) for $k = m > 1$ we use the inequality $\overline{w_1} < w_1$; now this follows from $w_1 = \beta_1 > M/2$ (see Remarks 3.6(i)). The proof of (6.4) for $k = m = 1$ (in which case q_1 is an integer) follows directly from Examples 3.22(i) below.

For the proof of (6.4) for $k \geq 2m$ we use the inequality $\gamma_{k+1} = 0 < w_1$ which also follows from $w_1 = \beta_1 > M/2$.

Since $\alpha(q_2)$ does not satisfy (3.4), we conclude that $q_2 \notin \overline{\mathcal{U}}$. □

Lemma 3.19. *$\overline{\mathcal{U}}$ is a Cantor set.*

Proof. It follows from Lemma 3.10 that $\overline{\mathcal{U}}$ has no isolated points. Lemma 3.10 combined with the preceding lemma imply that $\overline{\mathcal{U}}$ has no interior points either. □

Remark 3.20. Since $\overline{\mathcal{U}}$ is a non-empty perfect set, each neighborhood of each point of $\overline{\mathcal{U}}$ contains uncountably many elements of $\overline{\mathcal{U}}$. Since there are only countably many algebraic numbers and since $\overline{\mathcal{U}} \setminus \mathcal{U}$ is countable, this implies again that the transcendental univoque bases are dense in $\overline{\mathcal{U}}$.

Now we consider the following construction. Given $q_0 \in \{1\} \cup (\overline{\mathcal{U}} \setminus \mathcal{U})$, we let $(\beta_i) := \beta(q_1)$, where q_1 is the smallest element of $\mathcal{V}$ that is greater than q_0. The number q_1 exists by Lemma 3.5 and Lemma 3.18. Let β_m be the last non-zero digit of (β_i). Starting with the block $s_1 := \beta_1 \cdots \beta_m$, we define a sequence of blocks by the recursive formula

$$s_{n+1} := s_n \overline{s_n}^+, \quad n = 1, 2, \ldots.$$

Here the superscript $^+$ means that we increase the last digit of the block $s_n \overline{s_n}$ by one.

By Lemma 3.18 we obtain a strictly increasing sequence $q_0 < q_1 < q_2 < \cdots$, converging to some number q_0^*, and satisfying the condition (1.2) of Theorem 1.3. The following lemma completes the proof of Theorems 1.2 and 1.3:

Lemma 3.21. *The above intervals (q_0, q_0^*) are exactly the connected components of $(1, M+1] \setminus \overline{\mathcal{U}}$.*

The set $\mathcal{U}^*$ of right endpoints is a subset of $\mathcal{U}$.

Proof. Consider an interval of the form (q_0, q_0^*). Since $\mathcal{V}$ is closed, $q_0^* = \lim q_n \in \mathcal{V}$. Moreover, $q_0^* \in \mathcal{U}$. Indeed, it cannot belong to $\mathcal{V} \setminus \overline{\mathcal{U}}$ because this set is discrete, and it cannot belong to $\overline{\mathcal{U}} \setminus \mathcal{U}$ because $\mathcal{U}$ is closed from above.

Since $q_n \notin \overline{\mathcal{U}}$ by Lemma 3.18 for $n = 1, 2, \ldots$, (q_0, q_0^*) is a subset of $(1, M+1] \setminus \overline{\mathcal{U}}$. Since, furthermore, q_0 and q_0^* belong to $\{1\} \cup \overline{\mathcal{U}}$, (q_0, q_0^*) is a connected component of $(1, M+1] \setminus \overline{\mathcal{U}}$.

It remains to show that each $q \in (1, M+1] \setminus \overline{\mathcal{U}}$ belongs to one of the intervals (q_0, q_0^*). Let ℓ_0 be the smallest element of $\mathcal{V}$ such that $\ell_0 \ge q$. We have $\ell_0 \in \mathcal{V} \setminus \overline{\mathcal{U}}$ by Remark 3.13.

Applying Lemma 3.17 there is a greatest $\ell_1 \in \{1\} \cup \mathcal{V}$ that is smaller than ℓ_0, and $\ell_1 \in \{1\} \cup (\mathcal{V} \setminus \mathcal{U})$. Furthermore, by Lemma 3.4 and (3.20), $\alpha(\ell_1)$ has a *smaller* period than the smallest period of $\alpha(\ell_0)$ in case $\ell_1 \ne 1$. Also, $\ell_1 < q \le \ell_0$ by the definition of ℓ_0.

Continuing recursively, we obtain a *finite* sequence $\ell_k < \cdots < \ell_0$ such that $\ell_i \in \mathcal{V} \setminus \overline{\mathcal{U}}$ for $i = 0, \dots, k-1$, $\ell_k \in \{1\} \cup (\overline{\mathcal{U}} \setminus \mathcal{U})$, and

$$(\ell_{i+1}, \ell_i) \cap \mathcal{V} = \varnothing \text{ for } i = 0, \dots, k-1.$$

Setting $q_0 := \ell_k$, we conclude that $q \in (q_0, q_0^*)$. $\square$

We end this paper with some examples.

Examples 3.22.

(i) Consider the integers $k = 1, 2, \dots, M+1$. We have

$$\begin{aligned} k &= M+1 && \Longrightarrow k \in \mathcal{U}, \\ \frac{M}{2} + 1 &< k \le M && \Longrightarrow k \in \overline{\mathcal{U}} \setminus \mathcal{U}, \\ k &= \frac{M}{2} + 1 && \Longrightarrow k \in \mathcal{V} \setminus \overline{\mathcal{U}}, \\ k &< \frac{M}{2} + 1 && \Longrightarrow k \notin \mathcal{V}. \end{aligned}$$

(ii) We recall that the Thue–Morse sequence $(\tau_i)_{i=0}^{\infty}$ is defined by the formulas $\tau_0 := 0$, and

$$\tau_{2^N+i} = 1 - \tau_i \quad \text{for} \quad i = 0, \dots, 2^N - 1, \quad N = 0, 1, 2, \dots.$$

Starting with $q_0 = 1$ and using Lemma 3.5, the above construction leads to the q-expansion of the smallest univoque base:

$$\alpha(q_0^*) = \begin{cases} (m - 1 + \tau_i)_{i=1}^{\infty} & \text{if } M = 2m-1, \quad m = 1, 2, \dots; \\ (m + \tau_i - \tau_{i-1})_{i=1}^{\infty} & \text{if } M = 2m, \quad m = 1, 2, \dots. \end{cases}$$

They were derived in [16,17] for $M = 1$ and $M \ge 2$, respectively; see also [21]. Our results provide a uniform approach for the odd and even values of M.

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