Research Article

Vito Crismale, Manuel Friedrich and Francesco Solombrino*

Integral representation for energies in linear elasticity with surface discontinuities

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Abstract: In this paper we prove an integral representation formula for a general class of energies defined on the space of generalized special functions of bounded deformation ($GSBD^p$) in arbitrary space dimensions. Functionals of this type naturally arise in the modeling of linear elastic solids with surface discontinuities including phenomena as fracture, damage, surface tension between different elastic phases, or material voids. Our approach is based on the global method for relaxation devised in [G. Bouchitté, I. Fonseca and L. Mascarenhas, A global method for relaxation, *Arch. Ration. Mech. Anal.* **145** (1998), no. 1, 51–98] and a recent Korn-type inequality in $GSBD^p$, cf. [F. Cagnetti, A. Chambolle and L. Scardia, Korn and Poincaré–Korn inequalities for functions with a small jump set, preprint (2020)]. Our general strategy also allows to generalize integral representation results in SBD^p, obtained in dimension two [S. Conti, M. Focardi and F. Iurlano, Integral representation for functionals defined on SBD^p in dimension two, *Arch. Ration. Mech. Anal.* **223** (2017), no. 3, 1337–1374], to higher dimensions, and to revisit results in the framework of generalized special functions of bounded variation ($GSBV^p$).

Keywords: Integral representation, global method for relaxation, free discontinuity problems, generalized special functions of bounded deformation, Korn-type inequalities

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1 Introduction

Integral representation results are a fundamental tool in the abstract theory of variational limits by Γ-convergence or in relaxation problems (see [32]). The topic has attracted widespread attention in the mathematical community over the last decades, with applications in various contexts, such as homogenization, dimension reduction, or atomistic-to-continuum approximations. In this paper we contribute to this topic by proving an integral representation result for a general class of energies arising in the modeling of linear elastic solids with surface discontinuities.

Integral representation theorems have been provided with increasing generality, ranging from functionals defined on Sobolev spaces [1, 17–19, 33, 49] to those defined on spaces of functions of bounded variation [12, 14, 22, 30], in particular on the subspace SBV of special functions of bounded variation [13, 15, 16] and

***Corresponding author: Francesco Solombrino,** Dipartimento di Matematica e Applicazioni “Renato Caccioppoli”, Università degli Studi di Napoli“Federico II”, Via Cintia, Monte S. Angelo 80126 Napoli, Italy, e-mail: francesco.solombrino@unina.it. https://orcid.org/0000-0003-4251-3851
Vito Crismale, Centre de Mathématiques Appliquées, École Polytechnique, 91128 Palaiseau Cedex, France, e-mail: vito.crismale@polytechnique.edu. https://orcid.org/0000-0002-9719-0044
Manuel Friedrich, Applied Mathematics Münster, University of Münster, Einsteinstrasse 62, 48149 Münster, Germany, e-mail: manuel.friedrich@uni-muenster.de

on piecewise constant functions [2]. In recent years, this analysis has been further improved to deal with functionals and variational limits on $GSBV^p$ (generalized special functions of bounded variation with p-integrable bulk density), which is the natural energy space for the variational description of many problems with free discontinuities, see among others [6–9, 21, 37, 41]. A very general method for dealing with all the abovementioned classes of functionals, the so-called *global method for relaxation*, has been developed by Bouchitté, Fonseca, Leoni, and Mascarenhas in [13, 14]. It essentially consists in comparing asymptotic Dirichlet problems on small balls with different boundary data depending on the local properties of the functions and allows to characterize energy densities in terms of cell formulas.

When coming to the variational description of rupture phenomena in general linearly elastic materials, however, the functional setting to be considered becomes weaker. Indeed, problems need to be formulated in suitable subspaces of *functions of bounded deformation* (BD functions) for which the distributional symmetrized gradient is a bounded Radon measure.

In the mathematical description of linear elasticity, the elastic properties are determined by the *elastic strain*. For a solid in a (bounded) reference configuration $\Omega \subset \mathbb{R}^d$, whose *displacement field* with respect to the equilibrium is $u\colon \Omega \to \mathbb{R}^d$, the elastic strain is given by the symmetrized gradient $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}})$. In standard models, the corresponding linear elastic energy is a suitable quadratic form of $e(u)$, possibly depending on the material point, see, e.g., [38, Section 2.1]. However, this is often generalized to the case of p-growth for a power $p > 1$, see [46, Sections 10–11]. The presence of surface discontinuities is related to several dissipative phenomena, such as cracks, surface tension between different elastic phases, or internal cavities. In the energetic description, this is represented by a term concentrated on the *jump set* J_u. This set is characterized by the property that for $x \in J_u$, when blowing up around x, the jump set approximates a hyperplane with normal $\nu_u(x) \in \mathbb{S}^{d-1}$ and the displacement field is close to two suitable values $u^+(x), u^-(x) \in \mathbb{R}^d$ on the two sides of the material with respect to this hyperplane.

Prototypical examples of functionals described above are energies which are controlled from above and below by suitable multiples of

$$\int_\Omega |e(u)|^p \,\mathrm{d}x + \int_{J_u \cap \Omega} (1 + |[u]|) \,\mathrm{d}\mathcal{H}^{d-1}, \tag{1}$$

where $[u](x) = u^+(x) - u^-(x)$ denotes the jump opening, or which are controlled by multiples of *Griffith's energy* [45]

$$\int_\Omega |e(u)|^p \,\mathrm{d}x + \mathcal{H}^{d-1}(J_u \cap \Omega). \tag{2}$$

Whereas in case (1) the energy space is be given by SBD^p, a subspace of BD, problems with control of type (2) are naturally formulated on *generalized special functions of bounded deformation* $GSBD^p$, introduced by Dal Maso [31]. (We refer to Section 3.1 for more details.) The only available integral representation result in this context is due to Conti, Focardi, and Iurlano [28] who considered variational functionals controlled locally in terms of (1) in dimension $d = 2$. Let us mention that the behavior is quite different if linear growth on the symmetrized gradient is assumed (corresponding to $p = 1$), as suited for the description of plasticity. In that case, representation results in the framework of BD have been obtained, for instance, in [10, 36] and [23] (see also [35, 48], containing essential tools for the proof).

The goal of the present article is twofold: we generalize the results of [28] for energies with control of type (1) to arbitrary space dimensions and, more importantly, we extend the theory to encompass also problems of the form (2), which are most relevant from an applicative viewpoint. Indeed, already in dimension two, the extension of [28] to the case where only a control of type (2) is available is no straightforward task. This is a fundamental difference with respect to the BV-theory where problems for generalized functions of bounded variation can be reconducted to SBV by a *perturbation trick* (see for instance [21]): one considers a small perturbation of the functional, depending on the jump opening, to represent functionals on SBV^p. Then, by letting the perturbation parameter vanish and by truncating functions suitably, the representation can be extended to $GSBV^p$. Unfortunately, the trick of reducing problem (2) to (1) is not expedient in the linearly elastic context and does not allow to deduce an integral representation result in $GSBD^p$ from the one in SBD^p. This is mainly due to the fact that, given a control only on the symmetrized gradient, it is in principle not possible to use smooth truncations to decrease the energy up to a small error.

Let us also remark that, while in the majority of integral representation results in BV and BD the L^1-topology was considered, this is not the right choice when only a lower bound of the form (2) is at hand. Indeed, in this case, the available compactness results [25, 31] have been established with respect to the topology of the convergence in measure. This latter is also the topology where recently an integral representation result for the subspace $PR(\Omega)$ of piecewise rigid functions has been proved in [44].

In our main result (Theorem 2.1), we prove an integral representation for *variational functionals*

$$\mathcal{F}\colon \mathrm{GSBD}^p(\Omega) \times \mathcal{B}(\Omega) \to [0, +\infty)$$

($\mathcal{B}(\Omega)$ denoting the Borel subsets of Ω) that satisfy the standard abstract conditions to be Borel measures in the second argument, lower semicontinuous with respect to convergence in measure, and local in the first argument. Moreover, we require control of type (2), localized to any $B \in \mathcal{B}(\Omega)$.

Let us comment on the proof strategy. We follow the general approach of the *global method for relaxation* provided in [13, 14] for variational functionals in BV. The proof strategy recovers the integral bulk and surface densities as blow-up limits of cell minimization formulas. The steps to be performed are the following:

- one first shows that, for fixed $u \in \mathrm{GSBD}^p(\Omega)$, the set function $\mathcal{F}(u, \cdot)$ is asymptotically equivalent to its minimum $\mathbf{m}_{\mathcal{F}}(u, \cdot)$ over competitors attaining the same boundary conditions as u on the boundaries of small balls centered in $x_0 \in \Omega$ with vanishing radii. With this we mean that the two quantities have the same Radon–Nikodym derivative with respect to $\mu := \mathcal{L}^d\lfloor_\Omega + \mathcal{H}^{d-1}\lfloor_{J_u\cap\Omega}$ (Lemma 4.1),
- one then proves that the Radon–Nikodym derivative $\frac{\mathrm{d}\mathbf{m}_{\mathcal{F}}(u,\cdot)}{\mathrm{d}\mu}$ only depends on x_0, the value $u(x_0)$, and the (approximate) gradient $\nabla u(x_0)$ at a Lebesgue point x_0, while at a jump point x_0 it is uniquely determined by the one-sided traces $u^+(x_0)$, $u^-(x_0)$ and the normal vector $\nu_u(x_0)$ to J_u in x_0 (Lemmas 4.2 and 4.3).

When dealing with all of the abovementioned issues, a key ingredient is given by a *Korn-type inequality for special functions of bounded deformation*, established recently by Cagnetti, Chambolle, and Scardia [20], which generalizes a two-dimensional result in [28] (see also [39]) to arbitrary dimension. It provides a control of the full gradient in terms of the symmetrized gradient, up to an exceptional set whose perimeter has a surface measure comparable to that of the discontinuity set. In particular, this estimate is used to approximate the function u with functions u_ε, which have Sobolev regularity in a ball (around a Lebesgue point), or in half-balls oriented by the jump normal (around a jump point), and which converge to the purely elastic competitor $u(x_0) + \nabla u(x_0)(\cdot - x_0)$, or the two-valued function with values $u^-(x_0)$ and $u^+(x_0)$, respectively. This is done in Lemmas 5.1 and 6.1, respectively, and is used for proving Lemmas 4.2 and 4.3. Let us mention that this application of the Korn-type inequality is similar to the one in dimension two [28] (with the topology of convergence in measure in place of L^1), and constitutes the counterpart of the SBV-Poincaré inequality [34] used in the SBV-case [13]. We also point out that our construction for approximating two-valued functions in Lemma 4.3 slightly differs from the ones in [13, 28] in order to fix a possible flaw contained in these proofs, see Remark 6.2 for details.

In contrast to [28], the Korn inequality is also used in the proof of Lemma 4.1: at this point, one needs to show that functions of the form

$$v^\delta := \sum v_i^\delta \chi_{B_i^\delta}$$

approximate u in the topology of the convergence in measure, where B_i^δ is a fine cover of a given set with disjoint balls of radius smaller than δ and v_i^δ denote minimizers for $\mathbf{m}_{\mathcal{F}}(u, B_i^\delta)$. In [28], the lower bound in (1) allows to control the distributional symmetrized gradient Eu which along with a scaling argument and the classical Korn–Poincaré inequality in BD (see [50, Theorem 2.2]) shows that v_i^δ is close to u on each B_i^δ. (In [13], the SBV-Poincaré inequality is used.) Our weaker lower bound of the form (2), however, calls for novel arguments and we use the Korn-type inequality to show that v_i^δ are close to u in L^p up to exceptional sets ω_i^δ whose volumes scale like $\delta(\mathcal{F}(u, B_i^\delta) + \mu(B_i^\delta))$.

We also point out that, if instead a control of the type (1) is assumed, the arguments leading to Theorem 2.1 can be successfully adapted to extend the result for functionals on SBD^p (see [28]) to arbitrary space dimensions, see Theorem 7.1. This is done by exploiting the stronger blow-up properties of SBD functions.

We note that, in principle, this result could be also obtained by adapting the arguments in [28] to higher dimension by employing the Korn inequality [20]. We however preferred to give a self-contained proof of Theorem 7.1, which requires only slight modifications of the arguments used for Theorem 2.1 and nicely illustrates the differences between SBD^p and its generalized space.

For a related purpose, in Section 8 we discuss how our arguments can also provide a direct proof for integral representation results on GSBV^p, if a local control on the full deformation gradient of the form

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x + \mathcal{H}^{d-1}(J_u \cap \Omega)$$

is given, see Theorem 8.1. In particular, no perturbation or truncation arguments are needed in the proof. Therefore, we believe that this provides a new perspective and a slightly simpler approach to integral representation results in GSBV^p without necessity of the *perturbation trick* discussed before, relying on the SBV result. Let us, however, mention that in [21] a more general growth condition from above is considered: dealing with such a condition would instead require a truncation method in the proof.

We close the introduction by mentioning that in a subsequent work [42] we use the present result to obtain integral representation of Γ-limits for sequences of energies in linear elasticity with surface discontinuities. There, we additionally characterize the bulk and surface densities as blow-up limits of cell minimization formulas where the minimization is not performed on GSBD^p but more specifically on Sobolev functions (bulk density) and piecewise rigid functions [44] (surface density). The latter characterization particularly allows to identify integrands of relaxed functionals and to treat homogenization problems.

The paper is organized as follows. In Section 2 we present our main integral representation result in GSBD^p. Section 3 is devoted to some preliminaries about the function space. In particular, we present the Korn-type inequality established in [20] and prove a fundamental estimate. Section 4 contains the general strategy and the proof of Lemma 4.1. The identifications of the bulk and surface density (Lemmas 4.2 and 4.3) are postponed to Sections 5 and 6, respectively. In Section 7 we describe the modifications necessary to obtain the SBD^p-case. Finally, in Section 8 we explain how our method can be used to establish an integral representation result in GSBV^p.

2 The integral representation result

In this section we present our main result. We start with some basic notation. Let $\Omega \subset \mathbb{R}^d$ be open, bounded with Lipschitz boundary. Let $\mathcal{A}(\Omega)$ be the family of open subsets of Ω, and denote by $\mathcal{B}(\Omega)$ the family of Borel sets contained in Ω. For every $x \in \mathbb{R}^d$ and $\varepsilon > 0$ we indicate by $B_\varepsilon(x) \subset \mathbb{R}^d$ the open ball with center x and radius ε. For $x, y \in \mathbb{R}^d$, we use the notation $x \cdot y$ for the scalar product and $|x|$ for the Euclidean norm. Moreover, we let $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ and we denote by $\mathbb{M}^{d\times d}$ the set of $d \times d$ matrices. The m-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^m$ is indicated by γ_m for every $m \in \mathbb{N}$. We denote by $\mathcal{L}^d$ and $\mathcal{H}^k$ the d-dimensional Lebesgue measure and the k-dimensional Hausdorff measure, respectively.

For definition and properties of the space $\mathrm{GSBD}^p(\Omega)$, $1 < p < \infty$, we refer the reader to [31]. Some relevant properties are collected in Section 3 below. In particular, the approximate gradient is denoted by ∇u (it is well defined, see Lemma 3.5) and the (approximate) jump set is denoted by J_u with corresponding normal ν_u and one-sided limits u^+ and u^-. We also define $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}})$.

We consider functionals $\mathcal{F}\colon \mathrm{GSBD}^p(\Omega) \times \mathcal{B}(\Omega) \to [0, +\infty)$ with the following general assumptions:

(H1) $\mathcal{F}(u, \cdot)$ is a Borel measure for any $u \in \mathrm{GSBD}^p(\Omega)$,

(H2) $\mathcal{F}(\cdot, A)$ is lower semicontinuous with respect to convergence in measure on Ω for any $A \in \mathcal{A}(\Omega)$,

(H3) $\mathcal{F}(\cdot, A)$ is local for any $A \in \mathcal{A}(\Omega)$, in the sense that if $u, v \in \mathrm{GSBD}^p(\Omega)$ satisfy $u = v$ a.e. in A, then $\mathcal{F}(u, A) = \mathcal{F}(v, A)$,

(H4) there exist $0 < \alpha < \beta$ such that for any $u \in \mathrm{GSBD}^p(\Omega)$ and $B \in \mathcal{B}(\Omega)$ we have

$$\alpha\Big(\int_B |e(u)|^p \,\mathrm{d}x + \mathcal{H}^{d-1}(J_u \cap B)\Big) \le \mathcal{F}(u, B) \le \beta\Big(\int_B (1 + |e(u)|^p) \,\mathrm{d}x + \mathcal{H}^{d-1}(J_u \cap B)\Big).$$

We now formulate the main result of this article addressing integral representation of functionals $\mathcal{F}$ satisfying (H1)–(H4). To this end, we introduce some further notation: for every $u \in \mathrm{GSBD}^p(\Omega)$ and $A \in \mathcal{A}(\Omega)$ we define

$$\mathbf{m}_{\mathcal{F}}(u, A) = \inf_{v \in \mathrm{GSBD}^p(\Omega)} \{\mathcal{F}(v, A) : v = u \text{ in a neighborhood of } \partial A\}. \tag{2.1}$$

For $x_0 \in \Omega$, $u_0 \in \mathbb{R}^d$, and $\xi \in \mathbb{M}^{d\times d}$ we introduce the functions $\ell_{x_0,u_0,\xi} : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\ell_{x_0,u_0,\xi}(x) = u_0 + \xi(x - x_0). \tag{2.2}$$

Moreover, for $x_0 \in \Omega$, $a, b \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$ we introduce $u_{x_0,a,b,\nu} : \mathbb{R}^d \to \mathbb{R}^d$ by

$$u_{x_0,a,b,\nu}(x) = \begin{cases} a & \text{if } (x - x_0)\cdot \nu > 0, \\ b & \text{if } (x - x_0)\cdot \nu < 0. \end{cases} \tag{2.3}$$

In this paper, we will prove the following result.

Theorem 2.1 (Integral representation in GSBD^p). *Let $\Omega \subset \mathbb{R}^d$ be open, bounded with Lipschitz boundary and suppose that $\mathcal{F} : \mathrm{GSBD}^p(\Omega) \times \mathcal{B}(\Omega) \to [0, +\infty)$ satisfies (H1)–(H4). Then*

$$\mathcal{F}(u, B) = \int_B f(x, u(x), \nabla u(x))\,\mathrm{d}x + \int_{J_u\cap B} g(x, u^+(x), u^-(x), \nu_u(x))\,\mathrm{d}\mathcal{H}^{d-1}(x)$$

for all $u \in \mathrm{GSBD}^p(\Omega)$ and $B \in \mathcal{B}(\Omega)$, where f is given by

$$f(x_0, u_0, \xi) = \limsup_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(\ell_{x_0,u_0,\xi}, B_\varepsilon(x_0))}{\gamma_d \varepsilon^d} \tag{2.4}$$

for all $x_0 \in \Omega$, $u_0 \in \mathbb{R}^d$, $\xi \in \mathbb{M}^{d\times d}$, and g is given by

$$g(x_0, a, b, \nu) = \limsup_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(u_{x_0,a,b,\nu}, B_\varepsilon(x_0))}{\gamma_{d-1} \varepsilon^{d-1}} \tag{2.5}$$

for all $x_0 \in \Omega$, $a, b \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$.

Remark 2.2. We proceed with some remarks on the result.

(i) In general, if f is not convex in ξ, in spite of the growth conditions (H4), the functional may fully depend on ∇u and not just on the symmetric part $e(u)$. We refer to [28, Remark 4.14] for an example in this direction.

(ii) As $\mathcal{F}$ is lower semicontinuous on $W^{1,p}$ with respect to weak convergence, the integrand f is quasi-convex [47]. Since $\mathcal{F}$ is lower semicontinuous on piecewise rigid functions, the integrand g is BD-elliptic [43] (at least if one can ensure, for instance, that g has a continuous dependence in x). A fortiori, g is BV-elliptic [3].

(iii) If the functional $\mathcal{F}$ additionally satisfies $\mathcal{F}(u + a, A) = \mathcal{F}(u, A)$ for all affine functions $a : \mathbb{R}^d \to \mathbb{R}^d$ with $e(a) = 0$, then there are two functions $f : \Omega \times \mathbb{M}^{d\times d} \to [0, +\infty)$ and $g : \Omega \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0, +\infty)$ such that

$$\mathcal{F}(u, B) = \int_B f(x, e(u)(x))\,\mathrm{d}x + \int_{J_u\cap B} g(x, [u](x), \nu_u(x))\,\mathrm{d}\mathcal{H}^{d-1}(x),$$

where $[u](x) := u^+(x) - u^-(x)$.

(iv) A variant of the proof shows that, in the minimization problems (2.4)–(2.5), one may replace balls $B_\varepsilon(x_0)$ by cubes $Q^\nu_\varepsilon(x_0)$ with sidelength ε, centered at x_0, and two faces orthogonal to $\nu = \nu_u(x_0)$.

(v) An analogous result holds on the space $\mathrm{GSBV}^p(\Omega; \mathbb{R}^m)$ for $m \in \mathbb{N}$. We refer to Section 8 for details.

We will additionally discuss the minor modifications needed in order to deal with functionals

$$\mathcal{F} : \mathrm{SBD}^p(\Omega) \times \mathcal{B}(\Omega) \to [0, +\infty)$$

satisfying (H1)–(H3) and

(H4') there exist $0 < \alpha < \beta$ such that for any $u \in \mathrm{SBD}^p(\Omega)$ and $B \in \mathcal{B}(\Omega)$ we have

$$\alpha\Big(\int_B |e(u)|^p\,\mathrm{d}x + \int_{J_u\cap B} (1 + |[u]|)\,\mathrm{d}\mathcal{H}^{d-1}\Big) \le \mathcal{F}(u, B) \le \beta\Big(\int_B (1 + |e(u)|^p)\,\mathrm{d}x + \int_{J_u\cap B} (1 + |[u]|)\,\mathrm{d}\mathcal{H}^{d-1}\Big).$$

In this case, $\mathrm{SBD}^p(\Omega)$ (see Section 3.1) is the natural energy space for $\mathcal{F}$. Furthermore, sequences of competitors with bounded energy, which are converging in measure, are additionally L^1-convergent if we assume (H4'), due to the classical Korn–Poincaré inequality in BD (see [50, Theorem II.2.2]). Hence, in this latter case, (H2) is equivalent to requiring lower semicontinuity with respect to the L^1-convergence. The statement of the result in this setting, as well as of the changes needed in the proofs, will be given in Section 7.

3 Preliminaries

We start this preliminary section by introducing some further notation. For $E \subset \mathbb{R}^d$, $\varepsilon > 0$, and $x_0 \in \mathbb{R}^d$ we set

$$E_{\varepsilon,x_0} := x_0 + \varepsilon(E - x_0). \tag{3.1}$$

The diameter of E is indicated by $\mathrm{diam}(E)$. Given two sets $E_1, E_2 \subset \mathbb{R}^d$, we denote their symmetric difference by $E_1 \,\Delta\, E_2$. We write χ_E for the characteristic function of any $E \subset \mathbb{R}^d$, which is 1 on E and 0 otherwise. If E is a set of finite perimeter, we denote its essential boundary by $\partial^* E$, see [5, Definition 3.60]. We denote the set of symmetric and skew-symmetric matrices by $\mathbb{M}^{d\times d}_{\mathrm{sym}}$ and $\mathbb{M}^{d\times d}_{\mathrm{skew}}$, respectively.

3.1 BD and GBD functions

Let $U \subset \mathbb{R}^d$ be open. A function $v \in L^1(U;\mathbb{R}^d)$ belongs to the space of *functions of bounded deformation*, denoted by $\mathrm{BD}(U)$, if the distribution $\mathrm{E}v := \frac{1}{2}(\mathrm{D}v + (\mathrm{D}v)^{\mathrm{T}})$ is a bounded $\mathbb{M}^{d\times d}_{\mathrm{sym}}$-valued Radon measure on U, where $\mathrm{D}v = (\mathrm{D}_1 v, \dots, \mathrm{D}_d v)$ is the distributional differential. It is well known (see [4, 50]) that for $v \in \mathrm{BD}(U)$ the jump set J_v is countably $\mathcal{H}^{d-1}$-rectifiable (in the sense of [5, Definition 2.57]), and that

$$\mathrm{E}v = \mathrm{E}^a v + \mathrm{E}^c v + \mathrm{E}^j v,$$

where $\mathrm{E}^a v$ is absolutely continuous with respect to $\mathcal{L}^d$, $\mathrm{E}^c v$ is singular with respect to $\mathcal{L}^d$ and such that $|\mathrm{E}^c v|(B) = 0$ if $\mathcal{H}^{d-1}(B) < \infty$, while $\mathrm{E}^j v$ is concentrated on J_v. The density of $\mathrm{E}^a v$ with respect to $\mathcal{L}^d$ is denoted by $e(v)$.

The space $\mathrm{SBD}(U)$ is the subspace of all functions $v \in \mathrm{BD}(U)$ such that $\mathrm{E}^c v = 0$. For $p \in (1,\infty)$, we define $\mathrm{SBD}^p(U) := \{v \in \mathrm{SBD}(U) : e(v) \in L^p(U;\mathbb{M}^{d\times d}_{\mathrm{sym}}),\ \mathcal{H}^{d-1}(J_v) < \infty\}$. For a complete treatment of BD and SBD functions, we refer to to [4, 11, 50].

The spaces $\mathrm{GBD}(U)$ of *generalized functions of bounded deformation* and $\mathrm{GSBD}(U) \subset \mathrm{GBD}(U)$ of *generalized special functions of bounded deformation* have been introduced in [31] (cf. [31, Definitions 4.1 and 4.2]), and are defined as follows.

Definition 3.1. Let $U \subset \mathbb{R}^d$ be a bounded open set, and let $v\colon U \to \mathbb{R}^d$ be measurable. We introduce the notation

$$\Pi^\xi := \{y \in \mathbb{R}^d : y \cdot \xi = 0\}, \quad B^\xi_y := \{t \in \mathbb{R} : y + t\xi \in B\} \text{ for any } y \in \mathbb{R}^d \text{ and } B \subset \mathbb{R}^d$$

for fixed $\xi \in \mathbb{S}^{d-1}$, and for every $t \in B^\xi_y$ we let

$$v^\xi_y(t) := v(y + t\xi), \quad \widehat{v}^\xi_y(t) := v^\xi_y(t) \cdot \xi.$$

Then $v \in \mathrm{GBD}(U)$ if there exists $\lambda_v \in \mathcal{M}^+_b(U)$ such that $\widehat{v}^\xi_y \in \mathrm{BV}_{\mathrm{loc}}(U^\xi_y)$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$, and for every Borel set $B \subset U$,

$$\int_{\Pi^\xi} \big(|\mathrm{D}\widehat{v}^\xi_y|(B^\xi_y \setminus J^1_{\widehat{v}^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{\widehat{v}^\xi_y})\big)\,\mathrm{d}\mathcal{H}^{d-1}(y) \le \lambda_v(B), \quad J^1_{\widehat{v}^\xi_y} := \{t \in J_{\widehat{v}^\xi_y} : |[\widehat{v}^\xi_y]|(t) \ge 1\}.$$

Moreover, the function v belongs to $\mathrm{GSBD}(U)$ if $v \in \mathrm{GBD}(U)$ and $\widehat{v}^\xi_y \in \mathrm{SBV}_{\mathrm{loc}}(U^\xi_y)$ for every $\xi \in \mathbb{S}^{d-1}$ and for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$.

We recall that every $v \in \mathrm{GBD}(U)$ has an *approximate symmetric gradient* $e(v) \in L^1(U; \mathbb{M}^{d\times d}_{\mathrm{sym}})$ and an *approximate jump set* J_v which is still countably $\mathcal{H}^{d-1}$-rectifiable (cf. [31, Theorem 9.1, Theorem 6.2]).

The notation for $e(v)$ and J_v, which is the same as that one in the SBD case, is consistent: in fact, if v lies in $\mathrm{SBD}(U)$, the objects coincide (up to negligible sets of points with respect to $\mathcal{L}^d$ and $\mathcal{H}^{d-1}$, respectively). For $x \in J_v$ there exist $v^+(x), v^-(x) \in \mathbb{R}^d$ and $\nu_v(x) \in \mathbb{S}^{d-1}$ such that

$$\lim_{\varepsilon\to 0} \varepsilon^{-d}\mathcal{L}^d(\{y \in B_\varepsilon(x) : \pm(y-x)\cdot\nu_v(x) > 0\} \cap \{|v - v^\pm(x)| > \varrho\}) = 0 \tag{3.2}$$

for every $\varrho > 0$, and the function $[v] := v^+ - v^- : J_v \to \mathbb{R}^d$ is measurable. For $1 < p < \infty$, the space $\mathrm{GSBD}^p(U)$ is given by

$$\mathrm{GSBD}^p(U) := \{v \in \mathrm{GSBD}(U) : e(v) \in L^p(U; \mathbb{M}^{d\times d}_{\mathrm{sym}}), \mathcal{H}^{d-1}(J_v) < \infty\}.$$

Any function $v \in \mathrm{GSBD}(U)$ with $[v]$ integrable belongs to $\mathrm{SBD}(U)$, as follows from [27, Theorem 2.9] for $\mathbb{A}v = \mathrm{E}v$ (see [27, Remark 2.5]). This corresponds to the following proposition.

Proposition 3.2. *If $v \in \mathrm{GSBD}^p(U)$ is such that $[v] \in L^1(J_v; \mathbb{R}^d)$, then $v \in \mathrm{SBD}^p(U)$.*

If U has Lipschitz boundary, for each $v \in \mathrm{GBD}(U)$ the traces on ∂U are well defined (see [31, Theorem 5.5]), in the sense that for $\mathcal{H}^{d-1}$-a.e. $x \in \partial U$ there exists $\mathrm{tr}(v)(x) \in \mathbb{R}^d$ such that

$$\lim_{\varepsilon\to 0} \varepsilon^{-d}\mathcal{L}^d(U \cap B_\varepsilon(x) \cap \{|v - \mathrm{tr}(v)(x)| > \varrho\}) = 0 \qquad \text{for all } \varrho > 0. \tag{3.3}$$

3.2 Korn's inequality and fundamental estimate

In this subsection we discuss two important tools which will be instrumental for the proof of Theorem 2.1. We start by the following Korn and Korn–Poincaré inequalities in GSBD for functions with small jump sets, see [20, Theorem 1.1, Theorem 1.2]. In the following, we say that $a : \mathbb{R}^d \to \mathbb{R}^d$ is an *infinitesimal rigid motion* if a is affine with $e(a) = \frac{1}{2}(\nabla a + (\nabla a)^{\mathrm{T}}) = 0$.

Theorem 3.3 (Korn inequality for functions with small jump set). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $1 < p < +\infty$. Then there exists a constant $c = c(\Omega, p) > 0$ such that for all $u \in \mathrm{GSBD}^p(\Omega)$ there is a set of finite perimeter $\omega \subset \Omega$ with*

$$\mathcal{H}^{d-1}(\partial^*\omega) \le c\mathcal{H}^{d-1}(J_u), \quad \mathcal{L}^d(\omega) \le c(\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}} \tag{3.4}$$

and an infinitesimal rigid motion a such that

$$\|u - a\|_{L^p(\Omega\setminus\omega)} + \|\nabla u - \nabla a\|_{L^p(\Omega\setminus\omega)} \le c\|e(u)\|_{L^p(\Omega)}. \tag{3.5}$$

Moreover, there exists $v \in W^{1,p}(\Omega; \mathbb{R}^d)$ such that $v = u$ on $\Omega \setminus \omega$ and

$$\|e(v)\|_{L^p(\Omega)} \le c\|e(u)\|_{L^p(\Omega)}.$$

Note that the result is indeed only relevant for functions with sufficiently small jump set, as otherwise one can choose $\omega = \Omega$, and (3.5) trivially holds. Note that, in [20], $\mathcal{L}^d(\omega) \le c(\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}}$ has not been shown, but it readily follows from $\mathcal{H}^{d-1}(\partial^*\omega) \le c\mathcal{H}^{d-1}(J_u)$ by the isoperimetric inequality.

Remark 3.4 (Almost Sobolev regularity, constants, and scaling invariance). (i) More precisely, in [20] it is proved that there exists $v \in W^{1,p}(\Omega; \mathbb{R}^d)$ such that $v = u$ on $\Omega \setminus \omega$ and $\|e(v)\|_{L^p(\Omega)} \le c\|e(u)\|_{L^p(\Omega)}$, whence by Korn's and Poincaré's inequality in $W^{1,p}(\Omega; \mathbb{R}^d)$ we get

$$\|v - a\|_{L^p(\Omega)} + \|\nabla v - \nabla a\|_{L^p(\Omega)} \le c\|e(u)\|_{L^p(\Omega)}$$

for an infinitesimal rigid motion a. This directly implies (3.5), see [20, Theorem 4.1, Theorem 4.4].

(ii) Given a collection of bounded Lipschitz domains $(\Omega_k)_k$ which are related through bi-Lipschitzian homeomorphisms with Lipschitz constants of both the homeomorphism itself and its inverse bounded uniformly in k, in Theorem 3.3 we can choose a constant c uniformly for all Ω_k, see [20, Remark 4.2].

(iii) Recall definition (3.1). Consider a bounded Lipschitz domain Ω, $\varepsilon > 0$, and $x_0 \in \mathbb{R}^d$. Then for each $u \in \mathrm{GSBD}^p(\Omega_{\varepsilon,x_0})$ we find $\omega \subset \Omega_{\varepsilon,x_0}$ and a rigid motion a such that

$$\mathcal{H}^{d-1}(\partial^*\omega) \le C\mathcal{H}^{d-1}(J_u), \quad \mathcal{L}^d(\omega) \le C(\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}}$$

and

$$\varepsilon^{-1}\|u-a\|_{L^p(\Omega_{\varepsilon,x_0}\setminus\omega)} + \|\nabla u - \nabla a\|_{L^p(\Omega_{\varepsilon,x_0}\setminus\omega)} \le C\|e(u)\|_{L^p(\Omega_{\varepsilon,x_0})},$$

where $C = C(\Omega, p) > 0$ is independent of ε. This follows by a standard rescaling argument.

From Theorem 3.3, one can also deduce that for $u \in \mathrm{GSBD}^p(\Omega)$ the *approximate gradient* ∇u exists $\mathcal{L}^d$-a.e. in Ω, see [20, Corollary 5.2].

Lemma 3.5 (Approximate gradient). *Let $\Omega \subset \mathbb{R}^d$ be open, bounded with Lipschitz boundary, let $1 < p < +\infty$, and $u \in \mathrm{GSBD}^p(\Omega)$. Then for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$ there exists a matrix in $\mathbb{M}^{d\times d}$, denoted by $\nabla u(x_0)$, such that*

$$\lim_{\varepsilon\to 0} \varepsilon^{-d}\mathcal{L}^d\Big(\Big\{x \in B_\varepsilon(x_0) : \frac{|u(x)-u(x_0)-\nabla u(x_0)(x-x_0)|}{|x-x_0|} > \varrho\Big\}\Big) = 0 \quad \text{for all } \varrho > 0.$$

We point out that the result in Lemma 3.5 has already been obtained in [40] for $p = 2$, as a consequence of the embedding $\mathrm{GSBD}^2(\Omega) \subset (\mathrm{GBV}(\Omega))^d$, see [40, Theorem 2.9].

To control the affine mappings appearing in Theorem 3.3, we will make use of the following elementary lemma on affine mappings, see, e.g., [44, Lemma 3.4] or [29, Lemmas 4.3] for similar statements. (It is obtained by the equivalence of norms in finite dimensions and by standard rescaling arguments.)

Lemma 3.6. *Let $1 \le p < +\infty$, let $x_0 \in \mathbb{R}^d$, and let $R, \theta > 0$. Let $a\colon \mathbb{R}^d \to \mathbb{R}^d$ be affine, defined by $a(x) = A\,x + b$ for $x \in \mathbb{R}^d$, and let $E \subset B_R(x_0) \subset \mathbb{R}^d$ with $\mathcal{L}^d(E) \ge \theta\mathcal{L}^d(B_R(x_0))$. Then there exists a constant $c_0 > 0$ only depending on p and θ such that*

$$\|a\|_{L^p(B_R(x_0))} \le \gamma_d^{\frac{1}{p}} R^{\frac{d}{p}} \|a\|_{L^\infty(B_R(x_0))} \le c_0\|a\|_{L^p(E)}, \quad |A| \le c_0 R^{-1-\frac{d}{p}}\|a\|_{L^p(E)}.$$

We now proceed with another consequence of Theorem 3.3.

Corollary 3.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $1 < p < +\infty$. Then there exists a constant $C = C(\Omega, p) > 0$ such that for all $u \in \mathrm{GSBD}^p(\Omega)$ with trace $\mathrm{tr}(u) = 0$ on $\partial\Omega$ (see (3.3)) there is a set of finite perimeter $\omega \subset \mathbb{R}^d$ with*

$$\mathcal{H}^{d-1}(\partial^*\omega) \le C\mathcal{H}^{d-1}(J_u), \quad \mathcal{L}^d(\omega) \le C(\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}} \tag{3.6}$$

such that

$$\|u\|_{L^p(\Omega\setminus\omega)} + \|\nabla u\|_{L^p(\Omega\setminus\omega)} \le C\|e(u)\|_{L^p(\Omega)}. \tag{3.7}$$

Proof. We start by choosing a bounded Lipschitz domain $\Omega' \subset \mathbb{R}^d$ with $\Omega \subset\subset \Omega'$. Each $u \in \mathrm{GSBD}^p(\Omega)$ with $\mathrm{tr}(u) = 0$ on $\partial\Omega$ can be extended to a function $\tilde u \in \mathrm{GSBD}^p(\Omega')$ by $\tilde u = 0$ on $\Omega' \setminus \Omega$ such that $J_{\tilde u} = J_u$. We first note that it is not restrictive to assume that

$$\mathcal{H}^{d-1}(J_u) \le \Big(\frac{\mathcal{L}^d(\Omega'\setminus\Omega)}{2c}\Big)^{\frac{d-1}{d}}, \tag{3.8}$$

where $c = c(\Omega', p) > 0$ is the constant of Theorem 3.3. In fact, otherwise we could take $\omega = \Omega$ and the statement would be trivially satisfied since (3.7) is clearly trivial and for (3.6) we use that

$$\mathcal{H}^{d-1}(\partial^*\Omega) \le C\Big(\frac{\mathcal{L}^d(\Omega'\setminus\Omega)}{2c}\Big)^{\frac{d-1}{d}}, \quad \mathcal{L}^d(\Omega) \le C\frac{\mathcal{L}^d(\Omega'\setminus\Omega)}{2c}$$

for a sufficiently large constant $C > 0$ depending only on Ω and Ω'.

Now, consider a function u satisfying (3.8). We apply Theorem 3.3 on $\tilde u \in \mathrm{GSBD}^p(\Omega')$ and obtain a set $\omega \subset \Omega' \subset \mathbb{R}^d$ satisfying (3.4) as well as an infinitesimal rigid motion a such that

$$\|\tilde u - a\|_{L^p(\Omega'\setminus\omega)} + \|\nabla\tilde u - \nabla a\|_{L^p(\Omega'\setminus\omega)} \le c\|e(\tilde u)\|_{L^p(\Omega')} = c\|e(u)\|_{L^p(\Omega)}. \tag{3.9}$$

In particular, $\tilde{u} = 0$ on $\Omega' \setminus \Omega$ implies

$$\|a\|_{L^p(\Omega' \setminus (\Omega \cup \omega))} \le c\|e(u)\|_{L^p(\Omega)}. \tag{3.10}$$

By (3.4) and (3.8) we get $\mathcal{L}^d(\omega) \le \frac{1}{2}\mathcal{L}^d(\Omega' \setminus \Omega)$. In view of (3.10), we apply Lemma 3.6 on $E = \Omega' \setminus (\Omega \cup \omega)$ with $R = \operatorname{diam}(\Omega')$ and $\theta = \frac{1}{2}\mathcal{L}^d(\Omega' \setminus \Omega)/\gamma_d R^d$ to get

$$\|a\|_{L^p(\Omega')} \le c\|a\|_{L^p(\Omega' \setminus (\Omega \cup \omega))} \le c\|e(u)\|_{L^p(\Omega)},$$

and, in a similar fashion, $|\nabla a| \le c\|e(u)\|_{L^p(\Omega)}$, where $c > 0$ depends on Ω, Ω', and p. Then (3.7) follows from (3.9), the triangle inequality, and the fact that $u = \tilde{u}$ on Ω. □

We conclude this subsection with another important tool in the proof of the integral representation, namely a fundamental estimate in $GSBD^p$.

Lemma 3.8 (Fundamental estimate in $GSBD^p$). *Let $\Omega \subset \mathbb{R}^d$ be open, bounded with Lipschitz boundary, and let $1 < p < +\infty$. Let $\eta > 0$ and let $A, A', A'' \in \mathcal{A}(\Omega)$ with $A' \subset\subset A$. For every functional $\mathcal{F}$ satisfying* (H1), (H3), *and* (H4) *and for every $u \in GSBD^p(A)$, $v \in GSBD^p(A'')$ there exists a function $\varphi \in C^\infty(\mathbb{R}^d; [0, 1])$ such that $w := \varphi u + (1 - \varphi)v \in GSBD^p(A' \cup A'')$ satisfies*

$$\begin{cases} \text{(i)} & \mathcal{F}(w, A' \cup A'') \le (1 + \eta)(\mathcal{F}(u, A) + \mathcal{F}(v, A'')) + M\|u - v\|^p_{L^p((A \setminus A') \cap A'')} + \eta\mathcal{L}^d(A' \cup A''), \\ \text{(ii)} & w = u \quad \text{on } A' \qquad \text{and} \qquad w = v \quad \text{on } A'' \setminus A, \end{cases} \tag{3.11}$$

where $M = M(A, A', A'', p, \eta) > 0$ depends only on A, A', A'', p, η, but is independent of u and v. Moreover, if for $\varepsilon > 0$ and $x_0 \in \mathbb{R}^d$ we have $A_{\varepsilon,x_0}, A'_{\varepsilon,x_0}, A''_{\varepsilon,x_0} \subset \Omega$, then

$$M(A_{\varepsilon,x_0}, A'_{\varepsilon,x_0}, A''_{\varepsilon,x_0}, p, \eta) = \varepsilon^{-p} M(A, A', A'', p, \eta), \tag{3.12}$$

where we used the notation introduced in (3.1).

The same statement holds if $\mathcal{F}$ satisfies (H4'), *$u \in SBD^p(A)$, and $v \in SBD^p(A'')$.*

In the statement above, we intend that $\|u - v\|^p_{L^p((A \setminus A') \cap A'')} = +\infty$ if $u - v \notin L^p((A \setminus A') \cap A'')$.

Proof. The proof follows the lines of [16, Proposition 3.1]. Choose $k \in \mathbb{N}$ such that

$$k \ge \max\Big\{\frac{3^{p-1}\beta}{\eta\alpha}, \frac{\beta}{\eta}\Big\}. \tag{3.13}$$

Let $A_1, \dots, A_{k+1}$ be open subsets of $\mathbb{R}^d$ with

$$A' \subset\subset A_1 \subset\subset \cdots \subset\subset A_{k+1} \subset\subset A.$$

For $i = 1, \dots, k$ let $\varphi_i \in C_0^\infty(A_{i+1}; [0, 1])$ with $\varphi_i = 1$ in a neighborhood V_i of $\overline{A_i}$.

Consider $u \in GSBD^p(A)$ and $v \in GSBD^p(A'')$. We can clearly assume that $u - v \in L^p((A \setminus A') \cap A'')$ as otherwise the result is trivial. We define the function $w_i = \varphi_i u + (1 - \varphi_i)v \in GSBD^p(A' \cup A'')$, where u and v are extended arbitrarily outside A and A'', respectively. Letting $T_i = A'' \cap (A_{i+1} \setminus \overline{A_i})$ we get by (H1) and (H3)

$$\begin{aligned} \mathcal{F}(w_i, A' \cup A'') &\le \mathcal{F}(u, (A' \cup A'') \cap V_i) + \mathcal{F}(v, A'' \setminus \operatorname{supp}(\varphi_i)) + \mathcal{F}(w_i, T_i) \\ &\le \mathcal{F}(u, A) + \mathcal{F}(v, A'') + \mathcal{F}(w_i, T_i). \end{aligned} \tag{3.14}$$

For the last term, we compute using (H4) ($\odot$ denotes the symmetrized vector product)

$$\begin{aligned} \mathcal{F}(w_i, T_i) &\le \beta \int_{T_i} (1 + |e(w_i)|^p)\,dx + \beta\mathcal{H}^{d-1}(J_{w_i} \cap T_i) \\ &\le \beta \int_{T_i} (1 + |\varphi_i e(u) + (1 - \varphi_i)e(v) + \nabla\varphi_i \odot (u - v)|^p) + \beta\mathcal{H}^{d-1}((J_u \cup J_v) \cap T_i) \\ &\le \beta\mathcal{L}^d(T_i) + 3^{p-1}\beta \int_{T_i} (|e(u)|^p + |e(v)|^p + |\nabla\varphi_i|^p|u - v|^p) + \beta\mathcal{H}^{d-1}(J_u \cap T_i) + \beta\mathcal{H}^{d-1}(J_v \cap T_i) \\ &\le 3^{p-1}\beta\alpha^{-1}(\mathcal{F}(u, T_i) + \mathcal{F}(v, T_i)) + 3^{p-1}\beta\|\nabla\phi_i\|^p_\infty\|u - v\|^p_{L^p(T_i)} + \beta\mathcal{L}^d(T_i). \end{aligned}$$

Notice that we can obtain the same estimate also if $\mathcal{F}$ satisfies (H4'), $u \in \mathrm{SBD}^p(A)$, and $v \in \mathrm{SBD}^p(A'')$. (We refer to [16, proof of Proposition 3.1] for details.) Consequently, recalling (3.13) and using (H1), we find $i_0 \in \{1, \dots, k\}$ such that

$$\mathcal{F}(w_{i_0}, T_{i_0}) \le \frac{1}{k} \sum_{i=1}^{k} \mathcal{F}(w_i, T_i) \le \eta(\mathcal{F}(u, A) + \mathcal{F}(v, A'')) + M\|u - v\|^p_{L^p((A\setminus A')\cap A'')} + \eta\mathcal{L}^d((A \setminus A') \cap A''),$$

where $M := 3^{p-1}\beta k^{-1} \max_{i=1,\dots,k} \|\nabla\varphi_i\|^p_\infty$. This along with (3.14) concludes the proof of (3.11) by setting $w = w_{i_0}$. To see the scaling property (3.12), it suffices to use the cut-off functions $\varphi_i^\varepsilon \in C_0^\infty((A_{i+1})_{\varepsilon,x_0}; [0, 1])$, $i = 1, \dots, k$, defined by $\varphi_i^\varepsilon(x) = \varphi_i(x_0 + \frac{1}{\varepsilon}(x - x_0))$ for $x \in (A_{i+1})_{\varepsilon,x_0}$. This concludes the proof. □

4 The global method

This section is devoted to the proof of Theorem 2.1 which is based on three ingredients. First, we show that $\mathcal{F}$ is equivalent to $\mathbf{m}_\mathcal{F}$ (see (2.1)) in the sense that the two quantities have the same Radon–Nikodym derivative with respect to $\mu := \mathcal{L}^d\lfloor_\Omega + \mathcal{H}^{d-1}\lfloor_{J_u\cap\Omega}$.

Lemma 4.1. *Suppose that $\mathcal{F}$ satisfies* (H1)–(H4). *Let $u \in \mathrm{GSBD}^p(\Omega)$ and $\mu = \mathcal{L}^d\lfloor_\Omega + \mathcal{H}^{d-1}\lfloor_{J_u\cap\Omega}$. Then for μ-a.e. $x_0 \in \Omega$ we have*

$$\lim_{\varepsilon\to0} \frac{\mathcal{F}(u, B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} = \lim_{\varepsilon\to0} \frac{\mathbf{m}_\mathcal{F}(u, B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))}.$$

We prove this lemma in the final part of this section. The second ingredient is that, asymptotically as $\varepsilon \to 0$, the minimization problems $\mathbf{m}_\mathcal{F}(u, B_\varepsilon(x_0))$ and $\mathbf{m}_\mathcal{F}(\bar{u}^{\mathrm{bulk}}_{x_0}, B_\varepsilon(x_0))$ coincide for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$, where we write $\bar{u}^{\mathrm{bulk}}_{x_0} := \ell_{x_0,u(x_0),\nabla u(x_0)}$ for brevity, see (2.2).

Lemma 4.2. *Suppose that $\mathcal{F}$ satisfies* (H1) *and* (H3)–(H4) *and let $u \in \mathrm{GSBD}^p(\Omega)$. Then for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$ we have*

$$\lim_{\varepsilon\to0} \frac{\mathbf{m}_\mathcal{F}(u, B_\varepsilon(x_0))}{\gamma_d\varepsilon^d} = \limsup_{\varepsilon\to0} \frac{\mathbf{m}_\mathcal{F}(\bar{u}^{\mathrm{bulk}}_{x_0}, B_\varepsilon(x_0))}{\gamma_d\varepsilon^d}. \tag{4.1}$$

We defer the proof of Lemma 4.2 to Section 5. The third ingredient is that, asymptotically as $\varepsilon \to 0$, the minimization problems $\mathbf{m}_\mathcal{F}(u, B_\varepsilon(x_0))$ and $\mathbf{m}_\mathcal{F}(\bar{u}^{\mathrm{surf}}_{x_0}, B_\varepsilon(x_0))$ coincide for $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$, where we write $\bar{u}^{\mathrm{surf}}_{x_0} := u_{x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)}$ for brevity, see (2.3).

Lemma 4.3. *Suppose that $\mathcal{F}$ satisfies* (H1) *and* (H3)–(H4) *and let $u \in \mathrm{GSBD}^p(\Omega)$. Then for $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$ we have*

$$\lim_{\varepsilon\to0} \frac{\mathbf{m}_\mathcal{F}(u, B_\varepsilon(x_0))}{\gamma_{d-1}\varepsilon^{d-1}} = \limsup_{\varepsilon\to0} \frac{\mathbf{m}_\mathcal{F}(\bar{u}^{\mathrm{surf}}_{x_0}, B_\varepsilon(x_0))}{\gamma_{d-1}\varepsilon^{d-1}}. \tag{4.2}$$

We defer the proof of Lemma 4.3 to Section 6, and now proceed to prove Theorem 2.1.

Proof of Theorem 2.1. In view of (H4) on $\mathcal{F}$ and of the Besicovitch derivation theorem (cf. [5, Theorem 2.22]), we need to show that for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$ one has

$$\frac{\mathrm{d}\mathcal{F}(u,\cdot)}{\mathrm{d}\mathcal{L}^d}(x_0) = f(x_0, u(x_0), \nabla u(x_0)), \tag{4.3}$$

where f was defined in (2.4), and that for $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$ one has

$$\frac{\mathrm{d}\mathcal{F}(u,\cdot)}{\mathrm{d}\mathcal{H}^{d-1}\lfloor_{J_u}}(x_0) = g(x_0, u^+(x_0), u^-(x_0), \nu_u(x_0)), \tag{4.4}$$

where g was defined in (2.5).

By Lemma 4.1 and the fact that $\lim_{\varepsilon\to0}(\gamma_d\varepsilon^d)^{-1}\mu(B_\varepsilon(x_0)) = 1$ for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$ we deduce

$$\frac{\mathrm{d}\mathcal{F}(u,\cdot)}{\mathrm{d}\mathcal{L}^d}(x_0) = \lim_{\varepsilon\to0} \frac{\mathcal{F}(u, B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} = \lim_{\varepsilon\to0} \frac{\mathbf{m}_\mathcal{F}(u, B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} = \lim_{\varepsilon\to0} \frac{\mathbf{m}_\mathcal{F}(u, B_\varepsilon(x_0))}{\gamma_d\varepsilon^d} < \infty$$

for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$. Then (4.3) follows from (2.4) and Lemma 4.2. By Lemma 4.1 and the fact that

$$\lim_{\varepsilon\to 0}(\gamma_{d-1}\varepsilon^{d-1})^{-1}\mu(B_\varepsilon(x_0)) = 1$$

for $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$ we deduce

$$\frac{\mathrm{d}\mathcal{F}(u,\cdot)}{\mathrm{d}\mathcal{H}^{d-1}\lfloor_{J_u}}(x_0) = \lim_{\varepsilon\to 0}\frac{\mathcal{F}(u,B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} = \lim_{\varepsilon\to 0}\frac{\mathbf{m}_{\mathcal{F}}(u,B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} = \lim_{\varepsilon\to 0}\frac{\mathbf{m}_{\mathcal{F}}(u,B_\varepsilon(x_0))}{\gamma_{d-1}\varepsilon^{d-1}} < \infty$$

for $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$. Now, (4.4) follows from (2.5) and Lemma 4.3. □

In the remaining part of the section we prove Lemma 4.1. We basically follow the lines of [13, 14, 28], with the difference that the required compactness results are more delicate due to the weaker growth condition from below (see (H4)) compared to [13, 14, 28]. We start with some notation. For $\delta > 0$ and $A \in \mathcal{A}(\Omega)$, we define

$$\mathbf{m}^\delta_{\mathcal{F}}(u,A) = \inf\Big\{\sum_{i=1}^\infty \mathbf{m}_{\mathcal{F}}(u,B_i) : B_i \subset A \text{ pairwise disjoint balls, } \mathrm{diam}(B_i) \le \delta,\ \mu\Big(A\setminus\bigcup_{i=1}^\infty B_i\Big) = 0\Big\},$$

where, as before, $\mu = \mathcal{L}^d\lfloor_\Omega + \mathcal{H}^{d-1}\lfloor_{J_u\cap\Omega}$. As $\mathbf{m}^\delta_{\mathcal{F}}(u,A)$ is decreasing in δ, we can also introduce

$$\mathbf{m}^*_{\mathcal{F}}(u,A) = \lim_{\delta\to 0}\mathbf{m}^\delta_{\mathcal{F}}(u,A). \tag{4.5}$$

In the following lemma, we prove that $\mathcal{F}$ and $\mathbf{m}^*_{\mathcal{F}}$ coincide under our assumptions.

Lemma 4.4. *Suppose that $\mathcal{F}$ satisfies* (H1)–(H4) *and let $u \in GSBD^p(\Omega)$. Then for all $A \in \mathcal{A}(\Omega)$ there holds $\mathcal{F}(u,A) = \mathbf{m}^*_{\mathcal{F}}(u,A)$.*

Proof. We follow the lines of the proof of [28, Lemma 4.1] focusing on the necessary adaptions due to the weaker growth condition from below (see (H4)) compared to [28]. For each ball $B \subset A$, $\mathbf{m}_{\mathcal{F}}(u,B) \le \mathcal{F}(u,B)$ by definition. By (H1) we get $\mathbf{m}^\delta_{\mathcal{F}}(u,A) \le \mathcal{F}(u,A)$ for all $\delta > 0$. This shows $\mathbf{m}^*_{\mathcal{F}}(u,A) \le \mathcal{F}(u,A)$, cf. (4.5).

We now address the reverse inequality. We fix $A \in \mathcal{A}(\Omega)$ and $\delta > 0$. Let $(B_i^\delta)_i$ be balls as in the definition of $\mathbf{m}^\delta_{\mathcal{F}}(u,A)$ such that

$$\sum_{i=1}^\infty \mathbf{m}_{\mathcal{F}}(u,B_i^\delta) \le \mathbf{m}^\delta_{\mathcal{F}}(u,A) + \delta. \tag{4.6}$$

By the definition of $\mathbf{m}_{\mathcal{F}}$, we find $v_i^\delta \in GSBD^p(B_i^\delta)$ such that $v_i^\delta = u$ in a neighborhood of ∂B_i^δ and

$$\mathcal{F}(v_i^\delta, B_i^\delta) \le \mathbf{m}_{\mathcal{F}}(u,B_i^\delta) + \delta\mathcal{L}^d(B_i^\delta). \tag{4.7}$$

We define

$$v^{\delta,n} := \sum_{i=1}^n v_i^\delta\chi_{B_i^\delta} + u\chi_{N_0^{\delta,n}} \quad \text{for } n\in\mathbb{N}, \qquad v^\delta := \sum_{i=1}^\infty v_i^\delta\chi_{B_i^\delta} + u\chi_{N_0^\delta}, \tag{4.8}$$

where $N_0^{\delta,n} := \Omega\setminus\bigcup_{i=1}^n B_i^\delta$ and $N_0^\delta := \Omega\setminus\bigcup_{i=1}^\infty B_i^\delta$. By construction, we have that each $v^{\delta,n}$ lies in $GSBD^p(\Omega)$ and that $\sup_{n\in\mathbb{N}}(\|e(v^{\delta,n})\|_{L^p(\Omega)} + \mathcal{H}^{d-1}(J_{v^{\delta,n}})) < +\infty$ by (4.6)–(4.7) and (H4). Moreover, $v^{\delta,n} \to v^\delta$ pointwise a.e. in Ω. Then [25, Theorem 1.1] yields $v^\delta \in GSBD^p(\Omega)$.

$$\begin{aligned}\mathcal{F}(v^\delta,A) &= \sum_{i=1}^\infty \mathcal{F}(v_i^\delta,B_i^\delta) + \mathcal{F}(u,N_0^\delta\cap A)\\ &\le \sum_{i=1}^\infty(\mathbf{m}_{\mathcal{F}}(u,B_i^\delta) + \delta\mathcal{L}^d(B_i^\delta))\\ &\le \mathbf{m}^\delta_{\mathcal{F}}(u,A) + \delta(1+\mathcal{L}^d(A)),\end{aligned} \tag{4.9}$$

where we also used the fact that $\mu(N_0^\delta\cap A) = \mathcal{F}(u,N_0^\delta\cap A) = 0$ by the definition of $(B_i^\delta)_i$ and (H4). For later purpose, we also note by (H4) that this implies

$$\|e(v^\delta)\|^p_{L^p(A)} + \mathcal{H}^{d-1}(J_{v^\delta}\cap A) \le \alpha^{-1}(\mathbf{m}^\delta_{\mathcal{F}}(u,A) + \delta(1+\mathcal{L}^d(A))). \tag{4.10}$$

We now claim that $v^\delta \to u$ in measure on A. To this end, we apply Remark 3.4 (iii) and Corollary 3.7 on each B_i^δ for the function $u - v_i^\delta$ and we get sets of finite perimeter $\omega_i^\delta \subset B_i^\delta$ such that

$$\begin{cases} \text{(i)} & (\mathcal{L}^d(\omega_i^\delta))^{\frac{d-1}{d}} \leq C\mathcal{H}^{d-1}((J_u \cup v^\delta) \cap B_i^\delta), \\ \text{(ii)} & \|u - v_i^\delta\|^p_{L^p(B_i^\delta \setminus \omega_i^\delta)} \leq C\delta^p(\|e(u)\|^p_{L^p(B_i^\delta)} + \|e(v^\delta)\|^p_{L^p(B_i^\delta)}) \end{cases} \tag{4.11}$$

for a constant $C > 0$ only depending on p. Here, we used that $\operatorname{diam}(B_i^\delta) \leq \delta$ and $(u - v^\delta)\lfloor_{B_i^\delta} \in GSBD^p(B_i^\delta)$ with trace zero on ∂B_i^δ. We define $\psi\colon [0, +\infty) \to [0, +\infty)$ by $\psi(t) = \min\{t^p, 1\}$ and observe that $v^\delta \to u$ in measure on A is equivalent to $\int_A \psi(|u - v^\delta|)\,dx \to 0$ as $\delta \to 0$. In view of (4.8), we compute

$$\int_A \psi(|u - v^\delta|)\,dx = \sum_{i=1}^\infty \int_{B_i^\delta} \psi(|u - v_i^\delta|)\,dx \leq \sum_{i=1}^\infty (\|u - v_i^\delta\|^p_{L^p(B_i^\delta \setminus \omega_i^\delta)} + \mathcal{L}^d(\omega_i^\delta)). \tag{4.12}$$

By (4.11) (ii) and the fact that the balls $(B_i^\delta)_i$ are pairwise disjoint we get

$$\sum_{i=1}^\infty \|u - v_i^\delta\|^p_{L^p(B_i^\delta \setminus \omega_i^\delta)} \leq C\delta^p(\|e(u)\|^p_{L^p(A)} + \|e(v^\delta)\|^p_{L^p(A)}). \tag{4.13}$$

As $\omega_i^\delta \subset B^i_\delta$ and $\operatorname{diam}(B_i^\delta) \leq \delta$, we further get by (4.11) (i)

$$\sum_{i=1}^\infty \mathcal{L}^d(\omega_i^\delta) \leq \gamma_d^{\frac{1}{d}}\delta \sum_{i=1}^\infty (\mathcal{L}^d(\omega_i^\delta))^{\frac{d-1}{d}} \leq \gamma_d^{1/d} C\delta\mathcal{H}^{d-1}((J_u \cup J_{v^\delta}) \cap A). \tag{4.14}$$

Now, combining (4.12)–(4.14) and using (4.10), we find $\int_A \psi(|u - v^\delta|)\,dx \to 0$ as $\delta \to 0$. With this, using (H2), (4.5), and (4.9) we get the required inequality $\mathbf{m}^*_{\mathcal{F}}(u, A) \geq \mathcal{F}(u, A)$ in the limit as $\delta \to 0$. This concludes the proof. □

Proof of Lemma 4.1. The statement follows by repeating exactly the arguments in [28, proofs of Lemma 4.2 and Lemma 4.3]. We report a sketch for the reader's convenience.

The definition of $\mathbf{m}_{\mathcal{F}}$ gives readily that for every $x_0 \in \Omega$,

$$\limsup_{\varepsilon \to 0} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} \leq \limsup_{\varepsilon \to 0} \frac{\mathcal{F}(u, B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))}.$$

The converse inequality follows by proving that, for every $t > 0$, the set

$$E_t := \Big\{x \in \Omega : \liminf_{\varepsilon \to 0} \frac{\mathcal{F}(u, B_\varepsilon(x)) - \mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x))}{\mu(B_\varepsilon(x))} > t\Big\}$$

satisfies $\mu(E_t) = 0$. To this end, we fix $t > 0$. We introduce the family of balls (depending on t)

$$X^\delta := \{B_\varepsilon(x) : \varepsilon < \delta, \overline{B_\varepsilon(x)} \subset \Omega, \mu(\partial B_\varepsilon(x)) = 0, \mathcal{F}(u, B_\varepsilon(x)) > \mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x)) + t\mu(B_\varepsilon(x))\},$$

and we show that

$$U^* := \bigcap_{\delta > 0} \{x \in \Omega : B_\varepsilon(x) \in X^\delta \text{ for some } \varepsilon > 0\}$$

satisfies

$$E_t \subset U^*, \quad \mu(U^*) = 0. \tag{4.15}$$

Then $\mu(E_t) = 0$ indeed holds true and the proof is concluded.

We now confirm (4.15). The inclusion $E_t \subset U^*$ follows from the definition of E_t which permits to find, for any $x \in E_t$, $\varepsilon < \delta$ such that $\mathcal{F}(u, B_\varepsilon(x)) > \mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x)) + t\mu(B_\varepsilon(x))$. Note that, due to the left continuity of $\varepsilon \mapsto \mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x))$ (cf. [28, Lemma 4.2]) one can also ensure the additional property $\mu(\partial B_\varepsilon(x)) = 0$ by slightly varying ε. In order to prove that $\mu(U^*) = 0$, one fixes a compact set $K \subset U^*$, two positive numbers $\delta < \eta$, and defines

$$U^\eta := \bigcup\{B_\varepsilon(x) : B_\varepsilon(x) \in X^\eta\}, \quad Y^\delta := \{B_\varepsilon(x) : \varepsilon < \delta, \overline{B_\varepsilon(x)} \subset U^\eta \setminus K, \mu(\partial B_\varepsilon(x)) = 0\}.$$

By recalling also the definition of U^*, we see that X^δ and Y^δ are fine covers of K and $U^\eta \setminus K$, respectively. Thus there exists countable many pairwise disjoint $B_i \in X^\delta$, $\hat{B}_j \in Y^\delta$, and a set N with $\mu(N) = 0$ such that $U^\eta = \bigcup_i B_i \cup \bigcup_j \hat{B}_j \cup N$. In view of assumptions (H1), the definitions of X^δ, Y^δ (in particular the balls have radii smaller than δ) give that

$$\mathcal{F}(u, U^\eta) \geq \sum_i \mathbf{m}_{\mathcal{F}}(u, B_i) + \sum_j \mathbf{m}_{\mathcal{F}}(u, \hat{B}_j) + t\mu\Big(\bigcup_i B_i\Big) \geq \mathbf{m}^\delta_{\mathcal{F}}(u, U^\eta) + t\mu(K).$$

Passing to the limit in δ, (4.5) and Lemma 4.4 imply

$$\mathcal{F}(u, U^\eta) \geq \mathbf{m}^*_{\mathcal{F}}(u, U^\eta) + t\mu(K) = \mathcal{F}(u, U^\eta) + t\mu(K),$$

so that $\mu(K) = 0$. Then $\mu(U^*) = 0$ by the regularity of μ. □

To conclude the proof of Theorem 2.1, it remains to prove Lemmas 4.2 and 4.3. This is the subject of the following two sections.

5 The bulk density

This section is devoted to the proof of Lemma 4.2. We start by analyzing the blow-up at points with approximate gradient. The latter exists for $\mathcal{L}^d$-a.e. point in Ω by Lemma 3.5.

Lemma 5.1 (Blow-up at points with approximate gradient). *Let $u \in GSBD^p(\Omega)$. Let $\theta \in (0, 1)$. Then for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$ there exists a family $u_\varepsilon \in GSBD^p(B_\varepsilon(x_0))$ such that*

$$\begin{cases} \text{(i)} & u_\varepsilon = u \text{ in a neighborhood of } \partial B_\varepsilon(x_0), \quad \lim_{\varepsilon\to 0} \varepsilon^{-(d+1)}\mathcal{L}^d(\{u_\varepsilon \neq u\}) = 0, \\ \text{(ii)} & \lim_{\varepsilon\to 0} \varepsilon^{-(d+p)} \int_{B_{(1-\theta)\varepsilon}(x_0)} |u_\varepsilon(x) - u(x_0) - \nabla u(x_0)(x - x_0)|^p \,\mathrm{d}x = 0, \\ \text{(iii)} & \lim_{\varepsilon\to 0} \varepsilon^{-d} \int_{B_\varepsilon(x_0)} |e(u_\varepsilon)(x) - e(u)(x_0)|^p \,\mathrm{d}x = 0, \\ \text{(iv)} & \lim_{\varepsilon\to 0} \varepsilon^{-d}\, \mathcal{H}^{d-1}(J_{u_\varepsilon}) = 0. \end{cases} \tag{5.1}$$

Proof. Let $x_0 \in \Omega$ be such that

$$\begin{cases} \text{(i)} & \lim_{\varepsilon\to 0} \varepsilon^{-d} \int_{B_\varepsilon(x_0)} |e(u)(x) - e(u)(x_0)|^p \,\mathrm{d}x = 0, \\ \text{(ii)} & \lim_{\varepsilon\to 0} \varepsilon^{-d}\, \mathcal{H}^{d-1}(J_u \cap B_\varepsilon(x_0)) = 0, \\ \text{(iii)} & \lim_{\varepsilon\to 0} \varepsilon^{-d}\mathcal{L}^d\Big(\Big\{x \in B_\varepsilon(x_0) : \dfrac{|u(x) - u(x_0) - \nabla u(x_0)(x - x_0)|}{|x - x_0|} > \varrho\Big\}\Big) = 0 \text{ for all } \varrho > 0. \end{cases} \tag{5.2}$$

These properties hold for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$ by Lemma 3.5 and the facts that $|e(u)|^p \in L^1(\Omega)$ and J_u is countably $\mathcal{H}^{d-1}$-rectifiable. We use again the notation $\bar{u}^{\mathrm{bulk}}_{x_0} = \ell_{x_0,u_0,\nabla u(x_0)} = u(x_0) + \nabla u(x_0)(\cdot - x_0)$ for brevity, see (2.2).

Fix $\theta > 0$. We apply Theorem 3.3 and Remark 3.4 (i) for the function $u - \bar{u}^{\mathrm{bulk}}_{x_0}$ on the set $B_{(1-\theta)\varepsilon}(x_0)$ to obtain a set of finite perimeter $\omega_\varepsilon \subset B_{(1-\theta)\varepsilon}(x_0)$, a function $v_\varepsilon \in W^{1,p}(B_{(1-\theta)\varepsilon}(x_0); \mathbb{R}^d)$ with $v_\varepsilon = u - \bar{u}^{\mathrm{bulk}}_{x_0}$ in $B_{(1-\theta)\varepsilon}(x_0) \setminus \omega_\varepsilon$, and an infinitesimal rigid motion a_ε such that

$$\begin{cases} \text{(i)} & \mathcal{H}^{d-1}(\partial^*\omega_\varepsilon) \leq c\mathcal{H}^{d-1}(J_u \cap B_\varepsilon(x_0)), \quad \mathcal{L}^d(\omega_\varepsilon) \leq c(\mathcal{H}^{d-1}(J_u \cap B_\varepsilon(x_0)))^{\frac{d}{d-1}}, \\ \text{(ii)} & \|v_\varepsilon - a_\varepsilon\|_{L^p(B_{(1-\theta)\varepsilon}(x_0))} \leq c\,\varepsilon\|e(u - \bar{u}^{\mathrm{bulk}}_{x_0})\|_{L^p(B_\varepsilon(x_0))}, \\ \text{(iii)} & \|e(v_\varepsilon)\|_{L^p(B_{(1-\theta)\varepsilon}(x_0))} \leq c\|e(u - \bar{u}^{\mathrm{bulk}}_{x_0})\|_{L^p(B_\varepsilon(x_0))}, \end{cases} \tag{5.3}$$

where $c > 0$ depends only on p, cf. also Remark 3.4 (iii). We directly note by (5.2) (ii) and (5.3) (i) that

$$\lim_{\varepsilon\to 0} \varepsilon^{-\frac{d^2}{d-1}}\mathcal{L}^d(\omega_\varepsilon) = 0. \tag{5.4}$$

We define $u_\varepsilon \in GSBD^p(B_\varepsilon(x_0))$ as

$$u_\varepsilon := u\chi_{B_\varepsilon(x_0)\setminus B_{(1-\theta)\varepsilon}(x_0)} + (v_\varepsilon + \bar{u}^{\text{bulk}}_{x_0})\chi_{B_{(1-\theta)\varepsilon}(x_0)}, \tag{5.5}$$

and proceed by confirming the properties stated in (5.1). Notice that, by construction, $u_\varepsilon = u$ in $B_\varepsilon(x_0) \setminus \omega_\varepsilon$. First, (5.1) (i) follows directly from the fact that $\omega_\varepsilon \subset B_{(1-\theta)\varepsilon}(x_0)$, as well as (5.4)–(5.5). Moreover, (5.3) (i) and (5.2) (ii) imply (5.1) (iv). As for (5.1) (iii), we notice that by (5.3) (iii) and (5.2) (i) we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-d} \int_{B_{(1-\theta)\varepsilon}(x_0)} |e(v_\varepsilon)(x)|^p \,dx = 0.$$

Since, by a direct computation, $e(u_\varepsilon)(x) - e(u)(x_0) = e(v_\varepsilon)(x)$ for $x \in B_{(1-\theta)\varepsilon}(x_0)$, see (5.5), in combination with (5.2) (i) we obtain (5.1) (iii). It therefore remains to prove (5.1) (ii).

To this end, fix $\varrho > 0$ and define $\hat{\omega}_\varepsilon := \{x \in B_\varepsilon(x_0) : |u(x) - \bar{u}^{\text{bulk}}_{x_0}(x)| > \varrho\varepsilon\}$. In view of (5.2) (iii) and (5.4), we can choose $\varepsilon_0 > 0$ sufficiently small such that for all $0 < \varepsilon \le \varepsilon_0$ we have

$$\mathcal{L}^d(\omega_\varepsilon \cup \hat{\omega}_\varepsilon) \le \frac{1}{2}\mathcal{L}^d(B_{(1-\theta)\varepsilon}(x_0)). \tag{5.6}$$

By the definition of $\hat{\omega}_\varepsilon$ and the fact that $v_\varepsilon = u - \bar{u}^{\text{bulk}}_{x_0}$ in $B_{(1-\theta)\varepsilon}(x_0) \setminus \omega_\varepsilon$, we have $|v_\varepsilon(x)| \le \varrho\varepsilon$ for all $x \in B_{(1-\theta)\varepsilon}(x_0) \setminus (\omega_\varepsilon \cup \hat{\omega}_\varepsilon)$. Hence, (5.3) (ii) and the triangle inequality give

$$\|a_\varepsilon\|^p_{L^p(B_{(1-\theta)\varepsilon}(x_0)\setminus(\omega_\varepsilon\cup\hat{\omega}_\varepsilon))} \le C\varepsilon^p \|e(u - \bar{u}^{\text{bulk}}_{x_0})\|^p_{L^p(B_\varepsilon(x_0))} + C\mathcal{L}^d(B_\varepsilon(x_0))\varrho^p\varepsilon^p,$$

where $C > 0$ depends only on p. By (5.6) and Lemma 3.6 we get

$$\|a_\varepsilon\|^p_{L^p(B_{(1-\theta)\varepsilon}(x_0))} \le C\varepsilon^p \|e(u - \bar{u}^{\text{bulk}}_{x_0})\|^p_{L^p(B_\varepsilon(x_0))} + C\mathcal{L}^d(B_\varepsilon(x_0))\varrho^p\varepsilon^p.$$

Therefore, by using also (5.2) (i), we derive $\limsup_{\varepsilon\to 0} \varepsilon^{-(d+p)}\|a_\varepsilon\|^p_{L^p(B_{(1-\theta)\varepsilon}(x_0))} \le C\gamma_d\varrho^p$. As $\varrho > 0$ was arbitrary, we get

$$\lim_{\varepsilon\to 0} \varepsilon^{-(d+p)} \int_{B_{(1-\theta)\varepsilon}(x_0)} |a_\varepsilon|^p \,dx = 0. \tag{5.7}$$

Now, (5.2) (i) and (5.3) (ii) give that

$$\lim_{\varepsilon\to 0} \varepsilon^{-(d+p)}\|v_\varepsilon - a_\varepsilon\|^p_{L^p(B_{(1-\theta)\varepsilon}(x_0))} \le c\,\varepsilon^{-d}\|e(u - \bar{u}^{\text{bulk}}_{x_0})\|^p_{L^p(B_\varepsilon(x_0))} = 0.$$

As $u_\varepsilon - \bar{u}^{\text{bulk}}_{x_0} = v_\varepsilon$ in $B_{(1-\theta)\varepsilon}(x_0)$, this shows (5.1) (ii) by (5.7). □

We are now in a position to prove Lemma 4.2.

Proof of Lemma 4.2. It suffices to prove (4.1) for points $x_0 \in \Omega$ where the statement of Lemma 5.1 holds and we have $\lim_{\varepsilon\to 0} \varepsilon^{-d}\mu(B_\varepsilon(x_0)) = \gamma_d$. This holds true for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$. Then also $\lim_{\varepsilon\to 0} \varepsilon^{-d}\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0)) \in \mathbb{R}$ exists, see Lemma 4.1. As before, we write $\bar{u}^{\text{bulk}}_{x_0} = u(x_0) + \nabla u(x_0)(\cdot - x_0)$ for shorthand.

Step 1 (Inequality "≤" in (4.1)). We fix $\eta > 0$ and $\theta > 0$. Choose $z_\varepsilon \in GSBD^p(B_{(1-3\theta)\varepsilon}(x_0))$ with $z_\varepsilon = \bar{u}^{\text{bulk}}_{x_0}$ in a neighborhood of $\partial B_{(1-3\theta)\varepsilon}(x_0)$ and

$$\mathcal{F}(z_\varepsilon, B_{(1-3\theta)\varepsilon}(x_0)) \le \mathbf{m}_{\mathcal{F}}(\bar{u}^{\text{bulk}}_{x_0}, B_{(1-3\theta)\varepsilon}(x_0)) + \varepsilon^{d+1}. \tag{5.8}$$

We extend z_ε to a function in $GSBD^p(B_\varepsilon(x_0))$ by setting $z_\varepsilon = \bar{u}^{\text{bulk}}_{x_0}$ outside $B_{(1-3\theta)\varepsilon}(x_0)$. Let $(u_\varepsilon)_\varepsilon$ be the family given by Lemma 5.1. We apply Lemma 3.8 on z_ε (in place of u) and u_ε (in place of v) for η as above and the sets

$$A' = B_{1-2\theta}(x_0), \quad A = B_{1-\theta}(x_0), \quad A'' = B_1(x_0) \setminus \overline{B_{1-4\theta}(x_0)}. \tag{5.9}$$

By (3.11)–(3.12) there exist functions $w_\varepsilon \in GSBD^p(B_\varepsilon(x_0))$ such that $w_\varepsilon = u_\varepsilon$ on $B_\varepsilon(x_0) \setminus B_{(1-\theta)\varepsilon}(x_0)$ and

$$\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0)) \le (1+\eta)(\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0}) + \mathcal{F}(u_\varepsilon, A''_{\varepsilon,x_0})) + \frac{M}{\varepsilon^p}\|z_\varepsilon - u_\varepsilon\|^p_{L^p((A\setminus A')_{\varepsilon,x_0})} + \mathcal{L}^d(B_\varepsilon(x_0))\eta, \tag{5.10}$$

where $M > 0$ depends on θ and η, but is independent of ε. Here and in the following, we use notation (3.1). In particular, we have $w_\varepsilon = u_\varepsilon = u$ in a neighborhood of $\partial B_\varepsilon(x_0)$ by (5.1) (i). By (5.1) (ii), (5.9), and the fact that $z_\varepsilon = \bar{u}^{\mathrm{bulk}}_{x_0}$ outside $B_{(1-3\theta)\varepsilon}(x_0)$ we find

$$\lim_{\varepsilon\to 0} \varepsilon^{-(d+p)} \|z_\varepsilon - u_\varepsilon\|^p_{L^p((A\setminus A')_{\varepsilon,x_0})} = \lim_{\varepsilon\to 0} \varepsilon^{-(d+p)} \|u_\varepsilon - \bar{u}^{\mathrm{bulk}}_{x_0}\|^p_{L^p(B_{(1-\theta)\varepsilon}(x_0))} = 0. \tag{5.11}$$

This along with (5.10) shows that there exists a sequence $(\rho_\varepsilon)_\varepsilon \subset (0, +\infty)$ with $\rho_\varepsilon \to 0$ such that

$$\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0)) \le (1+\eta)(\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0}) + \mathcal{F}(u_\varepsilon, A''_{\varepsilon,x_0})) + \varepsilon^d \rho_\varepsilon + \gamma_d \varepsilon^d \eta. \tag{5.12}$$

On the one hand, by using that $z_\varepsilon = \bar{u}^{\mathrm{bulk}}_{x_0}$ on $B_\varepsilon(x_0) \setminus B_{(1-3\theta)\varepsilon}(x_0) \subset A''_{\varepsilon,x_0}$, (H1), (H4), and (5.8) we compute

$$\begin{aligned}\limsup_{\varepsilon\to 0} \frac{\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0})}{\varepsilon^d} &\le \limsup_{\varepsilon\to 0} \frac{\mathcal{F}(z_\varepsilon, B_{(1-3\theta)\varepsilon}(x_0))}{\varepsilon^d} + \limsup_{\varepsilon\to 0} \frac{\mathcal{F}(\bar{u}^{\mathrm{bulk}}_{x_0}, A''_{\varepsilon,x_0})}{\varepsilon^d} \\ &\le \limsup_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(\bar{u}^{\mathrm{bulk}}_{x_0}, B_{(1-3\theta)\varepsilon}(x_0))}{\varepsilon^d} + \beta\gamma_d[1-(1-4\theta)^d](1+|e(u)(x_0)|^p) \\ &\le (1-3\theta)^d \limsup_{\varepsilon'\to 0} \frac{\mathbf{m}_{\mathcal{F}}(\bar{u}^{\mathrm{bulk}}_{x_0}, B_{\varepsilon'}(x_0))}{(\varepsilon')^d} + \beta\gamma_d[1-(1-4\theta)^d](1+|e(u)(x_0)|^p), \end{aligned} \tag{5.13}$$

where in the last step we substituted $(1-3\theta)\varepsilon$ by ε'. On the other hand, by (H4) and (5.9) we also find

$$\begin{aligned}\mathcal{F}(u_\varepsilon, A''_{\varepsilon,x_0}) &\le \beta \int_{A''_{\varepsilon,x_0}} (1+|e(u_\varepsilon)|^p) + \beta\mathcal{H}^{d-1}(J_{u_\varepsilon} \cap A''_{\varepsilon,x_0}) \\ &\le \gamma_d \varepsilon^d \beta[1-(1-4\theta)^d](1+2^{p-1}|e(u)(x_0)|^p) + 2^{p-1}\beta\|e(u_\varepsilon) - e(u)(x_0)\|^p_{L^p(B_\varepsilon(x_0))} + \beta\mathcal{H}^{d-1}(J_{u_\varepsilon}).\end{aligned}$$

By (5.1) (iii)–(iv) this implies

$$\limsup_{\varepsilon\to 0} \frac{\mathcal{F}(u_\varepsilon, A''_{\varepsilon,x_0})}{\varepsilon^d} \le \beta\gamma_d[1-(1-4\theta)^d](1+2^{p-1}|e(u)(x_0)|^p). \tag{5.14}$$

Recall that $w_\varepsilon = u$ in a neighborhood of $\partial B_\varepsilon(x_0)$. This along with (5.12)–(5.14) and $\rho_\varepsilon \to 0$ yields

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\gamma_d \varepsilon^d} &\le \limsup_{\varepsilon\to 0} \frac{\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0))}{\gamma_d\varepsilon^d} \\ &\le (1+\eta)(1-3\theta)^d \limsup_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(\bar{u}^{\mathrm{bulk}}_{x_0}, B_\varepsilon(x_0))}{\gamma_d\varepsilon^d} \\ &\quad + 2(1+\eta)\beta[1-(1-4\theta)^d](1+2^{p-1}|e(u)(x_0)|^p) + \eta.\end{aligned}$$

Passing to $\eta, \theta \to 0$, we obtain inequality "≤" in (4.1).

Step 2 (Inequality "≥" in (4.1)). We fix $\eta,\ \theta > 0$ and let $(u_\varepsilon)_\varepsilon$ be again the family from Lemma 5.1. By (5.1) (i) and Fubini's Theorem, for each $\varepsilon > 0$ we can find $s_\varepsilon \in (1-4\theta, 1-3\theta)\varepsilon$ such that

$$\begin{cases} \text{(i)} & \lim_{\varepsilon\to 0} \varepsilon^{-d}\mathcal{H}^{d-1}(\{u \neq u_\varepsilon\} \cap \partial B_{s_\varepsilon}(x_0)) = 0, \\ \text{(ii)} & \mathcal{H}^{d-1}((J_u \cup J_{u_\varepsilon}) \cap \partial B_{s_\varepsilon}(x_0)) = 0 \quad \text{for all } \varepsilon > 0. \end{cases} \tag{5.15}$$

We consider $z_\varepsilon \in \mathrm{GSBD}^p(B_{s_\varepsilon}(x_0))$ such that $z_\varepsilon = u$ in a neighborhood of $\partial B_{s_\varepsilon}(x_0)$, and

$$\mathcal{F}(z_\varepsilon, B_{s_\varepsilon}(x_0)) \le \mathbf{m}_{\mathcal{F}}(u, B_{s_\varepsilon}(x_0)) + \varepsilon^{d+1}. \tag{5.16}$$

We extend z_ε to a function in $\mathrm{GSBD}^p(B_\varepsilon(x_0))$ by setting

$$z_\varepsilon = u_\varepsilon \quad \text{in } B_\varepsilon(x_0) \setminus B_{s_\varepsilon}(x_0). \tag{5.17}$$

We apply Lemma 3.8 on z_ε (in place of u) and $\bar{u}^{\mathrm{bulk}}_{x_0}$ (in place of v) for the sets indicated in equation (5.9). By (3.11)–(3.12) there exist functions $w_\varepsilon \in \mathrm{GSBD}^p(B_\varepsilon(x_0))$ such that $w_\varepsilon = \bar{u}^{\mathrm{bulk}}_{x_0}$ on $B_\varepsilon(x_0) \setminus B_{(1-\theta)\varepsilon}(x_0)$ and

$$\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0)) \le (1+\eta)(\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0}) + \mathcal{F}(\bar{u}^{\mathrm{bulk}}_{x_0}, A''_{\varepsilon,x_0})) + \frac{M}{\varepsilon^p}\|z_\varepsilon - \bar{u}^{\mathrm{bulk}}_{x_0}\|^p_{L^p((A\setminus A')_{\varepsilon,x_0})} + \mathcal{L}^d(B_\varepsilon(x_0))\eta.$$

By (5.17) and the choice of s_ε we get that $z_\varepsilon = u_\varepsilon$ outside $B_{(1-3\theta)\varepsilon}(x_0)$. Thus, similar to Step 1, cf. (5.11) and (5.12), we find a sequence $(\rho_\varepsilon)_\varepsilon \subset (0, +\infty)$ with $\rho_\varepsilon \to 0$ such that

$$\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0)) \le (1+\eta)(\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0}) + \mathcal{F}(\bar u^{\text{bulk}}_{x_0}, A''_{\varepsilon,x_0})) + \varepsilon^d \rho_\varepsilon + \gamma_d \varepsilon^d \eta. \tag{5.18}$$

Let us now estimate the terms in (5.18). We get by (H1), (H4), (5.16)–(5.17), and the choice of s_ε that

$$\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0}) \le \mathbf{m}_{\mathcal{F}}(u, B_{s_\varepsilon}(x_0)) + \varepsilon^{d+1} + \beta\mathcal{H}^{d-1}((\{u \ne u_\varepsilon\} \cup J_u \cup J_{u_\varepsilon}) \cap \partial B_{s_\varepsilon}(x_0)) + \mathcal{F}(u_\varepsilon, A''_{\varepsilon,x_0}). \tag{5.19}$$

Therefore, by (5.14), (5.15), and the fact that $s_\varepsilon \le (1-3\theta)\varepsilon$ we derive

$$\begin{aligned}\limsup_{\varepsilon\to 0} \frac{\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0})}{\varepsilon^d} &\le \Big(\frac{s_\varepsilon}{\varepsilon}\Big)^d \limsup_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(u, B_{s_\varepsilon}(x_0))}{s_\varepsilon^d} + \beta\gamma_d[1-(1-4\theta)^d](1+2^{p-1}|e(u)(x_0)|^p)\\ &\le (1-3\theta)^d \limsup_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\varepsilon^d} + \beta\gamma_d[1-(1-4\theta)^d](1+2^{p-1}|e(u)(x_0)|^p).\end{aligned} \tag{5.20}$$

Estimating $\mathcal{F}(\bar u^{\text{bulk}}_{x_0}, A''_{\varepsilon,x_0})$ as in (5.13), with (5.18)–(5.20) and $\rho_\varepsilon \to 0$, we then obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0} \frac{\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0))}{\varepsilon^d} &\le (1+\eta)(1-3\theta)^d \limsup_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\varepsilon^d}\\ &\quad + 2(1+\eta)\beta\gamma_d[1-(1-4\theta)^d](1+2^{p-1}|e(u)(x_0)|^p) + \gamma_d\eta.\end{aligned}$$

Passing to $\eta, \theta \to 0$ and recalling that $w_\varepsilon = \bar u^{\text{bulk}}_{x_0}$ in a neighborhood of $\partial B_\varepsilon(x_0)$, we derive

$$\begin{aligned}\limsup_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(\bar u^{\text{bulk}}_{x_0}, B_\varepsilon(x_0))}{\gamma_d\varepsilon^d} &\le \limsup_{\varepsilon\to 0} \frac{\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0))}{\gamma_d\varepsilon^d}\\ &\le \limsup_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\gamma_d\varepsilon^d} = \lim_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\gamma_d\varepsilon^d}.\end{aligned}$$

This shows inequality "≥" in (4.1) and concludes the proof. □

6 The surface density

This section is devoted to the proof of Lemma 4.3. We start by analyzing the blow-up at jump points. In the following, for any $x_0 \in J_u$ we adopt the notation $\bar u^{\text{surf}}_{x_0}$ for the function $u_{x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)}$, see (2.3), with $\nu_u(x_0) \in \mathbb{S}^{d-1}$ and $u^\pm(x_0) \in \mathbb{R}^d$ being the approximate normal to J_u and the traces on both sides of J_u at x_0, respectively. Recall also notation (3.1).

Lemma 6.1 (Blow-up at jump points). *Let $u \in \mathrm{GSBD}^p(\Omega)$ and let $\theta \in (0,1)$. For $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$ there exists a family $u_\varepsilon \in \mathrm{GSBD}^p(B_\varepsilon(x_0))$ such that*

$$\begin{cases} \text{(i)} & u_\varepsilon = u \text{ in a neighborhood of } \partial B_\varepsilon(x_0), \quad \lim_{\varepsilon\to 0} \varepsilon^{-d}\mathcal{L}^d(\{u_\varepsilon \ne u\}) = 0,\\ \text{(ii)} & \lim_{\varepsilon\to 0} \varepsilon^{-(d-1+p)} \int_{B_{(1-\theta)\varepsilon}(x_0)} |u_\varepsilon(x) - \bar u^{\text{surf}}_{x_0}|^p \,dx = 0,\\ \text{(iii)} & \lim_{\varepsilon\to 0} \varepsilon^{-(d-1)}\mathcal{H}^{d-1}(J_{u_\varepsilon} \cap E_{\varepsilon,x_0}) = \mathcal{H}^{d-1}(\Pi_0 \cap E) \quad \text{for all Borel sets } E \subset B_1(x_0),\\ \text{(iv)} & \lim_{\varepsilon\to 0} \varepsilon^{-(d-1)} \int_{B_\varepsilon(x_0)} |e(u_\varepsilon)(x)|^p \,dx = 0,\end{cases} \tag{6.1}$$

where Π_0 denotes the hyperplane passing through x_0 with normal $\nu_u(x_0)$.

Proof. We start by using the fact that J_u is countably $\mathcal{H}^{d-1}$-rectifiable and the blow-up properties of GSBD^p functions. Arguing as in, e.g., [24, proof of Theorem 2], [26, proof of Theorem 1.1], [28, Lemma 3.4], we infer that for $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$ there exist $\bar\varepsilon > 0$, $\nu_u(x_0) \in \mathbb{S}^{d-1}$, $u^\pm(x_0) \in \mathbb{R}^d$, and a hypersurface Γ which is a graph of a function h defined on Π_0, being C^1 and Lipschitz, such that $x_0 \in \Gamma$, Π_0 is tangent to Γ in x_0, $\Gamma \cap B_\varepsilon(x_0)$ separates $B_\varepsilon(x_0)$ in two open connected components $B_\varepsilon^{\Gamma,\pm}(x_0)$ for each $\varepsilon < \bar\varepsilon$, and

$$\begin{cases} \text{(i)} & \lim\limits_{\varepsilon\to 0} \varepsilon^{-(d-1)}\mathcal{H}^{d-1}((J_u \,\Delta\, \Gamma)\cap B_\varepsilon(x_0)) = 0, \\ & \lim\limits_{\varepsilon\to 0} \varepsilon^{-(d-1)}\mathcal{H}^{d-1}(\Gamma\cap E_{\varepsilon,x_0}) = \mathcal{H}^{d-1}(\Pi_0\cap E) \quad \text{for all Borel sets } E\subset B_1(x_0), \\ \text{(ii)} & \lim\limits_{\varepsilon\to 0} \varepsilon^{-(d-1)} \displaystyle\int_{B_\varepsilon(x_0)} |e(u)|^p\,dx = 0, \\ \text{(iii)} & \lim\limits_{\varepsilon\to 0} \varepsilon^{-d}\mathcal{L}^d(\{x\in B_\varepsilon(x_0) : |u-\bar u_{x_0}^{\mathrm{surf}}| > \varrho\}) = 0 \quad \text{for all } \varrho > 0. \end{cases} \tag{6.2}$$

In particular, (ii) follows from the fact that $|e(u)|^p \in L^1(\Omega)$ and (iii) from (3.2). Then, since Π_0 is tangent to Γ in x_0, $\Gamma\cap B_\varepsilon(x_0)$ is the graph of a Lipschitz function h_ε defined on a subset of Π_0, with Lipschitz constant L_ε such that $\lim_{\varepsilon\to 0} L_\varepsilon = 0$. Therefore, it holds that

$$\lim_{\varepsilon\to 0} \varepsilon^{-d}\mathcal{L}^d(B_\varepsilon^{\Gamma,\pm}(x_0)\,\Delta\, B_\varepsilon^{\pm}(x_0)) = 0, \tag{6.3}$$

where $B_\varepsilon^\pm(x_0) := \{y\in B_\varepsilon(x_0) : \pm(y-x_0)\cdot\nu_u(x_0) > 0\}$. By this and Fubini's Theorem, for each $\varepsilon > 0$ we can find $s_\varepsilon \in (1-\theta, 1-\frac{\theta}{2})\varepsilon$ such that

$$\lim_{\varepsilon\to 0} \varepsilon^{-(d-1)}\mathcal{L}^d\big((B_\varepsilon^{\Gamma,\pm}(x_0)\,\Delta\, B_\varepsilon^{\pm}(x_0))\cap \partial B_{s_\varepsilon}(x_0)\big) = 0. \tag{6.4}$$

For any $\varepsilon > 0$, we apply Theorem 3.3 and Remark 3.4 (i) on u in the two connected components $B_{s_\varepsilon}^{\Gamma,\pm}(x_0)$ for $\varepsilon < \bar\varepsilon$. This gives two functions $v_\varepsilon^\pm \in W^{1,p}(B_{s_\varepsilon}^{\Gamma,\pm}(x_0);\mathbb{R}^d)$, two sets of finite perimeter $\omega_\varepsilon^\pm \subset B_{s_\varepsilon}^{\Gamma,\pm}(x_0)$, and two infinitesimal rigid motions $a_\varepsilon^\pm$ such that

$$\begin{cases} \text{(i)} & v_\varepsilon^\pm = u \quad \text{in } B_{s_\varepsilon}^{\Gamma,\pm}(x_0)\setminus\omega_\varepsilon^\pm, \\ \text{(ii)} & \mathcal{H}^{d-1}(\partial^*\omega_\varepsilon^\pm) \le c\mathcal{H}^{d-1}(J_u\cap B_\varepsilon^{\Gamma,\pm}(x_0)), \quad \mathcal{L}^d(\omega_\varepsilon^\pm) \le c(\mathcal{H}^{d-1}(J_u\cap B_\varepsilon^{\Gamma,\pm}(x_0)))^{\frac{d}{d-1}}, \\ \text{(iii)} & \|v_\varepsilon^\pm - a_\varepsilon^\pm\|_{L^p(B_{s_\varepsilon}^{\Gamma,\pm}(x_0))} \le c\,\varepsilon\|e(u)\|_{L^p(B_\varepsilon^{\Gamma,\pm}(x_0))}, \\ \text{(iv)} & \|\nabla v_\varepsilon^\pm - \nabla a_\varepsilon^\pm\|_{L^p(B_{s_\varepsilon}^{\Gamma,\pm}(x_0))} \le c\|e(u)\|_{L^p(B_\varepsilon^{\Gamma,\pm}(x_0))}, \end{cases} \tag{6.5}$$

where $c > 0$ is independent of ε. (See Remark 3.4 (ii)–(iii) and recall that the Lipschitz constant of h_ε vanishes as $\varepsilon\to 0$.) By the Sobolev extension theorem we extend $v_\varepsilon^\pm$ to $\hat v_\varepsilon^\pm \in W^{1,p}(B_{s_\varepsilon}(x_0);\mathbb{R}^d)$, and (6.5) (iii)–(iv) along with the linearity of the extension operator yield

$$\varepsilon^{-1}\|\hat v_\varepsilon^\pm - a_\varepsilon^\pm\|_{L^p(B_{s_\varepsilon}(x_0))} + \|\nabla\hat v_\varepsilon^\pm - \nabla a_\varepsilon^\pm\|_{L^p(B_{s_\varepsilon}(x_0))} \le c\|e(u)\|_{L^p(B_\varepsilon^{\Gamma,\pm}(x_0))}, \tag{6.6}$$

where, as before, the constant is independent of ε. (Here, we used again the properties of the functions h_ε recalled below (6.2).) We define $u_\varepsilon \in GSBD^p(B_\varepsilon(x_0))$ as

$$u_\varepsilon := \begin{cases} \hat v_\varepsilon^+ & \text{in } B_{s_\varepsilon}^+(x_0), \\ \hat v_\varepsilon^- & \text{in } B_{s_\varepsilon}^-(x_0), \\ u & \text{in } B_\varepsilon(x_0)\setminus B_{s_\varepsilon}(x_0), \end{cases} \tag{6.7}$$

where $B_{s_\varepsilon}^\pm(x_0)$ is defined below (6.3). We now prove the properties in (6.1). First, by definition we have that $u_\varepsilon = u$ in a neighborhood of $\partial B_\varepsilon(x_0)$. By $B_{(1-\theta)\varepsilon}^{\Gamma,\pm}(x_0)\cap\Gamma = \emptyset$, (6.2) (i), (6.3), and (6.5) (i)–(ii) we obtain $\lim_{\varepsilon\to 0}\varepsilon^{-d}\mathcal{L}^d(\{u_\varepsilon \neq u\}) = 0$. This concludes (6.1) (i). Moreover, (6.2) (ii) and (6.6) imply (6.1) (iv). By the definition of u_ε and (6.5) (i) it holds that

$$J_{u_\varepsilon} \subset (\Pi_0\cap\overline{B_{s_\varepsilon}(x_0)}) \cup (J_u\cap(B_\varepsilon(x_0)\setminus B_{s_\varepsilon}(x_0))) \cup \partial^*\omega_\varepsilon^+ \cup \partial^*\omega_\varepsilon^- \cup ((B_\varepsilon^{\Gamma,\pm}(x_0)\,\Delta\, B_\varepsilon^\pm(x_0))\cap\partial B_{s_\varepsilon}(x_0)).$$

We now show (6.1) (iii). Concerning the "≤" inequality, for a fixed Borel set $E\subset B_1(x_0)$ we have to estimate the measure of the intersection with E_{ε,x_0} and any of the five sets in the right-hand side above: it holds that

$$\mathcal{H}^{d-1}(\Pi_0\cap\overline{B_{s_\varepsilon}(x_0)}\cap E_{\varepsilon,x_0}) = \varepsilon^{d-1}\mathcal{H}^{d-1}(\Pi_0\cap\overline{B_{s_\varepsilon/\varepsilon}(x_0)}\cap E)$$

for any $\varepsilon > 0$ by rescaling, that

$$\lim_{\varepsilon\to 0} |\varepsilon^{-(d-1)}\mathcal{H}^{d-1}(J_u \cap (E_{\varepsilon,x_0} \setminus B_{s_\varepsilon}(x_0))) - \mathcal{H}^{d-1}(\Pi_0 \cap (E \setminus B_{s_\varepsilon/\varepsilon}(x_0)))| = 0$$

by (6.2) (i), while the last three terms are estimated by (6.5) (ii) and (6.4). To see the converse, we first apply [25, Theorem 1.1] to the functions $u_\varepsilon(x_0 + \varepsilon\cdot)$, which converge in measure to $\bar u^{\mathrm{surf}}_{x_0}$ in $B_1(0)$ by (6.2), (6.3), and (6.5) (ii). Then we scale back to $B_\varepsilon(x_0)$. Hence, (6.1) (iii) holds.

It remains to prove (6.1) (ii). We notice that this easily follows from

$$\lim_{\varepsilon\to 0} \varepsilon^{-(d-1+p)} \int_{B^\pm_{s_\varepsilon}(x_0)} |a^\pm_\varepsilon - \bar u^{\mathrm{surf}}_{x_0}|^p \, dx = 0. \tag{6.8}$$

In fact, (6.2) (ii) and (6.6) give that

$$\lim_{\varepsilon\to 0} \varepsilon^{-(d-1+p)} \int_{B^\pm_{s_\varepsilon}(x_0)} |\hat v^\pm_\varepsilon - a^\pm_\varepsilon|^p \, dx = 0. \tag{6.9}$$

Then (6.8), (6.9), the triangle inequality, $s_\varepsilon \geq (1-\theta)\varepsilon$, and (6.7) imply (6.1) (ii).

Therefore, let us now confirm (6.8). We only address the "+" case, for the "−" case is analogous. We first observe that by (6.2) (iii), (6.3), and a diagonal argument, we may find a sequence $(\varrho_\varepsilon)_\varepsilon \subset (0, +\infty)$ with $\lim_{\varepsilon\to 0}\varrho_\varepsilon = 0$ such that the sets

$$\hat\omega^+_\varepsilon := \{x \in B_\varepsilon(x_0) : |u(x) - u^+(x_0)| > \varrho_\varepsilon\} \cap B^{\Gamma,+}_{s_\varepsilon}(x_0)$$

satisfy

$$\lim_{\varepsilon\to 0} \varepsilon^{-d}\mathcal{L}^d(\hat\omega^+_\varepsilon) = 0. \tag{6.10}$$

In view of (6.5) (i), (iii), we have that

$$\|u - a^+_\varepsilon\|_{L^p(B^{\Gamma,+}_{s_\varepsilon}(x_0)\setminus\omega^+_\varepsilon)} \leq c\varepsilon\|e(u)\|_{L^p(B^{\Gamma,+}_\varepsilon(x_0))}. \tag{6.11}$$

Then, by (6.11), the definition of $\hat\omega^+_\varepsilon$, and the triangle inequality we get that

$$\|u^+(x_0) - a^+_\varepsilon\|_{L^p(B^{\Gamma,+}_{s_\varepsilon}(x_0)\setminus(\omega^+_\varepsilon\cup\hat\omega^+_\varepsilon))} \leq c\varepsilon\|e(u)\|_{L^p(B^{\Gamma,+}_\varepsilon(x_0))} + \gamma_d^{\frac{1}{p}}\varepsilon^{\frac{d}{p}}\varrho_\varepsilon. \tag{6.12}$$

By (6.2) (i), (6.3), (6.5) (ii), and (6.10) we obtain $\mathcal{L}^d(\omega^+_\varepsilon \cup \hat\omega^+_\varepsilon) \leq \frac{1}{2}\mathcal{L}^d(B^{\Gamma,+}_{s_\varepsilon}(x_0))$ for ε sufficiently small. Then, by Lemma 3.6, we have that $\gamma_d\varepsilon^{\frac{d}{p}}\|u^+(x_0) - a^+_\varepsilon\|_{L^\infty(B_\varepsilon(x_0))}$ is less or equal than the right-hand side of (6.12), up to multiplication with a constant. This along with (6.2) (ii), $p \geq 1$, and the fact that $\varrho_\varepsilon \to 0$ implies

$$\lim_{\varepsilon\to 0} \|u^+(x_0) - a^+_\varepsilon\|_{L^\infty(B_\varepsilon(x_0))} = 0. \tag{6.13}$$

Let us consider $A^+_\varepsilon \in \mathbb{M}^{d\times d}_{\mathrm{skew}}$ and $b^+_\varepsilon \in \mathbb{R}^d$ such that $a^+_\varepsilon(x) = A^+_\varepsilon(x - x_0) + b^+_\varepsilon$. Then (6.8) follows by

$$\lim_{\varepsilon\to 0} \varepsilon|A^+_\varepsilon|^p = 0, \tag{6.14a}$$

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{1-p}{p}}|b^+_\varepsilon - u^+(x_0)| = 0. \tag{6.14b}$$

So we are left to prove (6.14) which corresponds to [28, equations (3.18)–(3.19)]. The proof goes in the same way with slight modifications that we indicate below. For fixed $\delta > 0$ small, by (6.2) (ii) there exists $\hat\varepsilon > 0$, depending on δ, such that

$$\varepsilon^{-(d-1)} \int_{B_\varepsilon(x_0)} |e(u)|^p \, dx \leq \delta^p \quad \text{for all } \varepsilon \leq \hat\varepsilon. \tag{6.15}$$

For $\tilde\varepsilon < \varepsilon < \hat\varepsilon$, we set $\varepsilon_k := \min\{2^k\tilde\varepsilon, \varepsilon\}$ and adopt the notation k in place of ε_k in the subscripts. We then obtain

$$\|a^+_k - a^+_{k+1}\|_{L^\infty(B^+_{\varepsilon_k}(x_0))} \leq c\gamma_d^{-\frac{1}{p}}\varepsilon_k^{-\frac{d}{p}}\|a^+_k - a^+_{k+1}\|_{L^p(B^{\Gamma,+}_{s_{\varepsilon_k}}(x_0)\setminus(\omega^+_k\cup\omega^+_{k+1}))} \leq c\delta\varepsilon_k^{\frac{p-1}{p}}. \tag{6.16}$$

In fact, the first inequality follows from (6.5) (ii) and Lemma 3.6, and the second one from (6.11), (6.15), and the triangle inequality. Similarly, employing (6.5) (i), (iv) in place of (6.11), and recalling $\nabla a^+_{\varepsilon_k} = A^+_k$, we obtain

$$|A^+_k - A^+_{k+1}| \leq c\delta\varepsilon_k^{-\frac{1}{p}}. \tag{6.17}$$

At this stage, (6.14) follow exactly as in [28]: for $\widehat{k}$ being the first index such that $\varepsilon_{\widehat{k}} = \varepsilon$, recalling $\tilde{\varepsilon} = \varepsilon_0$ and summing (6.17) gives

$$\tilde{\varepsilon}|A^+_{\tilde{\varepsilon}}|^p \leq \tilde{\varepsilon}(|A^+_{\widehat{k}}| + \sum_{k=0}^{\widehat{k}-1} |A^+_k - A^+_{k+1}|)^p \leq c\delta^p + c\tilde{\varepsilon}|A^+_{\widehat{k}}|^p.$$

The right-hand side vanishes as $\tilde{\varepsilon} \to 0$ and $\delta \to 0$, and this proves (6.14a). Moreover, summing (6.16) (and since $|b^+_k - b^+_{k+1}| \leq \|a^+_k - a^+_{k+1}\|_{L^\infty(B^+_{\varepsilon_k}(x_0))}$), we obtain

$$|b^+_{\tilde{\varepsilon}} - b^+_{\varepsilon}| \leq c\delta\varepsilon^{\frac{p-1}{p}}$$

for all $0 < \tilde{\varepsilon} \leq \varepsilon$. By passing to the limit as $\tilde{\varepsilon} \to 0$ together with (6.13), we get

$$\varepsilon^{\frac{1-p}{p}}|u^+(x_0) - b^+_{\varepsilon}| \leq c\delta$$

for $\varepsilon < \widehat{\varepsilon} = \widehat{\varepsilon}(\delta)$. Thus, (6.14b) follows by the arbitrariness of $\delta > 0$, concluding the proof. □

Remark 6.2 (Construction of u_ε). We point out that our definition of u_ε in (6.7) differs from the corresponding constructions in [13, Lemma 3] and [28, Lemma 3.4] in order to fix a possible flaw contained in these proofs. Roughly speaking, in our notation, in [13, 28], u_ε on $B_{s_\varepsilon}(x_0)$ is defined as

$$u_\varepsilon = \begin{cases} v^+_\varepsilon & \text{in } B^{\Gamma,+}_{s_\varepsilon}(x_0), \\ v^-_\varepsilon & \text{in } B^{\Gamma,-}_{s_\varepsilon}(x_0). \end{cases} \tag{6.18}$$

Then, instead of (6.8), one needs to check

$$\lim_{\varepsilon\to 0} \varepsilon^{-(d-1+p)} \int_{B^{\Gamma\pm}_{s_\varepsilon}(x_0)} |a^\pm_\varepsilon - \bar{u}^{\text{surf}}_{x_0}|^p \,dx = 0.$$

This, however, is in general false if $\liminf_{\varepsilon\to 0} \varepsilon^{-(d-1+p)}\mathcal{L}^d(B^{\Gamma,\pm}_\varepsilon(x_0) \,\Delta\, B^\pm_\varepsilon(x_0)) > 0$ (which is clearly possible). Let us also remark that, in contrast to our construction, (6.18) allows to prove an estimate of the form

$$\lim_{\varepsilon\to 0} \varepsilon^{-(d-1)}\mathcal{H}^{d-1}(J_{u_\varepsilon} \setminus J_u) = 0, \tag{6.19}$$

see [13, equation (24)] and [28, Lemma 3.4 (i)]. It is not clear to us if it is possible that for u_ε satisfying the fundamental blow-up property (6.1) (ii) one may still have an estimate of the form (6.19). The latter, however, is not needed for our proofs.

We now proceed with the proof of Lemma 4.3.

Proof of Lemma 4.3. The proof follows the same strategy of the proof of Lemma 4.2. We fix $x_0 \in J_u$ such that the statement of Lemma 6.1 holds at x_0 and $\lim_{\varepsilon\to 0} \varepsilon^{-(d-1)}\mu(B_\varepsilon(x_0)) = \gamma_{d-1}$. This is possible for $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$. Then also $\lim_{\varepsilon\to 0} \varepsilon^{-(d-1)}\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0)) \in \mathbb{R}$ exists, see Lemma 4.1. We prove (4.2) for x_0 of this type.

Step 1 (Inequality "≤" in (4.2)). We fix η, $\theta > 0$, and consider $z_\varepsilon \in GSBD^p(B_{(1-3\theta)\varepsilon}(x_0))$ with $z_\varepsilon = \bar{u}^{\text{surf}}_{x_0}$ in a neighborhood of $\partial B_{(1-3\theta)\varepsilon}(x_0)$ and

$$\mathcal{F}(z_\varepsilon, B_{(1-3\theta)\varepsilon}(x_0)) \leq \mathbf{m}_{\mathcal{F}}(\bar{u}^{\text{surf}}_{x_0}, B_{(1-3\theta)\varepsilon}(x_0)) + \varepsilon^d. \tag{6.20}$$

We extend z_ε to a function in $GSBD^p(B_\varepsilon(x_0))$ by setting $z_\varepsilon = \bar{u}^{\text{surf}}_{x_0}$ outside $B_{(1-3\theta)\varepsilon}(x_0)$. Let $(u_\varepsilon)_\varepsilon$ be the family given by Lemma 6.1. As in the proof of Lemma 4.2, we apply Lemma 3.8 on z_ε (in place of u) and u_ε (in place of v) for η fixed above and the sets

$$A' = B_{1-2\theta}(x_0), \quad A = B_{1-\theta}(x_0), \quad A'' = B_1(x_0) \setminus \overline{B_{1-4\theta}(x_0)}.$$

Recalling notation (3.1) and (3.11)–(3.12), we find a function $w_\varepsilon \in GSBD^p(B_\varepsilon(x_0))$ such that $w_\varepsilon = u_\varepsilon$ on $B_\varepsilon(x_0) \setminus B_{(1-\theta)\varepsilon}(x_0)$ and

$$\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0)) \leq (1+\eta)(\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0}) + \mathcal{F}(u_\varepsilon, A''_{\varepsilon,x_0})) + \frac{M}{\varepsilon^p}\|z_\varepsilon - u_\varepsilon\|^p_{L^p((A\setminus A')_{\varepsilon,x_0})} + \mathcal{L}^d(B_\varepsilon(x_0))\eta, \tag{6.21}$$

where $M > 0$ depends on θ and η, but is independent of ε. In particular, we have $w_\varepsilon = u_\varepsilon = u$ in a neighborhood of $\partial B_\varepsilon(x_0)$ by (6.1) (i). By (6.1) (ii) and the fact that $z_\varepsilon = \bar{u}^{\text{surf}}_{x_0}$ outside $B_{(1-3\theta)\varepsilon}(x_0)$ we find

$$\lim_{\varepsilon\to 0} \varepsilon^{-(d-1+p)}\|z_\varepsilon - u_\varepsilon\|^p_{L^p((A\setminus A')_{\varepsilon,x_0})} = \lim_{\varepsilon\to 0} \varepsilon^{-(d-1+p)}\|u_\varepsilon - \bar{u}^{\text{surf}}_{x_0}\|^p_{L^p(B_{(1-\theta)\varepsilon}(x_0))} = 0.$$

Inserting this in (6.21), we find that, for a suitable sequence $(\rho_\varepsilon)_\varepsilon \subset (0,+\infty)$ with $\rho_\varepsilon \to 0$,

$$\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0)) \le (1+\eta)(\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0}) + \mathcal{F}(u_\varepsilon, A''_{\varepsilon,x_0})) + \varepsilon^{d-1}\rho_\varepsilon + \gamma_d\varepsilon^d\eta. \tag{6.22}$$

We evaluate the first terms on the right-hand side of (6.22): since $z_\varepsilon = \bar{u}^{\mathrm{bulk}}_{x_0}$ on $B_\varepsilon(x_0) \setminus B_{(1-3\theta)\varepsilon}(x_0) \subset A''_{\varepsilon,x_0}$, by (H1), (H4), and (6.20) we have

$$\begin{aligned}\limsup_{\varepsilon\to0}\frac{\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0})}{\varepsilon^{d-1}} &\le \limsup_{\varepsilon\to0}\frac{\mathcal{F}(z_\varepsilon, B_{(1-3\theta)\varepsilon}(x_0))}{\varepsilon^{d-1}} + \limsup_{\varepsilon\to0}\frac{\mathcal{F}(\bar{u}^{\mathrm{surf}}_{x_0}, A''_{\varepsilon,x_0})}{\varepsilon^{d-1}}\\ &\le \limsup_{\varepsilon\to0}\frac{\mathbf{m}_{\mathcal{F}}(\bar{u}^{\mathrm{surf}}_{x_0}, B_{(1-3\theta)\varepsilon}(x_0))}{\varepsilon^{d-1}} + \beta\mathcal{H}^{d-1}(A''\cap\Pi_0)\\ &\le (1-3\theta)^{d-1}\limsup_{\varepsilon\to0}\frac{\mathbf{m}_{\mathcal{F}}(\bar{u}^{\mathrm{surf}}_{x_0}, B_\varepsilon(x_0))}{\varepsilon^{d-1}} + \beta\gamma_{d-1}(1-(1-4\theta)^{d-1}),\end{aligned} \tag{6.23}$$

where, as in Lemma 6.1, we denote by Π_0 the hyperplane passing through x_0 with normal $\nu_u(x_0)$. By (H4) and (6.1) (iii), (iv) we get

$$\begin{aligned}\limsup_{\varepsilon\to0}\frac{\mathcal{F}(u_\varepsilon, A''_{\varepsilon,x_0})}{\beta\varepsilon^{d-1}} &\le \limsup_{\varepsilon\to0}\frac{\int_{A''_{\varepsilon,x_0}}(1+|e(u_\varepsilon)|^p)\,\mathrm{d}x}{\varepsilon^{d-1}} + \limsup_{\varepsilon\to0}\frac{\mathcal{H}^{d-1}(J_{u_\varepsilon}\cap A''_{\varepsilon,x_0})}{\varepsilon^{d-1}}\\ &= \gamma_{d-1}(1-(1-4\theta)^{d-1}).\end{aligned} \tag{6.24}$$

Collecting (6.22), (6.23), (6.24), and recalling $\rho_\varepsilon \to 0$, as well as the fact that $w_\varepsilon = u$ in a neighborhood of $\partial B_\varepsilon(x_0)$, we obtain

$$\begin{aligned}\lim_{\varepsilon\to0}\frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\gamma_{d-1}\varepsilon^{d-1}} &\le \limsup_{\varepsilon\to0}\frac{\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0))}{\gamma_{d-1}\varepsilon^{d-1}}\\ &\le (1+\eta)(1-3\theta)^{d-1}\limsup_{\varepsilon\to0}\frac{\mathbf{m}_{\mathcal{F}}(\bar{u}^{\mathrm{surf}}_{x_0}, B_\varepsilon(x_0))}{\gamma_{d-1}\varepsilon^{d-1}} + 2(1+\eta)\beta(1-(1-4\theta)^{d-1}).\end{aligned}$$

Passing to the limit as $\eta, \theta \to 0$, we conclude inequality "$\le$" in (4.2).

Step 2 (Inequality "$\ge$" in (4.2)). We fix $\eta,\ \theta > 0$ and let, as in Step 1, $(u_\varepsilon)_\varepsilon$ be the family given by Lemma 6.1. By (6.1) (i) and Fubini's Theorem, for each $\varepsilon > 0$ we can find $s_\varepsilon \in (1-4\theta, 1-3\theta)\varepsilon$ such that

$$\begin{cases}\text{(i)} & \lim_{\varepsilon\to0}\varepsilon^{-(d-1)}\mathcal{H}^{d-1}(\{u \ne u_\varepsilon\}\cap\partial B_{s_\varepsilon}(x_0)) = 0,\\ \text{(ii)} & \mathcal{H}^{d-1}((J_u\cup J_{u_\varepsilon})\cap\partial B_{s_\varepsilon}(x_0)) = 0 \quad \text{for all } \varepsilon > 0.\end{cases} \tag{6.25}$$

We consider $z_\varepsilon \in GSBD^p(B_{s_\varepsilon}(x_0))$ such that $z_\varepsilon = u$ in a neighborhood of $\partial B_{s_\varepsilon}(x_0)$, and

$$\mathcal{F}(z_\varepsilon, B_{s_\varepsilon}(x_0)) \le \mathbf{m}_{\mathcal{F}}(u, B_{s_\varepsilon}(x_0)) + \varepsilon^d. \tag{6.26}$$

We extend z_ε to a function in $GSBD^p(B_\varepsilon(x_0))$ by setting

$$z_\varepsilon = u_\varepsilon \quad \text{in } B_\varepsilon(x_0)\setminus B_{s_\varepsilon}(x_0). \tag{6.27}$$

We apply Lemma 3.8 for z_ε (in place of u), $\bar{u}^{\mathrm{surf}}_{x_0}$ (in place of v), and for the sets A, A', B as in Step 1, in correspondence to ε. By (3.11)–(3.12), there exists $w_\varepsilon \in GSBD^p(B_\varepsilon(x_0))$ such that $w_\varepsilon = \bar{u}^{\mathrm{surf}}_{x_0}$ on $B_\varepsilon(x_0)\setminus B_{(1-\theta)\varepsilon}(x_0)$, and

$$\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0)) \le (1+\eta)(\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0}) + \mathcal{F}(\bar{u}^{\mathrm{surf}}_{x_0}, A''_{\varepsilon,x_0})) + \frac{M}{\varepsilon^p}\|z_\varepsilon - \bar{u}^{\mathrm{surf}}_{x_0}\|^p_{L^p((A\setminus A')_{\varepsilon,x_0})} + \mathcal{L}^d(B_\varepsilon(x_0))\eta.$$

We observe that $z_\varepsilon = u_\varepsilon$ outside $B_{(1-3\theta)\varepsilon}(x_0)$, by (6.27) and the choice of s_ε. Then, as done in Step 1, we may employ (6.1) (ii). This gives us a sequence $(\rho_\varepsilon)_\varepsilon \subset (0,+\infty)$ with $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$ such that

$$\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0)) \le (1+\eta)(\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0}) + \mathcal{F}(\bar{u}^{\mathrm{surf}}_{x_0}, A''_{\varepsilon,x_0})) + \varepsilon^{d-1}\rho_\varepsilon + \gamma_d\varepsilon^d\eta. \tag{6.28}$$

We estimate the first terms in (6.28). We get by (H1), (H4), (6.26)–(6.27), and the choice of s_ε that

$$\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0}) \le \mathbf{m}_\mathcal{F}(u, B_{s_\varepsilon}(x_0)) + \varepsilon^d + \beta\mathcal{H}^{d-1}((\{u \ne u_\varepsilon\} \cup J_u \cup J_{u_\varepsilon}) \cap \partial B_{s_\varepsilon}(x_0)) + \mathcal{F}(u_\varepsilon, A''_{\varepsilon,x_0}). \tag{6.29}$$

By (6.24), (6.25), and the fact that $s_\varepsilon \le (1-3\theta)\varepsilon$ we thus deduce that

$$\begin{aligned}\limsup_{\varepsilon\to 0}\frac{\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0})}{\varepsilon^{d-1}} &\le \limsup_{\varepsilon\to 0}\frac{\mathbf{m}_\mathcal{F}(u, B_{s_\varepsilon}(x_0))}{\varepsilon^{d-1}} + \beta\gamma_{d-1}(1-(1-4\theta)^{d-1})\\ &\le (1-3\theta)^{d-1}\limsup_{\varepsilon\to 0}\frac{\mathbf{m}_\mathcal{F}(u, B_\varepsilon(x_0))}{\varepsilon^{d-1}} + \beta\gamma_{d-1}(1-(1-4\theta)^{d-1}),\end{aligned} \tag{6.30}$$

and, similarly to (6.23),

$$\limsup_{\varepsilon\to 0}\frac{\mathcal{F}(\bar u^{\mathrm{surf}}_{x_0}, A''_{\varepsilon,x_0})}{\varepsilon^{d-1}} \le \beta\gamma_{d-1}(1-(1-4\theta)^{d-1}). \tag{6.31}$$

Collecting (6.28), (6.30), (6.31) and using $\rho_\varepsilon \to 0$, we derive

$$\limsup_{\varepsilon\to 0}\frac{\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0))}{\varepsilon^{d-1}} \le (1+\eta)\Big((1-3\theta)^{d-1}\limsup_{\varepsilon\to 0}\frac{\mathbf{m}_\mathcal{F}(u, B_\varepsilon(x_0))}{\varepsilon^{d-1}} + 2\beta\gamma_{d-1}(1-(1-4\theta)^{d-1})\Big).$$

Finally, recalling that $w_\varepsilon = \bar u^{\mathrm{surf}}_{x_0}$ in a neighborhood of $\partial B_\varepsilon(x_0)$, and using the arbitrariness of η, $\theta > 0$, we obtain

$$\limsup_{\varepsilon\to 0}\frac{\mathbf{m}_\mathcal{F}(\bar u^{\mathrm{surf}}_{x_0}, B_\varepsilon(x_0))}{\gamma_{d-1}\varepsilon^{d-1}} \le \limsup_{\varepsilon\to 0}\frac{\mathcal{F}(w_\varepsilon, B_\varepsilon(x_0))}{\gamma_{d-1}\varepsilon^{d-1}} \le \limsup_{\varepsilon\to 0}\frac{\mathbf{m}_\mathcal{F}(u, B_\varepsilon(x_0))}{\gamma_{d-1}\varepsilon^{d-1}} = \lim_{\varepsilon\to 0}\frac{\mathbf{m}_\mathcal{F}(u, B_\varepsilon(x_0))}{\gamma_{d-1}\varepsilon^{d-1}}.$$

This shows "≥" in (4.2) and concludes the proof. □

7 The SBD^p case

This section is devoted to the analysis of the integral representation result for $\mathcal{F}\colon \mathrm{SBD}^p(\Omega)\times\mathcal{B}(\Omega) \to [0,+\infty)$ satisfying (H1)–(H3) and (H4'). This case has been addressed, for $d = 2$, in [28]. On the one hand, the arguments there could be now generalized to general dimension by virtue of Theorem 3.3. On the other hand, as we are going to show, the result can also be obtained with minor changes of our more general strategy.

We start by pointing out that, under (H4'), only competitors in SBD^p may have finite energy. In fact, in view of Proposition 3.2, in the present setting definition (2.1) reads as

$$\mathbf{m}_\mathcal{F}(u, A) = \inf_{v\in\mathrm{SBD}^p(\Omega)}\{\mathcal{F}(v, A) : v = u \text{ in a neighborhood of } \partial A\}. \tag{7.1}$$

Then the following integral representation result holds.

Theorem 7.1 (Integral representation in SBD^p). *Let $\Omega \subset \mathbb{R}^d$ be open, bounded with Lipschitz boundary and suppose that $\mathcal{F}\colon \mathrm{SBD}^p(\Omega)\times\mathcal{B}(\Omega) \to [0,+\infty)$ satisfies* (H1)–(H3) *and* (H4'). *Then*

$$\mathcal{F}(u, B) = \int_B f(x, u(x), \nabla u(x))\,\mathrm{d}x + \int_{J_u\cap B} g(x, u^+(x), u^-(x), \nu_u(x))\,\mathrm{d}\mathcal{H}^{d-1}(x)$$

for all $u \in \mathrm{SBD}^p(\Omega)$ and $B \in \mathcal{B}(\Omega)$, where f is given by

$$f(x_0, u_0, \xi) = \limsup_{\varepsilon\to 0}\frac{\mathbf{m}_\mathcal{F}(\ell_{x_0,u_0,\xi}, B_\varepsilon(x_0))}{\gamma_d\varepsilon^d}$$

for all $x_0 \in \Omega$, $u_0 \in \mathbb{R}^d$, $\xi \in \mathbb{M}^{d\times d}$, and $\ell_{x_0,u_0,\xi}$ as in (2.2), *and g is given by*

$$g(x_0, a, b, \nu) = \limsup_{\varepsilon\to 0}\frac{\mathbf{m}_\mathcal{F}(u_{x_0,a,b,\nu}, B_\varepsilon(x_0))}{\gamma_{d-1}\varepsilon^{d-1}}$$

for all $x_0 \in \Omega$, $a, b \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, and $u_{x_0,a,b,\nu}$ as in (2.3).

The remainder of this section is devoted to the proof of Theorem 7.1 which follows along the lines of the proof of Theorem 2.1 devised in Section 4. First, the analogue of Lemma 4.1 holds essentially with the same proof.

Lemma 7.2. *Suppose that $\mathcal{F}$ satisfies* (H1)–(H3) *and* (H4'). *Let $u \in \mathrm{SBD}^p(\Omega)$ and let $\mu = \mathcal{L}^d\lfloor_\Omega + \mathcal{H}^{d-1}\lfloor_{J_u\cap\Omega}$. Then for μ-a.e. $x_0 \in \Omega$ we have*

$$\lim_{\varepsilon\to 0}\frac{\mathcal{F}(u, B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))} = \lim_{\varepsilon\to 0}\frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\mu(B_\varepsilon(x_0))}.$$

Proof. One can follow the same argument used to prove Lemma 4.1 through Lemma 4.4.

First, we remark that [28, Lemmas 4.2 and 4.3] is proved under the assumptions (H4') and $u \in \mathrm{SBD}^p(\Omega)$, hence it can be used to derive (4.15).

Concerning Lemma 4.4, as the lower bound of (H4') is stronger than the one of (H4), the GSBD compactness result [25, Theorem 1.1] is still applicable. First, this shows $v^\delta \in \mathrm{GSBD}^p(\Omega)$. Additionally, (4.9), (H4'), and Proposition 3.2 imply that the function v^δ belongs indeed to $\mathrm{SBD}^p(\Omega)$. The rest of the proof remains unchanged, upon noticing that, under assumption (H4'), (4.10) still holds. □

We now address the adaptions necessary for the bulk density. When $u \in \mathrm{SBD}^p(\Omega)$, we show that the approximating sequence constructed in Lemma 5.1 satisfies some additional properties.

Lemma 7.3. *Let $u \in \mathrm{SBD}^p(\Omega)$. Let $\theta \in (0, 1)$. For $\mathcal{L}^d$-a.e. $x_0 \in \Omega$ there exists a family $u_\varepsilon \in \mathrm{SBD}^p(B_\varepsilon(x_0))$ such that* (5.1) *holds, and additionally*

$$\begin{cases} \text{(i)} & \displaystyle\lim_{\varepsilon\to 0}\varepsilon^{-(d+1)}\int_{B_\varepsilon(x_0)}|u_\varepsilon - u|\,\mathrm{d}x = 0,\\ \text{(ii)} & \displaystyle\lim_{\varepsilon\to 0}\varepsilon^{-d}\int_{J_{u_\varepsilon}}|[u_\varepsilon]|\,\mathrm{d}\mathcal{H}^{d-1} = 0.\end{cases} \tag{7.2}$$

Proof. Since $u \in \mathrm{SBD}(\Omega)$, for $\mathcal{L}^d$-a.e. x_0 it holds that

$$\lim_{\varepsilon\to 0}\varepsilon^{-d}\int_{J_u\cap B_\varepsilon(x_0)}|[u]|\,\mathrm{d}\mathcal{H}^{d-1} = 0, \tag{7.3}$$

and (see [4, Theorem 7.4]) that

$$\lim_{\varepsilon\to 0}\varepsilon^{-(d+1)}\int_{B_\varepsilon(x_0)}|u - \bar{u}^{\mathrm{bulk}}_{x_0}|\,\mathrm{d}x = 0, \tag{7.4}$$

where for brevity we again let $\bar{u}^{\mathrm{bulk}}_{x_0} = \ell_{x_0,u(x_0),\nabla u(x_0)}$. Hence, with Fubini's Theorem we can fix $s_\varepsilon \in (1-\theta, 1-\frac{\theta}{2})\varepsilon$ so that $\mathcal{H}^{d-1}(J_u \cap \partial B_{s_\varepsilon}(x_0)) = 0$ and

$$\lim_{\varepsilon\to 0}\varepsilon^{-d}\int_{\partial B_{s_\varepsilon}(x_0)}|u - \bar{u}^{\mathrm{bulk}}_{x_0}|\,\mathrm{d}\mathcal{H}^{d-1} = 0. \tag{7.5}$$

We can now perform the same construction as in (5.5) with s_ε in place of $(1-\theta)\varepsilon$. Notice that in this case $u_\varepsilon \in \mathrm{SBD}^p(B_\varepsilon(x_0))$. By arguing exactly as in the proof of Lemma 5.1, we derive (5.1) (i), (iii), (iv) while (ii) holds in $B_{s_\varepsilon}(x_0)$ and a fortiori in $B_{(1-\theta)\varepsilon}(x_0)$. In particular, this in combination with Hölder's inequality, (7.4), and $u = u_\varepsilon$ in $B_\varepsilon(x_0)\setminus B_{s_\varepsilon}(x_0)$ yields (7.2) (i).

To see (7.2) (ii), observe that, since $\frac{s_\varepsilon}{\varepsilon}$ is bounded from above and from below, (5.1) (ii), (iii), the fact that $u_\varepsilon\lfloor_{B_{s_\varepsilon}(x_0)} \in W^{1,p}(B_{s_\varepsilon}(x_0);\mathbb{R}^d)$ (see (5.5)), and Korn's inequality imply that

$$\frac{u_\varepsilon(x_0 + s_\varepsilon\cdot) - \bar{u}^{\mathrm{bulk}}_{x_0}(x_0 + s_\varepsilon\cdot)}{s_\varepsilon} \to 0 \quad \text{in } W^{1,p}(B_1(0);\mathbb{R}^d).$$

Hence, by the trace inequality and by scaling back to $B_{s_\varepsilon}(x_0)$, we obtain

$$\lim_{\varepsilon\to 0}s_\varepsilon^{-(d-1)}\int_{\partial B_{s_\varepsilon}(x_0)}\tfrac{1}{s_\varepsilon}|u_\varepsilon - \bar{u}^{\mathrm{bulk}}_{x_0}|\,\mathrm{d}\mathcal{H}^{d-1} = 0.$$

With this, (7.5), and the fact that $\frac{s_\varepsilon}{\varepsilon}$ is bounded from below we then have

$$\lim_{\varepsilon\to 0} \varepsilon^{-d} \int_{\partial B_{s_\varepsilon}(x_0)} |u_\varepsilon - u| \,\mathrm{d}\mathcal{H}^{d-1} = 0.$$

Hence, we get by the construction of u_ε and (7.3) that

$$\lim_{\varepsilon\to 0} \varepsilon^{-d} \int_{J_{u_\varepsilon}} |[u_\varepsilon]| \,\mathrm{d}\mathcal{H}^{d-1} = \lim_{\varepsilon\to 0} \varepsilon^{-d} \int_{J_u \cap (B_\varepsilon(x_0)\setminus B_{s_\varepsilon}(x_0))} |[u]| \,\mathrm{d}\mathcal{H}^{d-1} = 0,$$

which concludes the proof. □

With the above lemma at our disposal, we can deduce the asymptotic equivalence of the minimization problems (7.1) for u and $\bar{u}^{\mathrm{bulk}}_{x_0} = \ell_{x_0,u(x_0),\nabla u(x_0)}$.

Lemma 7.4. *Suppose that $\mathcal{F}$ satisfies* (H1), (H3), *and* (H4'). *Let $u \in \mathrm{SBD}^p(\Omega)$. Then for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$ we have*

$$\lim_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\gamma_d \varepsilon^d} = \limsup_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(\bar{u}^{\mathrm{bulk}}_{x_0}, B_\varepsilon(x_0))}{\gamma_d \varepsilon^d}. \tag{7.6}$$

Proof. We argue as in the proof of Lemma 4.2.

For the ”≤” inequality, take u_ε satisfying (5.1) and (7.2), and perform the same construction as in Lemma 4.2. (Observe that the fundamental estimate also holds in this case, see Lemma 3.8.) Notice that, in this case, we have by (H4') and (5.9) that

$$\mathcal{F}(u_\varepsilon, A''_{\varepsilon,x_0}) \le \beta \int_{A''_{\varepsilon,x_0}} (1 + |e(u_\varepsilon)|^p) \,\mathrm{d}x + \beta \int_{J_{u_\varepsilon}\cap A''_{\varepsilon,x_0}} (1 + |[u_\varepsilon]|) \,\mathrm{d}\mathcal{H}^{d-1}.$$

Thus, using (5.1) (iii), (iv) and (7.2) (ii), we still get (5.14), and may deduce inequality “≤” in (7.6).

For the “≥” inequality, we start by observing that, if u_ε also satisfies (7.2), in addition to (5.15) we may require that

$$\lim_{\varepsilon\to 0} \varepsilon^{-d} \int_{\partial B_{s_\varepsilon}(x_0)} |u^+ - u^-_\varepsilon| \,\mathrm{d}\mathcal{H}^{d-1} = 0, \tag{7.7}$$

where u^-_ε and u^+ indicate the inner and outer traces at $\partial B_{s_\varepsilon}(x_0)$, respectively. We also have that (5.14) holds, as seen in the previous step. We then perform the same construction as in Lemma 4.2. In this case, inequality (5.19) is replaced by

$$\begin{aligned}\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0}) \le \mathbf{m}_{\mathcal{F}}(u, B_{s_\varepsilon}(x_0)) + \varepsilon^{d+1} + \beta\mathcal{H}^{d-1}((\{u \ne u_\varepsilon\} \cup J_u \cup J_{u_\varepsilon}) \cap \partial B_{s_\varepsilon}(x_0))\\ + \beta \int_{\partial B_{s_\varepsilon}(x_0)} |u^+ - u^-_\varepsilon| \,\mathrm{d}\mathcal{H}^{d-1} + \mathcal{F}(u_\varepsilon, A''_{\varepsilon,x_0}),\end{aligned}$$

so that, using (5.14), (5.15), (7.7), and the fact that $s_\varepsilon \le (1 - 3\theta)\varepsilon$, we are still in a position to deduce (5.20). The rest of the argument remains unchanged and we obtain inequality “≥” in (7.6). □

Similar changes have to be performed also for the surface density. We first deduce the analogue of Lemma 6.1. We again set $\bar{u}^{\mathrm{surf}}_{x_0} = u_{x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)}$ for brevity, see (2.3). We also recall the notation in (3.1).

Lemma 7.5. *Let $u \in \mathrm{SBD}^p(\Omega)$ and $\theta \in (0,1)$. For $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$ there exists a family $u_\varepsilon \in \mathrm{SBD}^p(B_\varepsilon(x_0))$ such that* (6.1) *holds, and additionally*

$$\begin{cases} \text{(i)} & \displaystyle\lim_{\varepsilon\to 0} \varepsilon^{-d} \int_{B_\varepsilon(x_0)} |u_\varepsilon - u| \,\mathrm{d}x = 0,\\ \text{(ii)} & \displaystyle\lim_{\varepsilon\to 0} \varepsilon^{-(d-1)} \int_{J_{u_\varepsilon}\cap E_{\varepsilon,x_0}} |[u_\varepsilon]| \,\mathrm{d}\mathcal{H}^{d-1} = |[\bar{u}^{\mathrm{surf}}_{x_0}]| \mathcal{H}^{d-1}(\Pi_0 \cap E) \quad \textit{for all Borel sets } E \subset B_1(x_0), \end{cases} \tag{7.8}$$

where Π_0 denotes the hyperplane passing through x_0 with normal $\nu_u(x_0)$.

Proof. Since $u \in \mathrm{SBD}(\Omega)$, for $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$ it holds that

$$\lim_{\varepsilon\to 0} \varepsilon^{-d} \int_{B_\varepsilon(x_0)} |u - \bar{u}^{\mathrm{surf}}_{x_0}|\,\mathrm{d}x = 0. \tag{7.9}$$

Hence, by Fubini's Theorem we find $s_\varepsilon \in (1-\theta, 1-\frac{\theta}{2})\varepsilon$ such that (6.4) holds, we have

$$\mathcal{H}^{d-1}(J_u \cap \partial B_{s_\varepsilon}(x_0)) = 0$$

and

$$\lim_{\varepsilon\to 0} \varepsilon^{-(d-1)} \int_{\partial B_{s_\varepsilon}(x_0)} |u - \bar{u}^{\mathrm{surf}}_{x_0}|\,\mathrm{d}\mathcal{H}^{d-1} = 0. \tag{7.10}$$

We can now perform the same construction as in (6.7) and derive (6.1). In particular, this in combination with Hölder's inequality, $p > 1$, (7.9), and $u = u_\varepsilon$ in $B_\varepsilon(x_0) \setminus B_{s_\varepsilon}(x_0)$ yields (7.8) (i).

To see (7.8) (ii), observe that, since $\frac{s_\varepsilon}{\varepsilon}$ is bounded from above and from below, (6.1) (ii), (iv), $p > 1$, the fact that $u_\varepsilon \lfloor_{B^\pm_{s_\varepsilon}(x_0)} \in W^{1,p}(B^\pm_{s_\varepsilon}(x_0);\mathbb{R}^d)$ (see (6.7)), and Korn's inequality imply that

$$\int_{s_\varepsilon^{-1}(B^\pm_{s_\varepsilon}(x_0)-x_0)} |u_\varepsilon(x_0 + s_\varepsilon y) - \bar{u}^{\mathrm{surf}}_{x_0}|^p\,\mathrm{d}y + \int_{s_\varepsilon^{-1}(B^\pm_{s_\varepsilon}(x_0)-x_0)} |\nabla_y u_\varepsilon(x_0 + s_\varepsilon y)|^p\,\mathrm{d}y \to 0.$$

Hence, by the trace inequality and by scaling back to $B^\pm_{s_\varepsilon}(x_0)$, we obtain

$$\lim_{\varepsilon\to 0} s_\varepsilon{}^{-(d-1)} \int_{\partial B^\pm_{s_\varepsilon}(x_0)} |u_\varepsilon - u^\pm(x_0)|\,\mathrm{d}\mathcal{H}^{d-1} = 0.$$

Since $\frac{s_\varepsilon}{\varepsilon}$ is bounded from below, we then get that

$$\lim_{\varepsilon\to 0} \varepsilon^{-(d-1)} \int_{\partial B^\pm_{s_\varepsilon}(x_0)} |u_\varepsilon - u^\pm(x_0)|\,\mathrm{d}\mathcal{H}^{d-1} = 0. \tag{7.11}$$

Given a Borel set $E \subset B_1(x_0)$, we define $E^\varepsilon = E \cap B_{\frac{s_\varepsilon}{\varepsilon}}(x_0)$ for every $\varepsilon > 0$. Then by (6.7) and (7.11) we get

$$\lim_{\varepsilon\to 0}\Big(\varepsilon^{-(d-1)} \int_{J_{u_\varepsilon}\cap E^\varepsilon_{\varepsilon,x_0}} |[u_\varepsilon]|\,\mathrm{d}\mathcal{H}^{d-1} - |[\bar{u}^{\mathrm{surf}}_{x_0}]|\,\mathcal{H}^{d-1}(\Pi_0 \cap E^\varepsilon)\Big) = 0. \tag{7.12}$$

By (7.10) and (7.11) we also have

$$\lim_{\varepsilon\to 0} \varepsilon^{-(d-1)} \int_{\partial B_{s_\varepsilon}(x_0)} |u^-_\varepsilon - u^+|\,\mathrm{d}\mathcal{H}^{d-1} = 0,$$

where u^-_ε and u^+ indicate the inner and outer traces at $\partial B_{s_\varepsilon}(x_0)$, respectively. Hence, by construction of u_ε in (6.7) and since $u \in \mathrm{SBD}(\Omega)$, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\Big(\varepsilon^{-(d-1)} \int_{J_{u_\varepsilon}\cap(E\setminus E^\varepsilon)_{\varepsilon,x_0}} |[u_\varepsilon]|\,\mathrm{d}\mathcal{H}^{d-1} - |[\bar{u}^{\mathrm{surf}}_{x_0}]|\,\mathcal{H}^{d-1}(\Pi_0 \cap (E \setminus E^\varepsilon))\Big) \\
&= \lim_{\varepsilon\to 0}\Big(\varepsilon^{-(d-1)} \int_{J_u\cap(E\setminus E^\varepsilon)_{\varepsilon,x_0}} |[u]|\,\mathrm{d}\mathcal{H}^{d-1} - |[\bar{u}^{\mathrm{surf}}_{x_0}]|\,\mathcal{H}^{d-1}(\Pi_0 \cap (E \setminus E^\varepsilon))\Big) = 0.
\end{aligned}$$

By combining with (7.12), this concludes the proof of (7.8) (ii). □

With this lemma at hand, we can address the equivalence of minimization problems for the surface scaling.

Lemma 7.6. *Suppose that* $\mathcal{F}$ *satisfies* (H1), (H3), *and* (H4'). *Let* $u \in \mathrm{SBD}^p(\Omega)$. *Then for* $\mathcal{H}^{d-1}$*-a.e.* $x_0 \in J_u$ *we have*

$$\lim_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\gamma_{d-1}\varepsilon^{d-1}} = \limsup_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(\bar{u}^{\mathrm{surf}}_{x_0}, B_\varepsilon(x_0))}{\gamma_{d-1}\varepsilon^{d-1}}. \tag{7.13}$$

Proof. We argue as in the proof of Lemma 4.3.

For the "≤" inequality, take u_ε satisfying (6.1) and (7.8), and perform the same construction as in Lemma 4.3. Estimates (6.22) and (6.23) continue to hold, provided one replaces β with the larger constant $\beta(1+|[\bar u^{\mathrm{surf}}_{x_0}]|)$. Then, with (H4'), (6.1) (iii), (iv), and (7.8) (ii) we get

$$\limsup_{\varepsilon\to 0}\frac{\mathcal{F}(u_\varepsilon, A''_{\varepsilon,x_0})}{\varepsilon^{d-1}} \le \beta(1+|[\bar u^{\mathrm{surf}}_{x_0}]|)\gamma_{d-1}(1-(1-4\theta)^{d-1}), \tag{7.14}$$

which is the analogue of (6.24). This is enough to derive inequality "≤" in (7.13).

For the reverse one, take again u_ε satisfying (6.1) and (7.8). Then, by (7.8) (i), in addition to (6.25) we may also require that

$$\lim_{\varepsilon\to 0}\varepsilon^{-(d-1)}\int_{\partial B_{s_\varepsilon}(x_0)}|u^+-u^-_\varepsilon|\,\mathrm{d}\mathcal{H}^{d-1}=0, \tag{7.15}$$

where u^-_ε and u^+ indicate the inner and outer traces at $\partial B_{s_\varepsilon}(x_0)$, respectively. We perform the same construction as in Lemma 4.3. In this case, inequality (6.29) is replaced by

$$\begin{aligned}\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0}) \le \mathbf{m}_{\mathcal{F}}(u, B_{s_\varepsilon}(x_0)) + \varepsilon^d + \beta\mathcal{H}^{d-1}((\{u\neq u_\varepsilon\}\cup J_u\cup J_{u_\varepsilon})\cap\partial B_{s_\varepsilon}(x_0))\\ + \beta\int_{\partial B_{s_\varepsilon}(x_0)}|u^+-u^-_\varepsilon|\,\mathrm{d}\mathcal{H}^{d-1} + \mathcal{F}(u_\varepsilon, A''_{\varepsilon,x_0}),\end{aligned}$$

so that, using (6.25), (7.14), (7.15), and the fact that $s_\varepsilon \le (1-3\theta)\varepsilon$ we deduce that

$$\limsup_{\varepsilon\to 0}\frac{\mathcal{F}(z_\varepsilon, A_{\varepsilon,x_0})}{\varepsilon^{d-1}} \le (1-3\theta)^{d-1}\limsup_{\varepsilon\to 0}\frac{\mathbf{m}_{\mathcal{F}}(u, B_\varepsilon(x_0))}{\varepsilon^{d-1}} + \beta(1+|[\bar u^{\mathrm{surf}}_{x_0}]|)\gamma_{d-1}(1-(1-4\theta)^{d-1}).$$

Then one concludes exactly in the same way, upon replacing β in equation (6.31) with the larger constant $\beta(1+|[\bar u^{\mathrm{surf}}_{x_0}]|)$. □

Proof of Theorem 7.1. The result follows from Lemmas 7.2, 7.4, and 7.6, arguing exactly as in the proof of Theorem 2.1. □

8 The GSBV^p case

In this section we briefly remark that the strategy devised in this paper allows also to establish an integral representation result in the space $GSBV^p(\Omega;\mathbb{R}^m)$ for $m\in\mathbb{N}$. We consider functionals

$$\mathcal{F}\colon GSBV^p(\Omega;\mathbb{R}^m)\times\mathcal{B}(\Omega)\to[0,+\infty)$$

with the following general assumptions:

(H1) $\mathcal{F}(u,\cdot)$ is a Borel measure for any $u\in GSBV^p(\Omega;\mathbb{R}^m)$,
(H2) $\mathcal{F}(\cdot,A)$ is lower semicontinuous with respect to convergence in measure on Ω for any $A\in\mathcal{A}(\Omega)$,
(H3) $\mathcal{F}(\cdot,A)$ is local for any $A\in\mathcal{A}(\Omega)$, in the sense that if $u,v\in GSBV^p(\Omega;\mathbb{R}^m)$ satisfy $u=v$ a.e. in A, then $\mathcal{F}(u,A)=\mathcal{F}(v,A)$,
(H4) there exist $0<\alpha<\beta$ such that for any $u\in GSBV^p(\Omega;\mathbb{R}^m)$ and $B\in\mathcal{B}(\Omega)$ we have

$$\alpha\Big(\int_B|\nabla u|^p\,\mathrm{d}x+\mathcal{H}^{d-1}(J_u\cap B)\Big)\le\mathcal{F}(u,B)\le\beta\Big(\int_B(1+|\nabla u|^p)\,\mathrm{d}x+\mathcal{H}^{d-1}(J_u\cap B)\Big).$$

In this setting, we replace definition (2.1) by

$$\mathbf{m}_{\mathcal{F}}(u,A)=\inf_{v\in GSBV^p(\Omega;\mathbb{R}^m)}\{\mathcal{F}(v,A): v=u\text{ in a neighborhood of }\partial A\}.$$

Moreover, as in (2.2)–(2.3), we define the functions $\ell_{x_0,u_0,\xi}(x)=u_0+\xi(x-x_0)$ and $u_{x_0,a,b,\nu}(x)=a$ on the set $\{(x-x_0)\cdot\nu>0\}$ and $u_{x_0,a,b,\nu}(x)=b$ on the set $\{(x-x_0)\cdot\nu<0\}$ for $x_0\in\Omega$, $u_0\in\mathbb{R}^m$, $\xi\in\mathbb{M}^{m\times d}$, $a,b\in\mathbb{R}^m$, and $\nu\in\mathbb{S}^{d-1}$.

Theorem 8.1 (Integral representation in GSBV^p). *Let $\Omega \subset \mathbb{R}^d$ be open, bounded with Lipschitz boundary, let $m \in \mathbb{N}$, and suppose that $\mathcal{F}\colon \mathrm{GSBV}^p(\Omega;\mathbb{R}^m) \times \mathcal{B}(\Omega) \to [0,+\infty)$ satisfies* ($\mathbb{H}$1)–($\mathbb{H}$4). *Then*

$$\mathcal{F}(u,B) = \int_B f(x,u(x),\nabla u(x))\,\mathrm{d}x + \int_{J_u\cap B} g(x,u^+(x),u^-(x),\nu_u(x))\,\mathrm{d}\mathcal{H}^{d-1}(x)$$

for all $u \in \mathrm{GSBV}^p(\Omega;\mathbb{R}^m)$ and $B \in \mathcal{B}(\Omega)$, where f is given by

$$f(x_0,u_0,\xi) = \limsup_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(\ell_{x_0,u_0,\xi}, B_\varepsilon(x_0))}{\gamma_d \varepsilon^d}$$

for all $x_0 \in \Omega$, $u_0 \in \mathbb{R}^m$, $\xi \in \mathbb{M}^{m\times d}$, and g is given by

$$g(x_0,a,b,\nu) = \limsup_{\varepsilon\to 0} \frac{\mathbf{m}_{\mathcal{F}}(u_{x_0,a,b,\nu}, B_\varepsilon(x_0))}{\gamma_{d-1}\varepsilon^{d-1}}$$

for all $x_0 \in \Omega$, $a, b \in \mathbb{R}^m$, and $\nu \in \mathbb{S}^{d-1}$.

We point out that integral representation results in GSBV^p have been used in several contributions, see, e.g., [6–9, 21, 37]. They all rely on [13] along with a perturbation and truncation argument as follows: first, one considers the regularization

$$\mathcal{F}_\sigma(u) := \mathcal{F}(u) + \sigma \int_{J_u} |[u]|\,\mathrm{d}\mathcal{H}^{d-1},$$

restricted to $u \in \mathrm{SBV}^p(\Omega;\mathbb{R}^m)$. Then the assumptions of the integral representation result in SBV^p [13] are satisfied and one obtains a representation of $\mathcal{F}_\sigma$. In a second step, this representation is extended to GSBV^p by a truncation argument which allows to approximate GSBV^p functions by SBV^p functions. Eventually, by sending $\sigma \to 0$, an integral representation result for the original functional can be obtained. We refer to [21, Theorem 4.3, Theorem 5.1] for details on this procedure. (We notice that in [21] a more general growth condition from above is allowed in the surface energy density, cf. [21, assumption (1.4)], analogous to the one in (H4').) With our result at hand, this method can be considerably simplified since no perturbation and truncation arguments are needed.

For the proof of the theorem we need the following Poincaré-type inequality, which can be directly deduced from Theorem 3.3.

Theorem 8.2 (Poincaré inequality for functions with small jump set). *Assume $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain and let $1 < p < +\infty$. Then there exists a constant $c = c(\Omega,p,m) > 0$ such that for all $u \in \mathrm{GSBV}^p(\Omega;\mathbb{R}^m)$ there is a set of finite perimeter $\omega \subset \Omega$ with*

$$\mathcal{H}^{d-1}(\partial^*\omega) \le c\mathcal{H}^{d-1}(J_u), \quad \mathcal{L}^d(\omega) \le c(\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}}$$

and $v \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $v = u$ on $\Omega \setminus \omega$ and

$$\|\nabla v\|_{L^p(\Omega)} \le c\|\nabla u\|_{L^p(\Omega)}. \tag{8.1}$$

In particular, for all $u \in \mathrm{GSBV}^p(\Omega;\mathbb{R}^m)$ there is a constant $b \in \mathbb{R}^m$ such that

$$\|u - b\|_{L^p(\Omega\setminus\omega)} \le c\|\nabla u\|_{L^p(\Omega)}.$$

Proof. It suffices to consider the case $m = 1$ and to prove (8.1). This can be obtained for instance by applying Theorem 3.3 to the function $\bar{u}\colon \Omega \to \mathbb{R}^d$ defined as $\bar{u} := (u, 0, \dots, 0)$ and using the Sobolev–Korn inequality to get ∇v on the left-hand side. □

Proof of Theorem 8.1. We follow the proof of Theorem 2.1 and only indicate briefly the necessary adaptions. First, we observe that a version of the fundamental estimate in Lemma 3.8 holds true in $\mathrm{GSBV}^p(\Omega;\mathbb{R}^m)$ by repeating the proof with ($\mathbb{H}$4) in place of (H4). (We refer also to [16, Proposition 3.1].) Recall that the result follows by combining Lemmas 4.1, 4.2, and 4.3.

Lemma 4.1: The result is proved via (4.15) and Lemma 4.4. Notice indeed that the derivation of (4.15), as well as the argument ensuing therefrom are the same, up to using the growth condition ($\mathbb{H}4$) instead of (H4). (We also refer to [13, Lemma 6] for the corresponding argument in SBV^p.) In the proof of Lemma 4.4, due to the (stronger) lower bound in ($\mathbb{H}4$) and Ambrosio's compactness theorem in $GSBV^p$ (see [5, Theorem 4.36]), one can ensure that the function v^δ defined in (4.8) now belongs to $GSBV^p(\Omega;\mathbb{R}^m)$. Then the result follows with the same argument, up to using Theorem 8.2 in place of Theorem 3.3.

Lemma 4.2: With the fundamental estimate in $GSBV^p$ at hand, we can follow the proof of Lemma 4.2 for each $u \in GSBV^p(\Omega;\mathbb{R}^m)$ with ($\mathbb{H}4$) instead of (H4). The family $(u_\varepsilon)_\varepsilon$ is defined as in Lemma 5.1 (see (5.5)) using Theorem 8.2 in place of Theorem 3.3. First, $u_\varepsilon \in GSBV^p(B_\varepsilon(x_0);\mathbb{R}^m)$ since $u \in GSBV^p(\Omega;\mathbb{R}^m)$, and $u_\varepsilon\lfloor_{B_{(1-\theta)\varepsilon}(x_0)} \in W^{1,p}(B_{(1-\theta)\varepsilon}(x_0);\mathbb{R}^m)$. Observing that

$$\lim_{\varepsilon\to 0} \varepsilon^{-d} \int_{B_\varepsilon(x_0)} |\nabla u(x) - \nabla u(x_0)|^p \,dx = 0 \tag{8.2}$$

for $\mathcal{L}^d$-a.e. $x_0 \in \Omega$ (as $\nabla u \in L^p(\Omega;\mathbb{M}^{m\times d})$) and using (8.1), we further get

$$\lim_{\varepsilon\to 0} \varepsilon^{-d} \int_{B_{(1-\theta)\varepsilon}(x_0)} |\nabla u_\varepsilon(x) - \nabla u(x_0)|^p \,dx = 0.$$

With (5.5), and using again (8.2), we deduce that (5.1) (iii) can be improved to

$$\lim_{\varepsilon\to 0} \varepsilon^{-d} \int_{B_\varepsilon(x_0)} |\nabla u_\varepsilon(x) - \nabla u(x_0)|^p \,dx = 0.$$

This adaption is enough to redo the proof of Lemma 4.2 in the present situation.

Lemma 4.3: Here, we can follow the proof of Lemma 4.3 for each $u \in GSBV^p(\Omega;\mathbb{R}^m)$ with ($\mathbb{H}4$) instead of (H4), and Theorem 8.2 in place of Theorem 3.3. The family $(u_\varepsilon)_\varepsilon$ defined in Lemma 6.1 needs to satisfy $u_\varepsilon \in GSBV^p(B_\varepsilon(x_0);\mathbb{R}^m)$ and (6.1) (iv) needs to be improved to

$$\lim_{\varepsilon\to 0} \varepsilon^{-(d-1)} \int_{B_\varepsilon(x_0)} |\nabla u_\varepsilon|^p \,dx = 0. \tag{8.3}$$

First, we use (6.7) to see that $u_\varepsilon \in GSBV^p(B_\varepsilon(x_0);\mathbb{R}^m)$. Arguing as in the proof of (6.1) (iv), with Theorem 8.2 at hand, we obtain

$$\lim_{\varepsilon\to 0} \varepsilon^{-(d-1)} \int_{B^\pm_{s\varepsilon}(x_0)} |\nabla u_\varepsilon|^p \,dx = 0.$$

This along with $\lim_{\varepsilon\to 0} \varepsilon^{-(d-1)} \int_{B_\varepsilon(x_0)} |\nabla u|^p \,dx = 0$ for $\mathcal{H}^{d-1}$-a.e. $x_0 \in J_u$ and (6.7) concludes the proof of equation (8.3). □

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