FEM FOR QUASILINEAR OBSTACLE PROBLEMS IN BAD DOMAINS *

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Abstract. We study obstacle problems involving the p-Laplace operator in domains with fractal boundary and the corresponding pre-fractals problems. We obtain error estimates for FEM solutions based on smoothness properties.

Résumé. On étudie des problèmes d'obstacle pour l'opérateur p-Laplace dans les domaines avec frontière fractale et les problèmes pré-fractals correspondants. On établit des estimations d'erreur pour les solutions FEM liées à les propriétés de régularité.

1991 Mathematics Subject Classification. 28A80, 35J70, 65N30.

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1. Introduction

In this paper we consider obstacle problems involving the p-Laplace operator in bad domains in $\mathbb{R}^2$. By bad domains we mean domains with irregular and possibly fractal boundary (see Figures 1 and 2).

The study of fractals have been faced in the mathematical literature to model various phenomena in different fields: in Biology, in Medicine, in Engineering applications and in many Applied Sciences. The principal feature is that fractals provide interesting examples of settings with large surfaces and small volumes like, for example,

Keywords and phrases: Fractals, Degenerate elliptic equations, FEM

* *The authors are members of GNAMPA (INdAM). The authors are partially supported by INdAM GNAMPA Project 2016 and Grant Ateneo "Sapienza" 2015.*

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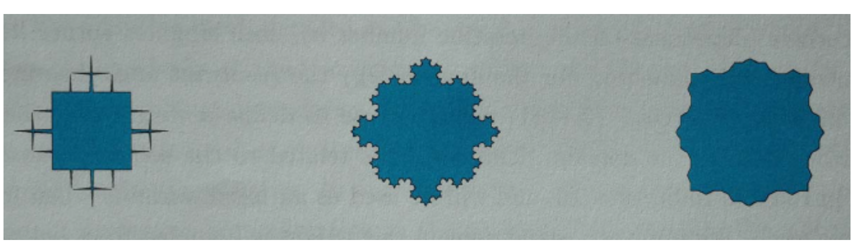

Figure 1. $\Omega^2_\alpha, \quad \alpha = 2.1, \alpha = 3, \alpha = 3.75$

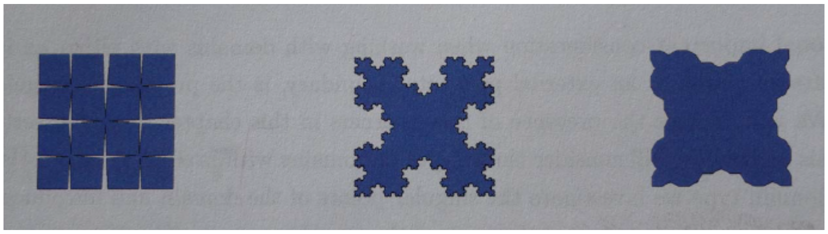

FIGURE 2. Ω^2_α, $\alpha = 2.1, \alpha = 3, \alpha = 3.75$

tortuous pulmonary acini, rough metallic electrodes, porous catalysts, irrigation tubes (see, for instance, [25] and the references quoted there).

We are principally interested in obstacle problems involving the p-Laplace operator in domains with fractal boundary, the corresponding pre-fractals problems, the smoothness properties, the FEM-approximation and the corresponding error estimates.

The interest in problems involving the p-Laplace operator comes from many applications: the non-Newtonian fluid mechanics, reaction-diffusion problems, flows through porous media. The p-Laplace operator also appears in modeling nonlinear elasticity, glaciology, and petroleum extraction (see [20] and reference therein). Moreover the p-Laplace operator also arises in the study of quasi-conformal mappings (see [27]) as well as in the study of stationary points of energy integrals defined on maps between Riemannian manifolds (for $p \geq 2$) (see [53]).

Let Ω_α denote a Koch-Island (see Section 2 for definitions and properties) and let us consider the two obstacle problem: find

$$u \in \mathcal{K}, \quad a_p(u, v-u) - \int_{\Omega_\alpha} f(v-u)\, dxdy \geq 0 \quad \forall v \in \mathcal{K} \tag{1.1}$$

where $a_p(u,v) = \int_{\Omega_\alpha} |\nabla u|^{p-2}\nabla u \nabla v\, dx\, dy$ and $\mathcal{K} = \{v \in W^{1,p}_0(\Omega_\alpha) : \varphi_1 \leq v \leq \varphi_2 \text{ in } \Omega_\alpha\}$.

Then, under *natural* assumptions (see (3.2) in Section 3), there exists a unique function u that solves problem (1.1). There is a huge literature about the regularity in the Hölder classes for both the solution u and the gradient ∇u (see [30] and the references quoted there). As far as we know there are no results concerning the global smoothness of the second derivatives of the solution u. Actually the smoothness of the second derivatives is little investigated also in the case of the solutions of linear equations (p=2) in such type of irregular domains. The Koch Islands belong to the class of the so called $(\varepsilon, \delta)-$domains in the terminology of P. W. Jones for which extension and trace results for functions in Sobolev (or Besov) spaces hold. However, to our knowledge, the only result concerning the smoothness of the second derivatives of the weak solutions of the Poisson equation is due to K. Nyström (see [48]). More precisely, the regularity result concerns the homogeneous Dirichlet problem for the Laplace operator in the snowflake domain and it is derived in the framework of the so called Non-Tangentially Accessible domains (NTA) introduced by D. S. Jerison and C. E. Kenig (see [29]). On the other hand many domains with a fractal boundary can be seen as limits of pre-fractal approximating domains, for which there exists a nowadays well established smoothness theory (P. Grisvard, V. A. Kondratiev, V. A. Koslov, V. G. Maz'ya, B. A. Plamenevskij, J. Rossman, V. A. Solonnikov (see [9] and the references quoted there). In this spirit, for the particular case of Laplace operator (p=2), we proved that the solution of the *fractal* obstacle problem belongs to the weighted Sobolev space $H^{2,\mu}(\Omega_\alpha)$ with weight δ^μ, where δ denotes the distance function from the boundary of Ω_α and μ depends on the Hausdorff dimension of the boundary $\partial\Omega_\alpha$. Moreover, we proved uniform estimates for the second derivatives of the solutions to the obstacle problem in the pre-fractal domains Ω^n_3 (approximating the snowflake domain) in the space $L^{2,\mu}(\Omega^n_3)$ where the weight is ρ^μ and $\rho = \rho_n$ denotes the distance function from the set $\mathcal{R}_n$ of the vertices of reentrant corners of Ω^n_3, and μ depends on the Hausdorff dimension of the boundary of the snowflake domain (see [14] and [15]).

As previously mentioned there are no results concerning the global smoothness of the second derivatives of the solution for the quasilinear case with $p > 2$, neither for the Koch-Island Ω_α nor for the approximating polygonal domains Ω_α^n.

Our guess is that, under suitable assumption on the data $(f, \varphi_1, \varphi_2)$, the solution u to problem (3.4) belongs to the weighted Sobolev Space $H^{2,\mu}(\Omega_\alpha^n)$ for any $\mu = \mu(p) > 1 - \gamma$ where the weight is ρ^μ and $\rho = \rho_n$ denotes the distance function from the set $\mathcal{R}_n$ of the vertices of reentrant corners of Ω_α^n and γ is defined in (4.24) (see Section 3 and also [18]).

Having in mind applications and numerical approaches, a crucial point is to construct *explicit, approximate* solutions and to show the convergence of the approximate solutions to the *fractal* solution.

In this paper we consider obstacle problems (3.4) in the pre-fractal approximating domains Ω_α^n. The domains Ω_α^n are polygonal, non convex and with an increasing number of sides. More precisely, the boundary is a polygonal curve, union of an increasing number of graphs, developing at the limit a fractal geometry. Under *natural* assumptions (see (3.5) in Section 3), there exists a unique function u_n that solves problem (3.4). Moreover, by using the Poincaré inequality and the monotonicity properties of the p-Laplacian we state energy estimates (3.6) where the constant does not depend on n. Then, by the nowadays well established variational convergence methods (see [44]), under *natural* assumptions on the data, we can easily show that the sequence of functions u_n^* converges (strongly in $W_0^{1,p}(\Omega_\alpha)$) to the solution of problem (1.1), where the function u_n^* is the extension of the function u_n by 0 on $\Omega_\alpha \setminus \Omega_\alpha^n$ (if we consider the case of outward Koch curves). Because of the tricky geometry of our domains it can be interesting to construct sequences of obstacles $\varphi_{i,n}$, $i = 1, 2$, that satisfy the *natural* assumptions (see (3.11), (3.12) of Proposition 3.3). Hence we construct *appropriate* approximating obstacles by using the iterative procedure and the family of similarities generating the fractal boundary of Ω_α (see Lemma 3.4).

We then introduce the triangulation of the domains Ω_α^n, the corresponding finite-dimensional spaces, the *discrete* obstacle problems (4.4) and we define the *discrete* solutions $u_{n,h}$ according to Galerkin method.

The more interesting and tricky part is then to establish error estimates that, as it is well known, are deeply based on the regularity properties of the solutions (similar questions for different problems and different approximation approach have been faced in [1], [2], [5], [19], [31], [32]).

In this paper we prove *sharp* error estimates according to the smoothness properties of the solution u_n (see Theorems 4.2, 4.5, 5.1).

In Proposition 3.7 we establish an estimate of the convergence of the pre-fractal solutions u_n to the solution of the fractal problem u in terms of the convergence of obstacles and of the domains. Combining estimate (3.30) with estimates (4.7) (or (4.25) or (5.7)) we deduce the convergence of the discrete solutions $u_{n,h}^*$ to the fractal solution u of Problem (1.1).

In order to establish FEM error estimates we need regularity results for the fractal solution u or, at least, uniform in n regularity results for the pre-fractal solution u_n. Unfortunately, up to now, these types of results are only proved for $p = 2$ (see [14] and [15]).

We have to mention that our results have been inspired by the papers of J. W. Barrett and W. B. Liu but we stress the fact that in our irregular domains we can not expect the regularity properties assumed in these papers (see [7] and [35]).

The plan of the paper is the following. In Section 2 we describe the geometry of our bad domains. In Section 3 we introduce the obstacle problems and we state existence, uniqueness, energy estimates, Lewy-Stampacchia inequalities and a regularity result for the solutions. In Section 4 we establish a first error estimate based on the regularity result proved in Section 3 that holds for any value of α (see Theorem 4.2) and a *sharper* error estimate that takes into account the geometrical properties of the domain Ω_α^n (see Theorem 4.5). More precisely, we assume that the solution belongs to a fractional Sobolev space $W^{\sigma,p}(\Omega_\alpha^n)$ with a smoothness index σ depending on α and we prove a convergence faster than the one established in Theorem 4.2. The smoothness assumption is justified by the expected regularity results mentioned before and by embedding of the weighted Sobolev space $H^{2,\mu}(\Omega_\alpha^n)$ in the fractional Sobolev space $W^{\sigma,p}(\Omega_\alpha^n)$, (see, *e.g.,* [51]).

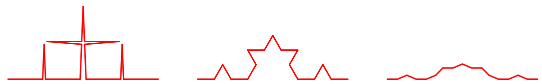

Figure 3. Second step

Finally in Section 5 we improve the previous error estimates assuming further hypotheses on the smoothness of the solution and we present a model obstacle problem whose solution satisfies the assumptions of Theorem 5.1.

2. Koch Islands

The Koch Islands Ω_α are domains in $\mathbb{R}^2$ having as boundary Koch curves. We can start by any regular polygon in $\mathbb{R}^2$ and we replace each side by a Koch curve.

Let us recall the definition of the contractive similarities generating the Koch curve with endpoints $A=(0,0)$, and $B=(1,0)$. Let $\Psi_\alpha=\{\psi_{1,\alpha}\dots,\psi_{4,\alpha}\}$ be the family of the contractive similarities $\psi_{i,\alpha}:\mathbb{C}\to\mathbb{C}$, $i=1,\dots,4$, with contraction factor α^{-1}, $2<\alpha<4$,

$$\psi_{1,\alpha}(z)=\frac{z}{\alpha}, \qquad \psi_{2,\alpha}(z)=\frac{z}{\alpha}e^{i\theta(\alpha)}+\frac{1}{\alpha},$$

$$\psi_{3,\alpha}(z)=\frac{z}{\alpha}e^{-i\theta(\alpha)}+\frac{1}{2}+i\sqrt{\frac{1}{\alpha}-\frac{1}{4}}, \qquad \psi_{4,\alpha}(z)=\frac{z-1}{\alpha}+1,$$

$$\theta(\alpha)=\arcsin\left(\frac{\sqrt{\alpha(4-\alpha)}}{2}\right). \tag{2.1}$$

By the general theory of self-similar fractals there exists a unique closed bounded set K_α which is *invariant* with respect to Ψ_α, that is,

$$K_\alpha=\cup_{i=1}^4\psi_{i,\alpha}(K_\alpha).$$

Moreover there exists a unique Borel regular measure ν_α with $supp\,\nu_\alpha=K_\alpha$ *invariant* with respect to Ψ_α which coincides with the normalized d_f-dimensional Hausdorff measure on K_α,

$$\nu_\alpha=(H(K_\alpha))^{-1}H^{d_f}|_{K_\alpha} \tag{2.2}$$

where Hausdorff dimension $d_f=\ln_\alpha 4$ (see [28]).

The pre-fractal Koch Islands Ω_α^n are polygonal domains having as sides pre-fractal Koch curves. We start by the regular polygon chosen to construct Ω_α and we replace each side by a pre-fractal Koch curve (see Figures 1 and 2).

Let K^0 be the line segment of unit length that has as endpoints $A=(0,0)$ and $B=(1,0)$. We set, for each n in $\mathbb{N}$,

$$K_\alpha^1=\bigcup_{i=1}^4\psi_{i,\alpha}(K^0), \qquad K_\alpha^2=\bigcup_{i=1}^4\psi_{i,\alpha}(K_\alpha^1),\dots \qquad K_\alpha^n=\bigcup_{i=1}^4\psi_{i,\alpha}(K_\alpha^{n-1})=\bigcup_{i|n}\psi_{i|n,\alpha}(K^0)$$

where $\psi_{i|n,\alpha}=\psi_{i_1,\alpha}\circ\psi_{i_2,\alpha}\circ\dots\circ\psi_{i_n,\alpha}$ for each integer $n>0$ is the map associated with arbitrary $n-$tuple of indices $i|n=(i_1,i_2,\dots,i_n)\in\{1,\dots,4\}^n$ and, if $n=0$, $\psi_{i|n,\alpha}$ is the identity map in $\mathbb{R}^2$.

K_α^n is the so-called n-th pre-fractal curve.

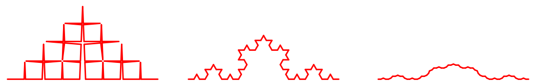

FIGURE 4. Third step

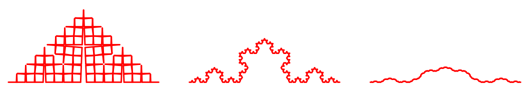

FIGURE 5. Fourth step

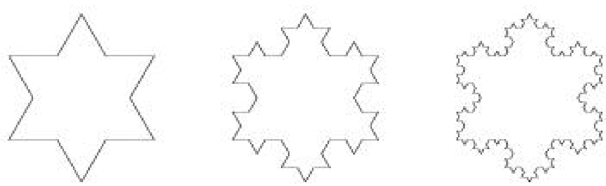

FIGURE 6. Ω_3^n

In Figure 3, we see the iterations at the step number 2 obtained by different choices of the contraction factors $\alpha = 2.2$, $\alpha = 3$ and $\alpha = 3.8$; in Figures 4 and 5 the iterations at the steps number 3 and number 4 respectively. Particular examples are the pre-fractal snowflakes: in Figure 6 we have chosen outward curves and $\alpha = 3$. The domains Ω_3^n are polygonal, non convex and with an increasing number of sides, the amplitude of the reentrant corners is equal to $\frac{4}{3}\pi$.

An important property of the Koch curves is that the family Ψ_α satisfies the so-called Open Set Condition that is there exists an open set T_0 such that $\psi_{i|n,\alpha}(T_0) \subset T_0$ for every $i|n$ and $\psi_{i|n,\alpha}(T_0) \cap \psi_{j|n,\alpha}(T_0) = \emptyset$ for every $i|n \neq j|n$. Here T_0 is the triangle of vertices $A = (0,0)$, $B = (1,0)$ and $C = (1/2, \delta_1/2)$, where $\delta_1 = \tan(\frac{\theta}{2})$, being θ the rotation angle (2.1). As consequence both the fractal curve and the pre-fractal curves (at any step of the iteration procedure) are contained in T_0 then if we start by a regular polygon with $m \geq 4$ sides, we can construct Koch Islands (and pre-fractal Koch Islands) by inward (and outward) curves for any $\alpha \in (2,4)$. If instead we start from a regular triangle we have to impose $\alpha > 3$ (for inward curves) in order to ensure that the Koch curves composing the boundary do not intersect.

Actually we can consider more general domains as the so-called *fluctuating Koch Island* having as boundary scale irregular Koch curves and the related pre-fractal fluctuating Islands having as sides scale irregular pre-fractal Koch curves (see Figures 7 and 8). Particular examples Ω_ξ and Ω_ξ^n can be constructed fixing two values of the contraction factor $2 < \alpha_1 < \alpha_2 < 4$ and the two families $\Psi_{\alpha_j} = \{\psi_{1,\alpha_j} \ldots, \psi_{4,\alpha_j}\}$ $(j = 1,2)$ of the relative contractive similarities in $\mathbb{R}^2$. Let ξ be the sequence $\xi = (\xi_1, \xi_2, \xi_3, \ldots), \xi_i \in \{1,2\}$.

The scale irregular Koch curve is

$$K^{(\xi)} = \overline{\bigcup_{n=1}^{+\infty} V_n^{(\xi)}}, \quad V_0^{(\xi)} = \Gamma = \{A, B\}, \quad V_n^{(\xi)} = \Psi_{\alpha_{\xi_1}} \circ \ldots \Psi_{\alpha_{\xi_n}}(\Gamma)$$

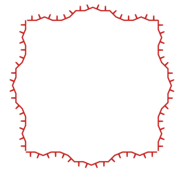

FIGURE 7. $\alpha_1 = 3.75, \alpha_1 = 3.75, \alpha_2 = 2.1$

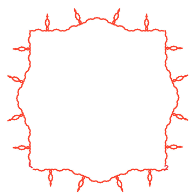

FIGURE 8. $\alpha_1 = 3.75, \alpha_2 = 2.1, \alpha_1 = 3.75$

where $\Psi_{\alpha_{\xi_l}}(\mathcal{O}) = \bigcup_{i=1}^{4} \psi_{i,\alpha_{\xi_l}}(\mathcal{O})$ for $\mathcal{O} \subset \mathbb{R}^2$ and $l = 1, ..., n$.

We remarks that the geometrical aspect of the curves depends not only on the contraction factors but also on the order by which we jump from a family to another one as we see in Figures 7 and 8 where the contraction factors are the same but the order is different (in Figure 7 in the first two steps we have used the family Ψ_{α_1} and in the third step the family Ψ_{α_2}; instead in Figure 8 in the first step we have used the family Ψ_{α_1}, in the second step the family Ψ_{α_2}, in the third step the family Ψ_{α_1} again).

In this construction the relevant parameter is the frequency of the occurrence. We denote by $h_j^{(\xi)}(n)$ the frequency of the occurrence of the family j in the finite sequence $\xi|n,\ n \geq 1$:

$$h_j^{(\xi)}(n) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{\xi_i = j\}},\ j = 1, 2.$$

The term $h_j^{(\xi)}(n)$ gives the frequency by which each family Ψ_{α_j} occurs, up to the step n, in our construction of the graph $V_n^{(\xi)}$ and, eventually as $n \to +\infty$, the frequency by which each family Ψ_{α_j} occurs in the construction of the limit set $K^{(\xi)}$. We suppose that there exists an asymptotic frequency of occurrence, p_j such that: $0 \leq p_j \leq 1,\ \ p_1 + p_2 = 1$,

$$|h_j^{(\xi)}(n) - p_j| \leq \frac{g(n)}{n}$$

(as $n \to +\infty$) $j = 1, 2$ where $|g(n)| \leq g_0 n^{1-\eta},\ \ g_0 > 0$. If $\eta = 1$, then the limit set $K^{(\xi)}$ is a d-set w.r.t. the $\mathcal{H}^d$-measure, with $d = d^{(\xi)} = \frac{\lg 4}{\lg \alpha^*}$ $\alpha^* = \alpha_1^{p_1} \alpha_2^{p_2}$. If instead $0 < \eta < 1$, then $c_0 r^{d+\iota} \leq \nu^{(\xi)}(K^{(\xi)} \cap B_r(x)) \leq c_0 r^{d-\iota}\ \ \forall r \leq r_0,\ \ \iota > 0$, where $\nu^{(\xi)}$ denotes a suitable measure constructed in terms on the maps (see [6] and [46]).

3. Two obstacle problems

Let Ω_α denote a Koch-Island and let us consider the two obstacle problem: find

$$u \in \mathcal{K}, \quad a_p(u, v-u) - \int_{\Omega_\alpha} f(v-u)\, dx\, dy \geq 0 \quad \forall v \in \mathcal{K} \tag{3.1}$$

where $a_p(u,v) = \int_{\Omega_\alpha} |\nabla u|^{p-2} \nabla u \nabla v \, dx\, dy$ and $\mathcal{K} = \{v \in W_0^{1,p}(\Omega_\alpha) : \varphi_1 \leq v \leq \varphi_2 \text{ in } \Omega_\alpha\}$.

By using the Poincaré inequality (see, *e.g.*, [41]), the monotonicity properties of the p-Laplacian and choosing as test function in (3.1) $v = \varphi_2 \wedge (\varphi_1 \vee 0)$ we can prove the following result.

Proposition 3.1. *Let*

$$\begin{cases} f \in L^{p'}(\Omega_\alpha), \quad \frac{1}{p} + \frac{1}{p'} = 1, \; \varphi_i \in W^{1,p}(\Omega_\alpha), \; i = 1, 2, \\ \varphi_1 \leq \varphi_2 \; \textit{in} \; \Omega_\alpha, \; \varphi_1 \leq 0 \leq \varphi_2 \; \textit{in} \; \partial\Omega_\alpha. \end{cases} \tag{3.2}$$

Then, there exists a unique function u that solves problem (3.1). *Moreover,*

$$||u||_{W^{1,p}(\Omega_\alpha)} \leq c\Big\{||f||^{\frac{p'}{p}}_{L^{p'}(\Omega_\alpha)} + ||\varphi_1||_{W^{1,p}(\Omega_\alpha)} + ||\varphi_2||_{W^{1,p}(\Omega_\alpha)}\Big\}. \tag{3.3}$$

Consider the two obstacle problems (3.4) in the pre-fractal approximating domains Ω_α^n : find

$$u \in \mathcal{K}_n, \quad a_{p,n}(u, v-u) - \int_{\Omega_\alpha^n} f(v-u)\, dx\, dy \geq 0 \;\; \forall v \in \mathcal{K}_n, \tag{3.4}$$

where $a_{p,n}(u,v) = \int_{\Omega_\alpha^n} |\nabla u|^{p-2} \nabla u \nabla v \, dx\, dy$ and $\mathcal{K}_n = \{v \in W_0^{1,p}(\Omega_\alpha^n) : \varphi_{1,n} \leq v \leq \varphi_{2,n} \text{ in } \Omega_\alpha^n\}$. As previously, by choosing $v = \varphi_{2,n} \wedge (\varphi_{1,n} \vee 0)$ as test function in (3.4), we can prove the following result.

Proposition 3.2. *Let*

$$\begin{cases} f \in L^{p'}(\Omega_\alpha^n), \quad \frac{1}{p} + \frac{1}{p'} = 1, \varphi_{i,n} \in W^{1,p}(\Omega_\alpha^n), \; i = 1, 2, \\ \varphi_{1,n} \leq \varphi_{2,n} \; \textit{in} \; \Omega_\alpha^n, \; \varphi_{1,n} \leq 0 \leq \varphi_{2,n} \; \textit{in} \; \partial\Omega_\alpha^n. \end{cases} \tag{3.5}$$

Then, there exists a unique function u_n that solves problem (3.4). Moreover

$$||u_n||_{W^{1,p}(\Omega_\alpha^n)} \leq c\Big\{||f||^{\frac{p'}{p}}_{L^{p'}(\Omega_\alpha^n)} + ||\varphi_{1,n}||_{W^{1,p}(\Omega_\alpha^n)} + ||\varphi_{2,n}||_{W^{1,p}(\Omega_\alpha^n)}\Big\} \tag{3.6}$$

where the constant c does not depend on n.

We recall that the solution u to problem (3.1) and the solutions u_n to problems (3.4) realize the minimum on the convex $\mathcal{K}$ of the functional $J_p(\cdot)$ and on the convex $\mathcal{K}_n$ of the functional $J_{p,n}(\cdot)$ respectively, *i.e.*,

$$J_p(u) = \min_{v \in \mathcal{K}} J_p(v), \;\; \text{where} \;\; J_p(v) = \frac{1}{p} a_p(v,v) - \int_{\Omega_\alpha} f v \, dx\, dy \tag{3.7}$$

$$J_{p,n}(u_n) = \min_{v \in \mathcal{K}_n} J_{p,n}(v), \;\; \text{where,} \;\; J_{p,n}(v) = \frac{1}{p} a_{p,n}(v,v) - \int_{\Omega_\alpha^n} f v \, dx\, dy. \tag{3.8}$$

We consider the case of outward Koch curves, the case of inward Koch curves can be treated in an analogous way. We define the functions $u_n^* \in W_0^{1,p}(\Omega_\alpha)$

$$u_n^* = \begin{cases} u_n & \text{in } \Omega_\alpha^n \\ 0 & \text{in } \Omega_\alpha \setminus \Omega_\alpha^n, \end{cases} \tag{3.9}$$

where u_n is the solution to problem (3.4). Then, by the well established variational convergence methods (see [44], and also Remark 4.1 in [16]) we can prove the following convergence result.

Proposition 3.3. *Let*

$$\begin{cases} f \in L^{p'}(\Omega_\alpha), \quad \frac{1}{p}+\frac{1}{p'}=1, \;\; \varphi_i \in W^{1,p}(\Omega_\alpha), \;\; i=1,2, \\ \varphi_1 \leq \varphi_2 \;\; in \;\; \Omega_\alpha, \;\; \varphi_1 \leq 0 \leq \varphi_2 \;\; in \;\; \partial\Omega_\alpha, \end{cases} \tag{3.10}$$

$$\begin{cases} f \in L^{p'}(\Omega_\alpha), \quad \frac{1}{p}+\frac{1}{p'}=1, \;\; \varphi_{i,n} \in W^{1,p}(\Omega_\alpha), \;\; i=1,2, \\ \varphi_{1,n} \leq \varphi_{2,n} \;\; in \;\; \Omega^n_\alpha, \;\; \varphi_{1,n} \leq 0 \leq \varphi_{2,n} \;\; in \;\; \partial\Omega^n_\alpha \end{cases} \tag{3.11}$$

and

$$\varphi_{i,n} \to \varphi_i \; in \;\; W^{1,p}(\Omega_\alpha), \;\; i=1,2. \tag{3.12}$$

Then, the sequence of functions u^*_n defined in (3.9) strongly converges in $W^{1,p}_0(\Omega_\alpha)$ to the solution of problem (3.1).

Because of the tricky geometry of our domains it can be interesting to construct sequences of obstacles $\varphi_{i,n}$, $i=1,2$, that satisfy assumptions (3.11) and (3.12) of Proposition 3.3. We consider obstacles φ_i, $i=1,2$, satisfying assumption (3.10) and we assume a further smoothness $\varphi_i \in Lip(\Omega_\alpha)$, $i=1,2$. We recall that, for any α, the domain Ω_α is a $(\varepsilon,\delta)-$domain in the terminology of P. W. Jones and hence extension and trace results for solutions in Sobolev (or Besov) spaces hold. We construct suitable arrays of fibers Σ^n around the boundary of the domain Ω^n_α, in the spirit of the construction in [17]. In order to fix the notation, we consider the Koch Island constructed starting from the domain Ω^0 that is the open triangle of vertices $A=(0,0)$, $B=(1,0)$ and $D=(1/2,-\sqrt{3}/2)$ (see Figure 6). By T_0 we denote the open triangle of vertices $A=(0,0)$, $B=(1,0)$ and $C=(1/2,\delta_1/2)$, where $\delta_1=\tan(\frac{\theta}{2})$, being θ the rotation angle (2.1). The triangle T_0 satisfies the open set condition with respect to the maps Ψ, that is, $\psi_{i|n}(T_0) \subset T_0$ for every $i|n$ and $\psi_{i|n}(T_0) \cap \psi_{j|n}(T_0) = \emptyset$ for every $i|n \neq j|n$. From now on, as we keep α fixed, we remove the index α in the maps and in similar expressions. By T^*_0 we denote the open triangle of vertices $A=(0,0)$, $B=(1,0)$ and $C^*=(1/2,-\delta_2/2)$, where $\delta_2=\tan(\theta^*)$ with $\theta^*=\min\{\pi/2-\theta,\theta/2\}$, and we set the reference fiber Σ^0_1 around the side AB putting $\Sigma^0_1 = T_0 \bigcup T^*_0 \bigcup K^0$.

We now iteratively transform the arrays Σ^0_1 into increasingly fine arrays, by the action, for each integer $n>0$ of the maps $\psi_{i|n}=\psi_{i_1}\circ\psi_{i_2}\circ\cdots\circ\psi_{i_n}$.

For every $n \geq 0$, we then define the arrays of fibers $\Sigma^n_1 = \Sigma^n_{1,+} \bigcup \Sigma^n_{1,-} \bigcup K^n$ by setting

$$\Sigma^n_{1,+} = \bigcup_{i|n} \Sigma^{i|n}_{1,+} \;, \quad \Sigma^{i|n}_{1,+} = \psi_{i|n}(T_0) \;, \tag{3.13}$$

$$\Sigma^n_{1,-} = \bigcup_{i|n} \Sigma^{i|n}_{1,-} \;, \quad \Sigma^{i|n}_{1,-} = \psi_{i|n}(T^*_0) \,. \tag{3.14}$$

We follow analogous procedures for the others sides AD and BD of the reference domain Ω^0 and we denote by $\Sigma^n_{2,+}, \Sigma^n_{3,+}, \Sigma^n_{2,-}$, and $\Sigma^n_{3,-}$ the relative arrays and we set

$$\Sigma^n = \bigcup_{j=1,2,3} \Sigma^n_j \;, \quad \Sigma^n_+ = \bigcup_{j=1,2,3} \Sigma^n_{j,+} \;, \quad \Sigma^n_- = \bigcup_{j=1,2,3} \Sigma^n_{j,-} . \tag{3.15}$$

We put

$$\hat{\Omega}^n_\alpha = int(\bar{\Omega}^n_\alpha \bigcup \Sigma^n_{j,+}) \;, \quad \breve{\Omega}^n_\alpha = \Omega^n_\alpha \setminus \bar{\Sigma}^n_{j,-} \tag{3.16}$$

We have the following inclusions

$$\breve{\Omega}^n_\alpha \subset \Omega^n_\alpha \subset \hat{\Omega}^n_\alpha, \;\; \text{and} \;\; \hat{\Omega}^{n+1}_\alpha \subset \hat{\Omega}^n_\alpha, \quad \breve{\Omega}^n_\alpha \subset \breve{\Omega}^{n+1}_\alpha . \tag{3.17}$$

For every n, for $i = 1, 2$, we define

$$\varphi_{i,n}(x,y) = \begin{cases} \varphi_i(x,y) & \text{if} \quad (x,y) \in \breve{\Omega}^n \\ \varphi_{i,j,n}(x,y) & \text{if} \quad (x,y) \in \overline{\Sigma_j^n}. \end{cases} \tag{3.18}$$

For every n, for $i = 1, 2$, we define $\varphi_{i,1,n}$ on $\overline{\Sigma_1^n} = \overline{\bigcup_{i|n} \Sigma_1^{i|n}}$

$$\varphi_{i,1,n}(x,y) = G_1(\varphi_i \circ \psi_{i|n}) \circ \psi_{i|n}^{-1}(x,y) \qquad \text{if} \quad (x,y) \in \overline{\Sigma_1^{i|n}} \tag{3.19}$$

where G_1 is the operator: $Lip(\overline{\hat{\Omega}^1}) \to Lip(\overline{\Sigma_1^0})$ defined in the following way. For every $x \in (0,1)$, we define $P_+ = P_+(x) = (x, x_+(x)) \in \partial T_0$ to be the intersection of $\partial T_0 \setminus K^0$ with the vertical line through the point $(x,0) \in K^0$ and $P_- = P_-(x) = (x, x_-(x)) \in \partial T_0^*$ to be the intersection of $\partial T_0^* \setminus K^0$ with the vertical line through the point $(x,0) \in K^0$. Then, for a given $g \in Lip(\overline{\hat{\Omega}^1})$ we put

$$G_1(g)(x,y) = \begin{cases} (1 - \frac{y}{x_+}) g_{I_0}(x,0) + \frac{y}{x_+} g(P_+) & \text{if} \ (x,y) \in \overline{T_0} \setminus \{A, B\} \\ (1 - \frac{y}{x_-}) g_{I_0}(x,0) + \frac{y}{x_-} g(P_-) & \text{if} \ (x,y) \in \overline{T_0^*} \setminus \{A, B\} \\ g(0,0) & \text{if} \quad (x,y) = (0,0) \\ g(1,0) & \text{if} \quad (x,y) = (1,0) \end{cases} \tag{3.20}$$

and by $g_{I_0}(x,0)$ we denote the affine function on K^0 that is equal to g at the endpoints A and B. The operators G_j on $\overline{\Sigma_j^0}$ and the functions $\varphi_{i,j,n}$ on $\overline{\Sigma_j^n}$ for $j = 2, 3$ with $i = 1, 2$ are constructed in a similar way.

Lemma 3.4. *Let $\varphi_i \in Lip(\Omega_\alpha)$, $i = 1, 2$, satisfy assumption (3.10). Then the obstacles $\varphi_{i,n}$ defined in (3.18) converge to φ_i in $W^{1,p}(\Omega_\alpha)$, $\;i = 1, 2$ and satisfy condition (3.11).*

Proof. As previously mentioned, for any α, the domain Ω_α is an $(\varepsilon, \delta)-$ domain and hence we can extend the obstacles φ_i as functions in the space $Lip\left(\overline{\hat{\Omega}_\alpha^1}\right)$ with equi-bounded norms. We note that for every n, two distinct copies $\Sigma_j^{i|n}$, $\Sigma_j^{l|n}$, if $i|n \neq l|n$, intersect each other at most at vertices of the polygonal curves $\partial\Omega_\alpha^n$ and analogously two distinct copies $\Sigma_j^{i|n}$, $\Sigma_i^{l|n}$, if $i \neq j$, meet at most only at vertices of the polygonal curves $\partial\Omega_\alpha^n$. Therefore, the obstacles $\varphi_{i,n}$ defined in (3.18) belong (in particular) to $W^{1,p}(\hat{\Omega}_\alpha^n)$ and satisfy conditions in (3.11) as assumption (3.10) holds and all the vertices of the polygonal curves $\partial\Omega_\alpha^n$ belong to $\partial\Omega_\alpha$. To show condition (3.12) we proceed as in the proof of Theorem 4.1 in [47] (see also comments in Section 5 in [17]): as the functions $\varphi_{i,n}$ belong to the space $Lip\left(\overline{\hat{\Omega}_\alpha^n}\right)$, we conclude that

$$\lim_{n \to +\infty} \int_{\Sigma^n} |\nabla \varphi_{i,n}|^p dxdy = 0. \tag{3.21}$$

Hence

$$\int_{\Omega_\alpha} |\nabla \varphi_{i,n}|^p dxdy \leq \int_{\hat{\Omega}_\alpha^n} |\nabla \varphi_{i,n}|^p dxdy = \int_{\breve{\Omega}_\alpha^n} |\nabla \varphi_i|^p dxdy + \int_{\Sigma^n} |\nabla \varphi_{i,n}|^p dxdy \leq c \tag{3.22}$$

with a constant c that does not depends on n. Moreover

$$\max_{\overline{\Sigma^n}} |\varphi_{i,n}| \leq |\varphi_i|_{C(\overline{\hat{\Omega}_\alpha^1})} \tag{3.23}$$

and, since the two-dimensional Lebesgue measure of Σ^n goes to zero as $n \to +\infty$, we have

$$\lim_{n \to +\infty} \int_{\hat{\Omega}_\alpha^n} |\varphi_{i,n}|^p dxdy = \int_{\Omega_\alpha} |\varphi_i|^p dxdy. \tag{3.24}$$

Then the sequence $\varphi_{i,n}$ weakly converges in $W^{1,p}(\Omega_\alpha)$ to φ_i. Actually we have strong convergence in $W^{1,p}(\Omega_\alpha)$ as

$$\begin{aligned}&\textstyle\int_{\Omega_\alpha} |\nabla\varphi_i|^p dxdy \geq \limsup_{n\to+\infty} \int_{\breve{\Omega}_\alpha^n} |\nabla\varphi_{i,n}|^p dxdy = \\ &\textstyle\limsup_{n\to+\infty} \big(\int_{\Sigma^n} |\nabla\varphi_{i,n}|^p dxdy + \int_{\breve{\Omega}_\alpha^n} |\nabla\varphi_{i,n}|^p dxdy \big) \geq \limsup_{n\to+\infty} \int_{\Omega_\alpha} |\nabla\varphi_{i,n}|^p dxdy.\end{aligned} \tag{3.25}$$

□

Now we introduce the Lewy-Stampacchia inequality that plays an important role in our approach to the regularity of the solutions. We denote

$$\triangle_p = div(|\nabla u|^{p-2}\nabla u).$$

Proposition 3.5. *Assume hypothesis (3.2) and*

$$\triangle_p\varphi_i \in L^{p'}(\Omega_\alpha),\ i = 1, 2,\ \frac{1}{p} + \frac{1}{p'} = 1. \tag{3.26}$$

Let u be solution of (3.1). Then

$$(-\triangle_p\varphi_2) \wedge f \leq (-\triangle_p u) \leq (-\triangle_p\varphi_1) \vee f \ \ in\ \ \Omega_\alpha. \tag{3.27}$$

Proposition 3.6. *Assume hypothesis (3.5) and*

$$\triangle_p\varphi_{i,n} \in L^{p'}(\Omega_\alpha^n),\ i = 1, 2,\ \frac{1}{p} + \frac{1}{p'} = 1. \tag{3.28}$$

Let u_n be solution of (3.4). Then

$$(-\triangle_p\varphi_{2,n}) \wedge f \leq (-\triangle_p u_n) \leq (-\triangle_p\varphi_{1,n}) \vee f \ \ in\ \ \Omega_\alpha^n. \tag{3.29}$$

Lewy-Stampacchia inequality was first proved in [33] for superharmonic functions which solve a minimum problem, the proof being deeply based on the properties of the Green function. This result was extended to more general (linear) operators and more general obstacles by U. Mosco and G. M. Troianiello in [43]. It is important to recall that the new proof in [43] is completely different, much more flexible and based on order methods. This new approach gives rise to many further extensions and in particular to Lewy-Stampacchia inequalities for $T-$monotone operators like the $p-$Laplacian. Let us mention only a few works on this topic [45], [52], [49] and let us refer to the bibliography cited there. Actually, inequalities (3.27) and (3.29) hold under assumptions weaker than (3.26) and (3.28) (respectively) according to Remark 1 in Chapter 4.5 in [52].

The convergence of the pre-fractal solutions to the solution of the fractal problem (see Proposition 3.3) can be improved by establishing an estimate of the norm in term of the convergence of obstacles and of the domains. More precisely, we have the following result.

Proposition 3.7. *In the notation and assumptions of Proposition 3.5 we have for any function $v_n \in \mathcal{K}_n$ and $v \in \mathcal{K}$*

$$||\nabla(u - u_n^*)||^p_{L^p(\Omega_\alpha)} \leq \tag{3.30}$$

$$c\{||\nabla(u - v_n^*)||^{p'}_{L^p(\Omega_\alpha)} + ||f + \triangle_p u||_{L^{p'}(\Omega_\alpha^n)}\big(||u - v_n||_{L^p(\Omega_\alpha^n)} + ||v - u_n||_{L^p(\Omega_\alpha^n)}\big) + \int_{\Omega_\alpha\setminus\Omega_\alpha^n} (f + \triangle_p u)(u - v)dxdy\}$$

where the constant c does not depend on n.

Here u_n^* is defined in (3.9), v_n^* is defined in the same way starting from v_n.

The proof is similar to the proof of following Lemma 4.3: actually, it is simpler because do not make use the quasi-norm introduced in (4.8).

In our setting, we can choose $v = \varphi_2 \wedge (u_n^* \vee \varphi_1) = u_n^* + (\varphi_1 - u_n^*)^+ - (u_n^* - \varphi_2)^+$ so we have in Ω_α^n that

$$|v - u_n| \leq (\varphi_1 - \varphi_{1,n})^+ \vee (\varphi_{2,n} - \varphi_2)^+.$$

In a similar way we can choose $v_n = \varphi_{2,n} \wedge (w_n \vee \varphi_{1,n}) = w_n + (\varphi_{1,n} - w_n)^+ - (w_n - \varphi_{2,n})^+$ where $w_n^* \in W_0^{1,p}(\Omega_\alpha^n)$ and $w_n \to u$ in $W^{1,p}(\Omega_\alpha)$. Then $v_n \in \mathcal{K}_n$ and as

$$v_n^* - u = w_n^* - u + (\varphi_{1,n} - w_n^*)^+ - (w_n^* - \varphi_{2,n})^+$$

by using Theorem 1.56 in [52] we have the convergence of v_n^* to u (w_n^* is defined as in (3.9) starting from w_n).

The functions w_n can be constructed as before according to the construction used for the obstacles $\varphi_{i,n}$ first by assuming more regularity on u ($C^1(\overline{\Omega_\alpha})$) and then by removing this additional assumption by density results and diagonalization arguments (see [4]).

By using Lewy-Stampacchia inequality, we state a regularity result in terms of Besov Spaces. We recall that

$$B_{p,q}^{1-\lambda}(\Omega_\alpha^n) := (W^{1,p}(\Omega_\alpha^n), L^p(\Omega_\alpha^n))_{\lambda,q},$$

$$B_{p,q}^{2-\lambda}(\Omega_\alpha^n) := (W^{2,p}(\Omega_\alpha^n), W^{1,p}(\Omega_\alpha^n))_{\lambda,q} = \{u \in W^{1,p}(\Omega_\alpha^n) : \nabla u \in B_{p,q}^{1-\lambda}(\Omega_\alpha^n : \mathbb{R}^2)\}$$

where $\lambda \in [0,1]$, $p, q \in [1, +\infty]$ and by $(\cdot,\cdot)_{\lambda,q}$ the real interpolation functor (see [8]).

Theorem 3.8. *Assume hypotheses (3.5) and (3.28). Let u_n be the solution of (3.4). Then u_n belongs to the Besov Space $B_{p,+\infty}^{1+1/p}(\Omega_\alpha^n)$. Moreover,*

$$||u_n||_{B_{p,+\infty}^{1+1/p}(\Omega_\alpha^n)} \leq c\Big\{||f||_{L^{p'}(\Omega_\alpha^n)}^{\frac{p'}{p}} + ||\triangle_p \varphi_{1,n}||_{L^{p'}\Omega_\alpha^n)}^{\frac{p'}{p}} + ||\triangle_p \varphi_{2,n}||_{L^{p'}\Omega_\alpha^n)}^{\frac{p'}{p}}\Big\}. \tag{3.31}$$

Note that, putting in the previous theorem $p = 2$, we get $u \in H^{3/2-\epsilon}(\Omega_\alpha^n)$ in the Sobolev scale.

Proof. From Lewy-Stampacchia inequality (see (3.29)) we derive that the solution u_n of problem (3.4) actually satisfies the homogeneous Dirichlet problem with datum $-\triangle_p u_n$ belonging in the space $L^{p'}(\Omega_\alpha^n)$. Then we can apply Theorem 2 in [50] to show the required regularity and estimate (3.31) (using again (3.29)). ☐

4. Error estimates

From now on, as we keep n fixed, we remove the index n in the solution (3.4) and in similar expressions.

We introduce the triangulation of the domain Ω_α^n in order to define the approximate solutions u_h according to Galerkin method. Let T_h be a partitioning of the domain Ω_α^n into disjoint, open regular triangles τ, each side being bounded by h so that $\Omega_\alpha^n = \bigcup_{\tau \in T_h} \tau$. Associate with T_h, we consider the finite dimensional spaces

$$S_h = \Big\{v \in C(\bar{\Omega}_\alpha^n) : v|_\tau \quad \text{is affine} \quad \forall \tau \in T_h\Big\} \text{ and } S_{h,0} = \Big\{v \in S_h : \ v = 0 \ \text{ on } \partial\Omega_\alpha^n\Big\}. \tag{4.1}$$

By π_h we denote the interpolation operator, $\pi_h : C(\bar{\Omega}_\alpha^n) \to S_h$ that $\pi_h v(P_i) = v(P_i)$ for any vertex P_i of the partitioning T_h. We recall the following standard approximation results (see [7]):

$$|v - \pi_h v|_{W^{1,q}(\tau)} \leq c h^{l-1} |v|_{W^{l,q}(\tau)}, \quad q > 2, \quad l = 2 \text{ or } l = 1, \tag{4.2}$$

and

$$||v - \pi_h v||_{L^p(\tau)} \leq ch^l |v|_{W^{l,p}(\tau)}, \;\; p > 2, \quad l = 2 \text{ or } l = 1, \tag{4.3}$$

where by $|\cdot|_X$ we denote the seminorm in the space X.

Consider the two obstacle problem in the finite dimensional space $S_{h,0}$: find

$$u \in \mathcal{K}_h, \quad a_p(u, v - u) - \int_{\Omega_\alpha^n} f(v - u)\, dx\, dy \geq 0 \;\forall v \in \mathcal{K}_h \tag{4.4}$$

where $a_p(u,v) = \int_{\Omega_\alpha^n} |\nabla u|^{p-2} \nabla u \nabla v \, dx\, dy$ and $\mathcal{K}_h = \{v \in S_{h,0} : \varphi_{1,h} \leq v \leq \varphi_{2,h} \text{ in } \Omega_\alpha^n\}$, with $\varphi_{1,h} = \pi_h \varphi_1$ and $\varphi_{2,h} = \pi_h \varphi_2$.

Proposition 4.1. *Let us assume hypothesis (3.5). Then, there exists a unique function u_h that solves problem* (4.4). *Moreover,*

$$||u_h||_{W^{1,p}(\Omega_\alpha^n)} \leq c \Big\{ ||f||_{L^{p'}(\Omega_\alpha^n)}^{\frac{p'}{p}} + ||\varphi_1||_{W^{1,p}(\Omega_\alpha^n)} + ||\varphi_2||_{W^{1,p}(\Omega_\alpha^n)} \Big\} \tag{4.5}$$

As previously the solution u_h to problem 4.4 realizes the minimum on the convex $\mathcal{K}_h$ of the functional $J_p(\cdot)$, *i.e.,*

$$J_p(u) = \min_{v \in \mathcal{K}_h} J_p(v), \;\; \text{where} \;\; J_p(v) = \frac{1}{p} a_p(v,v) - \int_{\Omega_\alpha^n} f v \, dx\, dy. \tag{4.6}$$

Theorem 4.2. *Assume hypotheses (3.5) and (3.28). We denote by u and u_h the solutions of problem (3.4) and (4.4). Then,*

$$||u - u_h||_{W^{1,p}(\Omega_\alpha^n)} \leq \, c\, h^{\eta^*} ||u||_{W^{1+\eta,p}(\Omega_\alpha^n)}, \;\; \eta < \frac{1}{p}, \; \eta^* = \frac{2\eta}{p} \tag{4.7}$$

where the constant c does not depend on h. Moreover

$$||u||_{W^{1+\eta,p}(\Omega_\alpha^n)} \leq c\{ ||f||_{L^{p'}(\Omega_\alpha^n)}^{\frac{p'}{p}} + ||\triangle_p \varphi_1||_{L^{p'}(\Omega_\alpha^n)}^{\frac{p'}{p}} + ||\triangle_p \varphi_2||_{L^{p'}(\Omega_\alpha^n)}^{\frac{p'}{p}} \}.$$

We note that putting in the previous theorem $p = 2$ we get $\eta^* = \eta < 1/2$.

We split the proof in some steps. We start by the analogous in this context of the famous approach of J. Céa and R. Falk (see [12], [13], and [24]). According to [35], for any function $v \in W^{1,p}(\Omega_\alpha^n)$ and $\sigma \geq 0$, we introduce the following quasi-norm

$$|v|_{(p,\sigma)} = \Big(\int_{\Omega_\alpha^n} (|\nabla u| + |\nabla v|)^{p-\sigma} |\nabla v|^\sigma \, dx\, dy \Big)^{\frac{1}{p}} \tag{4.8}$$

where u is the solution to problem (3.4).

Lemma 4.3. *We denote by u and u_h the solutions to problems (3.4) and (4.4). Then, for any function $v_h \in \mathcal{K}_h$ and $v \in \mathcal{K}$ we have*

$$||u - u_h||_{W^{1,p}(\Omega_\alpha^n)}^p \leq c\{ |u - v_h|_{(p,2)}^p + ||f + \triangle_p u||_{L^{p'}(\Omega_\alpha^n)} \big(||u - v_h||_{L^p(\Omega_\alpha^n)} + ||v - u_h||_{L^p(\Omega_\alpha^n)} \big) \} \tag{4.9}$$

where the constant c does not depend on h.

Proof. Proceeding as in Lemma 2.1 in [7], we can easily prove that for all $p \geq 2$ and $\delta \geq 0$ there exist positive constants c_1 and c_2 such that for all $\xi, \eta \in \mathbb{R}^2$

$$||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \leq c_1 |\xi - \eta|^{1-\delta} (|\xi| + |\eta|)^{p-2+\delta} \tag{4.10}$$

and

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta)_{\mathbb{R}^2} \geq c_2 |\xi - \eta|^{2+\delta} (|\xi| + |\eta|)^{p-2-\delta}. \tag{4.11}$$

Moreover

$$J_p(v_h)-J_p(u)=\int_0^1\int_{\Omega_\alpha^n}(|\nabla(u+s(v_h-u))|^{p-2}\nabla(u+s(v_h-u)),\nabla(v_h-u))\,dxdyds-\int_{\Omega_\alpha^n}f(v_h-u)\,dxdy=$$

$$=\int_0^1\big(a_p(u+s(v_h-u),v_h-u)-a_p(u,v_h-u)\big)ds+a_p(u,v_h-u)-\int_{\Omega_\alpha^n}f(v_h-u)\,dx=$$

$$=A(v_h)+a_p(u,v_h-u)-\int_{\Omega_\alpha^n}f(v_h-u)\,dxdy$$

where

$$A(v_h)=\int_0^1\Big(\int_{\Omega_\alpha^n}((|\nabla(u+s(v_h-u))|^{p-2}\nabla(u+s(v_h-u))-|\nabla u|^{p-2}\nabla u,\nabla(v_h-u))dxdy\Big)ds. \tag{4.12}$$

Then, by (4.11) with $\delta=p-2$, we obtain

$$||\nabla(u-u_h)||^p_{L^p(\Omega_\alpha^n)}\le cA(u_h) \tag{4.13}$$

where $c=\frac{p}{c_2}$. By using the characterization of the solution u_h (see (4.6))

$$\begin{aligned}A(u_h)=&J_p(u_h)-J_p(u)-a_p(u,u_h-u)+\int_{\Omega_\alpha^n}f(u_h-u)\,dx\,dy\le\\ &J_p(v_h)-J_p(u)-a_p(u,u_h-u)+\int_{\Omega_\alpha^n}f(u_h-u)\,dx\,dy\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}&J_p(v_h)-J_p(u)-a_p(u,u_h-u)+\int_{\Omega_\alpha^n}f(u_h-u)\,dx\,dy=\\ &=A(v_h)+a_p(u,v_h-u)-\int_{\Omega_\alpha^n}f(v_h-u)\,dxdy-a_p(u,u_h-u)+\int_{\Omega_\alpha^n}f(u_h-u)\,dx\,dy=\\ &A(v_h)+a_p(u,v_h-u_h)-\int_{\Omega_\alpha^n}f(v_h-u_h)\,dx\,dy\end{aligned} \tag{4.15}$$

and, as u solves problem (3.4),

$$\begin{aligned}&a_p(u,u-v+v_h+v-u-u_h)-\int_{\Omega_\alpha^n}f(u-v+v_h+v-u-u_h)\,dx\,dy\le\\ &a_p(u,v_h-u)+a_p(u,v-u_h)-\int_{\Omega_\alpha^n}f(v_h-u)dxdy-\int_{\Omega_\alpha^n}f(v-u_h)dx\,dy\end{aligned} \tag{4.16}$$

and by (4.10) with $\delta=0$ we have

$$|A(v_h)|\le c_1\int_0^1 s\int_{\Omega_\alpha^n}\big(|\nabla(u+s(v_h-u))|+|\nabla u|\big)^{p-2}|\nabla(v_h-u)|^2\,dxdyds\le c|v_h-u|^p_{(p,2)} \tag{4.17}$$

where $c=2^{p-3}c_1$.

We collect estimates (4.13), (4.14), (4.15), (4.16), and (4.17) and taking into account Lewy-Stampacchia inequality (3.29) and Poincaré inequality, we complete the proof (see (4.9)). □

Now we evaluate the terms in the right hand side in estimate (4.9) by choosing in an appropriate way the test functions $v_h \in \mathcal{K}_h$ and $v \in \mathcal{K}$.

Lemma 4.4. *We denote by u the solution to problem (3.4) and by u_h the solution to problem (4.4). Then there exist functions $v_h \in \mathcal{K}_h$ and $v \in \mathcal{K}$ such that*

$$||u - v_h||_{L^p(\Omega_\alpha^n)} \leq ch||u||_{W^{1,p}(\Omega_\alpha^n)}, \tag{4.18}$$

$$||v - u_h||_{L^p(\Omega_\alpha^n)} \leq ch\{||u||_{W^{1,p}(\Omega_\alpha^n)} + ||\varphi_1||_{W^{1,p}(\Omega_\alpha^n)} + ||\varphi_2||_{W^{1,p}(\Omega_\alpha^n)}\} \tag{4.19}$$

where the constant c does not depend on h. Moreover,

$$|u - v_h|^p_{(p,2)} \leq c|u - v_h|^2_{W^{1,p}(\Omega_\alpha^n)} \tag{4.20}$$

where the constant c depends on f, φ_1, φ_2 but it does not depend on h.

Proof. We choose $v_h = \pi_h u$ then, by estimate (4.3), we derive estimate (4.18). Then we choose $v = \varphi_2 \wedge (u_h \vee \varphi_1)$ that is $v = \varphi_2 - (\varphi_2 - \varphi_1 - (u_h - \varphi_1)^+)^+$ where by the symbol v^+ we denote the positive part of the function v, *i.e.*, $v^+ = \max\{v, 0\}$. Taking into account assumption (3.5) we can note that v belongs to the convex $\mathcal{K}$. Moreover where $\varphi_2 \geq u_h \geq \varphi_1$, we have $u_h - v = 0$, where $u_h > \varphi_2 \geq \varphi_1$ we have $u_h - v = u_h - \varphi_2 > 0$ and where $\varphi_2 \geq \varphi_1 > u_h$ we have $u_h - v = u_h - \varphi_1 < 0$. Hence, where $u_h - v > 0$ we have $0 < u_h - v = u_h - \varphi_2 \leq \pi_h\varphi_2 - \varphi_2$ and where $u_h - v < 0$ we have $0 > u_h - v = u_h - \varphi_1 \geq \pi_h\varphi_1 - \varphi_1$. In any case

$$||v - u_h||^p_{L^p(\Omega_\alpha^n)} \leq ||\pi_h\varphi_2 - \varphi_2||^p_{L^p(\Omega_\alpha^n)} + ||\pi_h\varphi_1 - \varphi_1||^p_{L^p(\Omega_\alpha^n)}$$

and again using estimate (4.3) we derive estimate (4.19). Finally we note that by Hölder inequality we have

$$|u - v_h|^p_{(p,2)} \leq c\Big(|u|_{W^{1,p}(\Omega_\alpha^n)} + |v_h|_{W^{1,p}(\Omega_\alpha^n)}\Big)^{p-2}|u - v_h|^2_{W^{1,p}(\Omega_\alpha^n)}, \tag{4.21}$$

hence we choose again $v_h = \pi_h u$ and taking into account (3.6) and (4.2), from estimate (4.21), we derive (4.20) where the constant c depends on f, φ_1, φ_2 but it does not depend on h. □

We are now in position to prove estimate (4.7).

Proof of Theorem 4.2. To evaluate the seminorm $|u - \pi_h u|_{W^{1,p}(\Omega_\alpha^n)}$, we make use of estimates (4.2), and of the regularity result (see (3.31)). More precisely we consider the linear bounded operator

$$\mathcal{F} \ : W^{2,p}(\Omega_\alpha^n) \to W^{1,p}(\Omega_\alpha^n), \ \mathcal{F}u = u - \pi_h u.$$

For any value $\lambda \in (0,1)$ there exists a constant c such that

$$||u - \pi_h u||_{W^{1,p}(\Omega_\alpha^n)} \leq ch^{1-\lambda}||u||^{1-\lambda}_{W^{2,p}(\Omega_\alpha^n)}||u||^{\lambda}_{W^{1,p}(\Omega_\alpha^n)}, \ \forall u \in W^{2,p}(\Omega_\alpha^n); \tag{4.22}$$

then, the operator $\mathcal{F}$ can be continuously extended between $W^{2-\lambda,p}(\Omega_\alpha^n) \to W^{1,p}(\Omega_\alpha^n)$ and the following estimate holds,

$$||u - \pi_h u||_{W^{1,p}(\Omega_\alpha^n)} \leq ch^{1-\lambda}||u||_{W^{2-\lambda,p}(\Omega_\alpha^n)}, \ \forall u \in W^{2-\lambda,p}(\Omega_\alpha^n) \tag{4.23}$$

(see, *e.g.,* Theorem 1.1.6 in [40], [3], and [34]).

We choose $\lambda = 1 - \frac{1}{p} + \epsilon$ and we conclude the proof by taking into account estimates (4.9), (4.18), (4.19), (4.20), (4.23), (3.31), Poincaré inequality and the inclusion of the Besov space $B^{1+1/p}_{p,\infty}(\Omega_\alpha^n)$ in the Sobolev space $W^{1+1/p-\epsilon,p}(\Omega_\alpha^n)$ (see, *e.g.,* [8] or [34]). □

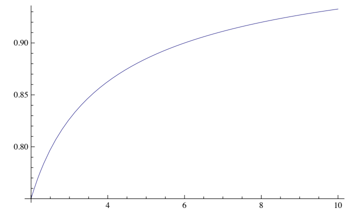

FIGURE 9. γ for $2 < p < 10$

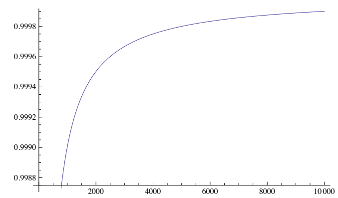

FIGURE 10. γ for $2 < p < 10.000$

We note that error estimates (4.7) is, in some sense, *natural* as it holds for any value of $\alpha \in (2,4)$ and we recall that, as $\alpha \to 2$, the Hausdorff dimension of the limit Koch curve (as $n \to \infty$) tends to 2 hence the pre-fractal curves are very fast oscillating and tend to fill a two dimensional region. A *natural* question is then if we can expect sharper error estimates based on better regularity results if we fix a value of α. We have approached the study of the regularity results in a paper in preparation ([18]) where we extend some results of M. Dobrowolski (see [21]). Our guess is that, under suitable assumption on the data $(f, \varphi_1, \varphi_2)$ the solution u to problem (3.4) belongs to the weighted Sobolev Space $H^{2,\mu}(\Omega_\alpha^n)$ for any $\mu = \mu(p) > 1 - \gamma$ where the weight is ρ^μ and $\rho = \rho_n$ denotes the distance function from the set $\mathcal{R}_n$ of the vertices of reentrant corners of Ω_α^n. Here by γ we denote the following expression

$$\gamma = \gamma(p,\chi) = \frac{p + \chi(2-\chi)(p-2) + (1-\chi)\sqrt{p^2 - \chi(2-\chi)(p-2)^2}}{2\chi(2-\chi)(p-1)} \tag{4.24}$$

where $\chi = \frac{\pi+\theta(\alpha)}{\pi}$ if the *sides* of the polygons are obtained by outward curves or $\chi = \frac{\pi+2\theta(\alpha)}{\pi}$ if the *sides* of the polygons are obtained by inward curves. We recall that by $\theta(\alpha)$ we denote the opening of the rotation angle (see (2.1)), then $\chi \in (1, \frac{3}{2})$ in the case of outward curves or $\chi \in (1,2)$ in the case of inward curves. We note that for any fixed value of $p > 2$ the function $\gamma(p,\cdot)$ decreases as the variable χ increases and it tends to the value $\frac{p-1}{p}$ as $\chi \to 2$. Similarly for any fixed value of $\chi < 2$ the function $\gamma(\cdot,\chi)$ increases as the variable p increases and it tends to the value 1 as $p \to +\infty$. Hence, in the setting of outward curves $\gamma(p,\chi)$ is always strictly greater than the value

$$\gamma(p, \frac{3}{2}) = 1 + \frac{p - \sqrt{p^2 + 12p - 12}}{6(p-1)}.$$

If we consider the famous example of the pre-fractal snowflake where $\theta = \theta(3) = \frac{4\pi}{3}$ (see Figure 6) then the expression for γ becomes

$$\gamma(p, \frac{4}{3}) = 1 + \frac{p - \sqrt{p^2 + 32p - 32}}{16(p-1)}.$$

Note that, putting in the previous formula $p = 2$, we get $\gamma = 3/4$ according to the by now classical results of Kondratiev for equations (see, *e.g.*, [9]). The behavior of γ for $2 < p < 10$ is showed in Figure 9 and for $2 < p < 10.000$ is showed in Figure 10.

The weighted Sobolev space $H^{2,\mu}(\Omega^n_\alpha)$ is continuously embedded in the fractional Sobolev space $W^{\sigma_2,2}(\Omega^n_\alpha)$ for any $\sigma_2 < 2-\mu$ (see, e-g., [51]). Hence, by Sobolev embedding (see, *e.g.,* [11]), the space $H^{2,\mu}(\Omega^n_\alpha)$ if $\mu=\mu(p)>1-\gamma$ is a proper subspace of the fractional Sobolev space $W^{\sigma,p}(\Omega^n_\alpha)$ for any $\sigma<\gamma+\frac{2}{p}$, $p\geq 2$.

We conclude this section by stating the corresponding error estimate that we can easily prove by proceeding as before (see Lemma 4.3, Lemma 4.4 and the proof of Theorem 4.2).

Theorem 4.5. *We denote by u and u_h the solutions of problem (3.4) and (4.4). Assuming $u\in W^{\sigma,p}(\Omega^n_\alpha)$, $\sigma=\gamma+\frac{2}{p}-\varepsilon$, then*

$$||u-u_h||_{W^{1,p}(\Omega^n_\alpha)} \leq c\,h^{\eta^{**}}||u||_{W^{\sigma,p}(\Omega^n_\alpha)},\ \gamma^*<\sigma-1,\ \eta^{**}=\frac{2\gamma^*}{p} \tag{4.25}$$

where the constant c does not depend on h.

Let us note that for any $p>2$ and any opening $\theta(\alpha)\in(0,\frac{\pi}{2})$ the exponent η^{**} in Theorem 4.5 is strictly greater than the exponent η^* in Theorem 4.2. In fact $\gamma(p,\chi)$ decreases as χ increases hence it is, in any case, strictly greater than $\frac{p-1}{p}$ then $\sigma-1>\frac{1}{p}$ for any value of $\alpha\in(2,4)$ and any $p>2$. In particular in the case of outward curves

$$\gamma(p,\chi)>\gamma(p,\frac{3}{2})=1+\frac{p-\sqrt{p^2+12p-12}}{6(p-1)}.$$

5. Comments and improvements

As we have remarked at the end of Section 4 we can expect sharper error estimates based on better regularity results if we fix a value of α. In particular an *interesting* question is then if we can prove sharper error estimates in norms or quasi-norms different from the *natural* norm $W^{1,p}$ (see [7], [35], [22], [23], [26], [36], [37], [38] and [39]). According to the approach of Barrett and Liu (see Theorem 4.1 and Corollary 4.1 in [7] and Theorem 4.4 in [35]) it could be interesting to establish error estimates in norms $W^{1,s}$, $s<p$. We note that in our geometry we can not expect that the solution u to Problem (3.4) belongs to the space $W^{1,\infty}(\Omega^n_\alpha)\bigcap W^{2,s}(\Omega^n_\alpha)$ for some $s>1$ as required in the above mentioned results and we can only assume weaker regularity properties. We give now a simple example to give a first highlight in this direction.

The function $u=\rho^\beta\sin(\beta\phi)$, with $1-\frac{2}{p}<\beta$ is the solution of the obstacle problem in the space $W^{1,p}(G)$ with datum $f=C\rho^{(\beta-1)(p-1)-1}\sin(\beta\phi)$ where $C=\beta^{p-1}(\beta-1)(p-2)$, obstacles $\varphi_1=0,\varphi_2=1$ with boundary datum $\Phi=\sin(\beta\phi)$ where G is the unit cone of opening $\frac{\pi}{\beta}$ *i.e.*, $G=\big\{(\rho,\phi),\ 0\leq\rho\leq 1, 0\leq\phi\leq\frac{\pi}{\beta}\big\}$. We assume $\frac{p^2-2p+2}{p^2-p}<\beta<1$, then $f\in L^{p'}(G)$ and, in particular, $|\nabla u|^{-1}$ is bounded. Moreover we can show by elementary calculation that the function

$$u^*=u(\rho,\phi)g(\rho), \tag{5.1}$$

where the cut function g is given by

$$g(\rho)=\begin{cases}1 & \text{if}\quad 0\leq\rho\leq 1\\ 2-\rho & \text{if}\quad 1\leq\rho\leq 2\\ 0 & \text{if}\quad 2\leq\rho,\end{cases} \tag{5.2}$$

is the solution of the obstacle problem in the space $W^{1,p}_0(G^*)$ with datum $f^*\in L^{p'}(G^*)$, obstacles $\varphi_1=0,\varphi_2=1$ where G^* is the cone of opening $\frac{\pi}{\beta}$ given by $G^*=\big\{(\rho,\phi),\ 0\leq\rho\leq 2, 0\leq\phi\leq\frac{\pi}{\beta}\big\}$. The function $f^*=-\Delta_p(u^*)$ is

$$f^*(\rho,\phi)=\begin{cases}f & \text{if}\quad 0\leq\rho<1\\ Smooth & \text{if}\quad 1\leq\rho\leq 2\\ 0 & \text{if}\quad 2<\rho.\end{cases} \tag{5.3}$$

More precisely, for $0\leq\rho<1$, $f^*=\beta^{p-1}(\beta-1)(p-2)\rho^{(\beta-1)(p-1)-1}\sin(\beta\phi)$, for $1\leq\rho\leq 2$, f^* is a combination of power of ρ having trigonometrical functions as coefficients.

Theorem 5.1. *We denote by u and u_h the solutions of problem (3.4) and (4.4) $(p>2)$ and we assume*

$$u \in W^{\sigma_2,2}(\Omega_\alpha^n), \;\; 2> \;\; \sigma_2 > 2 - \frac{1-2/p}{p-1}, \tag{5.4}$$

$$\varphi_i \in W^{\sigma,p}(\Omega_\alpha^n), \;\; i=1,2, \quad \sigma = \sigma_2 - 1 + 2/p \tag{5.5}$$

and

$$|\nabla u|^{-1} \in L^\infty(\Omega_\alpha^n) \quad if \quad q=t \quad or \quad |\nabla u|^{-\frac{(p-t)q}{t-q}} \in L^1(\Omega_\alpha^n) \quad if \quad q \in [1,t). \tag{5.6}$$

Then

$$||u-u_h||_{W^{1,q}(\Omega_\alpha^n)} \le c\, h^{\eta^{***}} (||u||^{2/t}_{W^{\sigma_2,2}(\Omega_\alpha^n)} + ||u||^{1/t}_{W^{\sigma,p}(\Omega_\alpha^n)} + ||\varphi_1||^{1/t}_{W^{\sigma,p}(\Omega_\alpha^n)} + ||\varphi_2||^{1/t}_{W^{\sigma,p}(\Omega_\alpha^n)}), \tag{5.7}$$

where $\eta^{***} = \frac{\sigma}{t}$, $t \in [2,p]$, $q \in [1,t]$, and the constant c does not depend on h.

As in the previous section we have to establish a preliminary estimate (see definition (4.8)).

Lemma 5.2. *We denote by u and u_h the solutions to problems (3.4) and (4.4). Then, for any function $v_h \in \mathcal{K}_h$ and $v \in \mathcal{K}$ we have*

$$|u-u_h|^p_{(p,t)} \le c\{|u-v_h|^p_{(p,r)} + ||f + \triangle_p u||_{L^{p'}(\Omega_\alpha^n)} \big(||u-v_h||_{L^p(\Omega_\alpha^n)} + ||v-u_h||_{L^p(\Omega_\alpha^n)}\big)\} \tag{5.8}$$

where $r \in [1,2], t \in [2,p]$, and the constant c does not depend on h.

Proof. As in the proof of Lemma 4.3 we use estimate (4.11) now with $\delta = t-2$ and we obtain

$$|u-u_h|^p_{(p,t)} \le cA(u_h) \tag{5.9}$$

where $A(\cdot)$ is defined in (4.12) and $c = \frac{2^{p-t}p}{c_2}$. By using the characterization of the solution u_h (see (4.6))

$$A(u_h) = J_p(u_h) - J_p(u) - a_p(u, u_h - u) + \int_{\Omega_\alpha^n} f(u_h - u)\, dx\, dy \le \tag{5.10}$$

$$J_p(v_h) - J_p(u) - a_p(u, u_h - u) + \int_{\Omega_\alpha^n} f(u_h - u)\, dxdy$$

and

$$J_p(v_h) - J_p(u) - a_p(u, u_h - u) + \int_{\Omega_\alpha^n} f(u_h - u)\, dx\, dy = \tag{5.11}$$

$$= A(v_h) + a_p(u, v_h - u) - \int_{\Omega_\alpha^n} f(v_h - u)\, dx - a_p(u, u_h - u) +$$

$$\int_{\Omega_\alpha^n} f(u_h - u)\, dx\, dy = A(v_h) + a_p(u, v_h - u_h) - \int_{\Omega_\alpha^n} f(v_h - u_h)\, dx\, dy$$

and, as u solves problem (3.4),

$$a_p(u, u - v + v_h + v - u - u_h) - \int_{\Omega_\alpha^n} f(u - v + v_h + v - u - u_h)\, dx\, dy \le \tag{5.12}$$

$$\le a_p(u, v_h - u) + a_p(u, v - u_h) - \int_{\Omega_\alpha^n} f(v_h - u)\, dx\, dy - \int_{\Omega_\alpha^n} f(v - u_h)\, dx\, dy$$

by (4.10) with $\delta = 2 - r$

$$|A(v_h)| \leq c_1 \int_0^1 s^{r-1} \int_{\Omega_\alpha^n} \big(|\nabla(u + s(v_h - u))| + |\nabla u|\big)^{p-r} |\nabla(v_h - u)|^r \, dx\, dy\, ds \leq c|v_h - u|^p_{(p,r)}, \tag{5.13}$$

where $c = \frac{2^{p-r} c_1}{r}$.

We collect estimates (5.9), (5.10), (5.11), (5.12), and (5.13) and taking into account Lewy-Stampacchia inequality (3.29) we complete the proof (see (5.8)). □

We are now in position to prove estimate (5.7).

Proof of Theorem 5.1. By assumption (5.4), $u \in W^{\sigma_2,2}(\Omega_\alpha^n)$. Then we derive from Sobolev embeddings that

$$\begin{cases} u \in W^{\sigma,p}(\Omega_\alpha^n) \bigcap W^{\sigma^*,p^*}(\Omega_\alpha^n) \bigcap W^{1,r^*}(\Omega_\alpha^n), \text{where} \\ \ \sigma \leq \sigma_2 - 1 + \frac{2}{p},\ \ \sigma^* \leq \sigma_2 - 1 + \frac{2}{p^*},\ \ p^* > 2,\ \ r^* \leq \frac{2}{2-\sigma_2}, \end{cases} \tag{5.14}$$

and we note that, as $\sigma_2 > 2 - \frac{2}{p}$, then we can choose $\sigma > 1$.

We proceed as in Lemma 4.4 and we choose $v_h = \pi_h u$ then by estimates (4.3) and (4.23) with $\lambda = 2 - \sigma$ we derive

$$||u - \pi_h u||_{L^p(\Omega_\alpha^n)} \leq ch^\sigma ||u||_{W^{\sigma,p}(\Omega_\alpha^n)}, \ \ \forall u \in W^{\sigma,p}(\Omega_\alpha^n). \tag{5.15}$$

Then we choose $v = \varphi_2 \wedge (u_h \vee \varphi_1)$ and as in Lemma 4.4 we have

$$||v - u_h||^p_{L^p(\Omega_\alpha^n)} \leq ||\pi_h \varphi_2 - \varphi_2||^p_{L^p(\Omega_\alpha^n)} + ||\pi_h \varphi_1 - \varphi_1||^p_{L^p(\Omega_\alpha^n)}.$$

Again using estimate (4.3) and assumption (5.5) we derive

$$||v - u_h||_{L^p(\Omega_\alpha^n)} \leq ch^\sigma. \tag{5.16}$$

To evaluate the seminorm $|u - u_h|_{W^{1,q}(\Omega_\alpha^n)}$, we suppose first $q = t$ then by assumption (5.6)

$$|u - u_h|^t_{W^{1,t}(\Omega_\alpha^n)} \leq ||\nabla u||^{t-p}_{L^\infty(\Omega_\alpha^n)} \cdot \int_{\Omega_\alpha^n} |\nabla(u - u_h)|^t |\nabla u|^{p-t} dx dy \leq C|u - u_h|^p_{(p,t)}. \tag{5.17}$$

If instead $q < t$, by applying Hölder inequality we obtain

$$|u - u_h|^t_{W^{1,q}(\Omega_\alpha^n)} \leq |||\nabla u|^{-\frac{(p-t)q}{t-q}}||^{\frac{t-q}{q}}_{L^1(\Omega_\alpha^n)} \cdot \int_{\Omega_\alpha^n} |\nabla(u - u_h)|^t |\nabla u|^{p-t} dx dy \leq C|u - u_h|^p_{(p,t)}. \tag{5.18}$$

We now evaluate the term $|u - v_h|^p_{(p,2)}$, where $v_h = \pi_h u$ as in previous Lemma 4.4; we have by Hölder inequality and taking into account that $|\nabla u| \in L^{r*}(\Omega_\alpha^n)$

$$|u - v_h|^p_{(p,2)} \leq c|u - \pi_h u|^2_{W^{1,p^*}(\Omega_\alpha^n)}, \tag{5.19}$$

where

$$p^* = \frac{2r^*}{r^* + 2 - p}. \tag{5.20}$$

We choose $\sigma = \sigma_2 - 1 + 2/p,\ \ \sigma^* = \sigma_2 - 1 + 2/p^*$ where p^* is such that

$$\sigma = 2(\sigma^* - 1). \tag{5.21}$$

Hence by (5.21) $p^* = \frac{4}{3+2/p-\sigma_2}$ and we have $p^* > 2$ as assumption (5.4) implies that $\sigma_2 > 1 + \frac{2}{p}$. Moreover, by (5.20), $r^* = \frac{p^*(p-2)}{p^*-2}$: then $r^* = \frac{2(p-2)}{\sigma_2-1-2/p}$ satisfies the inequality required in Sobolev embedding (see (5.14)).

By taking into account estimates (5.8), (5.19), (5.16), (5.17) (or (5.18)) and (5.21), we conclude the proof using once again Poincaré inequality. □

We stress the fact that estimate (5.7) improves estimate (4.25) (also for $q = p$) as $2(\sigma - 1) < \sigma$. Moreover, Theorem 5.1 provides *appropriate* error estimates in the norm $W^{1,q}$, $q < p$; by the term *appropriate* we mean that the rate of convergence is greater than the rate of convergence in the norm $W^{1,p}$.

We conclude this section by noting that if $1 - \frac{1-2/p}{p-1} < \beta < 1$ then the solution u^* (see 5.1) of our model problem satisfies the assumptions of Theorem 5.1 with $\sigma_2 = \beta + 1 - \varepsilon$ and $r^* = \frac{2(p-2)}{\beta-\varepsilon-2/p}$. As long as we keep n fixed, we expect that the smoothness properties of the solution of obstacle problem in the polygonal domains Ω^n_α do not depend to the numbers of the (reentrant) angles.

On the other hand, according to the discussion in Section 4 before Theorem 4.5 and having in mind the results of M. Dobrowolski (see [21]), we guess that under suitable assumptions on the data $(f, \varphi_1, \varphi_2)$ the solution of obstacle problem in the polygonal domains Ω^n_α belongs to the fractional Sobolev space $W^{1+\gamma,2}(\Omega^n_\alpha)$ where γ is defined in formula (4.24). It is easy to check that γ is an increasing function of p for a fixed value of χ that is for a fixed opening of the reentrant angles. For example, in the setting of outward curves, $\gamma(p, \chi)$ is always strictly greater than the value

$$\gamma(p, \frac{3}{2}) = 1 + \frac{p - \sqrt{p^2 + 12p - 12}}{6(p-1)}$$

and it is easy to check that $1 + \gamma > 2 - \frac{1-2/p}{p-1}$ for any $p > 3 + \sqrt{3}$.

The situation drastically changes in the fractal domain Ω_α. To our knowledge, global regularity results in terms of Sobolev (or Besov) spaces with smoothness index greater than 1 for the fractal solution u are up to now only established for $p = 2$ (see [15]). In particular, recent results of L. Brasco, F. Santambrogio in [10] and C. Mercuri, G. Riey, B. Sciunzi in [42] do not seem to work for obstacles problems in fractal domains.

Acknowledgement The authors wish to thank the reviewers for valuable comments and suggestions.

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