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Linearization via input-output injection of time delay systems

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This paper deals with the problem of linearization of systems with constant commensurable delays by input-output injection using algebraic control tools based on the theory of non-commutative rings. Solutions for the problem of linearization free of delays, and with delays of an observable nonlinear time-delay systems are presented based on the analysis of the input-output equation. These results are achieved by means of constructive algorithms that use the *n-th* derivative of the output expressed in terms of the state-space variables instead of the explicit computation of the input-output representation of the system. Necessary and sufficient conditions are established in both cases by means of an invertible change of coordinates.

Keywords: Nonlinear systems; Time-delay systems; Linearization; Algebraic tools; Output injection; Input-Output representation

1. Introduction

The importance of the study of time-delay systems in control theory lies in the large number of dynamical systems that are affected by the delay phenomenon. This kind of systems can be found in several fields such as mechanics, optics, medicine, chemistry, economy, electronics, computer science, among others, as it is addressed in Erneux (2009); Kolmanovskii & Myshkis (1999); Smith, & Thieme. (2012); L. Yang, & X. Yang (2012), and other references.
The state-space observers are of fundamental importance in applications like monitoring and control of dynamical systems. In Zheng, Barbot, Boutat, Floquet, & Richard (2010a,b, 2011) conditions are given to transform a time-delay system into a canonical form as well as for causal and non causal observability. Furthermore, the problem of linearization via input-output injection is investigated in Márquez-Martínez, Moog, & Velasco-Villa (2002) where an algorithm for the linearization free of delays is proposed together with sufficient conditions for the design of an observer. Using the geometric framework proposed in Califano, Márquez-Martínez, & Moog (2011), necessary and sufficient conditions for the existence of a bicausal change of coordinates that takes a nonlinear time-delay system into a linear weakly-observable time-delay system, modulo input-output injection, are presented in Califano, Márquez-Martínez, & Moog (2013).
In the present paper, the problem of linearization of a time-delay system up to input and output injection is addressed in an algebraic context. The problem is investigated both starting from its input-output equation and its state-space representation. More precisely, the results obtained are the following:

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With respect to the input-output equation,

- it is shown, as a direct consequence of the results presented in Halás and Anguelova (2013), that the input-output representation of a system linearizable up to input-output equation is of retarded type;
- necessary and sufficient constructive conditions are given for a system represented through its input-output equation to admit a linear delay state-space representation up to input and output injection.

With respect to the state-space representation,

- the class of changes of coordinates considered is the invertible one as defined in Definition 3. The bicausality property of the change of coordinates is dropped in this context since the equivalence to such a structure is useful in the observer design procedure. It should be noted that if the given dynamics is linked to the dynamics

$$\begin{aligned}\dot{z}(t) &= \sum_{i=0}^{\mu} A_i z(t-i) + \Phi(y(t), \cdots, y(t-\mu), u(t), \cdots, u(t-\mu))\\ y(t) &= \sum_{i=0}^{\mu} C_i z(t-i),\end{aligned}$$

  through an invertible transformation of the form

$$x(t) = \phi(z(t+p), \ldots, z(t-j)), \quad p, j \geq 0,$$

  then, extending the arguments in Andrieu, & Praly (2006) to time-delay systems, if the linear part is stabilizable through output injection, and ϕ satisfies proper conditions, $z(t)$ of the stabilized system can be used to estimate $x(t-p)$. In this sense the results presented in this paper are more general than those presented in Califano, Márquez-Martínez, & Moog (2013), and Zheng, Barbot, Boutat, Floquet, & Richard (2011).
- Necessary and sufficient conditions are given to transform a state-space representation into a linear delay one, up to a non linear input and output injection, through an invertible change of coordinates. A constructive procedure is given to obtain it. The case in which the linear part is delay-free is discussed in detail.

The paper is organized as follows. In Section 2, notations and definitions of the algebraic framework used throughout the paper are recalled, as well as some basic results about input-output realizations of time-delay systems. The main problems covered in this paper are stated. Preliminary new results are given in Section 2.3. In Section 3, necessary and sufficient conditions are given for a system represented through its input and output equation to admit a linear delay state-space representation up to input and output injection. Section 4 is devoted to the solution of the linearization problem up to input and output injection through invertible changes of coordinates. Examples show the main issues in solving the proposed problems.

2. Definitions and algebraic setting

Consider the nonlinear dynamical causal time-delay system with constant commensurable delays represented by the equations

$$\begin{array}{rcl} \dot{x}(t) & = & F(x(t), x(t-1), \ldots, x(t-s)) + \sum_{j=0}^{s} G_i(x(t), x(t-1), \ldots, x(t-s))u(t-j) \\ y(t) & = & H(x(t), x(t-1), \ldots, x(t-s)). \end{array} \tag{1}$$

The state variable $x(t) \in \mathbb{R}^n$, the output $y(t) \in \mathbb{R}$, and the input $u(t) \in \mathbb{R}$. The initial condition $\chi : [-s, 0] \to \mathbb{R}^n$ is assumed to be smooth. The following notation is taken from Califano, Márquez-Martínez, & Moog (2011): $\mathcal{K}$ denotes the field of meromorphic functions of the symbols $\{x(t-i), u(t-i), \ldots, u^{(k)}(t-i), i \in \mathbb{Z}, k \in \mathbb{Z}^+\}$; d is the differential operator that maps elements from $\mathcal{K}$ to $\mathcal{E} = span_{\mathcal{K}}\{dx(t-i), du(t-i), \ldots, du^{(k)}(t-i), i \in \mathbb{Z}, k \in \mathbb{N}\}$; δ is the time-shift operator defined as follows: if $a(\cdot), f(\cdot) \in \mathcal{K}$, then $\delta(a(t)df(t)) = a(t-1)\delta df(t) = a(t-1)df(t-1)$. Using the time-shift operator δ as indeterminate, the non-commutative Euclidean (left) ring of polynomials with coefficients over $\mathcal{K}$ is denoted as $\mathcal{K}(\delta]$; $\mathbb{R}[\delta]$ is the ring of polynomials in δ with coefficients in $\mathbb{R}$. $\mathcal{M}$ is the defined as the left-module over the ring $\mathcal{K}(\delta]$: $\mathcal{M} = span_{\mathcal{K}(\delta]}\{d\xi \mid \xi \in \mathcal{K}\}$. Denoting by $deg(\cdot)$ the polynomial degree in δ of its argument, the elements of $\mathcal{K}(\delta]$ may be written as $\alpha(\delta] = \sum_{i=0}^{r_\alpha} \alpha_i(t)\delta^i$, with $\alpha_i \in \mathcal{K}$, and $r_\alpha = deg(\alpha(\delta])$. Addition and multiplication on this ring are defined by $\alpha(\delta] + \beta(\delta] = \sum_{i=0}^{max\{r_\alpha, r_\beta\}} (\alpha_i(t) + \beta_i(t))\delta^i$, and $\alpha(\delta]\beta(\delta] = \sum_{i=0}^{r_\alpha} \sum_{j=0}^{r_\beta} \alpha_i(t)\beta_j(t-i)\delta^{i+j}$. Let us define for $p, s \geq 0$, by $(\mathbf{x}_{[p,s]}) = (x(t+p), \ldots, x(t-s))$; $(\mathbf{z}_{[p,s]})$, and $(\mathbf{u}_{[p,s]})$, are defined similarly. We will use $x_{[s]}$ for $x_{[0,s]}$. Define $\bar{\mathbf{u}} = (u(t), \dot{u}(t), \ldots, u^{(n-1)}(t))^T$, and $\bar{\mathbf{y}}$ is defined in a similar way. For simplicity $y(t)$, $x(t)$, and $u(t)$ will stand for $\mathbf{y}_{[0]}$, $\mathbf{x}_{[0]}$ and $\mathbf{u}_{[0]}$. A matrix $M(\mathbf{x}_{[p,s]}, \delta) \in \mathcal{K}^{n \times n}(\delta]$ is called unimodular if it has a polynomial inverse. It is called polymodular if there exists a polynomial matrix $M'(\mathbf{x}_{[p,s]}, \delta)$ such that $M(\mathbf{x}_{[p,s]}, \delta)M'(\mathbf{x}_{[p,s]}, \delta) = diag\{\delta^{k_1}, \ldots \delta^{k_n}\}$, for some $k_i \in \mathbb{Z}^+$. Consider also $\bar{f}(\mathbf{x}_{[l]}) \mid_{\mathbf{x}_{[l]}(-j)} := \bar{f}(x(t-j), x(t-j-1), \ldots, x(t-j-l))$. Then, it is possible to rewrite equation (1) as

$$\begin{array}{rcl} \dot{x}(t) & = & F(\mathbf{x}_{[s]}) + \sum_{j=0}^{s} G_j(\mathbf{x}_{[s]})u(t-j) \\ y(t) & = & H(\mathbf{x}_{[s]}). \end{array} \tag{2}$$

The corresponding differential-form representation is given by

$$d\dot{x}(t) = f(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta)dx(t) + g(\mathbf{x}_{[s]}, \delta)du(t) \tag{3}$$

where

$$\begin{array}{rcl} f(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) & = & \sum_{i=0}^{s} \left(\frac{\partial F(\mathbf{x}_{[s]})}{\partial x(t-i)} + \sum_{j=0}^{s} u(t-j) \frac{\partial G_j(\mathbf{x}_{[s]})}{\partial x(t-i)} \right) \delta^i \\ g(\mathbf{x}_{[s]}, \delta) & = & \sum_{j=0}^{s} G_j(\mathbf{x}_{[s]})\delta^j \end{array} \tag{4}$$

$$dy(t) = \sum_{i=0}^{s} \frac{\partial H(\mathbf{x}_{[s]})}{\partial x(t-i)} \delta^i dx(t) = h(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta)dx(t). \tag{5}$$

$\Omega = span_{\mathcal{K}(\delta]}\{\omega_i(\mathbf{x},\delta),\ i = 1,\ldots,p\}$ is the module spanned over $\mathcal{K}(\delta]$ by the row vectors $\omega_1(\mathbf{x},\delta), \omega_2(\mathbf{x},\delta), \ldots, \omega_p(\mathbf{x},\delta) \in \mathcal{K}^n(\delta]$.
Let us consider the definition of the extended Lie derivative for nonlinear time-delay systems, given in the framework of Xia, Márquez-Martínez, Zagalak, & Moog (2002), and expressed in Zheng, Barbot, Boutat, Floquet, & Richard (2011) as

$$L_F H(\mathbf{x}_{[s]}) = \sum_{i=0}^{s} \frac{\partial H(\mathbf{x}_{[s]})}{\partial x(t-i)} \delta^i F(\mathbf{x}_{[s]}), \tag{6}$$

$L_F^l H(\mathbf{x}_{[s]})$ the l-th extended Lie derivative, and the observability matrix

$$\mathcal{O}(\mathbf{x}_{[s]},\delta)d\mathbf{x} = \begin{pmatrix} dH(\mathbf{x}) \\ dL_F H(\mathbf{x}_{[p]}) \\ \vdots \\ dL_F^{n-1} H(\mathbf{x}_{[p]}) \end{pmatrix} = \begin{pmatrix} d\mathbf{y} \\ d\dot{\mathbf{y}} \\ \vdots \\ d\mathbf{y}^{(n-1)} \end{pmatrix}. \tag{7}$$

The characterization of the algebraic observability property is stated by the next definitions (Califano, Márquez-Martínez, & Moog (2013))

Definition 1: System (2) is said to be weakly-observable if the matrix $\mathcal{O}$ has full rank around $\mathbf{x}(0)$.

Definition 2: System (2) is said to be strongly observable if the matrix $\mathcal{O}$ is unimodular around $\mathbf{x}(0)$.

Definition 3: Given the system defined by (2), $z(t) = \phi(\mathbf{x}_{[p,s]})$ is an invertible change of coordinates if there exists a differentiable function $\bar{\phi}(\mathbf{z}_{[p',s']}) \in \mathcal{K},\ p,s,p',s' \in \mathbb{N}$, such that $\bar{\phi}(\mathbf{z}_{[p',s']}) \mid_{z(t)=\phi(\mathbf{x}_{[p,s]})} = x(t)$.

To the invertible change of coordinates $z(t) = \phi(\mathbf{x}_{[p,s]})$ we can associate a list of integers $r_i = max\{l \in \mathbb{Z} \mid \frac{\partial \phi_i(\mathbf{x}_{[p,s]})}{\partial x(t+l)} \equiv 0\}$. Its differential representation can be written as

$$\begin{pmatrix} \delta^{r_1} & 0 & \ldots \\ 0 & \ddots & \ldots \\ \vdots & & \delta^{r_n} \end{pmatrix} dz(t) = N(\mathbf{x}_{[0,\bar{s}]},\delta)dx(t)$$

For the inverse transformation, the corresponding indexes are defined by $k_i = max\{l' \in \mathbb{Z} \mid \frac{\partial \bar{\phi}(\mathbf{z}_{[p',s']})}{\partial z(t+l')} \equiv 0\}$. The differential representation is

$$dx(t) = \tilde{N}(\mathbf{z}_{[p',s']},\delta) \begin{pmatrix} dz(t+k_1) \\ \vdots \\ dz(t+k_n) \end{pmatrix}$$

Consequently

$$\begin{pmatrix} \delta^{r_1} & 0 & \dots \\ 0 & \ddots & \dots \\ \vdots & & \delta^{r_n} \end{pmatrix} dz(t) = N(\mathbf{x}_{[0,\bar{s}]}, \delta)|_{x=\psi(\mathbf{z}_{[p',s']})} \tilde{N}(\mathbf{z}_{[p',s']}, \delta) \begin{pmatrix} dz(t+k_1) \\ \vdots \\ dz(t+k_n) \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} \delta^{r_1+k_1} & 0 & \dots \\ 0 & \ddots & \dots \\ \vdots & & \delta^{r_n+k_n} \end{pmatrix} = N(\mathbf{x}_{[0,\bar{p}]}, \delta)|_{x=\psi(\mathbf{z}_{[p',s']})} \tilde{N}(\mathbf{z}_{[p',s']}, \delta)$$

If $p = j = 0$ the transformation is a bicausal change of coordinates and the associated differential representation is characterized by a unimodular matrix.

Remark 1: Note that the use of a bicausal transformation allows to keep invariant the strong and weak observability properties of the dynamical system. A non bicausal change of coordinates can modify these properties. Consider, for instance, next example:

Example 1: The strong observable system

$$\begin{aligned} \dot{x}(t) &= ax(t), \\ y(t) &= x(t). \end{aligned}$$

The change of coordinates $z = x(t+1)$ takes the system into the form

$$\begin{aligned} \dot{z}(t) &= az(t), \\ y(t) &= z(t-1), \end{aligned}$$

which is weakly-observable.

Let us end this section with the notion of normalized vector which will be used in the sequel.

Definition 4: Let $\lambda(x, u, \delta) = [\lambda_1, \cdots, \lambda_n] \in \mathcal{K}^{\bar{n}}(\delta]$. λ is called a normalized covector if $\lambda_i = 0$, for $i \in [1, j-1]$ and $\lambda_j = 1$.

2.1 *Problem statement*

As well known a dynamical system can be represented either through its input-output equation or through its state-space representation. Accordingly in this context the following problems can be set:

Problem 1: Given the input-output equation

$$\psi(\mathbf{y}_{[s]}^{(n)}, \mathbf{y}_{[s]}^{(n-1)}, \dots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-1)}, \dots, \mathbf{u}_{[s]}) = 0, \tag{8}$$

where the differential ideal generated by ψ is prime, find, if possible, a realization of the form

$$\begin{aligned}\dot{z}(t) &= \sum_{i=0}^{s} A_i z(t-i) + \varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}), \\ y(t) &= \sum_{j=0}^{s} C_j z(t-j).\end{aligned} \tag{9}$$

where $\mathbf{z} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}$, $y \in \mathbb{R}$, $A_i \in \mathbb{R}^{n\times n}$ for $i = 0, \ldots, s$, and $C_j \in \mathbb{R}^{1\times n}$ for $j = 0, \ldots, s$.

Starting from the state-space representation of the system then we have the following problem

Problem 2: Given the observable time-delay dynamical system (2) find, if possible, an invertible change of coordinates $z(t) = \phi(\mathbf{x}_{[p,s]})$ such that (2) is transformed into (9).

Remark 2: As an important remark note that, while in the delay-free case Problem 1 and Problem 2 are equivalent, this does not happen in the same way for nonlinear time-delay systems, as shown by next example

Example 2: The dynamical system defined by the delay differential equation

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)x_2(t-2) + (x_1(t-1) + x_1(t-2))^2, \\ \dot{x}_2(t) &= 0, \\ y(t) &= x_1(t) + x_1(t-1),\end{aligned} \tag{10}$$

has an input-output equation

$$\ddot{y}(t) = 2y(t-1)\dot{y}(t-1) + 2y(t-2)\dot{y}(t-2). \tag{11}$$

Note that setting $z_1(t) = y(t)$ and $z_2(t) = \dot{y}(t) - y(t-1)^2 - y(t-2)^2$ it is possible to take the input-output equation (11) into the linear state-space representation

$$\begin{aligned}\dot{z}_1(t) &= z_2(t) + z_1(t-1)^2 + z_1(t-2)^2, \\ \dot{z}_2(t) &= 0, \\ y(t) &= z_1(t).\end{aligned} \tag{12}$$

Nevertheless, (10) and (12) are not related by an invertible change of coordinates.

2.2 *Recalls on the Input-Output representation*

Let us consider the input-output representation of the dynamical system (1) which is of the form

$$\psi(\mathbf{y}_{[s]}^{(n)}, \mathbf{y}_{[s]}^{(n-1)}, \ldots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-1)}, \ldots, \mathbf{u}_{[s]}) = 0. \tag{13}$$

The next notation and results are issued from Halás and Anguelova (2013).
Let us define the r dimensional vector $(\nu_1, \ldots, \nu_r) := (\nu_1 \ \ldots \ \nu_r)^T \in \mathcal{K}^r$, and let $\frac{\partial(\nu_1,\ldots,\nu_r)}{\partial x} \in \mathcal{K}^{r\times n}(\delta]$ denote the matrix with entries

$$\left(\frac{\partial(\nu_1, \ldots, \nu_r)}{\partial x}\right)_{j,i} = \sum_{\iota=0}^{s} \frac{\partial \nu_j}{\partial x_i(t-\iota)} \delta^\iota \in \mathcal{K}(\delta] \tag{14}$$

The observability index $\bar{d}$ is defined as the least nonnegative integer that fulfills

$$rank_{\mathcal{K}(\delta]} \frac{\partial(H, \ldots, H^{(\bar{d}-1)})}{\partial x} = rank_{\mathcal{K}(\delta]} \frac{\partial(H, \ldots, H^{(\bar{d})})}{\partial x}, \tag{15}$$

with $\bar{d} \leq n$.

Definition 5: Let $\bar{d}$ be the observability index. Then, the input-output equation (13) is said to be retarded if

$$\frac{\partial \psi(\cdot)}{\partial y^{(\bar{d})}(t-i)} = 0$$

for all $i \geq 1$.

Definition 6: (Halás and Anguelova (2013)) Let $\bar{d}$ be the observability index then, the input-output equation (13) is said to be neutral if there exist $i_1 \neq i_2$ such that

$$\frac{\partial \psi(\cdot)}{\partial y^{(\bar{d})}(t-i)} \neq 0, \qquad for\ i = i_1,\ i = i_2$$

In the rest of this paper we assume $\bar{d} = n$. Considering Definition 6, the next example is presented.

Example 3: Let us consider the system defined by the set of equations:

$$\begin{array}{rcl} \dot{x}(t) & = & x(t-1)u(t) \\ y(t) & = & x(t) + x(t-1), \end{array} \tag{16}$$

which has an input-output representation that is written as

$$\dot{y}(t) - \dot{y}(t-1)\alpha(\mathbf{u}_{[s]}) = y(t-1)u(t) - y(t-2)u(t-2)\alpha(\mathbf{u}_{[s]}) \tag{17}$$

with $\alpha(\mathbf{u}_{[s]}) = \frac{u(t-1)-u(t)}{u(t-1)-u(t-2)}$. According to Definition 6, the input-output equation (17) is of neutral type. $\triangleleft$

Examples like the one defined by equation (16) show that it is possible to find systems which have a retarded state-space representation with a neutral input-output representation. Consider the next theorem taken from Halás and Anguelova (2013).

Theorem 1: *It is possible to find a retarded type input-output representation for system* (2) *if and only if*

$$\frac{\partial H^{(\bar{d})}}{\partial x} \in span_{\mathcal{K}(\delta]} \left\{ \frac{\partial(H, \ldots, H^{(\bar{d}-1)})}{\partial x} \right\}, \tag{18}$$

Note that, due to Theorem 1, it is possible to find out that system (16) does not have an input-output equation of retarded type.

2.3 *Preliminary results on the Input-Output realization*

In the present section, a preliminary result is presented which shows that any system which admits a linear state-space representation up to input and output injection, must be characterized by an input-output equation of retarded type. Such a result is fundamental in the solution of any of the problems investigated.

Lemma 1: *Problem 1 and Problem 2 are solvable, only if the given system admits an input-output equation of retarded type, and of the form*

$$y^{(n)}(t) = \sum_{i=1}^{n} \Phi_i(\mathbf{y}_{[\mathbf{s}]}, \mathbf{u}_{[\mathbf{s}]})^{(i-1)}. \tag{19}$$

Proof. Consider the differential form of (9) which is given by

$$\begin{array}{rcl} d\dot{z}(t) & = & A(\delta)dz(t) + d\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}) \\ dy(t) & = & C(\delta)dz(t). \end{array} \tag{20}$$

Consider the differential form for the first derivative of the output given by

$$d\dot{y} \;=\; C(\delta) \cdot \big(A(\delta)dz(t) + d\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})\big) \tag{21}$$

and iteratively

$$dy^{(k)} \;=\; C(\delta)A^k(\delta) \cdot dz(t) + \sum_{i=0}^{k-1} C(\delta)A^i(\delta)d\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})^{(k-i-1)} \tag{22}$$

with $k = 0, \ldots n$. Because of the commutativity of $\mathbb{R}[\delta]$, the use of the Cayley-Hamilton's theorem is allowed, so it is possible to find $\sigma_i \in \mathbb{R}[\delta],\ i = 0, \ldots n$ such that

$$\sum_{i=0}^{n} \sigma_i C(\delta)A^i(\delta) = 0, \tag{23}$$

with $\sigma_n = 1$. Then the differential form of the input-output representation of (9) can be written as

$$dy^{(n)} = -\sum_{i=0}^{n-1} \sigma_i dy^{(i)}(t) + \sum_{k=1}^{n}\sum_{i=0}^{k-1} \sigma_k C(\delta)A^i(\delta)d\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})^{(k-i-1)}, \tag{24}$$

which has the structure (24), with $d\Phi_{k+1}(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}) = -\sigma_k dy(t) + \sum_{j=0}^{n-k-1} \sigma_{n-j} C(\delta)A^{n-k-j-1}(\delta)d\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})$ for all $k = 0, \ldots, n-1$. □

The effective computation of the functions $\Phi_i\ (\mathbf{y}_{[s]}, \mathbf{u}_{[s]})$ can be performed using the linearization algorithm presented in Márquez-Martínez, Moog, & Velasco-Villa (2002).

To check whether a given system can be characterized by an input-output equation of retarded type, the following result can be used.

Lemma 2: *Assume that the given system has an input-output equation of the form*

$$\psi(\mathbf{y}_{[s]}^{(n)}, \mathbf{y}_{[s]}^{(n-1)}, \ldots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-1)}, \ldots, \mathbf{u}_{[s]}) = 0$$

where the differential ideal generated by ψ is prime. Then it can be represented also by an equation of retarded type, if and only if there exist a normalized covector $\lambda(\mathbf{y}_{[s]}, \cdots, \mathbf{y}_{[s]}^{(n-1)}, \mathbf{u}_{[s]}, \cdots, \mathbf{u}_{[s]}^{(n-1)}, \delta)$, *as defined in Definition 4, and a coefficient* $\alpha(\mathbf{y}_{[s]},\cdots,\mathbf{y}_{[s]}^{(n-1)},\mathbf{u}_{[s]},\cdots,\mathbf{u}_{[s]}^{(n-1)},\delta)$*, such that*

$$d\psi = \alpha(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta)\lambda \begin{bmatrix} dy^{(n)} \\ \vdots \\ dy \\ du^{(n-1)} \\ \vdots \\ du \end{bmatrix} = 0. \tag{25}$$

Proof. The differential of the input-output equation ψ is given by

$$d\psi(y^{(n)},\cdots,y,u^{(n-1)}\cdots,u) = \sum_{j=0}^{n}\ell_j(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta)dy^{(j)} + \sum_{j=0}^{n-1}\chi_j(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta)du^{(j)} = 0$$

On the other hand, since the system admits an input-output equation of retarded type then

$$y^{(n)} = \bar{\psi}(y^{(n-1)},\cdots,y,u^{(n-1)}\cdots,u)$$

so that

$$dy^{(n)} = d\bar{\psi}(y^{(n-1)},\cdots,y,u^{(n-1)}\cdots,u) = \sum_{j=0}^{n-1}q_j(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta)dy^{(j)} + \sum_{j=0}^{n-1}p_j(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta)du^{(j)}.$$

Consequently

$$\ell_n(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta)dy^{(n)} = \sum_{j=0}^{n-1}\ell_n(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta)q_j(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta)dy^{(j)} + \sum_{j=0}^{n-1}\ell_n(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta)p_j(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta)du^{(j)},$$

which shows that

$$\begin{aligned} \ell_j(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta) &= -\ell_n(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta)q_j(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta) \\ \chi_j(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta) &= -\ell_n(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta)p_j(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta). \end{aligned}$$

Accordingly $\alpha = \ell_n$ and

$$\lambda = [1, -q_{n-1}(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta),\cdots,-q_0(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta),-p_{n-1}(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta),\cdots,-p_0(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta)]$$

is the desired normalized covector.

Assume that the input–output equation satisfies (25). Dividing both sides by α, one immediately gets, due to the structure of λ, that

$$dy^{(n)} = \sum_{j=0}^{n-1}p_j(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta)dy^{(j)} + \sum_{j=0}^{n-1}q_j(\bar{\mathbf{y}},\bar{\mathbf{u}},\delta)du^{(j)}$$

Since the left side is an exact differential, such is also the right hand side. Since the differential of ψ generates a prime ideal, then we can compute a retarded type equation to describe the input-output relation. □

If instead we are starting from the state-space representation of the given system the following result can be stated

Lemma 3: *Assume that the given system is given in its state-space representation, and let* $\bar{A}(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) \in \mathcal{K}^{(2n+1)\times 2n}(\delta]$ *be*

$$\bar{A}(\mathbf{x}_{[s]}, \bar{\mathbf{u}}_{[s]}, \delta) = \begin{pmatrix} \hat{A}(\mathbf{x}_{[s]}, \bar{\mathbf{u}}_{[s]}, \delta) & \hat{B}(\mathbf{x}_{[s]}, \bar{\mathbf{u}}_{[s]}, \delta) \\ 0 & \mathbb{I} \end{pmatrix} \tag{26}$$

where setting $\bar{u} = (u, \dot{u}, \cdots, u^{(n-1)})$,

$$\hat{A}(\mathbf{x}_{[s]}, \bar{\mathbf{u}}_{[s]}, \delta) \quad = \quad \sum_{i=0}^{s} \frac{\partial(H^{(n)}, H^{(n-1)}, \ldots, H)}{\partial x(t-i)} \delta^i, \quad \hat{B}(\mathbf{x}_{[s]}, \bar{\mathbf{u}}_{[s]}, \delta) = \sum_{i=0}^{s} \frac{\partial(H^{(n)}, H^{(n-1)}, \ldots, H)}{\partial \bar{u}(t-i)} \delta^i. \tag{27}$$

Then the given system admits an input-output equation of retarded type, if and only if the left-annihilator of the matrix $\bar{A}(\mathbf{x}_{[s]}, \bar{\mathbf{u}}_{[s]}, \delta)$ *is generated by a normalized covector* $\lambda(\mathbf{x}_{[s]}, \bar{\mathbf{u}}_{[s]}, \delta)$, *as defined in Definition 4.*

Proof. Consider the set of equation

$$0 = \lambda(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) \begin{pmatrix} dy^{(n)} \\ dy^{(n-1)} \\ \vdots \\ d\dot{y} \\ dy \\ du^{(n-1)} \\ \vdots \\ d\dot{u} \\ du \end{pmatrix} = \lambda(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) \bar{A}(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) \begin{pmatrix} dx \\ d\bar{u} \end{pmatrix} \tag{28}$$

Since the dimension of the columns of $\bar{A}(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta)$ is $2n+1$, and the system is claimed to be observable, $rank(\bar{A}(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta)) = 2n$, so that there is one solution in the left kernel. If the system admits an input-output equation of retarded type, then there exists a $\lambda = [\chi_n, \cdots, \chi_0, \mu_{n-1}, \cdots, \mu_0]$ with $\chi_n = 1$ satisfying equation (28). Conversely if λ is a normalized vector, then $\chi_n = 1 \neq 0$

$$dy^{(n)} = \sum_{j=0}^{n-1} p_j(\bar{\mathbf{y}}, \bar{\mathbf{u}}, \delta) dy^{(j)} + \sum_{j=0}^{n-1} q_j(\bar{\mathbf{y}}, \bar{\mathbf{u}}, \delta) du^{(j)}$$

which, as before, ensures that the input–output equation is of retarded type. □

3. Linear realization up to input-output injection of a nonlinear input-output equation

Let us now discuss a solution to Problem 1, which consists in finding a linear delay state-space realization, up to injections of nonlinear time-delay functions of the input, and the output, from the input-output equation that describes the system dynamics.

Before entering in the details of the solution of the problem, let us recall that in Márquez-Martínez, Moog, & Velasco-Villa (2002) Problem 1 was addressed and solved in the particular case in which the required state-space realization had a delay free linear part, that is $A_i = 0$ and $C_i = 0$ for $i \in [1, s]$ in (9).

Define

$$\begin{array}{rcl} E^0 & = & 0 \\ E^k & = & span_{\mathcal{K}(\delta]}\{dy(t), \dots, dy^{(k-1)}(t), du(t), \dots, du^{(k-1)}\} \end{array}$$

and assume that $dim_{\mathcal{K}(\delta]} E^n = 2n$.

Algorithm 1:
STEP 0: Set $\psi_1 = \psi$, and compute the differential form of equation (8)

$$dy^{(n)}(t) = d\left(\psi_1(\mathbf{y}_{[s]}^{(n-1)}, \dots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-1)}, \dots, \mathbf{u}_{[s]})(t)\right) \tag{29}$$

STEP 1: By assumption $dy^{(n)} \in E^n$. Compute $\lambda_{n-1}^0 = \sum_{i=0}^{s} \frac{\partial \psi_1(\cdot)}{\partial y(t-i)^{(n-1)}} \delta^i$ (the coefficient of $dy^{(n-1)}(t)$) and $\mu_{n-1}^0 = \sum_{i=0}^{s} \frac{\partial \psi_1(\cdot)}{\partial u(t-i)^{(n-1)}} \delta^i$ (the coefficient of $du^{(n-1)}(t)$). Now Set

$$\omega_1 := \lambda_{n-1}^0 dy + \mu_{n-1}^0 du,$$

if $d\omega_1 \neq 0$ then STOP! there is no solution,
Compute $\Phi_1(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})$ such that $\omega_1 = d\Phi_1(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})$. Set

$$\begin{array}{rcl} \psi_2(\mathbf{y}_{[s]}^{(n-2)}, \dots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-2)}, \dots, \mathbf{u}_{[s]}) & := & \psi_1(\mathbf{y}_{[s]}^{(n-1)}, \dots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-1)}, \dots, \mathbf{u}_{[s]}) - \\ & & \Phi_1^{(n-1)}(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}), \end{array}$$

Compute the differential form of equation $\psi_2(\cdot)$

$$\begin{array}{rcl} d\left(\psi_2(\mathbf{y}_{[s]}^{(n-2)}, \dots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-2)}, \dots, \mathbf{u}_{[s]})\right) & = & d\left(\psi_1(\mathbf{y}_{[s]}^{(n-1)}, \dots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-1)}, \dots, \mathbf{u}_{[s]}) - \right. \\ & & \left. \Phi_1^{(n-1)}(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})\right) \end{array} \tag{30}$$

Check: $d\psi_2(t) \in E^{n-1}$?
NO: Stop, YES: Continue to the next step
STEP k: Define $\lambda_{n-k}^{k-1} = \sum_{i=0}^{s} \frac{\partial \psi_k(\cdot)}{\partial y(t-i)^{(n-k)}} \delta^i$ and $\mu_{n-k}^{k-1} = \sum_{i=0}^{s} \frac{\partial \psi_k(\cdot)}{\partial u(t-i)^{(n-k)}} \delta^i$ as the coefficient of $du^{(n-k)}(t)$ from the last equation in step $k-1$. Now Set

$$\omega_k := \lambda_{n-k}^{k-1} dy + \mu_{n-k}^{k-1} du,$$

if $d\omega_k \neq 0$ then STOP! there is no solution,
if $d\omega_k = 0$ then compute $\Phi_k(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})$ such that $\omega_k = d\Phi_k(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})$, and set

$$\begin{array}{rcl} \psi_{k+1}(\mathbf{y}_{[s]}^{(n-k+1)}, \dots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-k+1)}, \dots, \mathbf{u}_{[s]}) & := & \psi_k(\mathbf{y}_{[s]}^{(n-k)}, \dots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-k)}, \dots, \mathbf{u}_{[s]}) - \\ & & \Phi_k^{(n-k)}(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}) \end{array}$$

$$\begin{array}{rcl} d\left(\psi_{k+1}(\mathbf{y}_{[s]}^{(n-k+1)}, \dots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-k+1)}, \dots, \mathbf{u}_{[s]})\right) & = & d\left(\psi_k(\mathbf{y}_{[s]}^{(n-k)}, \dots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-k)}, \dots, \mathbf{u}_{[s]}) - \right. \\ & & \left. \Phi_k^{(n-k)}(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})\right) \end{array} \tag{31}$$

Check: $d\psi_{k+1}(t) \in E^{n-k}$?
For $k = 2, \ldots, n$. $\triangleleft$

If Algorithm 1 can be completed for each step k, for $k = 1, \ldots, n$, then it is possible to establish necessary and sufficient conditions for the solution of Problem 1, as it is stated in the next proposition.

Proposition 1: *Problem 1 with $A_i = 0$ and $C_i = 0$ for $i \in [1, s]$ in (9), is solvable if and only if ω_i, as defined in the Algorithm 1, are exact for all $i = 1, \ldots, n$.*

We include the proof of Proposition 1 considering that it is not presented in Márquez-Martínez, Moog, & Velasco-Villa (2002).

Proof. Since the algorithm is constructive, we only need to prove the necessity. To this end, note that if the system admits the form (9) with $A_i = 0$, $C_i = 0$ for $i \in [1, s]$, then at step k.

$$\psi_k = y^{(n)}(t) - \sum_{j=1}^{k} \Phi_j^{k-j}(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}),$$

consequently

$$\omega_k = \sum_{i=0}^{s} \frac{\partial \Phi_k(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})}{\partial y(t-i)} dy(t-i) + \sum_{i=0}^{s} \frac{\partial \Phi_k(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})}{\partial u(t-i)} du(t-i) = d\Phi_k(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}).$$

which proves the necessity of the exactness of the ω_i $i = 0, \ldots n-2$. $\square$

The following result can now be stated

Theorem 2: *The input-output equation (8) admits a linear state-space representation up to input-output injection of the form (9) if and only if*

i) The system can be represented by an input-output equation of retarded type.
ii) The linearization Algorithm 1 ends with n exact one-forms ω_i

Then the state-space representation is obtained by setting

$$\begin{aligned} z_1(t) &= y(t) \\ z_2(t) &= \dot{y}(t) - \Phi_1(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}) \\ &\vdots \\ z_{n-1}(t) &= y^{n-1}(t) - \sum_{i=1}^{n-1} \Phi_i^{(n-i-1)}(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}) \end{aligned} \tag{32}$$

Proof. Since the procedure is constructive, we only need to prove the necessity. To this end, recall that if the given system represented through its input-output equation can be written in the form (9) then, due to Lemma 1, necessarily the system must admit an input-output equation of retarded type given by

$$dy^{(n)} = -\sum_{i=0}^{n-1} \sigma_i dy^{(i)}(t) + \sum_{k=1}^{n} \sum_{i=0}^{k-1} \sigma_k C(\delta) A^i(\delta) d\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})^{(k-i-1)}$$

which proves the necessity of i). Applying the linearization algorithm, one gets that at the generic step $k \leq n-1$

$$\omega_k = d\Phi_{k+1}(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}) = -\sigma_k dy(t) + \sum_{j=0}^{n-k-1} \sigma_{n-j} C(\delta) A^{n-k-j-1}(\delta) d\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})$$

which shows that the algorithm ends up with n exact differentials ω_k, that is, ii) must be satisfied. With the position (32), one thus gets that the state-space representation is given by

$$\begin{array}{rcl} \dot{z}_1(t) & = & z_2(t) + \Phi_1(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}), \\ \dot{z}_2(t) & = & z_3(t) + \Phi_2(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}), \\ & \vdots & \\ \dot{z}_n(t) & = & \Phi_n(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}), \\ y(t) & = & z_1(t), \end{array} \tag{33}$$

which ends the proof. □

As a corollary, one gets the following result.

Corollary 1: *If the input-output equation* (8) *admits a retarded linear state-space representation, up to input-output injection, then it can be written in the form* (33).

Example 4: This example illustrates the use of Algorithm 1 in the solution of Problem 1. Let us consider the input-output equation

$$\ddot{y}(t) = y(t-1)u(t-3) + y(t)\dot{y}(t-1) + \dot{y}(t)y(t-1) \tag{34}$$

$$\begin{array}{rcl} d\ddot{y}(t) & = & y(t-1)du(t-3) + u(t-3)dy(t-1) + y(t)d\dot{y}(t-1) + \\ & & \dot{y}(t-1)dy(t) + \dot{y}(t)dy(t-1) + y(t-1)d\dot{y}(t) \end{array}$$

Following the algorithm, we define

$$\begin{array}{rcl} \omega_1 & = & y(t)dy(t-1) + y(t-1)dy(t) = d(y(t)y(t-1)) \\ \omega_2 & = & u(t-3)dy(t-1) + y(t-1)du(t) = d(y(t-1)u(t-3)), \end{array}$$

then a realization of the equation (34) is

$$\begin{array}{rcl} \dot{z}_1(t) & = & z_2(t) + y(t)y(t-1) \\ \dot{z}_2(t) & = & y(t-1)u(t-3) \\ y(t) & = & z_1(t) \end{array} \tag{35}$$

which is in the form (9). ◁

4. Equivalence to a linear system up to input-output injections through invertible changes of coordinates

In this Section given the state-space equations of a given system, the problem of its equivalence to a linear system up to input and output equation through an invertible change of coordinates is addressed.

Differently to the previous case it is not always possible to get a state-space equation linear up to input and output injection, and whose linear part is not affected by the delay. This can be easily understood through the following example.

Example 5: Let us consider the dynamical system

$$\begin{array}{rcl} \dot{x}_1(t) & = & x_2(t) + x_1(t-2) + y(t)^2 \\ \dot{x}_2(t) & = & y(t)y(t-1) \\ y(t) & = & x_1(t)x_1(t-1) \end{array} \tag{36}$$

The given system is already in the form (9). There doesn't exists an invertible change of coordinates which transforms the given system into the form (9) with $A_i = 0$, $C_i = 0$ for $i \in [1, s]$.

Examples like Example 5 show that weaker conditions are needed for testing the existence of a solution for the problem of linearization with delays. In the rest of this Section, necessary and sufficient conditions are discussed for the existence of a solution to this problem.

Now the solution of Problem 2 is discussed. To this end, an Algorithm is proposed which does not need the computation of the input-output representation but is based on the normalized covector $\lambda(x, \bar{u}, \delta)$ satisfying Lemma 3.

Algorithm 2:
Let $\lambda(\mathbf{x}, \bar{\mathbf{u}}, \delta) := [1, \chi^0_{n-1}, \cdots, \chi^0_0, \mu^0_{n-1}, \cdots, \mu^0_0]$ be a normalized covector satisfying Lemma 3.
Set

$$\Psi_1 := -\sum_{i=0}^{n-1} \chi^0_i(\mathbf{x}, \bar{\mathbf{u}}, \delta) dy^{(i)}(t) - \sum_{i=0}^{n-1} \mu^0_i(\mathbf{x}, \bar{\mathbf{u}}, \delta) du^{(i)}(t). \tag{37}$$

and set

$$dh_0 := dH(\mathbf{x}_{[s]}) \tag{38}$$

STEP 1. Set $\omega_1 := -\chi^0_{n-1}(\mathbf{x}, \bar{\mathbf{u}}, \delta) dy(t) - \mu^0_{n-1}(\mathbf{x}, \bar{\mathbf{u}}, \delta) du(t)$
Check: $d\omega_1 = 0$?
NO: Stop, YES: Compute $\Phi_1(\mathbf{x}, \mathbf{u}, \delta)$ such that $\omega_1 = d\Phi_1(\mathbf{x}, \mathbf{u}, \delta)$, and set

$$dh_1(x) := d\dot{H}(x(t)) - d\Phi_1(\mathbf{x}, \mathbf{u}, \delta) \tag{39}$$

and

$$\Psi_2 \;\; := \;\; -\textstyle\sum_{i=0}^{n-2} \chi^1_i(\mathbf{x}, \bar{\mathbf{u}}, \delta) dy^{(i)}(t) - \sum_{i=0}^{n-2} \mu^1_i(\mathbf{x}, \bar{\mathbf{u}}, \delta) du^{(i)}(t). \tag{40}$$

with

$$\begin{array}{rcl} \chi^1_i(\mathbf{x}, \bar{\mathbf{u}}, \delta) & = & \chi^0_i(\mathbf{x}, \bar{\mathbf{u}}, \delta) - \begin{pmatrix} n-1 \\ n-1-i \end{pmatrix} (\chi^0_{n-1}(\mathbf{x}, \bar{\mathbf{u}}, \delta))^{(n-1-i)} \\ \mu^1_i(\mathbf{x}, \bar{\mathbf{u}}, \delta) & = & \mu^0_i(\mathbf{x}, \bar{\mathbf{u}}, \delta) - \begin{pmatrix} n-1 \\ n-1-i \end{pmatrix} (\mu^0_{n-1}(\mathbf{x}, \bar{\mathbf{u}}, \delta))^{(n-1-i)} \end{array}$$

STEP k. Set $\omega_k := -\chi^{k-1}_{n-k}(\mathbf{x}, \bar{\mathbf{u}}, \delta) dy(t) - \mu^{k-1}_{n-k}(\mathbf{x}, \bar{\mathbf{u}}, \delta) du(t)$
Check: $d\omega_k = 0$?

NO: Stop, YES: Compute $\Phi_k(\mathbf{x},\mathbf{u},\delta)$ such that $\omega_k = d\Phi_k(\mathbf{x},\mathbf{u},\delta)$. Set

$$dh_k(x) := dH(x(t))^{(k)} - \sum_{j=0}^{k-1} d\Phi_{k-j}(\mathbf{x},\bar{\mathbf{u}},\delta)^{(j)}, \tag{41}$$

and

$$\Psi_{k+1} \;\; := \;\; -\textstyle\sum_{i=0}^{n-k-1} \chi_i^k(\mathbf{x},\bar{\mathbf{u}},\delta)dy^{(i)}(t) - \sum_{i=0}^{n-k-1} \mu_i^k(\mathbf{x},\bar{\mathbf{u}},\delta)du^{(i)}(t), \tag{42}$$

with

$$\begin{aligned}
\chi_i^k(\mathbf{x},\bar{\mathbf{u}},\delta) &= \chi_i^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta) - \binom{n-k}{n-k-i}(\chi_{n-k}^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta))^{(n-k-i)} \\
\mu_i^k(\mathbf{x},\bar{\mathbf{u}},\delta) &= \mu_i^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta) - \binom{n-k}{n-k-i}(\mu_{n-k}^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta))^{(n-k-i)}
\end{aligned}$$

$\triangleleft$

Proposition 2: *Assume that for ω_i in Algorithm 2 is an exact differential, $i = 1, \cdots k$. Then*

i) $\omega_i = d\Phi_i(y(t), \cdots, y(t-s), u(t), \cdots, u(t-s))$
ii) $\Psi_i = y^{(n)}(t) - \sum_{l=1}^{i-1} \Phi_l^{(n-l)}(\bar{\mathbf{y}},\bar{\mathbf{u}})$

Proof. By construction

$$\omega_i = -\chi_{n-i}^{i-1}(x,\bar{u},\delta)dy - \mu_{n-i}^{i-1}(x,\bar{u},\delta)du$$

Since ω_i is an exact differential, then necessarily it is only a function of $y(t), u(t)$ and their delays, which proves i).

As for ii), the proof is iterative. Ψ_1 is computed starting from the normalized covector λ and thus $\Psi_1 = y^{(n)}(t)$. Assume that ii) is true from k, then

$$\omega_k = -\chi_{n-k}^{k-1}(x,\bar{u},\delta)dy - \mu_{n-k}^{k-1}(x,\bar{u},\delta)du = d\Phi_k(y,u)$$

Accordingly

$$d\Phi_k^{(n-k)} = -\sum_{\ell=0}^{n-k} \binom{n-k}{\ell} \left[\left(\chi_{n-k}^{k-1}(x,\bar{u},\delta)\right)^{(\ell)} dy^{(n-k-\ell)} - \left(\mu_{n-k}^{k-1}(x,\bar{u},\delta)\right)^{(\ell)} du^{(n-k-\ell)} \right].$$

It follows that

$$
\begin{aligned}
\Psi_{k+1} &= -\sum_{i=0}^{n-k-1} \chi_i^k(\mathbf{x},\bar{\mathbf{u}},\delta)dy^{(i)}(t) - \sum_{i=0}^{n-k-1} \mu_i^k(\mathbf{x},\bar{\mathbf{u}},\delta)du^{(i)}(t) \\
&= -\sum_{i=0}^{n-k-1} \left(\chi_i^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta) - \binom{n-k}{n-k-i}(\chi_{n-k}^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta))^{(n-k-i)}\right) dy^{(i)} \\
&\quad -\sum_{i=0}^{n-k-1} \left(\mu_i^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta) - \binom{n-k}{n-k-i}(\mu_{n-k}^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta))^{(n-k-i)}\right) du^{(i)} \\
&= \Psi_k + \sum_{i=0}^{n-k} \left(\binom{n-k}{n-k-i}(\chi_{n-k}^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta))^{(n-k-i)}dy^{(i)} + \binom{n-k}{n-k-i}(\mu_{n-k}^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta))^{(n-k-i)}du^{(i)}\right) \\
&= \Psi_k - d\Phi_k^{(n-k)} = y^{(n)}(t) - \sum_{j=1}^{k} d\Phi_j^{(n-j)}
\end{aligned}
$$

□

Accordingly, the following result can be stated

Theorem 3: *Problem 2 is solvable if and only if*

i) the system admits an input–output equation of retarded type
ii) The one-forms ω_i defined by Algorithm 2 are exact for all $i = 1, \dots, n$.
iii) There exists a polymodular matrix $T(\mathbf{x}_{[p,j]}, \delta)$ and a full rank matrix $Q(\delta) \in \mathbb{R}[\delta]$ such that $Q(\delta)T(\mathbf{x}_{[p,j]}, \delta)dx(t+p) = P(\mathbf{x}_{[s]}, \delta)dx(t) = (dh_0^T, \dots, dh_{n-1}^T)^T$ from Algorithm 2.

Proof. From Lemma 1, it follows that system (2) is linearizable by additive input-output injections only if $i)$ stands. Assume now that the system is already in the form (9), and apply Algorithm 2. Because of its structure, the differential of its input-output equation is given by (24), that is,

$$
dy^{(n)} = -\sum_{i=0}^{n-1} \sigma_i dy^{(i)}(t) + \sum_{k=1}^{n}\sum_{i=0}^{k-1} \sigma_k C(\delta)A^i(\delta)d\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})^{(k-i-1)},
$$

Accordingly one gets that, starting from $dh_0 = C(\delta)dz$, at the first step

$$
\begin{aligned}
\omega_1 &= -\sigma_{n-1}dy + C(\delta)d\varphi = d\Phi_1 \\
dh_1 &= C(\delta)A(\delta)dz + \sigma_{n-1}dy
\end{aligned}
$$

and at step k

$$
\begin{aligned}
\omega_k &= -\sigma_{n-k}dy + \sum_{j=0}^{k-1} \sigma_{n-k+1+j}C(\delta)A^j(\delta)d\varphi = d\Phi_k \\
dh_k &= \sum_{j=0}^{k} \sigma_{n-j}C(\delta)A(\delta)^j dz(t)
\end{aligned}
$$

which proves that the ω_i's must be exact one-forms. Furthermore, in the x coordinates one thus gets

$$d\hat{\mathbf{h}} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \sigma_{n-1} & 1 & 0 & \dots & 0 \\ \sigma_{n-2} & \sigma_{n-1} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_1 & \sigma_2 & \sigma_3 & \dots & 1 \end{pmatrix} \begin{pmatrix} C(\delta) \\ C(\delta)A(\delta) \\ C(\delta)A(\delta)^2 \\ \vdots \\ C(\delta)A(\delta)^{n-1} \end{pmatrix} dz(t) = Q(\delta)dz(t). \tag{43}$$

Since by assumption $z(t) = \phi(x(t+p), \cdots, x(t-j))$, $dz(t) = T(\mathbf{x}_{[p,j]}, \delta)dx(t+p)$, we have that

$$d\hat{\mathbf{h}} = Q(\delta)T(\mathbf{x}_{[p,j]}, \delta)dx(t+p) = P(\mathbf{x}_{[0,s]}, \delta)dx(t)$$

which proves the necessity of iii).

For the sufficiency, according to iii) there exists $z(t) = \phi(x(t+p), \cdots, x(t-j))$, such that $dz(t) = T(\mathbf{x}_{[p,j]}, \delta)dx(t+p)$. Since conditions $i)$ and $ii)$ are verified, in the z–coordinates the output of the Algorithm 2 is given by

$$\begin{pmatrix} dy \\ d\dot{y} - d\varphi_1(y,u) \\ dy^{(2)} - d\dot{\varphi}_1(y,u) - d\varphi_2(y,u) \\ \vdots \\ dy^{(n-1)} - d\varphi_1^{(n-2)}(y,u) - \dots - d\varphi_{n-1}(y,u) \end{pmatrix} = Q(\delta)dz. \tag{44}$$

Differentiating equation (44) and denoting by $q_i(\delta)$ the i-th row of the matrix $Q(\delta)$

$$\begin{aligned} Q(\delta)d\dot{z} &= \begin{pmatrix} d\dot{y} \\ d\ddot{y} - d\dot{\varphi}_1(y,u) \\ dy^{(3)} - d\ddot{\varphi}_1(y,u) - d\dot{\varphi}_2(y,u) \\ \vdots \\ dy^{(n)} - d\varphi_1^{(n-1)}(y,u) - \dots - d\dot{\varphi}_{n-1}(y,u) \end{pmatrix} = \begin{pmatrix} q_2(\delta)dz + d\varphi_1 \\ \vdots \\ q_n(\delta)dz + d\varphi_{n-1} \\ d\varphi_n \end{pmatrix} \\ &= \begin{pmatrix} q_2(\delta) \\ \vdots \\ q_n(\delta) \\ 0 \end{pmatrix} dz + d\varphi = \bar{A}(\delta)dz + d\varphi \end{aligned} \tag{45}$$

Multiplying by the adjunct matrix $Q^{(a)}(\delta)$ we get

$$\begin{pmatrix} \bar{q}_1(\delta) & & \\ & \ddots & \\ & & \bar{q}_n(\delta) \end{pmatrix} d\dot{z} = \hat{A}(\delta)dz + d\hat{\varphi}$$

Using the identity of polynomials one thus gets that

$$d\dot{z} = A(\delta)dz + d\Psi(y,u)$$

which ends the proof. $\square$

5. Conclusions and open questions

In the present paper necessary and sufficient conditions under which a nonlinear time-delay system can be transformed into a linear time-delay system up to input and output injection were derived. Moreover, it is proven that if an observable system has an input-output equation of retarded type, a normalized left annihilator, as defined in Section 2.3, exists. This normalized vector allowed to develop a linearization algorithm that does not need the explicit computation of the input-output equation. The linearization algorithm was used to settle conditions for the existence of solutions for Problems 1 and 2. Sufficient and necessary conditions, established by Theorem 3, for the equivalence up to input and output injection to a linear system with delays were presented based on the computation of an invertible change of coordinates. The results presented can be successfully used in the observer design context. Further investigation will concern the wider class of neutral systems.

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