

# ON THE $L^2$ - $\bar{\partial}$ -COHOMOLOGY OF CERTAIN COMPLETE KÄHLER METRICS

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ABSTRACT. Let  $V$  be a compact and irreducible complex space of complex dimension  $v$  whose regular part is endowed with a complete Hermitian metric  $h$ . Let  $\pi : M \rightarrow V$  be a resolution of  $V$ . Under suitable assumptions on  $h$  we prove that

$$H_{2,\bar{\partial}}^{v,q}(\text{reg}(V), h) \cong H_{\bar{\partial}}^{v,q}(M), \quad q = 0, \dots, v.$$

Then we show that the previous isomorphism applies to the case of Saper-type Kähler metrics, see [10], and to the case of complete Kähler metrics with finite volume and pinched negative sectional curvatures.

**Keywords:** Saper-type Kähler metrics, complete Kähler metrics with finite volume and pinched negative sectional curvatures,  $L^2$ -Dolbeault cohomology, complex spaces, resolution of singularities.

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## CONTENTS

Introduction and main results	1
1. $d$ -bounded Kähler forms and $L^2$ - $\bar{\partial}$ -cohomology	3
2. Applications	10
2.1. Saper metrics	10
2.2. Further remarks on Saper metrics	12
2.3. Negatively curved Kähler manifolds with finite volume	13
References	14

## INTRODUCTION AND MAIN RESULTS

Let  $V$  be a complex projective variety in  $\mathbb{C}\mathbb{P}^m$  of complex dimension  $v$ . Let  $\text{reg}(V)$  be its regular part and let  $\text{sing}(V)$  be its singular locus. More generally, let  $N$  be a compact Kähler manifold of complex dimension  $n$  with Kähler form  $\omega$  and let  $V$  be an analytic subvariety in  $N$  of complex dimension  $v$ . One of the questions raised in [5] concerns the existence of a complete Kähler metric on  $\text{reg}(V)$  whose  $L^2$ -cohomology is isomorphic to the middle perversity intersection cohomology of  $V$ . This problem has been investigated by Saper in [22] who provided an affirmative answer in the setting of isolated singularities. Labeling with  $g_S$  the Kähler metric constructed by Saper, a problem related to the previous one, is to understand the  $L^2$ - $\bar{\partial}$ -cohomology of  $g_S$ . For the case of  $(v, q)$ -forms this latter problem has been addressed again by Saper in the setting of isolated singularities. He showed in [22] that

$$(1) \quad H_{2,\bar{\partial}}^{v,q}(\text{reg}(V), g_S) \cong H_{\bar{\partial}}^{v,q}(M)$$

where  $\pi : M \rightarrow V$  is any resolution of  $V$ . An analogous question, but for the  $L^2\bar{\partial}$ -cohomology of the incomplete Kähler metric  $g$  on  $\text{reg}(V)$  induced by the Kähler metric on  $N$ , has been considered by Pardon [19], Haskell [15], Brüning-Peyerimhoff-Schröder [4] and Pardon-Stern [20] who finally solved the MacPherson conjecture [17] by showing, more generally, that

$$H_{2,\bar{\partial}_{\min}}^{v,q}(\text{reg}(V), g) \cong H_{\bar{\partial}}^{v,q}(M).$$

Subsequently, inspired by the work of Saper [22], Grant Melles and Milman have constructed a new family of complete Kähler metrics on  $\text{reg}(V)$ , see [10] and [11], without any assumption on the singular set of  $V$ . The complete Kähler metrics built by Grant Melles and Milman on  $\text{reg}(V)$  are called *Saper-type Kähler metrics*. The purpose of this paper is to investigate the  $L^2\bar{\partial}$ -cohomology groups of various complete Hermitian metrics that are defined on the regular part of a compact and irreducible complex space; Saper-type Kähler metrics are examples of such Hermitian metrics. The paper is structured in the following way. In the first section we provide a general result in the setting of compact and irreducible complex space whose regular part is endowed with a complete Hermitian metric. Our theorem reads as follow:

**Theorem 1.** *Let  $V$  be a compact and irreducible complex space of complex dimension  $v$ . Let  $\pi : M \rightarrow V$  be a resolution of  $V$  with  $D := \pi^{-1}(\text{sing}(V))$  a normal crossings divisor of  $M$ . Let  $h$  be a complete Hermitian metric on  $\text{reg}(V)$  and let  $\sigma$  be the complete Hermitian metric on  $M \setminus D$  defined as  $\sigma := (\pi|_{M \setminus D})^*h$ . Assume that:*

- $\pi_*\mathcal{C}_{D,\sigma}^{v,q}$  is a fine sheaf for each  $q = 0, \dots, v$ ,
- for each  $p \in \text{sing}(V)$  there is an open neighborhood  $U$  of  $p$  and a  $d$ -bounded Kähler metric  $g_U$  on  $\text{reg}(U)$  such that  $h|_U$  and  $g_U$  are quasi-isometric.

Then we have the following isomorphism for each  $q = 0, \dots, v$ :

$$H_{2,\bar{\partial}}^{v,q}(\text{reg}(V), h) \cong H_{\bar{\partial}}^{v,q}(M).$$

The sheaves  $\mathcal{C}_{D,\sigma}^{v,q}$  on  $M$  are obtained by sheafification from the presheaves  $C_{D,\sigma}^{v,q}$  that assign to an open set  $U \subset M$  the maximal domain of  $\bar{\partial}_{v,q,\max}$  on  $(U \setminus U \cap D, \sigma|_{U \setminus U \cap D})$ . For more on this definition and for the notion of  $d$ -bounded Kähler metric appearing in the second condition we refer the reader to (1.2) and Def. 1.1 respectively. As a consequence of Th. 1 we obtain that

$$\bar{\partial}_v + \bar{\partial}_v^* : L^2\Omega^{v,\bullet}(\text{reg}(V), h) \rightarrow L^2\Omega^{v,\bullet}(\text{reg}(V), h)$$

is a Fredholm operator on its domain endowed with the graph norm, see (1.8). Moreover, under suitable assumptions, the groups  $H_{2,\bar{\partial}}^{v,q}(\text{reg}(V), h)$  are invariant under bimeromorphic maps.

The second section of this paper is devoted to some applications of Th. 1. Our goal here is to provide some examples of complete Kähler metrics that obey the hypothesis of Th. 1. As a first application we discuss the case of Saper-type Kähler metrics. Our main result is the following

**Theorem 2.** *Let  $M$  be a compact Kähler manifold with Kähler form  $\omega$  and let  $V$  be an analytic subvariety of  $M$  with complex dimension  $v$ . Denote now by  $\pi : \tilde{V} \rightarrow V$  a resolution of  $V$ . Finally let  $g_S$  be a Saper-type metric on  $\text{reg}(V)$  as in [10]. Then the following isomorphism holds:*

$$(2) \quad H_{2,\bar{\partial}}^{v,q}(\text{reg}(V), g_S) \cong H_{\bar{\partial}}^{v,q}(\tilde{V})$$

for every  $q = 0, \dots, v$ .

The second example that we present are complete Kähler manifolds with finite volume and pinched negative sectional curvatures. According to a result of Siu-Yau [23], a complex manifold that carries a Kähler metric with these properties is biholomorphic to the regular part of a complex

projective variety with only isolated singularities. In this setting, using our Th. 1, we prove the following result:

**Theorem 3.** *Let  $(M, h)$  be a complete Kähler manifold of complex dimension  $m$  with finite volume. Assume that the sectional curvatures of  $(M, h)$  satisfies  $-b^2 \leq \text{sec}_h \leq -a^2$  for some constants  $0 < a \leq b$ . Let  $V \subset \mathbb{C}\mathbb{P}^n$  be the Siu–Yau compactification of  $M$  and let  $\pi : \tilde{V} \rightarrow V$  be a resolution of  $V$ . Then we have the following isomorphism for each  $q = 0, \dots, m$*

$$H_{2, \bar{\partial}}^{m, q}(M, h) \cong H_{\bar{\partial}}^{m, q}(\tilde{V}).$$

In the second section we also collect various consequences of Th. 2 and Th. 3.

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### 1. $d$ -BOUNDED KÄHLER FORMS AND $L^2$ - $\bar{\partial}$ -COHOMOLOGY

We start with the following remarks about our notation. Let  $(M, h)$  be a complex Hermitian manifold. For any  $(p, q)$  the maximal extension of  $\bar{\partial}_{p, q}$ , labeled by  $\bar{\partial}_{p, q, \max} : L^2\Omega^{p, q}(M, h) \rightarrow L^2\Omega^{p, q+1}(M, h)$ , is the closed extension defined in the distributional sense:  $\omega \in \mathcal{D}(\bar{\partial}_{p, q, \max})$  if  $\omega \in L^2\Omega^{p, q}(M, h)$  and  $\bar{\partial}_{p, q}\omega$ , applied in the distributional sense, lies in  $L^2\Omega^{p, q+1}(M, h)$ . The minimal extension of  $\bar{\partial}_{p, q}$ , labeled by  $\bar{\partial}_{p, q, \min} : L^2\Omega^{p, q}(M, h) \rightarrow L^2\Omega^{p, q+1}(M, h)$ , is defined as the graph closure of  $\Omega_c^{p, q}(M)$  in  $L^2\Omega^{p, q}(M, h)$  with respect to graph norm of  $\bar{\partial}_{p, q}$ . It is easy to check that in both cases we get a complex whose corresponding cohomology is denoted by  $H_{2, \bar{\partial}, \max/\min}^{p, q}(M, h)$ . If  $(M, h)$  is complete then it is well known that  $\bar{\partial}_{p, q, \min} = \bar{\partial}_{p, q, \max}$  and we label this unique closed extension simply with  $\bar{\partial}_{p, q} : L^2\Omega^{p, q}(M, h) \rightarrow L^2\Omega^{p, q+1}(M, h)$ . Finally analogous notations and considerations hold for the operator  $\bar{\partial} + \bar{\partial}^t$ . Now we go on with the following definition.

**Definition 1.1.** *Let  $(M, g)$  be a Kähler manifold and let  $\omega$  be the corresponding Kähler form. We will say that  $\omega$  is  $d$ -bounded if there exists a 1-form  $\eta \in L^\infty\Omega^1(M, g)$  such that  $d\eta = \omega$  where  $d\eta$  is understood in the distributional sense.*

**Remark 1.2.** *The notion of  $d$ -bounded Kähler form has been introduced by Gromov in [14]. The definition that we stated above is slightly more general than the original one because we do not require  $\eta$  to be a smooth form.*

**Definition 1.3.** *Let  $(M, g)$  be a Kähler manifold and let  $\omega$  be the corresponding Kähler form. We will say that  $g$  satisfies the Ohsawa condition if there exists a function  $f \in C^\infty(M)$  such that  $\frac{i}{2\pi}\partial\bar{\partial}f = \omega$  with  $\partial f \in L^\infty\Omega^{1, 0}(M, g)$ . See for example [10].*

**Remark 1.4.** *It is clear that if  $g$  satisfies the Ohsawa condition then the corresponding Kähler form  $\omega$  is  $d$ -bounded.*

We recall now from [14] the following important result.

**Theorem 1.5.** *In the setting of definition 1.1. Assume moreover that  $(M, g)$  is complete and let  $m$  be the complex dimension of  $m$ . Then for any  $(p, q)$  with  $p + q \neq m$  we have*

$$H_{2, \bar{\partial}}^{p, q}(M, g) = 0.$$

*Proof.* When  $\eta$  is smooth this theorem is Th.1.4.A in [14]. A careful look at the arguments used there shows that the same proof applies also in our slightly more general setting. Finally we point out that if  $g$  satisfies the Ohsawa condition then the theorem had been already proved in [7] and [18].  $\square$

Furthermore, concerning  $d$ -bounded Kähler metrics, we have also the following basic properties.

**Proposition 1.6.** *Let  $f : M \rightarrow N$  be an holomorphic immersion between complex manifolds. Let  $g$  be a  $d$ -bounded Kähler metric on  $N$ . Then  $f^*g$  is a  $d$ -bounded Kähler metric on  $M$ .*

*Proof.* Let  $\omega$  be the Kähler form of  $g$  and let  $h := f^*g$ . It is easy to check that in general the pullback through a smooth map commutes with the distributional action of the de Rham differential. Therefore the above statement follows immediately noticing that if  $\eta \in L^\infty \Omega^1(N, g)$  with  $d\eta = \omega$  then  $d(f^*\eta) = f^*\omega$  and  $|f^*\eta|_h \leq |\omega|_g$  where  $|\cdot|_h$  and  $|\cdot|_g$  denote respectively the pointwise norm on  $T^*M \otimes \mathbb{C}$  induced by  $h$  and  $g$ .  $\square$

**Proposition 1.7.** *Let  $M$  be a complex manifold and let  $g$  and  $h$  be two  $d$ -bounded Kähler metrics on  $M$ . Then the Kähler metric  $\rho := g + h$  is  $d$ -bounded as well.*

In order to prove the above proposition we need the following elementary result.

**Proposition 1.8.** *Let  $M$  be a manifold and let  $g_1$  and  $g_2$  be two Riemannian metrics on  $M$  such that  $g_2 \leq g_1$ . Let  $g_1^*$  be the metric that  $g_1$  induces on  $T^*M$  and analogously let  $g_2^*$  be the metric that  $g_2$  induces on  $T^*M$ . Then we have  $g_1^* \leq g_2^*$ .*

*Proof.* Let  $A \in \text{End}(TM)$  such that  $g_2(\cdot, \cdot) = g_1(A \cdot, \cdot)$ . Then, for each  $p \in M$ ,  $A_p : T_p M \rightarrow T_p M$  is positive, symmetric with respect to  $g_1$  and its eigenvalues are bounded above by 1. Let  $A^{-1}$  be the inverse of  $A$  and let  $(A^{-1})^t \in \text{End}(T^*M)$  be the transposed endomorphism of  $A^{-1}$ . An easy calculation of linear algebra shows that  $g_2^*(\cdot, \cdot) = g_1^*((A^{-1})^t \cdot, \cdot)$ . Now, for each  $p \in M$ ,  $(A^{-1})_p^t : T_p^* M \rightarrow T_p^* M$  is positive, symmetric with respect to  $g_1^*$  and its eigenvalues are bounded below by 1. This in turn implies immediately that  $g_1^* \leq g_2^*$  as required.  $\square$

Now we give a proof of Prop. 1.7

*Proof.* Let  $\omega$  be the Kähler form of  $g$  and analogously let  $\tau$  be the Kähler form of  $h$ . According to the assumptions there exist 1-forms  $\alpha \in L^\infty \Omega^1(M, g)$ ,  $\beta \in L^\infty \Omega^1(M, h)$  such that  $d\alpha = \omega$  and  $d\beta = \tau$ . Let us label by  $\rho^*$ ,  $g^*$  and  $h^*$  the metrics on  $T^*M \otimes \mathbb{C}$  induced respectively by  $\rho$ ,  $g$  and  $h$ . Clearly  $g \leq \rho$  and  $h \leq \rho$ . Then, using Prop. 1.8, we have

$$(\rho^*(\alpha + \beta, \alpha + \beta))^{\frac{1}{2}} \leq (\rho^*(\alpha, \alpha))^{\frac{1}{2}} + (\rho^*(\beta, \beta))^{\frac{1}{2}} \leq (g^*(\alpha, \alpha))^{\frac{1}{2}} + (h^*(\beta, \beta))^{\frac{1}{2}}$$

Since  $\alpha \in L^\infty \Omega^1(M, g)$  and  $\beta \in L^\infty \Omega^1(M, h)$  we can conclude that  $\alpha + \beta \in L^\infty \Omega^1(M, \rho)$  as desired.  $\square$

We go on by spending a few words about resolution of singularities. We recall only what is strictly necessary for our purposes and we refer to the seminal work [16] and also to [2] for in-depth treatments of this topic. For a throughout discussion about complex spaces we refer to the monographs [8] and [12].

Consider a compact and irreducible complex space  $V$ . Then, thanks to the fundamental work of Hironaka, we know that the singularities of  $V$  can be resolved. More precisely there exists a compact complex manifold  $M$ , a divisor with only normal crossings  $D \subset M$ , a surjective and holomorphic map  $\pi : M \rightarrow V$  such that  $\pi^{-1}(\text{sing}(V)) = D$  and  $\pi|_{M \setminus D} : M \setminus D \rightarrow \text{reg}(V)$  is a biholomorphism. Moreover if  $V \subset N$  is an analytic subvariety of a compact complex manifold then there exists a compact complex manifold  $M$ , a compact complex submanifold  $Z \subset M$ , a surjective holomorphic map  $\pi : M \rightarrow N$  and a divisor with only normal crossings  $D \subset M$  such that  $\pi^{-1}(\text{sing}(V)) = D$ ,  $\pi|_{M \setminus D} : M \setminus D \rightarrow N \setminus \text{sing}(V)$  is a biholomorphism and  $\pi|_{Z \setminus (Z \cap D)} : Z \setminus (Z \cap D) \rightarrow V \setminus \text{sing}(V)$  is a biholomorphism. The latter is the so-called embedded desingularization.

We introduce now some presheaves and the corresponding sheaves arising by sheafification. Let  $M$  be a compact complex manifold,  $D \subset M$  a divisor with only normal crossings and  $g$  any Hermitian metric on  $M \setminus D$ . Consider the presheaves  $C_{D,g}^{p,q}$  on  $M$  given by the assignments

$$(1.1) \quad C_{D,g}^{p,q}(U) := \{\mathcal{D}(\bar{\partial}_{p,q,\max}) \text{ on } (U \setminus U \cap D, g|_{U \setminus U \cap D})\};$$

in other words to every open subset  $U$  of  $M$  we assign the maximal domain of  $\bar{\partial}_{p,q}$  over  $U \setminus (U \cap D)$  with respect to the Hermitian metric  $g|_{U \setminus U \cap D}$ . The sheafification of  $C_{D,g}^{p,q}$  is denoted by  $\mathcal{C}_{D,g}^{p,q}$  and its sections over an open subset  $U \subset M$  are

$$(1.2) \quad \mathcal{C}_{D,g}^{p,q}(U) := \{s \in L_{\text{loc}}^2 \Omega^{p,q}(U \setminus U \cap D, g|_{U \setminus U \cap D}) \text{ such that for each } p \in U \text{ there exists an open neighborhood } W \text{ with } p \in W \subset U \text{ such that } s|_{W \setminus W \cap D} \in \mathcal{D}(\bar{\partial}_{p,q,\max}) \text{ on } (W \setminus W \cap D, g|_{W \setminus W \cap D})\}.$$

We have now all the ingredients to state the main result of this section.

**Theorem 1.9.** *Let  $V$  be a compact and irreducible complex space of complex dimension  $v$ . Let  $\pi : M \rightarrow V$  be a resolution of  $V$  with  $D := \pi^{-1}(\text{sing}(V))$  a normal crossings divisor in  $M$ . Let  $h$  be a complete Hermitian metric on  $\text{reg}(V)$  and let  $\sigma$  be the complete Hermitian metric on  $M \setminus D$  defined as  $\sigma := (\pi|_{M \setminus D})^* h$ . Assume that:*

- $\pi_* \mathcal{C}_{D,\sigma}^{v,q}$  is a fine sheaf for each  $q = 0, \dots, v$ ,
- for each  $p \in \text{sing}(V)$  there is an open neighborhood  $U$  and a  $d$ -bounded Kähler metric  $g_U$  on  $\text{reg}(U)$  such that  $h|_U$  and  $g_U$  are quasi-isometric.

Then we have the following isomorphism for each  $q = 0, \dots, v$ :

$$H_{2,\bar{\partial}}^{v,q}(\text{reg}(V), h) \cong H_{\bar{\partial}}^{v,q}(M).$$

Before tackling the proof we recall the following properties.

**Proposition 1.10.** *Let  $M$  be a complex manifold of complex dimension  $m$ , let  $(E, \rho)$  be a Hermitian vector bundle on  $M$  and let  $g$  and  $h$  be two Hermitian metrics on  $M$ . Then we have an equality of Hilbert spaces*

$$L^2 \Omega^{m,0}(M, E, g) = L^2 \Omega^{m,0}(M, E, h).$$

Assume now that  $cg \geq h$  for some  $c > 0$ . Then for each  $q = 1, \dots, m$  there exists a constant  $\xi_q > 0$  such that for every  $s \in \Omega_c^{m,q}(M, E)$  we have

$$(1.3) \quad \|s\|_{L^2 \Omega^{m,q}(M, E, g)}^2 \leq \xi_q \|s\|_{L^2 \Omega^{m,q}(M, E, h)}^2.$$

Therefore the identity  $\Omega_c^{m,q}(M, E) \rightarrow \Omega_c^{m,q}(M, E)$  induces a continuous inclusion

$$L^2 \Omega^{m,q}(M, E, h) \hookrightarrow L^2 \Omega^{m,q}(M, E, g)$$

for each  $q = 1, \dots, m$ .

*Proof.* The statement follows by the computations carried out in [11] pag. 145.  $\square$

We are now in the position to prove Th. 1.9.

*Proof.* Let  $\pi : M \rightarrow V$  be a resolution of  $V$ . Let  $\mathcal{K}_M$  be the canonical sheaf of  $M$ , that is, the sheaf whose sections over any open subset  $U$  of  $M$  are the holomorphic  $(n, 0)$ -forms over  $U$ . Let us consider the following sheaf  $\mathcal{K}_V := \pi_*\mathcal{K}_M$ . This is the so-called Grauert-Riemenschneider canonical sheaf introduced in [13]. By the Takegoshi vanishing theorem, see [24], we get that  $H^q(M, \mathcal{K}_M) \cong H^q(V, \mathcal{K}_V)$  for each  $q = 0, \dots, v$ , see for instance [21] for the details. We are therefore left with the task of showing that  $H^q(V, \mathcal{K}_V) \cong H_{2, \bar{\partial}}^{n, q}(\text{reg}(V), h)$  for each  $q = 0, \dots, n$ . To this end consider the complex of sheaves  $\{\pi_*\mathcal{C}_{D, \sigma}^{v, q}, q \geq 0\}$ , see (1.2), whose morphisms are those induced by the distributional action of  $\bar{\partial}_{v, q}$ . It is clear that the cohomology groups of the complex given by the global sections of  $\{\pi_*\mathcal{C}_{D, \sigma}^{v, q}, q \geq 0\}$ , that is

$$(1.4) \quad 0 \rightarrow \pi_*\mathcal{C}_{D, \sigma}^{n, 0}(V) \rightarrow \dots \rightarrow \pi_*\mathcal{C}_{D, \sigma}^{n, q}(V) \rightarrow \dots \rightarrow \pi_*\mathcal{C}_{D, \sigma}^{n, n}(V) \rightarrow 0$$

are  $H_{2, \bar{\partial}}^{n, q}(\text{reg}(V), h)$ ,  $q = 0, \dots, n$ . Therefore our goal is to show that the complex  $\{\pi_*\mathcal{C}_{D, \sigma}^{v, q}, q \geq 0\}$  is a fine resolution of  $\mathcal{K}_V$ . Since we assumed that  $\pi_*\mathcal{C}_{D, \sigma}^{v, q}$  is a fine sheaf for each  $q = 0, \dots, v$  we have only to prove that  $\{\pi_*\mathcal{C}_{D, \sigma}^{v, q}, q \geq 0\}$  is a resolution of  $\mathcal{K}_V$ . We start to tackle this problem by showing that  $\{\pi_*\mathcal{C}_{D, \sigma}^{v, q}, q \geq 0\}$  is an exact sequence of sheaves. Let  $p$  be any point in  $V$ . It is clear that if  $p \in \text{reg}(V)$  then the induced sequence at the level of stalks,  $\{(\pi_*\mathcal{C}_{D, \sigma}^{v, q})_p, q \geq 0\}$ , is exact. Hence we can assume that  $p \in \text{sing}(V)$ . As a first step in order to show that  $\{(\pi_*\mathcal{C}_{D, \sigma}^{v, q})_p, q \geq 0\}$  is exact for any  $p \in \text{sing}(V)$  we need to introduce an auxiliary complete Kähler metric on a neighborhood of  $p$  that satisfies the assumptions of Th. 1.5. This is done as follows.

According to the assumptions we know that there exists a sufficiently small open neighbourhood  $U$  of  $p$  such that the restriction of  $h$  to the regular part of  $U$  is quasi-isometric to a Kähler metric  $g_U$  which is  $d$ -bounded. Now, taking  $U$  even smaller if necessary, we can assume that there exists a positive constant  $c$ , an integer  $n > v$  and a proper holomorphic embedding  $\phi : U \rightarrow B(0, c)$  where  $B(0, c)$  is the ball in  $\mathbb{C}^n$  centered in 0 with radius  $c$ . Let  $\psi : B(0, c) \rightarrow \mathbb{R}$  be defined as  $\psi := -(\log(c^2 - |z|^2))$  and let  $g$  be the Kähler metric on  $B(0, c)$  whose Kähler form is given by  $\sqrt{-1}\partial\bar{\partial}\psi$ . It is easy to check that  $g$  satisfies the Ohsawa condition and therefore in particular is  $d$ -bounded, see for instance [20] pag. 613. This in turn implies that  $\rho_U := (\phi|_{\text{reg}(U)})^*g$  is a  $d$ -bounded Kähler metric on  $U$ , see Prop. 1.6. Now we introduce the following Kähler metric on  $\text{reg}(U)$ :

$$(1.5) \quad \gamma_U := \rho_U + g_U$$

Next we check that  $\gamma_U$  satisfies the hypothesis of Th. 1.5. We begin by proving that  $\gamma_U$  is complete. According to Gordon's Theorem, [9] Theorem 2, this is equivalent to showing the existence of a positive, smooth and proper function  $f : \text{reg}(U) \rightarrow \mathbb{R}$  with bounded gradient. Let  $b : B(0, c) \rightarrow \mathbb{R}$  be a smooth function which satisfies the condition of Gordon's Theorem with respect to the complete Kähler metric associated to the  $(1, 1)$ -form  $\sqrt{-1}\partial\bar{\partial}\psi$ . Let  $\beta_U : \text{reg}(U) \rightarrow \mathbb{R}$  be defined as  $(b \circ \phi)|_{\text{reg}(U)}$ . Let  $\tau : \text{reg}(V) \rightarrow \mathbb{R}$  be a smooth function which satisfies the condition of Gordon's Theorem with respect to the metric  $h$ . Let us label by  $\tau_U$  the restriction of  $\tau$  to  $\text{reg}(U)$ . We claim that  $\beta_U + \tau_U$  is smooth, proper and with bounded gradient with respect to  $\gamma_U$ . We first show that  $\beta_U + \tau_U$  has bounded gradient with respect to  $\gamma_U$ . Labeling by  $\gamma_U^*$ ,  $g_U^*$  and  $\rho_U^*$  the metrics induced respectively by  $\gamma_U$ ,  $g_U$  and  $\rho_U$  on  $T^*\text{reg}(U)$ , our task is equivalent to showing that  $\beta_U + \tau_U$  has bounded differential with respect to  $\gamma_U^*$ . By Prop. 1.8 we have  $|d\tau_U|_{\gamma_U^*} \leq |d\tau_U|_{g_U^*}$  and  $|d\beta_U|_{\gamma_U^*} \leq |d\beta_U|_{\rho_U^*}$ . Therefore we have:

$$|d\tau_U + d\beta_U|_{\gamma_U^*} \leq |d\tau_U|_{\gamma_U^*} + |d\beta_U|_{\gamma_U^*} \leq |d\tau_U|_{g_U^*} + |d\beta_U|_{\rho_U^*}.$$

Finally  $|d\tau_U|_{g_U^*}$  is bounded because  $|d\tau|_{h^*}$  is bounded and  $g_U$  and  $h|_{\text{reg}(U)}$  are quasi-isometric on  $\text{reg}(U)$ . Analogously  $|d\beta_U|_{\rho_U^*}$  is bounded because  $\beta_U = (b \circ \phi)|_{\text{reg}(U)}$ ,  $b$  satisfy the conditions of Gordon's Theorem with respect to  $g$  and  $\rho_U = (\phi|_{\text{reg}(U)})^*g$ . Clearly  $\beta_U + \tau_U$  is smooth. It remains to show that it is proper. However, this is clear because it is easy to see, from the very definition of  $\beta_U$  and  $\tau_U$ , that if  $\{p_j\}$  is a sequence of points in  $\text{reg}(U)$  converging to a point  $p$  in  $\overline{\text{reg}(U)} \setminus \text{reg}(U)$ , then  $(\beta_U + \tau_U)(p_j) \rightarrow +\infty$  as  $j \rightarrow +\infty$ . Furthermore, according to Prop. 1.7, we know that  $\gamma_U$  is a  $d$ -bounded Kähler metric because it is defined as the sum of two  $d$ -bounded Kähler metrics. Hence, by Th. 1.5, we can conclude that

$$H_{2,\bar{\partial}}^{v,q}(\text{reg}(U), \gamma_U) = 0$$

for  $q > 0$ . Now, equipped with the above vanishing result, we can come back to the complex of sheaves  $\{\pi_*\mathcal{C}_{D,\sigma}^{v,q}, q \geq 0\}$ . Let  $p \in \text{sing}(V)$ . In order to conclude that the complex  $\{(\pi_*\mathcal{C}_{D,\sigma}^{v,q})_p, q \geq 0\}$  is exact, it is enough to show that given any open neighborhood  $A$  of  $p$  every cohomology class in  $H_{2,\bar{\partial}_{\max}}^{v,q}(\text{reg}(A), h|_{\text{reg}(A)})$  admits a representative that becomes exact when restricted to some open subset  $W \subset A$  with  $p \in W$ . This is done as follows. Consider again any point  $p \in \text{sing}(V)$  and any open neighborhood  $A$  of  $p$ . Let  $[\nu] \in H_{2,\bar{\partial}_{\max}}^{v,q}(\text{reg}(A), h|_{\text{reg}(A)})$ . According to [3] Th. 3.5 we know that  $[\nu]$  admits a smooth representative that we label by  $\nu$ . Let  $U \subset A$  be an open neighborhood of  $p$  such that  $h|_{\text{reg}(U)}$  is quasi isometric to a  $d$ -bounded Kähler metric  $g_U$ . Clearly we have  $h|_{\text{reg}(U)} \leq \gamma_U$ , where  $\gamma_U$  is defined as in (1.5). Hence by Prop. 1.10 we get that  $\nu|_{\text{reg}(U)} \in \ker(\bar{\partial}_{v,q}) \subset L^2\Omega^{v,q}(\text{reg}(U), \gamma_U)$ . Thus, according to what we have just shown above, there exists  $\mu \in \mathcal{D}(\bar{\partial}_{v,q-1}) \subset L^2\Omega^{v,q-1}(\text{reg}(U), \gamma_U)$  such that  $\bar{\partial}_{v,q-1}\mu = \nu|_{\text{reg}(U)}$  in  $L^2\Omega^{v,q}(\text{reg}(U), \gamma_U)$ . Let now  $W$  be any open subset of  $U$  such that  $\overline{W} \subset U$  and  $p \in W$ . It is immediate to check that  $\gamma_U|_{\text{reg}(W)}$  and  $h|_{\text{reg}(W)}$  are quasi-isometric. Therefore we have  $\mu|_{\text{reg}(W)} \in \mathcal{D}(\bar{\partial}_{v,q-1,\max}) \subset L^2\Omega^{v,q-1}(\text{reg}(W), h|_{\text{reg}(W)})$  and  $\bar{\partial}_{v,q-1,\max}(\mu|_{\text{reg}(W)}) = \nu|_{\text{reg}(W)}$  in  $L^2\Omega^{v,q}(\text{reg}(W), h|_{\text{reg}(W)})$  and so we can conclude that  $\{\pi_*\mathcal{C}_{D,\sigma}^{v,q}, q \geq 0\}$  is an exact sequence of sheaves as required.

Finally we finish the proof of Th. 1.9 by showing that  $\pi_*\mathcal{K}_M$  is equal to the kernel of the morphism  $\pi_*\mathcal{C}_{D,\sigma}^{v,0} \rightarrow \pi_*\mathcal{C}_{D,\sigma}^{v,1}$  induced by  $\bar{\partial}_{v,0}$ . To this aim we work on  $M$  and indeed we show that  $\mathcal{K}_M$  is equal to the kernel of the morphism  $\mathcal{C}_{D,\sigma}^{v,0} \rightarrow \mathcal{C}_{D,\sigma}^{v,1}$  induced by  $\bar{\partial}_{v,0}$ . Consider any open subset  $U$  of  $M$  and let  $p$  be any point in  $U$ . Let  $\alpha \in \mathcal{C}_{D,\sigma}^{v,0}(U)$  be such that  $\alpha \in \ker(\mathcal{C}_{D,\sigma}^{v,0} \rightarrow \mathcal{C}_{D,\sigma}^{v,1})$ . Hence, there exists an open neighbourhood  $W$  of  $p$ , with closure contained in  $U$ , such that  $\alpha$  restricted to  $W \setminus (W \cap D)$ ,  $\alpha|_{W \setminus (W \cap D)}$ , lies in  $L^2\Omega^{v,0}(W \setminus (W \cap D), \sigma|_{W \setminus (W \cap D)})$  and satisfies

$$(1.6) \quad \bar{\partial}_{v,0,\max} \alpha|_{W \setminus (W \cap D)} = 0.$$

Thus in turn implies that

$$(1.7) \quad (\bar{\partial}_{v,0,\max})^*(\bar{\partial}_{v,0,\max} \alpha|_{W \setminus (W \cap D)}) = 0.$$

Consider now the complex  $\{\Omega_c^{v,*}(W \setminus (W \cap D)), \bar{\partial}_{v,*}\}$ ; it is well known that this is an elliptic complex and thus the associated laplacians  $\Delta_{\bar{\partial}_{v,q}}$  are elliptic for each  $q$ . By (1.7) we know in particular that  $\alpha|_{W \setminus (W \cap D)}$  is in the null space of the maximal extension of  $\Delta_{\bar{\partial}_{v,0}} : L^2\Omega^{v,0}(W \setminus (W \cap D), \sigma|_{W \setminus (W \cap D)}) \rightarrow L^2\Omega^{v,0}(W \setminus (W \cap D), \sigma|_{W \setminus (W \cap D)})$ . Hence by elliptic regularity we can conclude that  $\alpha|_{W \setminus (W \cap D)}$  is smooth, and thus, by (1.6), holomorphic on  $W \setminus (W \cap D)$ . Summarizing:  $\omega$  lies in  $L^2\Omega^{v,0}(W \setminus (W \cap D), \sigma|_{W \setminus (W \cap D)})$  and it is holomorphic on  $W \setminus (W \cap D)$ . Now, if  $W \cap D = \emptyset$  we can already conclude that  $\omega$  is holomorphic in all of  $W$ . If  $W \cap D \neq \emptyset$  let  $\lambda$  be an arbitrary

Hermitian metric on  $M$  and let us consider  $\lambda|_{W \setminus (W \cap D)}$ . According to Prop. 1.10 we know that

$$L^2\Omega^{v,0}(W \setminus (W \cap D), \sigma|_{W \setminus (W \cap D)}) = L^2\Omega^{v,0}(W \setminus (W \cap D), \lambda|_{W \setminus (W \cap D)}).$$

Thus  $\alpha$  is in  $L^2\Omega^{v,0}(W \setminus (W \cap D), \lambda|_{W \setminus (W \cap D)})$  and it is holomorphic on  $W \setminus (W \cap D)$ . Finally using an  $L^2$ -extension theorem as in [21] we can conclude that  $\alpha$  extends as an holomorphic  $(v, 0)$ -form in all of  $W$ . Replacing  $p$  with any other point in  $U$  and repeating the same argument we can conclude that  $\alpha$  is holomorphic in  $U$  and, therefore, that it is an element of  $\mathcal{K}_M(U)$ . Clearly the other inclusion is trivial, that is:  $\mathcal{K}_M$  is a sub-sheaf of the kernel of the morphism  $\mathcal{C}_{D,\sigma}^{v,0} \rightarrow \mathcal{C}_{D,\sigma}^{n,1}$  which is induced by the distributional action of  $\bar{\partial}_{v,0}$ . Indeed if  $\omega \in \mathcal{K}_M(U)$  then  $\omega \in \mathcal{C}_{D,\sigma}^{v,0}(U)$  and the distributional action of  $\bar{\partial}_{v,0}$  applied to  $\omega$  is equal to 0. We can thus conclude that  $\{\pi_*\mathcal{C}_{D,\sigma}^{v,q}, q \geq 0\}$  is a fine resolution of  $\pi_*\mathcal{K}_M$  as desired.  $\square$

We give now a criterion which assures that the sheaves  $\{\pi_*\mathcal{C}_{D,\sigma}^{v,q}, q \geq 0\}$  are fine.

**Lemma 1.11.** *In the setting of Th. 1.9. Assume that given any open cover  $\mathcal{U} = \{U_i\}_{i \in I}$  of  $V$  there exists a continuous partition of unity  $\{\lambda_j\}_{j \in J}$  subordinate to  $\mathcal{U}$  such that for each  $j \in J$*

- (1)  $\lambda_j|_{\text{reg}(V)}$  is smooth
- (2)  $\|d(\lambda_j|_{\text{reg}(V)})\|_{L^\infty\Omega^1(\text{reg}(V),h)} < \infty$ .

Then  $\pi_*\mathcal{C}_{D,\sigma}^{v,q}$  is a fine sheaf for each  $q = 0, \dots, v$ .

*Proof.* This follows immediately from the description given in (1.2).  $\square$

We proceed by showing some corollaries of Th. 1.9.

**Corollary 1.12.** *The unique closed extension of the operator  $\bar{\partial}_v + \bar{\partial}_v^t : \Omega_c^{v,\bullet}(\text{reg}(V), h) \rightarrow \Omega_c^{v,\bullet}(\text{reg}(V), h)$ , denoted here*

$$(1.8) \quad \bar{\partial}_v + \bar{\partial}_v^* : L^2\Omega^{v,\bullet}(\text{reg}(V), h) \rightarrow L^2\Omega^{v,\bullet}(\text{reg}(V), h)$$

*is a Fredholm operator on its domain endowed with the graph norm.*

*Proof.* This follows immediately from the finite dimensionality of  $H_{2,\bar{\partial}}^{v,q}(\text{reg}(V), h)$  and Theorem 2.4 in [3].  $\square$

**Corollary 1.13.** *For each  $q = 0, \dots, v$  we have the following isomorphism:*

$$H_{2,\bar{\partial}}^{0,q}(\text{reg}(V), h) \cong H_{\bar{\partial}}^{0,q}(M).$$

*In particular we have*

$$\chi(M, \mathcal{O}_M) = \chi_2(\text{reg}(V), h)$$

*where the term on the right-hand side is defined as  $\sum (-1)^q \dim(H_{2,\bar{\partial}}^{0,q}(\text{reg}(V), h))$ .*

*Proof.* This follows from Th. 1.9 and the  $L^2$  version of Serre duality, see [21], which tells us that  $H_{2,\bar{\partial}}^{v,q}(\text{reg}(V), h) \cong H_{2,\bar{\partial}}^{0,q}(\text{reg}(V), h)$ .  $\square$

**Corollary 1.14.** *The unique closed extension of the operator  $\bar{\partial}_0 + \bar{\partial}_0^t : \Omega_c^{0,\bullet}(\text{reg}(V), h) \rightarrow \Omega_c^{0,\bullet}(\text{reg}(V), h)$ , denoted here*

$$\bar{\partial}_0 + \bar{\partial}_0^* : L^2\Omega^{0,\bullet}(\text{reg}(V), h) \rightarrow L^2\Omega^{0,\bullet}(\text{reg}(V), h)$$

*is a Fredholm operator on its domain endowed with the graph norm.*

*Proof.* As for Cor. 1.12 this follows immediately by the finite dimensionality of  $H_{2,\bar{\partial}}^{0,q}(\text{reg}(V), h)$  and Theorem 2.4 in [3].  $\square$



**Corollary 1.15.** *In the setting of Th. 1.9. Assume moreover that  $h$  is Kähler. Then  $H_{2,\bar{\partial}}^{q,0}(\text{reg}(V), h)$  is finite dimensional for each  $q = 0, \dots, v$ . Moreover*

$$\bar{\partial}_{q,0} : L^2\Omega^{q,0}(\text{reg}(V), h) \rightarrow L^2\Omega^{q,0}(\text{reg}(V), h)$$

that is the unique closed extension of  $\bar{\partial}_{q,0} : \Omega_c^{q,0}(\text{reg}(V)) \rightarrow \Omega_c^{q,1}(\text{reg}(V))$ , has closed range.

*Proof.* We consider  $\Delta_{v,q,\bar{\partial}} : \Omega_c^{v,q}(\text{reg}(V)) \rightarrow \Omega_c^{v,q}(\text{reg}(V))$ , with  $\Delta_{v,q,\bar{\partial}} := \bar{\partial}_{v,q}^t \bar{\partial}_{v,q} + \bar{\partial}_{v,q} \bar{\partial}_{v,q}^t$ . According to Th. 2.4 in [3] we know that the unique closed extension of this operator, labeled here by

$$(1.9) \quad \Delta_{v,q,\bar{\partial}} : L^2\Omega^{v,q}(\text{reg}(V), h) \rightarrow L^2\Omega^{v,q}(\text{reg}(V), h),$$

is a Fredholm operator on its domain endowed with the graph norm. Consider now

$$(1.10) \quad \Delta_{v-q,0,\bar{\partial}} : L^2\Omega^{v-q,0}(\text{reg}(V), h) \rightarrow L^2\Omega^{v-q,0}(\text{reg}(V), h)$$

that is, the operator defined as the unique closed extension of  $\Delta_{v-q,0,\bar{\partial}}$  on  $\Omega_c^{v-q,0}(\text{reg}(V))$ . It is easy to see that the Hodge star operator  $*$  :  $L^2\Omega^{v,q}(\text{reg}(V), h) \rightarrow L^2\Omega^{v-q,0}(\text{reg}(V), h)$  makes (1.9) and (1.10) unitarily equivalent. On the other hand since  $(\text{reg}(V), h)$  is Kähler we have  $\Delta_{v-q,0,\bar{\partial}} = \Delta_{v-q,0,\bar{\partial}}$  on  $\Omega_c^{v-q,0}(\text{reg}(V))$ . This in turn implies that we have an equality of operators acting on  $L^2\Omega^{v-q,0}(\text{reg}(V), h)$  that is  $\Delta_{v-q,0,\bar{\partial}} : L^2\Omega^{v-q,0}(\text{reg}(V), h) \rightarrow L^2\Omega^{v-q,0}(\text{reg}(V), h)$  coincides with  $\Delta_{v-q,0,\bar{\partial}} : L^2\Omega^{v-q,0}(\text{reg}(V), h) \rightarrow L^2\Omega^{v-q,0}(\text{reg}(V), h)$ . Hence we can conclude that  $\Delta_{v-q,0,\bar{\partial}} : L^2\Omega^{v-q,0}(\text{reg}(V), h) \rightarrow L^2\Omega^{v-q,0}(\text{reg}(V), h)$  is a Fredholm operator on its domain endowed with the graph norm. Moreover the completeness of  $(\text{reg}(V), h)$  assures us that  $\Delta_{v-q,0,\bar{\partial}} : L^2\Omega^{v-q,0}(\text{reg}(V), h) \rightarrow L^2\Omega^{v-q,0}(\text{reg}(V), h)$  coincides with

$$\bar{\partial}_{v-q,0}^* \circ \bar{\partial}_{v-q,0} : L^2\Omega^{v-q,0}(\text{reg}(V), h) \rightarrow L^2\Omega^{v-q,1}(\text{reg}(V), h)$$

where  $\bar{\partial}_{v-q,0} : L^2\Omega^{v-q,0}(\text{reg}(V), h) \rightarrow L^2\Omega^{v-q,1}(\text{reg}(V), h)$  is the unique closed extension of  $\bar{\partial}_{v-q,0} : \Omega_c^{v-q,0}(\text{reg}(V)) \rightarrow \Omega_c^{v-q,1}(\text{reg}(V))$  and  $\bar{\partial}_{v-q,0}^* : L^2\Omega^{v-q,1}(\text{reg}(V), h) \rightarrow L^2\Omega^{v-q,0}(\text{reg}(V), h)$  is its Hilbert space adjoint. In conclusion this shows that on  $L^2\Omega^{v-q,0}(\text{reg}(V), h)$  we have

$$\ker(\Delta_{v-q,0,\bar{\partial}}) = \ker(\bar{\partial}_{v-q,0}) = H_{2,\bar{\partial}}^{v-q,0}(\text{reg}(V), h).$$

We can thus conclude that  $H_{2,\bar{\partial}}^{v-q,0}(\text{reg}(V), h)$  is finite dimensional. Finally that

$$\bar{\partial}_{v-q,0} : L^2\Omega^{v-q,0}(\text{reg}(V), h) \rightarrow L^2\Omega^{v-q,1}(\text{reg}(V), h)$$

has closed range can be easily shown using the fact that  $\Delta_{v-q,0,\bar{\partial}} : L^2\Omega^{v-q,0}(\text{reg}(V), h) \rightarrow L^2\Omega^{v-q,0}(\text{reg}(V), h)$  is a Fredholm operator on its domain endowed with the graph norm arguing for instance as in [1] Cor. 4.1.  $\square$

We end this section with the following consequence.

**Corollary 1.16.** *Let  $V$  and  $W$  be a pair of compact and irreducible complex spaces of complex dimension  $v$  whose regular parts are endowed with complete Hermitian metrics  $h$  and  $g$  respectively. Assume that both  $(V, h)$  and  $(W, g)$  satisfy the assumptions of Th. 1.9. Assume moreover that  $V$  and  $W$  are bimeromorphic. Then for each  $q = 0, \dots, v$ ,*

$$H_{2,\bar{\partial}}^{v,q}(\text{reg}(V), h) \cong H_{2,\bar{\partial}}^{v,q}(\text{reg}(W), g) \quad H_{2,\bar{\partial}}^{0,q}(\text{reg}(V), h) \cong H_{2,\bar{\partial}}^{0,q}(\text{reg}(W), g).$$

*Proof.* Let  $\pi : M \rightarrow V$  be a resolution of  $V$  and analogously let  $p : N \rightarrow W$  be a resolution of  $W$ . Since we assumed that  $V$  and  $W$  are bimeromorphic we have  $H_{\bar{\partial}}^{v,q}(M) \cong H_{\bar{\partial}}^{v,q}(N)$  for each  $q = 0, \dots, v$ . Now the conclusion follows by Th. 1.9 and Cor. 1.13.  $\square$

## 2. APPLICATIONS

The aim of this section is to collect some examples of Hermitian complex spaces where Th. 1.9 and its corollaries can be applied. In the first part we treat Saper-type Kähler metrics while in last part we consider complete Kähler manifolds with finite volume and pinched negative sectional curvatures.

**2.1. Saper metrics.** These kind of metrics have been defined in [10] and [11] in order to extend to the case of arbitrary analytic subvarieties of Kähler manifolds the construction carried out by Saper in [22] in the setting of isolated singularities.

In the next definition we recall from [10] pag. 741 the definition of Saper-type metric.

**Definition 2.1.** *Let  $V$  be a singular subvariety of a compact complex manifold  $M$  and let  $\omega$  be the fundamental  $(1,1)$ -form of a Hermitian metric on  $M$ . Let  $\pi : \widetilde{M} \rightarrow M$  be a holomorphic map of a compact complex manifold  $\widetilde{M}$  to  $M$  whose exceptional set  $E$  is a divisor with normal crossing in  $\widetilde{M}$  and such that the restriction*

$$\pi|_{\widetilde{M} \setminus E} : \widetilde{M} \setminus E \longrightarrow M \setminus \text{sing}(V)$$

*is a biholomorphism. Let  $L_E$  be the line bundle on  $\widetilde{M}$  associated to  $E$  and let  $h$  be a Hermitian metric on  $L_E$ . Let  $s : \widetilde{M} \rightarrow L_E$  be a global holomorphic section whose associated divisor  $(s)$  equals  $E$  (in particular  $s$  vanishes exactly on  $E$ ) and let  $\|s\|_h$  be the norm of  $s$  with respect to  $h$ .*

*A metric on  $\widetilde{M} \setminus E$  which is quasi-isometric to a metric with fundamental  $(1,1)$ -form*

$$l\pi^*\omega - \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(\log \|s\|_h^2)^2$$

*for  $l$  a positive integer, will be called a **Saper-type metric**, distinguished with respect to the map  $\pi$ . The corresponding metric on  $M \setminus \text{sing} V \cong \widetilde{M} \setminus E$  and its restriction to  $V \setminus \text{sing} V$  are also called **Saper-type metric**.*

We follow the convention used in [10]: thus, when it is clear from the context, we omit the sentence "distinguished with respect to  $\pi$ ".

Now we go on stating a fundamental existence result for Saper-type metrics proved by Grant Melles and Milman in [10], Theorem 8.6 pag. 746.

**Theorem 2.2.** *Let  $V$  be a singular subvariety of a compact Kähler manifold  $M$  and let  $\omega$  be the Kähler  $(1,1)$ -form of a Kähler metric on  $M$ . There exists a  $C^\infty$  function  $F$  on  $M$ , vanishing on  $\text{sing}(V)$ , such that the  $(1,1)$ -form*

$$(2.1) \quad \omega_S = \omega - \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(\log F)^2$$

*is the Kähler form of a complete Saper-type metric on  $M \setminus \text{sing}(V)$  and hence on  $V \setminus \text{sing}(V)$ . Furthermore the function  $F$  can be constructed to be of the form*

$$F = \prod_{\alpha} F_{\alpha}^{\rho_{\alpha}}$$

*where  $\{\rho_{\alpha}\}$  is a  $C^\infty$  partition of unity subordinate to an open cover  $\{U_{\alpha}\}$  of  $M$ ,  $F_{\alpha}$  is a function on  $U_{\alpha}$  of the form*

$$F_{\alpha} = \sum_{j=1}^r |f_j|^2$$

and  $f_1, \dots, f_r$  are holomorphic functions on  $U_\alpha$ , vanishing exactly on  $U_\alpha \cap \text{sing}(V)$ . More specifically  $f_1, \dots, f_r$  are local holomorphic generators of a coherent ideal sheaf  $\mathcal{I}$  on  $M$  such that blowing up  $\widetilde{M}$  along  $\mathcal{I}$  desingularizes  $V$ ,  $\mathcal{I}$  is supported on  $\text{sing}(V)$  and the exceptional divisor in the blow-up  $\widetilde{M}$  along  $\mathcal{I}$  has normal crossing and has also normal crossing with the strict transform  $\widetilde{V}$  of  $V$  in  $\widetilde{M}$  (the so called embedded desingularization of  $X$ .)

Concerning Saper-type metric we have the following property.

**Proposition 2.3.** *In the setting of Th. 2.2. For each  $p \in \text{sing}(V)$  there exists an open neighborhood  $U$  and a Kähler metric  $g_U$  such that  $g_S|_U$  is quasi-isometric to  $g_U$  and  $g_U$  satisfies the Ohsawa condition.*

*Proof.* This follows from [10] Prop. 8.10 and Prop. 9.11.  $\square$

We have now all the ingredients to apply Th. 1.9 to Saper-type Kähler metrics.

**Theorem 2.4.** *Let  $M$  be a compact Kähler manifold with Kähler form  $\omega$  and let  $V$  be an analytic subvariety in  $M$  of complex dimension  $v$ . Let  $\pi : \widetilde{V} \rightarrow V$  be a resolution of  $V$ . Finally let  $g_S$  be a Saper-type metric on  $\text{reg}(V)$  as constructed in Th. 2.2. Then the following isomorphism holds:*

$$(2.2) \quad H_{2,\bar{\partial}}^{v,q}(\text{reg}(V), g_S) \cong H_{\bar{\partial}}^{v,q}(\widetilde{V})$$

for every  $q = v, \dots, n$ .

*Proof.* Let  $D \subset \widetilde{V}$  be the divisor with only normal crossing such that  $\pi^{-1}(\text{sing}(V)) = D$ . Let  $\sigma_S := (\pi|_{\widetilde{V} \setminus D})^* g_S$ . Thanks to Prop. 2.3, in order to deduce the above theorem by Th. 1.9, we have only to check that the sheaves  $\{\pi_* \mathcal{C}_{D,\sigma_S}^{v,q}, q \geq 0\}$  are fine. This is done as follows. In order to prove that  $\pi_* \mathcal{C}_{D,\sigma_S}^{v,q}$  is fine it is enough to show that given a cover  $\mathcal{U} = \{U_i\}_{i \in I}$  of  $V$  there exists a continuous partition of unity  $\{f_\gamma\}_{\gamma \in G}$  subordinated to  $\mathcal{U}$  such that for each  $\gamma \in G$

- $f_\gamma|_{\text{reg}(V)}$  is smooth
- If  $A$  is an open subset of  $V$  and  $\omega \in \pi_* \mathcal{C}_{D,g_S}^{v,q}(A)$  then  $f_\gamma \omega \in \pi_* \mathcal{C}_{D,g_S}^{v,q}(A)$ .

To this end we recall [11, Proposition 10.2.1]:

**Proposition 2.5.** *Let  $p \in \text{sing}(V)$ , let  $U$  be an open neighborhood of  $p$  in  $V$ , let  $f$  be a smooth function on  $M$  and let  $\omega \in L^2 \Omega^k(\text{reg}(U), g_S|_{\text{reg}(U)})$ . Then  $d(f|_{\text{reg}(U)}) \wedge \omega \in L^2 \Omega^{k+1}(\text{reg}(U), g_S|_{\text{reg}(U)})$ .*

Clearly  $d(f|_{\text{reg}(U)}) \wedge \omega = \partial(f|_{\text{reg}(U)}) \wedge \omega + \bar{\partial}(f|_{\text{reg}(U)}) \wedge \omega$ . Therefore, thanks to Prop. 2.5, if  $\omega \in L^2 \Omega^{p,q}(\text{reg}(U), g_S|_{\text{reg}(U)})$ ,  $p+q = k$ , we can conclude that  $\bar{\partial}(f|_{\text{reg}(U)}) \wedge \omega \in L^2 \Omega^{p,q+1}(\text{reg}(U), g_S|_{\text{reg}(U)})$ . In particular if  $\omega \in \pi_* \mathcal{C}_{D,g_S}^{v,q}(U)$  then we can conclude that  $f|_{\text{reg}(U)} \omega \in \pi_* \mathcal{C}_{D,g_S}^{v,q}(U)$ . Let now  $\mathcal{V} = \{V_j\}_{j \in J}$  be an open cover of  $M$  such that the induced cover on  $V$  is equal to  $\mathcal{U}$ . Considering a smooth partition of unity subordinated to  $\mathcal{V}$  and restricting it to  $V$  and using the remark that we have just made, it is clear that we have built a partition of unity on  $V$  subordinated to  $\mathcal{U}$  with the required properties.  $\square$

**Remark 2.6.** *In the particular case of isolated singularities the isomorphism above was established by Saper as a corollary of his main result in [22], namely the isomorphism of  $H_2^\ell(\text{reg}(V), g_S) \cong I^m H^\ell(V, \mathbb{R})$ , and the identification of the induced Hodge structure on  $I^m H^\ell(V, \mathbb{R})$  with the one constructed by Saito. Our proof, on the other hand, is direct and rests solely on analytic arguments.*

We conclude this subsection with the following corollaries.

**Corollary 2.7.** *In the setting of Th. 2.4. Then Cor. 1.12–Cor. 1.15 hold for  $(\text{reg}(V), g_S)$ .*

**Corollary 2.8.** *Let  $(M, h)$  and  $(N, g)$  be compact Kähler manifolds and let  $V \subset M$ ,  $W \subset N$  be analytic subvarieties of complex dimension  $v$ . Assume that  $V$  and  $W$  are bimeromorphic. Then for each  $q = 0, \dots, v$*

$$H_{2, \bar{\partial}}^{v, q}(\text{reg}(V), h_S) \cong H_{2, \bar{\partial}}^{v, q}(\text{reg}(W), g_S) \quad H_{2, \bar{\partial}}^{0, q}(\text{reg}(V), h_S) \cong H_{2, \bar{\partial}}^{0, q}(\text{reg}(W), g_S)$$

where  $h_S$  and  $g_S$  are Saper-type metrics as constructed in Th. 2.2 on  $V$  and  $W$  respectively.

*Proof.* This follows from Cor. 1.16.  $\square$

**2.2. Further remarks on Saper metrics.** In this subsection we collect some byproducts of Th. 1.9 in the framework of Saper-type metrics. Consider again the setting of Theorem 2.4. To avoid any confusion with the notations let us now label by  $\omega'_S$  the  $(1, 1)$ -form on  $M \setminus \text{sing}(V)$  given by

$$\omega'_S = \omega - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(\log F)^2$$

where  $\omega$  is the fundamental form of a Kähler metric on  $M$  and  $F$  is defined in (2.1). Let  $g'_S$  be the Saper-type metric on  $M \setminus \text{sing}(V)$  whose fundamental form is  $\omega'_S$ . If we label by  $i : \text{reg}(V) \rightarrow M$  the inclusion of  $\text{reg}(V)$  in  $M$ , then we have  $g_S = i^*(g'_S)$  where  $g_S$  is the Saper-type metric considered in Theorem 2.4. We have the following

**Theorem 2.9.** *In the setting described above. We have the following isomorphisms:*

$$(2.3) \quad H_{2, \bar{\partial}}^{m, q}(M \setminus \text{sing}(V), g'_S) \cong H_{2, \bar{\partial}}^{m, q}(M)$$

$$(2.4) \quad H_{2, \bar{\partial}}^{0, q}(M \setminus \text{sing}(V), g'_S) \cong H_{2, \bar{\partial}}^{0, q}(M)$$

where  $m$  is the complex dimension of  $M$ .

*Proof.* The proof is similar to the one given for Th. 2.4 and for the sake of brevity we omit the details. Let us label by  $V_s$  the singular locus of  $V$ . Consider on  $M$  the presheaf  $\mathcal{C}_{V_s, g'_S}^{p, q}$  defined by  $\mathcal{C}_{V_s, g'_S}^{p, q}(U) := \{\mathcal{D}(\bar{\partial}_{p, q, \max}) \text{ on } (U \setminus U \cap V_s, g'_S|_{U \setminus U \cap V_s})\}$  where  $U$  is any open subset of  $M$ . The sheafification of  $\mathcal{C}_{V_s, g'_S}^{p, q}$  is denoted by  $\mathcal{C}_{V_s, g'_S}^{p, q}$ . Analogously to the proof of Theorem 1.9 we want to show that the complex  $\{\mathcal{C}_{V_s, g'_S}^{m, q}, q \geq 0\}$ , whose morphisms are induced by the distributional action of  $\bar{\partial}_{m, q}$ , is a fine resolution of  $\mathcal{K}_M$ . In particular that  $\mathcal{C}_{V_s, g'_S}^{p, q}$  is fine for each  $p, q$  follows using a partition of unity of  $M$ . Now let  $p \in V_s$  and let  $W$  be a sufficiently small neighborhood of  $p$ . We can assume that there exists a positive constant  $c$ , an and a biholomorphism  $\phi : W \rightarrow B(0, c)$  where  $B(0, c)$  is the ball in  $\mathbb{C}^m$  centered in 0 with radius  $c$ . Let  $\psi : B(0, c) \rightarrow \mathbb{R}$  be defined as  $\psi := -(\log(c^2 - |z|^2))$ , let  $g$  be the Kähler metric on  $B(0, c)$  whose Kähler form is given by  $\sqrt{-1} \partial \bar{\partial} \psi$  and let  $\rho_W := (\phi|_{\text{reg}(W \setminus (W \cap V_s))})^* g$ . By [10] Prop. 8.10 and 9.11 we know that  $g_S|_{W \setminus (W \cap V_s)}$  is quasi isometric to a Kähler metric  $g_W$  which satisfies the Ohsawa condition. Now we introduce the following Kähler metric on  $W \setminus (W \cap V_s)$ :

$$(2.5) \quad \gamma_W := \rho_W + g_W$$

Using  $\gamma_W$  and arguing as in the proof of Th. 1.9 we can conclude that  $\{\mathcal{C}_{V_s, g'_S}^{m, q}, q \geq 0\}$  is an exact sequence of sheaves. Finally we are left to show that the kernel of the sheaves morphism  $\mathcal{C}_{V_s, g'_S}^{m, 0} \xrightarrow{\bar{\partial}_{m, 0}} \mathcal{C}_{V_s, g'_S}^{m, 1}$  is  $\mathcal{K}_M$ . This can be seen again as in the proof of Th. 1.9. In conclusion  $\{\mathcal{C}_{V_s, g'_S}^{m, q}, q \geq 0\}$  is a fine resolution of  $\mathcal{K}_M$  and thus we can conclude that  $H_{2, \bar{\partial}}^{m, q}(M \setminus \text{sing}(V), g'_S) \cong H_{2, \bar{\partial}}^{m, q}(M)$ . Applying  $L^2$ -Serre duality we finally get  $H_{2, \bar{\partial}}^{0, q}(M \setminus \text{sing}(V), g'_S) \cong H_{2, \bar{\partial}}^{0, q}(M)$  and this completes the proof.  $\square$

Next we consider a Hermitian holomorphic line bundle  $L$  on the resolution  $\tilde{V}$ . We can restrict  $L$  to  $\tilde{V} \setminus D$  and push it forward to  $\pi_*L$  on  $V \setminus \text{sing}(V)$  through the biholomorphism  $\pi|_{\tilde{V} \setminus D} : \tilde{V} \setminus D \rightarrow V \setminus \text{sing}(V)$  (where  $\pi : \tilde{V} \rightarrow V$  is the map appearing in the resolution of  $V$ ):

$$\pi_*L := (\pi|_{\tilde{V} \setminus D})^{-1}{}^*L|_{\tilde{V} \setminus D}.$$

We shall say that  $L$  is semipositive with respect to the base  $V$  if for each  $p \in V$  there exists a neighbourhood  $U_p$  in  $V$  such that  $L$  is positive on  $\pi^{-1}(U_p)$ .

**Theorem 2.10.** *Consider the setting of Theorem 2.4. Let  $L \rightarrow \tilde{V}$  be a Hermitian holomorphic line bundle which is semipositive with respect to  $V$ . Then there exist isomorphisms:*

$$(2.6) \quad H_{2,\bar{\partial}}^{v,q}(V \setminus \text{sing}(V), \pi_*L, g_S) \cong H_{2,\bar{\partial}}^{v,q}(\tilde{V}, L) \quad H_{2,\bar{\partial}}^{0,q}(V \setminus \text{sing}(V), \pi_*L^*, g_S) \cong H_{2,\bar{\partial}}^{0,q}(\tilde{V}, L^*).$$

*Proof.* Also in this case the proof is similar to the one given for Theorem 2.4 but with some modifications due to the presence of the line bundle  $L$ . These modifications are as follows: first of all we can introduce with self-explanatory notation the complex of sheaves  $\{\mathcal{C}_{D,g_S,L}^{v,q}\}$  on  $\tilde{V}$ . As showed in [21] we have  $H^q(\tilde{V}, \mathcal{K}_{\tilde{V}}(L)) = H^q(V, \pi_*\mathcal{K}_{\tilde{V}}(L))$  where  $\mathcal{K}_{\tilde{V}}(L)$  is the sheaf of holomorphic sections of the holomorphic line bundle  $K_{\tilde{V}} \otimes L$ . Hence, as in the proof of Th. 2.4, our purpose now is to show that  $\{\pi_*\mathcal{C}_{D,g_S,L}^{v,q}\}$  is a fine resolution of  $\pi_*(\mathcal{K}_{\tilde{V}}(L))$ . To this aim, using a results proved by Ruppenthal, see [21, Theorem 3.2], we know that for Kähler metrics satisfying the Ohsawa condition we can extend Th. 1.5 to the  $L^2$ - $\bar{\partial}$ -cohomology of forms with bi-degree  $(v, q)$  and with coefficients in any semipositive Hermitian holomorphic line bundle. Clearly since  $L$  is semipositive with respect to  $V$  it obeys the conditions of Th. 3.2 in [21]. Therefore, with an analogous strategy to the one used in the proof of Theorem 1.9, this vanishing result can in turn be used in order to show that  $\{\pi_*\mathcal{C}_{D,g_S,L}^{v,q}\}$  is a fine resolution of  $\pi_*(\mathcal{K}_{\tilde{V}} \otimes L)$ . All this gives us the first isomorphism in (2.6); using the  $L^2$ -version of Serre duality we get the second isomorphism.  $\square$

For the next result we begin by recalling that a Hermitian holomorphic line bundle  $L$  over an irreducible compact complex space  $V$  is *almost positive* if the curvature form is semipositive on  $\text{reg}(V)$  and positive on an open subset of  $\text{reg}(V)$ .

**Theorem 2.11.** *Let  $\pi : \tilde{V} \rightarrow V$  be as in Th. 2.4. Let  $L$  be an almost positive Hermitian holomorphic line bundle over  $V$ . Then for  $q > 0$  we have*

$$(2.7) \quad H_{2,\bar{\partial}}^{v,q}(V \setminus \text{sing}(V), L, g_S) \cong H_{2,\bar{\partial}}^{0,v-q}(V \setminus \text{sing}(V), L^*, g_S) = 0$$

*Proof.* Let us define  $F := \pi^*L$ . Then  $F$  is an almost positive line bundle over  $\tilde{V}$ . Using the Bochner-Kodaira-Nakano inequality, see [6] 13.3, and the fact that  $F$  is positive on an open subset of  $\tilde{V}$ , we easily get  $H_{2,\bar{\partial}}^{v,q}(\tilde{V}, F) = 0$  for  $q > 0$ . Now the conclusion follows immediately by applying Theorem 2.10.  $\square$

**2.3. Negatively curved Kähler manifolds with finite volume.** Let  $(M, h)$  be a complete Kähler manifold with finite volume and pinched negative sectional curvatures  $-b^2 \leq \text{sec}_h \leq -a^2$  for some constants  $0 < a \leq b$ . An important result concerning the geometry of such manifolds is the one proved in [23] by Siu and Yau. This result provides the existence of a compactification of  $M$  in terms of a complex projective variety with only isolated singularities. More precisely if  $(M, h)$  is a Kähler manifold as above then there exists a projective variety  $V \subset \mathbb{C}P^n$  with only isolated singularities such that  $\text{reg}(V)$  and  $M$  are biholomorphic. The purpose of this subsection is

to investigate the  $L^2\text{-}\bar{\partial}$ -cohomology of such Kähler manifolds with the help of our Th. 1.9 and the Siu-Yau compactification. Concerning this task the main result of this subsection reads as follows:

**Theorem 2.12.** *Let  $(M, h)$  be a complete Kähler manifold of complex dimension  $m$  with finite volume. Assume that the sectional curvatures of  $(M, h)$  satisfies  $-b^2 \leq \text{sec}_h \leq -a^2$  for some constants  $0 < a \leq b$ . Let  $V \subset \mathbb{C}\mathbb{P}^n$  be the Siu-Yau compactification of  $M$  and let  $\pi : \tilde{V} \rightarrow V$  be a resolution of  $V$ . Then we have the following isomorphism for each  $q = 0, \dots, m$*

$$H_{2,\bar{\partial}}^{m,q}(M, h) \cong H_{2,\bar{\partial}}^{m,q}(\tilde{V}).$$

*Proof.* Let  $\psi : M \rightarrow \text{reg}(V)$  be a biholomorphism between  $M$  and  $\text{reg}(V)$ . Let us label by  $v$  the Kähler metric  $(\psi^{-1})^*h$ . Henceforth we will indentify  $(M, h)$  and  $(\text{reg}(V), v)$ . According to [25] Lemma 3.2 we know that there exists a compact subset  $D \subset M$  and a bounded continuous 1-form  $\theta$  such that on  $M \setminus D$  we have  $d\theta = \omega$  where  $\omega$  is the Kähler form of  $h$ . Hence the second condition in the statement of Th. 1.9 is fulfilled. We are left to show that the sheaves  $\{\pi_*\mathcal{C}_{D,\eta}^{m,q}, q \geq 0\}$  are fine where  $D \subset \tilde{V}$  is the divisor with only normal crossings given by  $D = \pi^{-1}(\text{sing}(V))$  and  $\eta := (\pi|_{\tilde{V} \setminus D})^*v$ . Let  $\mathcal{U} := \{U_i\}_{i \in I}$  be an open cover of  $V$ . Since  $V$  is compact there exists a finite open cover of  $V$ ,  $\mathcal{W} := \{W_1, \dots, W_r\}$  for some positive integer  $r$ , such that  $\mathcal{W}$  is subordinate to  $\mathcal{U}$  and such that for any  $i, j \in \{1, \dots, r\}$  with  $i \neq j$  we have  $\text{sing}(V) \cap W_i \cap W_j = \emptyset$ . Now we can easily construct a partition of unity  $\{\phi_1, \dots, \phi_r\}$  subordinated to  $\mathcal{W}$  such that the following properties hold:

- $\phi_i : V \rightarrow [0, 1]$  is continuous for each  $i = 1, \dots, r$
- $\phi_i|_{\text{reg}(V)} : \text{reg}(V) \rightarrow [0, 1]$  is smooth for each  $i = 1, \dots, r$
- if  $p \in \text{sing}(V) \cap \text{supp}(\phi_i)$  then there exists a neighborhood  $A$  of  $p$  which is open in  $V$  and such that  $\phi_i|_A = 1$ .

It is immediate to check that  $\|d\phi_i\|_{L^\infty \Omega^1(\text{reg}(V), v)} < \infty$ . Hence by Lemma 1.11 we can conclude that  $\{\pi_*\mathcal{C}_{D,\eta}^{m,q}, q \geq 0\}$  is a complex of fine sheaves. The theorem is thus established.  $\square$

We have now some direct applications of Th. 2.12.

**Corollary 2.13.** *In the setting of Th. 2.12. Then Corollaries 1.12–1.15 hold for  $(M, h)$ .*

**Corollary 2.14.** *Let  $(M, h)$  and  $(N, g)$  be as in Th. 2.12. Let  $V \subset \mathbb{C}\mathbb{P}^s$ ,  $W \subset \mathbb{C}\mathbb{P}^r$  be the corresponding Siu-Yau compactification. Assume that  $V$  and  $W$  are birationally equivalent. Then for each  $q = 0, \dots, m$  we have*

$$H_{2,\bar{\partial}}^{m,q}(M, h) \cong H_{2,\bar{\partial}}^{m,q}(N, g) \quad H_{2,\bar{\partial}}^{0,q}(M, h) \cong H_{2,\bar{\partial}}^{0,q}(N, g)$$

*Proof.* This follows from Cor. 1.16.  $\square$

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