Optimal control of SIR epidemic model with state dependent switching cost index

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ABSTRACT— The problem of epidemic control has to face with the resources allocation; a change in the strategy should be advisable during the epidemic spread in view of a rational use of the limited resources. The SIR epidemic model, which describes the dynamics of Susceptible, Infected and Removed subjects, is considered and an optimal vaccination strategy is proposed by introducing a cost index that weights differently the control depending on the severity of the disease. The introduced weight is a step-wise one and the switching instants are not known in advance. The meaningfulness of this approach has been tested and compared with the case of a constant weight for the control, showing a more efficient resource allocation in the proposed approach.

Keywords: epidemic models, optimal control, SIR model, switching cost index.

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1. Introduction

Epidemic models are a mathematical representation of diseases whose spread, if not controlled, may be particularly dangerous all over the world due mostly to the increased number of travelling people. These models are characterized by variables representing different status of the subjects. The most common are: the Susceptibles (S), that are subjects that may catch the disease, the Infected (I), that are the subjects that already caught the disease, the Removed (R), that are the subjects that aren't infected anymore, the Quarantine subjects (Q) that are the ones that can't have contacts with others, the Cross-immune subjects (C) that represent the subjects that may caught the disease again. Sometimes it may be useful to distinguish among the infected subjects (I) the ones that are infected but not yet infectious (E).

Therefore, depending on the specific classes of subjects considered, the models usually studied are the SIR, the SIRS, the SIRC, the SEIR, the SEIQR and so on, [1-11].

The SIR model, originally formulated by Kermack and McKendrick [12], considers three classes of subjects, the Susceptible, the Infected and the Removed. To control an epidemic model means to introduce an external action aiming at the reduction of the effects of the disease: the vaccine, the quarantine, the drug distribution, for example. Among the different strategies the ones that rely in the framework of optimal control theory have received increasing attention [13]. The choice of the specific cost index is related to the aims to be pursued, generally a decrease in the number of infected subjects, with as less resources as possible. Then, a central aspect is the definition of the cost index, [9-14].

The approach proposed in this paper introduces in the cost index a state dependent weight for the control depending on the number of the infected subjects, therefore changing intervention strategy on the basis of the varied conditions. The switching instants aren't known in advance but are determined as consequence of the dynamic variables evolution and of the optimization process. The final time of the optimization process has to be minimized too. The proposed cost index is applied to a generic SIR model. The paper is organized as follows: in Section 2 the optimal control strategy is proposed for a SIR model after some recalls of its dynamics; in Section 3 the impact of the proposed cost index on the control strategy is numerically analyzed. Conclusions and future developments are outlined in Section 4.

2. Materials and Methods

The SIR model is one of the most versatile one; it splits the population into three groups, the susceptibles (S), the infected (I) and the removed (R), indicated in the following equations by x, y and z respectively. A general description with birth term μ, that takes into account all the effects that make the number of susceptible subjects increasing (immunes, newborns, new comings, and so on), and a control aiming at a prevention action (a vaccination strategy, for example) is [13]:

$$\dot{x}(t) = -\beta x(t) y(t) - x(t)u(t) + \mu \tag{1}$$

$$\dot{y}(t) = \beta x(t) y(t) - \gamma y(t) \tag{2}$$

$$\dot{z}(t) = \gamma y(t) \tag{3}$$

$$x(t_0) = x_0 \quad y(t_0) = y_0 \tag{4}$$

with β and γ parameters chosen depending on the specific epidemic disease and $u(t)$ a bounded control.

Be $y_m \geq 0$ the minimum threshold over which a control action is needed, the aim is to determine the optimal control u, continuous almost everywhere and satisfying the box constraint $0 \leq u(t) \leq U$, and the final time $T > 0$ that minimize the cost index

$$J(u,T) = \int_0^T \left[K_1 + K_2 x(t)u(t) + K_3 x(t) + K_4 y(t) + P\big(y(t)\big)u^2(t)\right]dt,$$
$$K_1, K_2, K_3, K_4 \in R_+ \tag{5}$$

which brings the number of infected subjects to the threshold value y_m.

The constraint $0 \leq u(t) \leq U$ can be also written, in view of the approach proposed, as

$$q_1(t) = -u(t) \leq 0\,, \quad q_2(t) = u(t) - U \leq 0$$

The weight $P\big(y(t)\big)$ changes in nonlinear way depending on the number of the infected subjects. the interval $\left[y_m, +\infty\right)$ is divided into N subintervals $[y_i, y_{i+1}), i=1,...,N$, $y_1 = y_m$ and $y_{N+1} = +\infty$. $P(y(t))$ is assumed such that:

$$P(y(t)) = \alpha_i \quad for \quad y(t) \in [y_i \ \ y_{i+1}), \ \ \alpha_i \in R_+, \ i=1,...,N$$

The quantities α_i are chosen so that the higher is the severity of the disease, the lower is the cost of the control action in the cost index: $\alpha_1 > \alpha_2 > \Lambda > \alpha_N$.

In the cost index (5) the term K_1 weights the time to eradicate the epidemic as soon as possible. Two terms depend directly on the number of the susceptibles and infected subjects, weighted respectively by K_3 and K_4; there is also a mixed term that is the control weighted by the number of susceptibles, therefore the assumed vaccination effort is also susceptible-dependent. The control acts when $y(t) \geq y_m$, so, for the present application, it can be assumed that $y(t_0) = y_0 \geq y_m$; if $y_0 \in [y_i, y_{i+1})$ for some i, the cost index to be firstly minimized is:

$$J(u,T) = \int_{t_j}^{T} \left[K_1 + K_2 x(t)u(t) + K_3 x(t) + K_4 y(t) + \alpha_i u^2(t)\right] dt,$$

To solve the problem the classical optimal control theory is applied; let's define the Hamiltonian in the normal case:

$$\begin{aligned} H(x(t), y(t), u(t), \bar{t}_j, \lambda_1(t), \lambda_2(t)) &= K_1 + K_2 x(t)u(t) + K_3 x(t) \\ &+ K_4 y(t) + \alpha_i u^2(t) + \lambda_1(t)\left(-\beta x(t)y(t) - x(t)u(t) + \mu\right) \\ &+ \lambda_2(t)\left(\beta x(t)y(t) - \gamma y(t)\right) \end{aligned}$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are the Lagrange multipliers.

The necessary optimality conditions are, [13]

$$\dot{\lambda}_1(t) = -\frac{\partial H}{\partial x} =$$
$$-K_2 u(t) - K_3 + \beta y(t)\lambda_1(t) + \lambda_1(t)u(t) - \beta y(t)\lambda_2(t)$$

$$\dot{\lambda}_2(t) = -\frac{\partial H}{\partial y} = -K_4 + \beta x(t)\lambda_1(t) - \beta x(t)\lambda_2(t) + \gamma\lambda_2(t)$$

$$\dot{\lambda}_3(t) = -\frac{\partial H}{\partial z} = 0$$

$$0 = \frac{\partial H}{\partial u} + \frac{\partial q_1}{\partial u_1}\eta_1 + \frac{\partial q_2}{\partial u_1}\eta_2$$
$$= 2\alpha_i u(t) + K_2 x(t) - \lambda_1(t)x(t) - \eta_1(t) + \eta_2(t)$$

$$\eta_1(t), \eta_2(t) \in R, \qquad \eta_1, \eta_2 \in C^0 \ \ almost \ \ everywhere$$
$$q_1(t)\eta_1(t) = 0, \qquad q_2(t)\eta_2(t) = 0$$
$$\eta_1(t) \geq 0, \qquad \eta_2(t) \geq 0$$

$$H|_T = 0 \tag{6}$$

$$\lambda_1(T) = 0, \quad \lambda_2(T) = -\varsigma, \quad \lambda_3(T) = 0 \qquad \varsigma \in R \tag{7}$$

The control obtained is

$$u(t) = \begin{cases} 0 & if \ \ x(t)(\lambda_1(t) - K_2) < 0 \\ \dfrac{x(t)(\lambda_1(t) - K_2)}{2\alpha_i} & if \ \ 0 < \dfrac{x(t)(\lambda_1(t) - K_2)}{2\alpha_i} < U \\ U & if \ \ \dfrac{x(t)(\lambda_1(t) - K_2)}{2\alpha_i} < U \end{cases} \tag{8}$$

If there exists an instant $t_{j+1} < T$ such that $y(t_{j+1}) = y_{i+1}$ or $y(t_{j+1}) = y_i$, then the obtained solution is a feasible one limited to the interval $\lfloor t_j, t_{j+1})$ and, consequently, t_{j+1} is a switching instant.

Setting $j = j+1$, the procedure is iterated. $j = 0$

Otherwise, when $y(t) \in [y_1, y_2)$, $\forall t \in \lfloor t_j, T \rfloor$, and therefore $y(T) = y_1 = y_m$, the procedure ends and the optimal solution $\left(x^o(t), \ y^o(t), \ z^o(t), u^o(t), T^o\right)$ is obtained by composing the expressions defined in each subintervals $\lfloor t_j, t_{j+1})$.

In the solution represented in (8) the costate $\lambda_1(t)$ and the state $x(t)$ appear explicitly; they can be determined with $y(t)$ and the costate $\lambda_2(t)$ taking into account the initial conditions (4) with Equations (6) and (7).

3. Numerical Results

In this Section some numerical results are used to show the behavior of the proposed approach and to compare it with the classical approach which makes use of constant weights coefficients in the cost function. Consider the model described in Equations (1)-(4), with $\beta = 0.005$, $\gamma = 0.7$, $\mu = 80$, $U = 1$ and initial conditions, at time $t_0 = 0$, chosen as $x_0 = 500$, $y_0 = 101$ and $z_0 = 0$.

In the simulations, for sake of simplicity only two subintervals have been fixed. They correspond, with respect to the infection propagation, to two operative conditions: a low dangerous situation, in which the infected subjects are less than 40% of the initial population of susceptibles, and a very serious condition, in which they exceed such a threshold.

So, according to the problem formulation, one may set

$$\begin{cases} P(y(t)) = \alpha_1 = 10 & for\ \ y(t) \in [y_1, y_2) = [y_m, 200) \\ P(y(t)) = \alpha_2 = 1 & for\ \ y(t) \in [y_2, +\infty) = [200, \infty) \end{cases}$$

Moreover, $y_m = 100$ has been chosen, meaning that once the infected becomes less than 20% of the initial population of susceptibles further action is no longer required. As far as the parameters in the cost index (5), the values $K_1 = 10$, $K_2 = 0.1$, $K_3 = 0.1$, $K_4 = 0.1$ are chosen.

The simulation results are reported hereafter. Time history of the susceptibles as well as the infected subjects are depicted in Figure 1 while Figure 2 shows the behavior of the optimal control.

In order to put in evidence if, how and where the proposed switching cost function represents an improvement with respect to more traditional approaches, a comparison between this optimal solution and the solutions with constant weights, $\alpha = 1$ and $\alpha = 10$, have been performed. Reporting the behavior of the number of infected $y(t)$ and the control action $u(t)$, Figures 3 and 4 are obtained.

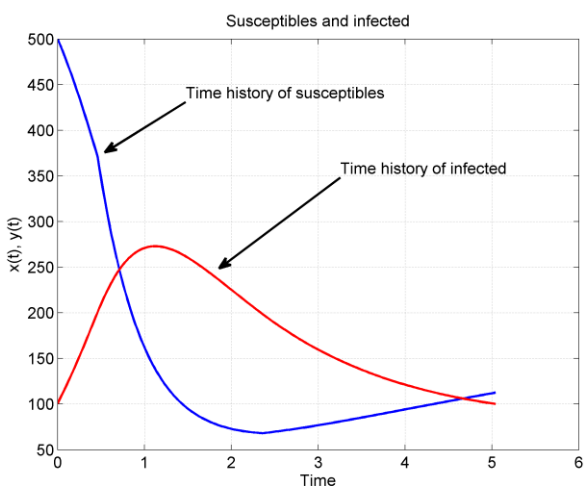

Fig. 1. Time history of susceptibles and infected subjects.

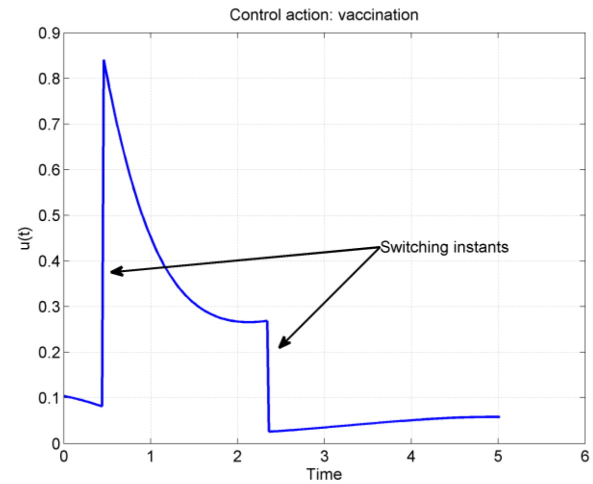

Fig. 2. Optimal control with switching instants.

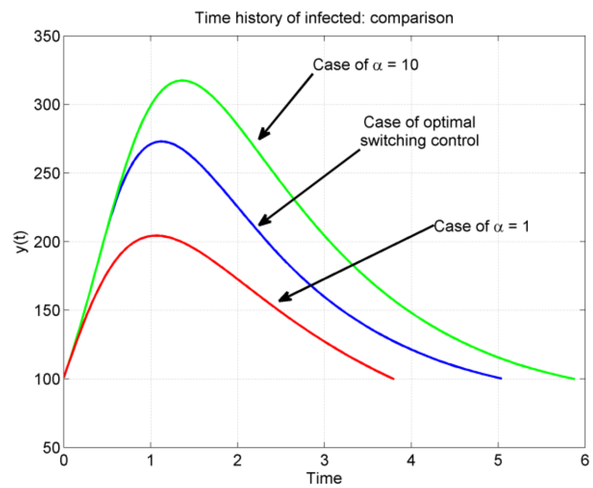

Fig. 3. Comparison of the infected subjects' evolutions.

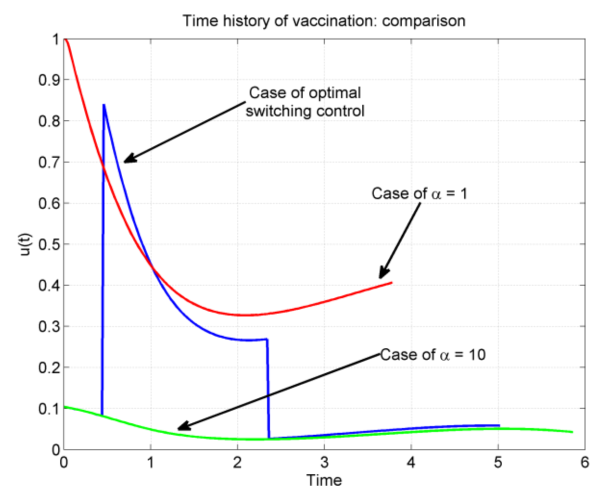

Fig. 4. Comparison of the control actions.

Clearly, making use of the fixed large weight $\alpha = 10$ a small control action is obtained producing a high number of infected at each time. Obviously, the contrary also holds: to the lower weight $\alpha = 1$, a higher control effort and a lower number of infected follow. The switching optimal solution seems to have an intermediate behavior in the infected evolution. The optimal switching control seems to follow better what one can expect in this kind of situation: a low control action is required, with corresponding reduced social and economic cost, if the epidemic diffusion is under the prefixed dangerous entity; a higher control is more appropriate when the dangerous level is exceeded, but limiting its action to the time in which the infection is more critical.

A quantification of such behaviors, with the aim of giving some references for quality comparison, can make use of the computation of the quantities $\int_0^t u(\tau)d\tau$ and $\int_0^t y(\tau)d\tau$ for $t \in [0,T]$. The first one corresponds to a sort of energy consumption evaluation for the control and it is directly related to the actual cost of the intervention; the second gives a measure of the number of infected subjects taking into account how long such a condition holds. This latter quantity can give a measure of the social and treatment costs. Results are in Figures 5 and 6.

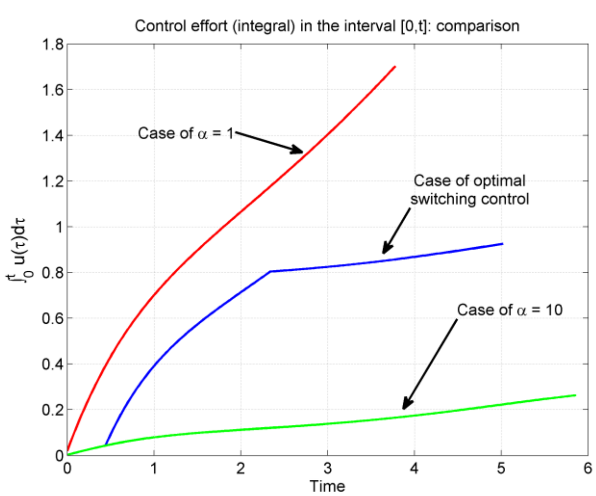

Fig. 5. Control energy estimation: $\int_0^t u(\tau)d\tau$ vs. time t

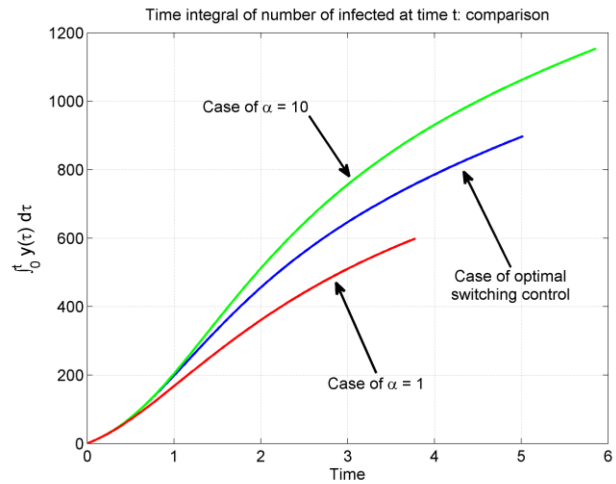

Fig. 6. $\int_0^t y(\tau)d\tau$ vs. time t

As far as the optimal final time, for the switching case one gets $T = 5.06$ while $T = 5.86$ is for the fixed weight $\alpha = 1$ and $T = 3.82$ for the fixed weight $\alpha = 10$.

In Table 1, the quantities $\int_0^T u(\tau)d\tau$ and $\int_0^T y(\tau)d\tau$ are compared with the corresponding ones obtained with the switching optimal control, $\int_0^T u^o(\tau)d\tau$ and $\int_0^T y^o(\tau)d\tau$ respectively, assuming the latter as reference.

	$\int_0^T u(\tau)d\tau$	$\int_0^T y(\tau)d\tau$	$\int_0^T u^o(\tau)d\tau$	$\int_0^T y^o(\tau)d\tau$
Optimal switching control	0.92	880	0	0
$\alpha = 1$	1.7	600	+84%	-32%
$\alpha = 10$	0.26	1160	-71%	+32%

Table 1. Evaluation of the total values for $\int_0^T u(\tau)d\tau$ and $\int_0^T y(\tau)d\tau$; comparison with $\int_0^T u^o(\tau)d\tau$ and $\int_0^T y^o(\tau)d\tau$ assumed as reference ones

The use of a low ($\alpha = 1$) constant weight in the cost function instead of a state dependent switching one produces a reduction of 32% in the amount of infected, but paying this with an increment of 84% of the control energy. On the other hand, once one tries to reduce the control effort making use of a large weight value ($\alpha = 10$), a reduction of 71% produces an epidemic 32% more severe.

The effectiveness of a switching approach can be derived also comparing the minimum value for the number of susceptibles and the time in which this happens. Figure 7 shows that for the switching control the minimum value is reached before and it is lower than both constant cases.

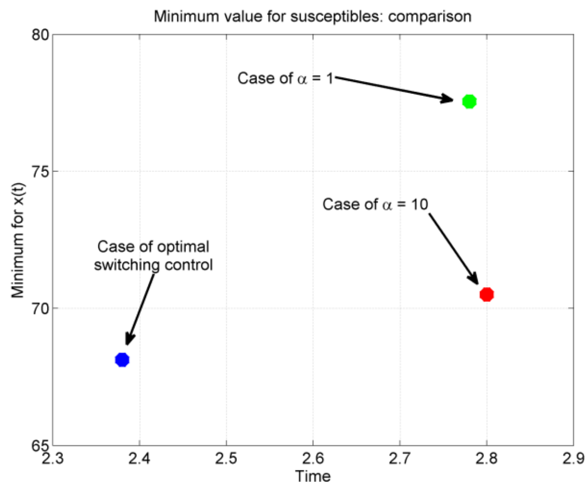

Fig. 7. Comparison of the minimum values for susceptibles

4. Conclusions

In this paper an optimal control approach with a state dependent switching cost index is proposed to determine the best strategy to control via a vaccination the epidemic disease by using a SIR model. Therefore, it is possible to change control strategy whenever new different conditions in the epidemic spread are present. Analytical solution is studied and numerical results are discussed. The proposed methodology, here presented with reference to a simple single input SIR model, could be generalized to the case of double control (both on susceptibles and infected subjects) and to more complex models.

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