

Backstepping control under multi-rate sampling

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Abstract—The paper deals with the design of sampled-data controllers which preserve the stabilizing performance of a continuous-time backstepping control strategy. This is achieved through matching of the control Lyapunov functions evolutions at the sampling instants. The method is developed for systems in strict-feedback form. The results are discussed and compared with similar strategies through simulated examples.

Index Terms—digital backstepping, multi-rate control, nonlinear systems, Lyapunov redesign

I. INTRODUCTION

AS well known, the use of digital controllers promoted the development of design procedures to overcome the loss of performance issued from sampling, quantization and computational delays. Disregarding the quantization and computational delays effects and assuming the existence of a continuous-time stabilizing backstepping controller, we propose in this work two sampled-data control schemes preserving stabilization.

Backstepping, introduced in [1] and widely developed in continuous time [2], [3], is a powerful tool for stabilizing systems in strict-feedback form. The approach is constructive and goes through the design of successive control Lyapunov functions up to the computation of a controller which asymptotically stabilizes the full dynamics. Removing the obstruction of *relative degree one*, backstepping also relies to the so called passivity based approach which achieves stabilization through output feedback and damping injection [4].

In the sampled-data context, backstepping, as other design techniques which take advantage of properties linked to the system structure, deserves special attention and ad hoc investigations because, in general, sampling destroys the structure.

In this paper we assume, as usual, that the controller is fed by sampled measures and provides piecewise constant signals; i.e. the input to the plant results from a zero order holding device - ZOH - of a discrete-time signal computed by the control algorithm. Following [5], [6], three different sampled-data control approaches can be distinguished:

- emulation - when the continuous-time control law itself, evaluated at the sampled time instants $t = k\delta, k \geq 0$, is

implemented through ZOH over intervals of length δ , the sampling period. Such an approach does not require extra effort. Improvements can be obtained by setting the design on a slightly modified model of the plant or taking into account the sampled-data architecture (redesign techniques);

- discrete-time control - when the controller is designed on a discrete-time equivalent model of the process defined for a fixed sampling period. In this case, major difficulties rely on the possible complexity of the discrete-time equivalent model which might even be not computable, on the lost through sampling of suitable structural properties (the strict feedback triangular form in the present context), on the possible lack of discrete-time design procedures [7], [8].

- sampled-data control - when the control is designed on a sampled-data equivalent model which is parametrized by the sampling period δ so that the δ -dependency can take part of the design procedure. Exact or approximate design procedures can be pursued, based on the exact or approximate sampled model (e.g. Euler in [9], higher order approximations in [10], [11]). In these cases, attention should be directed towards the best trade-off between computational complexity, sampling period length and performance.

Digital backstepping strategies have been investigated in several recent papers. In [9], a discrete-time design based on the Euler approximate discrete-time model is used to get a solution which achieves semiglobal-practical stability. In [12], assuming the existence of a continuous-time Lyapunov design, two sampled-data controllers are proposed: the first one increases negativity of the Lyapunov function increment, the second one reduces the mismatches between the continuous and the sampled Lyapunov evolutions for a given order of approximations of the Lyapunov function increment. A similar approach is pursued in [13] (see also [9], [12], [14], [15]) where the authors, with reference to a particular strict-feedback structure, look for matching the Lyapunov function increment making use of higher order approximations of the sampled model.

In the present work we propose to match, under sampled-data control, the behaviors of the Lyapunov functions used in the continuous-time design process. The idea of matching at the sampling times target behaviors which characterize the continuous-time closed loop dynamics was proposed in [16] and [17] to get sampled-data solutions to feedback linearization and stabilization problems, respectively. An approach, also pursued in [18], for the stabilization of a single-machine infinity bus system (SMIB) and in [19], to get stabilization in the context of Interconnection and Damping Assignment - Passivity Based Controller - (IDA-PBC) (see [20]).

Regarding properly backstepping, a preliminary solution proposed in [21] is further developed hereinafter and extended to a multi-rate scheme to face some typical drawbacks of

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the sampling process. As a matter of fact, apart from the lost of *lower triangular* strict-feedback structure, the relative degree fails to one under sampling so generating unstable *zero dynamics* [22]; a phenomenon which is responsible for degradation of internal stability. Matching the continuous Lyapunov functions evolutions through multi-rate controllers enables us to preserve, at least in the sense of the first approximation, the internal stability performance [23]. It must be recalled that multi-rate digital control, which corresponds to multiple variations of the control between two measures, has been introduced in the nonlinear context to overcome the mentioned degradation of performance [24]. A different multi-rate control scheme is proposed in [25] where multi-rate sampled-data measures are used to preserve stability of a slow sampled-data controller.

Starting from a continuous-time dynamics in strict-feedback form and assuming that a continuous-time backstepping procedure has been designed to provide a control Lyapunov function V and a stabilizing feedback $u_c(\cdot)$, we show in Theorem 3, the existence of a piecewise constant control law $u^\delta(\cdot)$ ensuring global asymptotic stabilization at the sampling instants. To better understand how the requested Lyapunov matching influences the internal state dynamics, it is sufficient to interpret the continuous-time design as a passifying design with respect to the function V which exhibits relative degree one and is locally minimum phase [26]. As a matter of fact, the sampled-data feedback computed to match the evolution of V at the sampling instants maintains the minimum phase property when the relative degree is equal to one [23].

The improvement proposed with the multi-rate solution takes advantage of the cascade structure of the strict-feedback dynamics and is thus specific to the backstepping procedure. Recalling that for m -cascade connected dynamics, $m + 1$ Lyapunov functions are instrumental tools, the idea is to design $m + 1$ sampled-data controllers to match, at the sampling instants, the behaviors of these $m + 1$ control Lyapunov functions. Arguing so, the existence of a multi-rate strategy of order $m + 1$ under which global asymptotic stabilization at the sampling instants is preserved with improved internal stability is shown in Theorem 16 when $m = 1$ and in Theorem 18 for a generic m . To have an insight on the theoretical issue which is behind such an improvement, it must be noticed that the successive Lyapunov functions involved in the design depend on partial state components and have decreasing relative degrees from $m + 1$ to 1. Following [27], the need of a multi-rate design of order $m + 1$ to maintain the minimum phase property is a direct consequence of these relative degree values. It results that the proposed sampled-data controller achieves both matching of the successive Lyapunov functions while guaranteeing stability in first approximation of their respective zero dynamics.

Solutions in closed form of the proposed controllers do not exist in general. Executable algorithms for computing approximate solutions at any prefixed order can be described taking advantage of the dependency in δ of the exact controller representation as a series around the continuous-time solution. The proposed design procedure can be supported by computer aided design tools which exploit the combinatoric properties of

the series expansions as proposed in [28] and tested on various examples in the electrical and mechanical domains [29], [10], [16], [17], [30], [19].

How to quantify the stabilizing performance under approximate control is a challenging problem. The case of emulated control with Euler sampled-data model is discussed in the literature due to its computational simplicity but its performance remains constrained to small enough sampling period. The formalism proposed in [31] to investigate the stability properties obtained under digital controllers designed on approximated sampled-data models is adapted to the present context to prove practical stability under approximate controller under Lipschitz condition on the Lyapunov control function. In fact, an important benefit of the here proposed multi-rate schemes is to guarantee full state matching of the continuous-time state evolutions in $O(\delta^3)$ up to $O(\delta^{m+3})$ on partial state components, a property which can be interpreted as one-step consistency following the lines in [31]. A first analysis reported in the present paper for the single-rate control strategy suggests to investigate the stabilizing properties of approximate controllers in the context of Input-to-State ISS stability (see for example in the continuous-time [32], discrete-time [33] or hybrid [34] contexts) under control error (the neglected terms). Provided ISS stability of the closed loop continuous-time design is ensured, ISS stability of the sampled-data design is verified from which conditions ensuring stability can be deduced in terms of the order of approximation and the sampling period length. Work is progressing in this direction to better quantify the expected advantages of the strategies and the Maximum Allowable Sampling Period - MASP.

The paper is organized as follows. Section II deals with notations and a brief recall of backstepping design. Single-rate and multi-rate control laws are proposed in sections III and IV, respectively. Simulation results are illustrated and discussed in sections V where a comparison of the proposed schemes with similar strategies is also provided. The proposed controllers are illustrated through two examples; the first example, used as case study in [12], [9], [13], provides a meaningful comparative evaluation of the performance.

II. RECALLS AND BASIC FACTS

A. Notations

Throughout the paper, maps and vector fields are assumed smooth (i.e. infinitely differentiable of class C^∞) and the associated dynamics forward complete to guarantee the existence of solutions and prevent from finite escape time. Given a vector field f , L_f denotes the associated Lie derivative operator, $L_f = \sum_{i=1}^n f_i(\cdot) \frac{\partial}{\partial x_i}$, e^{L_f} (or e^f when no confusion is possible) denotes the associated Lie series operator, $e^f := 1 + \sum_{i \geq 1} \frac{L_f^i}{i!}$. Given two vector fields f, g on \mathbb{R}^n , their Lie bracket is defined as $ad_f g := [f, g] := [L_f, L_g] := L_f \circ L_g - L_g \circ L_f$ and in an iterative way, $ad_f^i g := [f, ad_f^i g]$ with $ad_f^0 g := g$. To simplify the notations when no ambiguity is possible, $L_f L_g$ stands for $L_f \circ L_g$. For any smooth real valued function h , the following result holds $e^f h(x) = e^f h|_x = h(e^f x)$ where $e^f x$ stands for $e^{L_f} I_d | x$ with I_d the identity function on \mathbb{R}^n

and (x) (or equivalently $|_x$) denotes the evaluation at a point x of a generic map. The evaluation of a function at time $t = k\delta$ indicated by “ $|_{t=k\delta}$ ” is omitted, when it is obvious from the context. Given two exponential operators e^X and e^Y where X and Y are formal variables which stand for any two vector fields, the associated Baker-Campbell-Hausdorff exponent, denoted as $\mathcal{BCH}^2(\cdot)$, describes the exponent series of the non-commutative composition of the two operators:

$$\begin{aligned} e^X \circ e^Y &= e^{\mathcal{BCH}^2(X,Y)} \\ &= e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]+[Y,[Y,X]])+\dots} \end{aligned} \quad (1)$$

This formula can be iteratively generalized to the composition of m exponential series so defining $\mathcal{BCH}^m(X_1, \dots, X_m)$; further details can be found in [35]. We notice that all the manipulations performed over series of operators described by their asymptotic expansions are formal ones in the sense that no convergence study is performed.

As usual, $|\cdot|$ indicates the Euclidian norm and $\|\cdot\|$ the supremum norm of a function, typically an input. Any function or vector λ is said to satisfy the Lipschitz condition if for each compact set \mathcal{X} of $\mathbb{R}^n - \{0\}$ there exists a constant $M > 0$ such that $|V(z) - V(x)| \leq M|z - x|$ for all $x, z \in \mathcal{X}$. A positive function ρ is said of class \mathcal{K} if it is continuous, strictly increasing and zero at zero, it is of class \mathcal{K}_∞ when it is unbounded. A positive function β is said of class \mathcal{KL} if it is continuous and for each s , $\beta(\cdot, s) \in \mathcal{K}$ and for each r , $\beta(r, \cdot)$ is decreasing with respect to s and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$. A function $R(x, \delta)$ is of order δ^p , $p \geq 1$ and we write $R(x, \delta) = O(\delta^p)$, if whenever R is defined it can be written as $R(x, \delta) = \delta^{p-1} \tilde{R}(x, \delta)$ and there exist a function $\theta \in \mathcal{K}_\infty$ and $\delta^* > 0$ such that for each $\delta \leq \delta^*$, $|\tilde{R}(x, \delta)| \leq \theta(\delta)$.

B. The class of strict-feedback dynamics

We consider strict-feedback dynamics [2] with m -cascade connections of the form:

$$\begin{aligned} \dot{z}(t) &= f(z) + g(z)\xi_1 \\ \dot{\xi}_1(t) &= a_1(z, \xi_1) + b_1(z, \xi_1)\xi_2 \\ &\vdots \\ \dot{\xi}_{m-1}(t) &= a_{m-1}(z, \xi_{1/m-1}) + b_{m-1}(z, \xi_{1/m-1})\xi_m \\ \dot{\xi}_m(t) &= a_m(z, \xi_{1/m}) + b_m(z, \xi_{1/m})u(t) \end{aligned} \quad (2)$$

where $z \in \mathbb{R}^n$, $\xi_i \in \mathbb{R}$, $\xi_{1/j} = (\xi_1, \dots, \xi_j)$ and the $b_i(\cdot)$ are assumed different from 0. We also write in a compact way:

$$\dot{x}(t) = f_c(x) + g_c(x)u \quad (3)$$

with $x = (z', \xi_{1/m})' \in \mathbb{R}^{n+m}$ and adequately defined vector fields f_c and g_c .

C. Sampled-data equivalent models

Assuming the control $u(t)$ constant over time intervals of length δ , $u(t+\tau) = u(t) = u_k$ for $0 \leq \tau < \delta$, $t = k\delta$, $k \geq 0$, the discrete-time dynamics which describes the evolutions of (3) at the sampling instants $t = k\delta$ defines the *equivalent sampled-data model*. It takes the form of a map $F^\delta(\cdot, u_k)$,

parameterized by δ , which admits as asymptotic expansion the Lie exponential series.

$$x_{k+1} = F^\delta(x_k, u_k) = e^{\delta(f_c + u_k g_c)} x_k. \quad (4)$$

Multi-rate sampling of order m - MR^m - refers to the actuation of the control variable m -times over each time interval. More precisely, $u(t)$ is maintained constant at values u_{ik} , over intervals of length $\bar{\delta} = \frac{\delta}{m}$ for all $t \in [k\delta + (i-1)\bar{\delta}, k\delta + i\bar{\delta}]$, $i = 1/m$. Over time intervals of length $\delta = m\bar{\delta}$, one gets the *equivalent MR^m -sampled-data model* of (3):

$$\begin{aligned} x_{k+1} &= F^{m\bar{\delta}}(x_k, u_{1k}, \dots, u_{mk}) \\ &= e^{\bar{\delta}(f_c + u_{1k} g_c)} \circ \dots \circ e^{\bar{\delta}(f_c + u_{mk} g_c)} x_k. \end{aligned} \quad (5)$$

(5) admits an exponential Lie series representation in terms of the $\mathcal{BCH}^m(\dots)$ exponent [35]; i.e. $e^{\mathcal{BCH}^m(\dots)} := e^{\bar{\delta}(f_c + u_{1k} g_c)} \circ \dots \circ e^{\bar{\delta}(f_c + u_{mk} g_c)}$ so generalizing (4).

Besides (4) and (5), the so-called *exact single-rate (resp. multi-rate) sampled-data dynamics* represented by the pairs (u, F^δ) (resp. $(u_{i=1/m}, F^{m\bar{\delta}})$), one defines the *approximate single-rate (resp. multi-rate) sampled-data model of degree p* as the truncations of the expansions (4) (resp. (5)) at finite order p in δ ; i.e.

$$\begin{aligned} F^\delta(x, u) &= F_a^{[p]}(x, u) + O(\delta^{p+1}) \\ F^{m\bar{\delta}}(x, u_{i=1/m}) &= F_a^{[p]}(x, u_{i=1/m}) + O(\delta^{p+1}). \end{aligned}$$

Remark 1. : For $p = 1$, one recovers the well known Euler sampled-data dynamics $x_{k+1} \cong x_k + \delta f_c(x_k) + \delta u_k g_c(x_k)$ with the same nonlinearities as the continuous-time ones so explaining why most of the results set in continuous time are maintained through Euler sampling.

D. Continuous-time backstepping - recalls

Some instrumental steps are recalled below.

1) *one-cascade connection*: consider (2) with $m = 1$:

$$\dot{z}(t) = f(z) + g(z)\xi \quad (6)$$

$$\dot{\xi}(t) = a(z, \xi) + b(z, \xi)u. \quad (7)$$

Theorem 2. [2] - *Continuous-time backstepping* - Let the dynamics (6-7) and assume there exist a smooth function $\phi(z)$ with $\phi(0) = 0$ and $W(z) > 0$, radially unbounded, such that:

$$\frac{\partial W}{\partial z}(f(z) + g(z)\phi(z)) < 0, \forall z \in \mathbb{R}^n / \{0\} \quad (8)$$

then the state feedback control law:

$$u_c = b^{-1}(z, \xi) \left(\dot{\phi}(z) - \frac{\partial W}{\partial z} g(z) - a(z, \xi) + v \right) \quad (9)$$

with $v = -K_y(\xi - \phi(z))$, $K_y > 0$ and $\dot{\phi}(z) = \frac{\partial \phi}{\partial z}(f(z) + g(z)\xi)$ globally asymptotically stabilizes the origin - GAS.

Proof: Assuming $\xi = \phi(z)$, asymptotic stabilization of the z -dynamics follows from (8). Setting $y = \xi - \phi(z)$, (6-7) rewrite as:

$$\dot{z}(t) = \tilde{f}(z) + g(z)y \quad (10)$$

$$\dot{y}(t) = \tilde{a}(z, y) + \tilde{b}(z, y)u \quad (11)$$

so that the feedback (9) with $v = -K_y y$, $K_y > 0$ achieves GAS because of (8) and $W(z)$ radially unbounded; i.e.

$$\dot{V} = \frac{\partial W}{\partial z}(f(z) + g(z)\phi(z)) - K_y y^T y < 0 \quad (12)$$

with control Lyapunov function:

$$V(z, \xi) = W(z) + \frac{1}{2}(\xi - \phi(z))^2. \quad (13)$$

A local result does not request radial unboundedness of W and condition (8) can be relaxed to negative semi-definiteness under zero state detectability of y .

2) *multiple cascade connexions*: with reference to the general form (2), the design procedure can be reduced to the iterative application of the result recalled in Theorem 2.

• 1st step: set $y_1 = \xi_1 - \phi(z)$ and $z^1 = (z', y_1)'$. Under the assumptions of Theorem 2, the fictitious controller:

$$\phi_1(z^1) = \tilde{b}_1^{-1}(z^1) \left(-\frac{\partial W}{\partial z} g(z) - \tilde{a}_1(z^1) + v_1 \right) \quad (14)$$

with $v_1 = -K_1 y_1$, $K_1 > 0$ asymptotically stabilizes the first cascade connection with control Lyapunov function $V_1(z^1) = W(z) + \frac{1}{2}y_1^2$.

• ...

• i^{th} step - $i \geq 2$: set $y_i = \xi_i - \phi_{i-1}(z^{i-1})$, $z^i = (z', y_1/i)'$, the controller:

$$\phi_i(z^i) = \tilde{b}_i^{-1}(z^i) \left(-\frac{\partial V_{i-1}}{\partial y_{i-1}} \tilde{b}_{i-1}(z^{i-1}) - \tilde{a}_i(z^i) + v_i \right) \quad (15)$$

with $v_i = -K_i y_i$, $K_i > 0$, asymptotically stabilizes the first i connections with control Lyapunov function:

$$V_i(z^i) = V_{i-1}(z^{i-1}) + \frac{1}{2}y_i^2 = W(z) + \frac{1}{2} \sum_{j=1}^i y_j^2 \quad (16)$$

• ...

• Final m^{th} step: the controller $u_c = \phi_m(z^m)$ with $v_m = -K_m y_m$, $K_m > 0$ achieves global asymptotic stabilization of the whole control system with Lyapunov function:

$$V(z^m) = W(z) + \frac{1}{2} \sum_{j=1}^m y_j^2. \quad (17)$$

In the sequel, we denote by $(V_i(z^i))_{i=0/m}$ with $V_0(z) := W(z)$, $V_m(z^m) := V(z^m)$, the family of $(m+1)$ -control Lyapunov functions involved in the backstepping design.

III. SAMPLED-DATA BACKSTEPPING UNDER SINGLE-RATE

Given the strict-feedback dynamics (2), we assume that the continuous-time backstepping procedure has been preliminarily worked out up to the last step to get equations (10-11).

A. *Input-Lyapunov matching under piecewise constant control*
Consider (10-11) rewritten in compact form with $z^1 = (z', y)'$:

$$\dot{z}^1(t) = f_c(z^1) + g_c(z^1) \cdot u_c(z^1). \quad (18)$$

Given u_c in (9) with Lyapunov function V in (13) satisfying (12), integration over one sampling interval gives:

$$V(z^1|_{t=(k+1)\delta}) - V(z^1|_{t=k\delta}) = \int_{k\delta}^{(k+1)\delta} \dot{V}(z^1(\tau)) d\tau \quad (19)$$

where $z^1(t)$ indicates the closed loop continuous-time z^1 -dynamics under u_c . More in detail, (19) specifies the one-step ahead difference assumed by the Lyapunov function between two successive sampling times under the action of the continuous-time controller; it describes the target difference that we will match under piecewise constant control.

Sampled-Data Input-Lyapunov Matching - SD-ILM - means the design of a piecewise constant control law which assures one-step matching of the Lyapunov function; i.e. find $u_k^\delta = u^\delta(z_k^1)$ so that one-step ahead:

$$V(F^\delta(z_k^1, u_k^\delta)) - V(z_k^1) = \int_{k\delta}^{(k+1)\delta} \dot{V}(z^1(\tau)) d\tau \quad (20)$$

when $z_k^1 = z^1|_{t=k\delta}$. The *Lyapunov Matching Error - LME* - $E_V(z_k^1, \delta)$ - is the one-step mismatch between the values at time $t = (k+1)\delta$ of $V(z^1)$ under continuous-time $u_c(t)$ and constant controls u^δ , when $z_k^1 = z^1|_{t=k\delta}$; i.e.

$$E_V(z_k^1, \delta) := V(z^1|_{t=(k+1)\delta}) - V(z_k^1) := \sum_{p \geq 1} \delta^p E_V^p(z_k^1) \quad (21)$$

with by definition:

$$\begin{aligned} V(z^1|_{t=(k+1)\delta}) &= V(e^{f_c + u_c g_c} z^1)|_{t=k\delta} \\ &= (V + \delta L_{f_c + u_c g_c} V + \frac{\delta^2}{2!} L_{f_c + u_c g_c}^2 V)(z^1)|_{t=k\delta} + O(\delta^3) \end{aligned}$$

and

$$\begin{aligned} V(z_{k+1}^1) &= V(e^{\delta(f_c + u_k^\delta g_c)} z_k^1) \\ &= (V + \delta L_{f_c + u_k^\delta g_c} V + \frac{\delta^2}{2!} L_{f_c + u_k^\delta g_c}^2 V)(z_k^1) + O(\delta^3). \end{aligned}$$

B. *Digital backstepping under single-rate for a 1-connection*

Theorem 3. Consider a strict-feedback dynamics (6-7) under the assumptions of Theorem 2, then there exist $T^* > 0$ and for each $\delta \in]0, T^*[$, a digital feedback $u^\delta(z^1)$ of the form :

$$u^\delta(z^1) = u^0(z^1) + \sum_{i \geq 1} \frac{\delta^i}{(i+1)!} u^i(z^1) \quad (22)$$

with $u^0(z^1) = u_c(z^1)$ which ensures SD-ILM of V defined in (13) and guarantees SD-GAS.

Proof: The proof works out showing that there exists a solution u_k^δ to the SD-ILM equality (20). The index "k" is omitted when clear from the context. For, one rewrites (20) as a formal series equality:

$$\delta Q(z^1, \delta, u^\delta) = e^{\delta(f_c + u^\delta g_c)} V|_{z^1} - e^{\delta(f_c + u_c g_c)} V|_{z^1} = 0 \quad (23)$$

and looks for u^δ which satisfies for all $z^1 \in R^n$ the equality:

$$Q(z^1, \delta, u^\delta) = 0 \quad (24)$$

with by definition $Q(z^1, \delta, \cdot) := Q^0(z^1, \cdot) + \sum_{i \geq 1} \delta^i Q^i(z^1, \cdot)$.

At first, it is easily verified that setting $u^0 = u_c(z^1)$, one satisfies (24) for $\delta = 0$; i.e.

$$Q^0(z^1, u^0) = (L_{f_c} + u^0 L_{g_c})V|_{z^1} - (L_{f_c} + u_c L_{g_c})V|_{z^1} = 0.$$

Then, provided the rank condition:

$$\left. \frac{\partial Q(z^1, \delta, u)}{\partial u} \right|_{\delta=0, u=u^0} \neq 0 \quad (25)$$

holds true, one concludes from the Implicit Function Theorem [36, Th.7.9], the existence of $T^* > 0$ small enough and, for each $\delta \in]0, T^*[$, the existence of a solution to (24) in the form of an asymptotic expansion (22) around $u_c(z^1)$. Condition (25) is an immediate consequence of the strict feedback structure: $\frac{\partial Q(z^1, \delta, u)}{\partial u}|_{\delta=0} = L_{g_c}V(z^1) = y \cdot \tilde{b}(z, y) \neq 0$ when $y \neq 0$ with $b(z, y) = b(z, \xi)$. It results that SD-GAS is satisfied in the absence of finite escape time (forward completeness assumption) with, by construction, Lyapunov function V ; i.e.

$$V(z_{k+1}^1) - V(z_k^1) = \int_{k\delta}^{(k+1)\delta} (L_{f_c} + u_c L_{g_c})V(z^1(\tau))d\tau < 0$$

since $(L_{f_c} + u_c L_{g_c})V(z^1) < 0$. ■

Remark 4. T^* , which defines the Maximal Allowable Sampling Period - MASP - exists in a neighborhood of zero but has to be determined.

Denoting by (u^δ, F^δ) for any $\delta \in]0, T^*[$ the sampled dynamics (4) under state feedback (22), the following results hold.

Proposition 5. Under the assumptions of Theorem 2, there exist functions $(\rho_1, \rho_2) \in \mathcal{K}_\infty, \rho_3 \in \mathcal{K}$ and control Lyapunov function V in (13) such that under $u^\delta(z)$ in (22), the pair (u^δ, F^δ) satisfies:

$$\rho_1(|z_k^1|) \leq V(z_k^1) \leq \rho_2(|z_k^1|) \quad (26)$$

$$V(z_{k+1}^1) - V(z_k^1) \leq -\delta \rho_3(|z_k^1|). \quad (27)$$

Proof: The assumptions set in Theorem 2 guarantee [2] the existence of a continuous-time Lyapunov function V and functions $(\rho_1, \rho_2) \in \mathcal{K}_\infty, \rho \in \mathcal{K}$ such that $\forall z^1 \in R^{n+1}$:

$$\rho_1(|z^1(t)|) \leq V(z^1(t)) \leq \rho_2(|z^1(t)|) \quad (28)$$

$$\dot{V}(z^1(t)) \leq -\rho(|z^1(t)|). \quad (29)$$

As $u^\delta(\cdot)$ is properly designed to match at the sampling time instants the evolution of V under $u_c(\cdot)$, condition (26) immediately follows from (28). Then, from (29) and the Mean Value Theorem, one firstly deduces the existence of $\tau_m \in [k\delta, (k+1)\delta]$ so that

$$\int_{k\delta}^{(k+1)\delta} \dot{V}(z^1(\tau))d\tau \leq -\delta \rho(|z^1(\tau_m)|)$$

and a positive definite function $W_a(z^1)$ so that:

$$\int_{k\delta}^{(k+1)\delta} \dot{V}(z^1(\tau))d\tau = -\delta W_a(z_k^1).$$

Moreover, from [2, Lemma 4.3], there exist functions $\rho_3, \rho_4 \in \mathcal{K}_\infty$ such that:

$$\rho_3(|z_k^1|) \leq W_a(z_k^1) \leq \rho_4(|z_k^1|)$$

and thus (27) follows with $z^1|_{t=k\delta} = z_k^1$ and since $E_V(z_k^1, \delta) = 0$ under u^δ . ■

Remark 6. According to Lemma 4 in [31], Proposition 5 can be reformulated as (β, R^{n+1}) -stability of the pair (u^δ, F^δ) ; i.e. there exists $\beta \in \mathcal{KL}$ such that

$$|z_k^1| \leq \beta(|z_0^1|, k\delta) \quad \forall z_0^1 \in R^{n+1} \quad k \geq 0.$$

Remark 7. According to a result in [23] and by construction of $u^\delta(\cdot)$, the local minimum phase property with respect to V is maintained under sampling.

C. On approximate solutions

Theorem 3 states the existence of a sampled-data controller $u^\delta(\cdot)$ ensuring GAS of the equilibrium. However, a closed form solution does not exist in general and to compute its asymptotic series expansion is a difficult task. The design is thus limited to approximate solutions. An insight in the computation and performance of approximate solutions is given. For, the p -th approximate controller is denoted by:

$$u^{[p]}(z^1) := u^0(z^1) + \sum_{i=1}^p \frac{\delta^i}{(i+1)!} u^i(z^1). \quad (30)$$

Two interesting results can be proven. A first result refers to the one-step consistency property accordingly to [31]. In our context, it prevents from a large mismatch between the state evolutions of system (18) under digital controller $u^\delta(\cdot)$ and under continuous-time control $u_c(\cdot)$, at the sampling instants.

Proposition 8 (one-step consistency). Under the assumptions of Theorem 2, the pair (u^δ, F^δ) designed in Theorem 3 is one-step consistent with the continuous-time closed loop dynamics $(u_c, e^{f_c+u_c g_c} z^1)$; i.e. there exists a function $\theta_1 \in \mathcal{K}_\infty$ such that for each $\delta \in]0, T^*[$, for all $z_k^1 \in R^{n+1}$ with $z_k^1 = z^1(t)|_{t=k\delta}$, one has:

$$|F^\delta(z_k^1, u_k^\delta) - e^{\delta(f_c+u_c g_c)} z^1(t)|_{t=k\delta}| \leq \delta^2 \theta_1(\delta). \quad (31)$$

Moreover, (31) holds true under p -th approximate controller with $p \geq 1$.

Proof: It is enough to recall that one step matching of any function $\lambda : R^{n+1} \rightarrow R$ with relative degree 1, ensures state matching in $O(\delta^3)$ at least [17]. Since V exhibits relative degree 1, one immediately verifies:

$$F^\delta(z_k^1, u_k^{[p]}) = e^{\delta(f_c+u_c g_c)} z^1(t)|_{t=k\delta} + O(\delta^3)$$

and the result follows as soon as $p \geq 1$. ■

Practical stability under approximate controllers can now be deduced from the results in [31] according to the definition below when assuming V to be Lipschitz.

Definition 9. Let u denote a generic piecewise constant control, let $\beta \in \mathcal{KL}$ and \mathcal{N} an open (not necessarily bounded) set in R^{n+1} containing the origin. The pair (u, F^δ) is said

to be (β, \mathcal{N}) -practical stable if for each $R > 0$, there exist $T^* > 0$ such that for any $\delta \in]0, T^*]$

$$|z_k^1| \leq \beta(|z_0^1|, k\delta) + R \quad \forall z_0^1 \in \mathcal{N}, \quad k \geq 0.$$

Proposition 10 (Practical stability under p -th approximate controller). *Let u^δ be designed as in Theorem 3 under SD-ILM of a Lipschitz control Lyapunov function V ; then for any integer $p \geq 1$ the pair $(u^{[p]}, F^\delta)$ is (β, R^{n+1}) -practically stable.*

Proof. The proof follows directly from Theorem 2 in [31] when verifying first that the pair $(u^{[p]}, F^\delta)$ is one-step consistent with (u^δ, F^δ) (from Proposition 8 above) and secondly that the pair (u^δ, F^δ) is GAS and satisfies for any $\delta \in]0, T^*]$ Proposition 5 with control Lyapunov function V assumed Lipschitz. \square

A second result directly specifies Proposition 5 under $u^{[p]}(\cdot)$.

Proposition 11. *Under the assumptions of Theorem 2, let $u^\delta(\cdot)$ designed as in Theorem 3 with control Lyapunov function V , then for any $p \geq 1$, there exists $\theta_p \in \mathcal{K}_\infty$ such that the pair $(u^{[p]}, F^\delta)$ satisfies for each $\delta \in]0, T^*], \forall z_k^1 \in R^{n+1}$:*

$$V(z_{k+1}^1) - V(z_k^1) \leq -\delta \left(\rho_3(|z_k^1|) - \delta^p \theta_p(\delta) \right). \quad (32)$$

i.e. $u^{[p]}(\cdot)$ ensures SD-ILM of V in $O(\delta^{p+2})$.

Proof. By construction, $u^{[p]}(z_k^1)$ satisfies:

$$V(z^1|_{t=(k+1)\delta}) = V(z_{k+1}^1) + E_V^{[p+2]}(z_k^1, \delta) \quad (33)$$

with an error $E_V^{[p+2]}(z^1, \delta) := \sum_{j \geq 0} \delta^{j+p+2} E_V^{p+2+j}(z^1)$ in $O(\delta^{p+2})$ so immediately deducing (32) from (27). \square

Remark 12. *Condition (32) highlights the interplay between the length of δ , the order p of approximation and the closeness to zero of z_k^1 when guaranteeing negativity of the Lyapunov function increment $V(z_{k+1}^1) - V(z_k^1)$. Under p -th approximate controller, negativity of the Lyapunov difference is ensured provided by $\rho_3(|z_k^1|) > \delta^p \theta_p(\delta)$. A condition which specifies as $\rho_3(|z_k^1|) > \theta_0(\delta)$ under emulated control, $\rho_3(|z_k^1|) > \delta \theta_1(\delta)$ under approximate control at order one ($p = 1$), and so on. Such a condition provides a first characterization of the possible benefit by increasing p to ensure negativity of the Lyapunov difference.*

To better quantify condition (32), we define besides the p -th approximate controller in (30), the control error variable $e_p(z^1, \delta)$ as:

$$e_p^\delta(z^1) := u^{[p]}(z^1) - u^\delta(z^1) := \delta^{p+1} \epsilon_p^\delta(z^1)$$

and rewrite the dynamics (18) under p -th approximate controller as an hybrid dynamics for $t \in [k\delta, (k+1)\delta[; k \geq 0$

$$\dot{z}^1(t) = f_c(z^1) + u^\delta(z_k^1) g_c(z^1) + e_p^\delta(z_k^1) g_c(z^1) \quad (34)$$

with stable "drift" term $f_c + u^\delta(z_k^1) g_c$ and forced term weighted by δ^{p+1} .

The result in Proposition 11 can be further specified in the formalism of input-to-state stability - ISS - of hybrid dynamics (see for example [32] in continuous-time, [33] in discrete-time or [34] in the hybrid context). Provided the Lyapunov function V in Theorem 2 is an ISS Lyapunov function for (18) under the feedback $u_c(\cdot) + v(t)$, it can be easily shown that V is an ISS Lyapunov function for the sampled-data dynamics under state feedback $u^\delta(\cdot) + v_k$ with $v_k = v(t)$ for $t \in [k\delta, (k+1)\delta[; k \geq 0$ and $u^\delta(\cdot)$ designed in Theorem 3.

Specifying this result on the dynamics (34), one gets:

$$V(z_{k+1}^1) - V(z_k^1) \leq -\delta \rho_3(|z_k^1|) + \delta^{p+2} \sigma_p(|\epsilon_p^\delta|) \quad (35)$$

with $\sigma_p \in \mathcal{K}$ and ρ_3 , assumed to be \mathcal{K}_∞ , are deduced from the corresponding functions which specify the ISS property of (18). Since V is an ISS Lyapunov function for the sampled-data system under feedback $u^\delta(\cdot) + e_p^\delta(\cdot) := u^{[p]}(\cdot)$, the ISS stability of the pair $(u^{[p]}(\cdot), F^\delta)$ for any $p \geq 1$ follows; i.e. there exist functions $\beta \in \mathcal{K}\mathcal{L}, \sigma \in \mathcal{K}$ such that for all $z_0^1 \in R^{n+1}$, for all $e_p^\delta \in l_\infty$

$$|z_k^1| \leq \beta(|z_0^1|, k) + \sigma(|e_p^\delta|) \quad \forall k \geq 0. \quad (36)$$

Remark 13. *As (32), condition (35) highlights the interplay between the length of δ , the closeness to zero of z_k^1 and the error control variable ϵ_p in guaranteeing negativity of the Lyapunov function increment \dot{V} . Under p -th approximate controller negativity of the Lyapunov function increment is ensured provided $\rho_3(|z_k^1|) > \delta^{p+1} \sigma_p(|\epsilon_p^\delta(z^1, \delta)|)$. Such a condition can be rewritten as:*

$$\{|z_k^1| > \delta^{p+1} \bar{\sigma}_p(|\epsilon_p^\delta|)\} \rightarrow \{V(z_{k+1}^1) - V(z_k^1) \leq -\delta \rho_3(|z_k^1|)\}$$

with $\bar{\sigma}_p := \rho_3^{-1} \circ \sigma_p \in \mathcal{K}$ and $\lim_{p \rightarrow \infty} \epsilon_p^\delta = 0$.

Remark 14. *Easy computations (see [33] in discrete time) show that the gain function σ in (36) is of the form*

$$\sigma(|\epsilon_p^\delta|) = \delta^{p+1} \bar{\sigma}(|\epsilon_p^\delta|) \text{ with } \bar{\sigma} = \rho_1^{-1} \circ \rho_2 \circ \rho_3^{-1} \circ \rho^{-1} \circ \sigma_p \in \mathcal{K}$$

with ρ_1 in Proposition 5, $\rho \in \mathcal{K}_\infty$ such that $(I_d - \rho) \in \mathcal{K}_\infty$ so qualifying the interest of increasing the order p .

Remark 15. *The emulated controller $u^0(z_k^1) = u_c(z^1)|_{t=k\delta}$ brings to an error on V - matching in $O(\delta^2)$; i.e.*

$$E_V^{[2]}(z_k^1, \delta) = -\frac{\delta^2}{2!} \dot{u}_c L_g V(z_k^1) + O(\delta^3)$$

so that the Lyapunov difference verifies

$$V(z_{k+1}^1) - V(z_k^1) \leq -\delta \rho_3(|z_k^1|) - \frac{\delta^2}{2!} \dot{u}_c L_g V(z_k^1) + O(\delta^3)$$

putting in light that its negativity depends on the sign of $\dot{u}_c L_g V(z_k^1)$; i.e. the condition $\rho_3(|z_k^1|) \geq -\frac{\delta}{2!} \dot{u}_c L_g V(z_k^1) + O(\delta^2)$ has to be satisfied.

Further investigations along these lines must be developed for better quantifying the improvements, in terms of performance and/or MASP increments, when higher order single/multiple rate sampled-data controllers are used.

D. Some computational aspects

By replacing u_k^δ with its series expansion (22) into the SD-ILM equality (20) (equivalently (24)) and by equating the terms of the same power in δ in both sides of the equality, one reduces the computation of each successive term u^i to solving a linear equality. More in detail, $Q(z^1, \delta, u^\delta)$ in (24) rewrites:

$$Q(z^1, \delta, u^\delta) = - \sum_{i \geq 0} \frac{\delta^i}{(i+1)!} (f_c + u_c g_c)^i V|_{z^1} + \sum_{i \geq 0} \sum_{j_0 + \dots + j_{i+1} = i} \frac{\delta^i}{(j_0+1)! \dots (j_{i+1}+1)!} f_{j_1} \circ \dots \circ f_{j_{i+1}} V|_{z^1}$$

with $f_0 := f_c + u^0 g_c$; $f_{i \geq 1} := u^i g_c$ and the summation over the indices $j_0, \dots, j_{i+1} \geq 0$ such that $\sum_{l=0}^{i+1} j_l = i$. One easily verifies that Q^i depends nonlinearly on the terms u^0, \dots, u^{i-1} but is affine in u^i ($j_0 = 0, j_{i+1} = i$) through the term $f_i V|_{z^1} := u^i g_c V|_{z^1} \neq 0$ since V has relative degree one. Thus, each u^i solves a linear equality depending on the previously computed terms and the algorithm is said constructive. Based on strong combinatoric properties of the manipulated exponential series investigated in [35], a computer aided design algorithm has been developed in [28]. For the first terms, one computes:

$$\begin{aligned} u_k^0 &= u_c(z^1) \Big|_{t=k\delta} \\ u_k^1 &= \dot{u}_c(z^1) \Big|_{t=k\delta} = \frac{\partial u_c(z^1)}{\partial z} (f(z) + g(z)(\phi(z) + y)) \Big|_{t=k\delta} + \\ &+ \frac{\partial u_c(z^1)}{\partial y} (a(z, y + \phi(z)) - \dot{\phi}(z)) \Big|_{t=k\delta} \\ u_k^2 &= \ddot{u}_c(z^1) \Big|_{t=k\delta} + \frac{u_k^1}{2b(z, y + \phi(z))y} ad_{[f,g]} V(z^1) \Big|_{t=k\delta} \end{aligned} \quad (37)$$

where \dot{u}_c and \ddot{u}_c represent the derivatives with respect to t of the continuous-time control $u_c(t)$ and the function $ad_{[f,g]} V$ is described in (38).

E. Single rate solution for multiple cascade connexions

The result stated in Theorem 3 can be iteratively applied to design a SD controller for the m -cascade connections case (2). For, it is enough to note that the continuous-time backstepping procedure ends with a dynamics in the form (3) with $z^m = (z', y_{1/m})'$ and:

$$f_c = \begin{bmatrix} \tilde{f}(z^1) \\ \tilde{a}_1(z^1) + \tilde{b}_1(z^1)(y_2 + \phi_1(z^1)) \\ \vdots \\ \tilde{a}_{m-1}(z^{m-1}) + \tilde{b}_{m-1}(z^{m-1})(y_m + \phi_{m-1}(z^{m-1})) \\ \tilde{a}_m(z^m) \end{bmatrix}$$

$$g_c = \begin{bmatrix} 0 & \dots & b_m(z, y_1 + \phi(z), \dots, y_m + \phi_{m-1}(z^{m-1})) \\ 0 & 0 & \dots & \tilde{b}_m(z^m) \end{bmatrix}^T$$

where $\tilde{f}(z^1) = f(z) + g(z)(y_1 + \phi(z))$ and $\tilde{a}_i(z^i) = a_i(z, \dots, y_i + \phi_{i-1}(z^{i-1})) - \dot{\phi}_{i-1}(z^{i-1})$ for $i = 1/m$. Setting further $\zeta = z^{m-1} \in R^{n+m-1}$, $\eta = y_m \in R$, $\phi = \phi_{m-1}$, $\tilde{b}(\zeta, \eta) = \tilde{b}(z^{m-1}, y_m)$, $\tilde{g}(\zeta) =$

$\begin{bmatrix} 0 & \dots & 0 & \tilde{b}_{m-1}(z^{m-1}) \end{bmatrix}^T$, $\tilde{a}(\zeta, \eta) = \tilde{a}(z^{m-1}, y_m) - \frac{\partial \phi}{\partial \zeta} \dot{\zeta}$, one recovers a dynamics of the form of (6-7) which is the structure assumed in Theorem 3; i.e.

$$\dot{\zeta}(t) = \tilde{f}(\zeta) + \tilde{g}(\zeta)\eta \quad (39)$$

$$\dot{\eta}(t) = \tilde{a}(\zeta, \eta) + \tilde{b}(\zeta, \eta)u_c \quad (40)$$

with $u_c = \phi_m(z^m)$ and $v_m = -K_m y_m$. It follows that Theorem 3 can be applied to dynamics (39-40), equivalent to (6-7), to get the result.

F. Example 1

1) *Continuous-time design*: let the 1-cascade connection example treated in [9], [13]:

$$\dot{z}(t) = z^2 + \xi, \quad \dot{\xi}(t) = u.$$

Setting $W(z) = \frac{1}{2}z^2$, $\phi(z) = -z - z^2$ and $V(z, y) = W(z) + \frac{1}{2}y^2 = \frac{1}{2}(z^2 + y^2)$ with $y = \xi - \phi(z)$, one satisfies (8) and the dynamics take the form (10-11):

$$\dot{z}(t) = -z + y \quad (41)$$

$$\dot{y}(t) = -(1 + 2z)(z - y) + u.$$

According to Theorem 2, GAS is achieved under the state feedback (9); i.e. with $K_y = 1$

$$u_c = \dot{\phi}(z) - z - K_y y = 2(z^2 - y - zy). \quad (42)$$

2) *Sampled-data design*: starting from (41) with:

$$f_c = \begin{bmatrix} -z + y \\ -(1 + 2z)(z - y) \end{bmatrix}, \quad g_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and u_c in (42), the 2^{nd} -approximate controller which ensures (20) in $O(\delta^3)$ takes the form $u_a^{[2]} = u^0 + \frac{\delta}{2}u^1 + \frac{\delta^2}{6}u^2$ with:

$$\begin{aligned} u^0 &= 2(z^2 - y - zy) \\ u^1 &= \dot{u}_c(z, y) = 2(z + y - z^2 - y^2 + 4zy) \\ u^2 &= \frac{-u^1(z - 3y - 6zy + 4z^2)}{2y} - 4(z + 2zy - 2y^2). \end{aligned} \quad (43)$$

G. Example 2

1) *Continuous-time design*: adding an integrator to the previous dynamics, one gets a 2-cascade connection dynamics:

$$\dot{z}(t) = z^2 + \xi_1, \quad \dot{\xi}_1(t) = \xi_2, \quad \dot{\xi}_2(t) = u.$$

A continuous-time backstepping procedure is firstly applied to get a state representation in the form (6-7). Setting $y_1 = \xi_1 - \phi(z)$, $\phi(z) = -z - z^2$, $y_2 = \xi_2 - \phi_1(z, y_1)$, with:

$$\phi_1(z, y_1) = \dot{\phi}(z) - \frac{\partial W}{\partial z} + v_1 = \dot{\phi}(z) - z + v_1$$

one obtains (39-40) with $\zeta = y_2$ and $\eta = (z, y_1)'$:

$$\begin{aligned} \dot{z}(t) &= -z + y_1 \\ \dot{y}_1(t) &= \phi_1(z, y_1) - \dot{\phi}(z) + y_2 \\ \dot{y}_2(t) &= -\dot{\phi}_1(z, y_1) + u. \end{aligned} \quad (44)$$

From (15), the stabilizing feedback takes the form:

$$\begin{aligned} u_c &= \dot{\phi}_1(z, y_1) - y_1 + v_2 \\ &= 2(4zy_1 - z^2 - y_1^2 - zy_2 + z) - 3y_2 + y_1 \end{aligned} \quad (45)$$

with $v_i = -y_i$ for $i = 1, 2$ and $V = \frac{1}{2}(z^2 + y_1^2 + y_2^2)$.

$$ad_{[f,g]}V(z^1)\Big|_{t=k\delta} = y \frac{\partial b(z, y + \phi(z))}{\partial z} (f(z) + g(z)(\phi(z) + y)) + y \frac{\partial b(z, y + \phi(z))}{\partial y} a(z, y + \phi(z)) - \left(\frac{\partial W}{\partial z} g(z) + y \frac{\partial a(z, y + \phi(z))}{\partial y} \right) b(z, y + \phi(z)) \Big|_{t=k\delta} \quad (38)$$

2) *Sampled data design:* (44) can be rewritten in compact form with:

$$f_c = \begin{bmatrix} -z + y_1 \\ -z - y_1 + y_2 \\ -2(4zy_1 - z^2 - y_1^2 - zy_2 - y_2 + z + y_1) \end{bmatrix}$$

$$g_c = [0 \ 0 \ 1]^T$$

so getting $u_a^{[2]} = u^0 + \frac{\delta}{2}u^1 + \frac{\delta^2}{6}u^2$ with:

$$\begin{aligned} u^0 &= 2(4zy_1 - z^2 - y_1^2 - zy_2 + z) - 3y_2 + y_1 \\ u^1 &= 4y_1 - 3z + 4y_2 - 14zy_1 \\ &\quad + 12zy_2 - 6y_1y_2 - 4z^2 + 12y_1^2 \\ u^2 &= 2z - 17y_1 - 12zy_1 - y_2(48z - 54y_1 \\ &\quad - 6zy_1 + 12z^2 + 4) - 12zy_1^2 + 14z^2y_1 \\ &\quad + 29z^2 + 4z^3 - 41y_1^2 - 6y_2^2 \\ &\quad + \frac{y_1}{2y_2}(4z^2 + 14zy_1 + 3z - 12y_1^2 - 4y_1). \end{aligned} \quad (46)$$

IV. BACKSTEPPING UNDER MULTI-RATE DESIGN

To motivate the multi-rate approach, let us consider the simplest strict-feedback form:

$$\begin{aligned} \dot{z}(t) &= f(z) + g(z)\xi_1 \\ \dot{\xi}_1(t) &= \xi_2; \quad \dots; \quad \dot{\xi}_m(t) = u \end{aligned} \quad (47)$$

and assume the z -components to be globally asymptotically stabilizable under a feedback $\phi(z)$, with control Lyapunov function $W(z)$. Since the relative degree of $W(z)$ with respect to (47) is $m+1$, Lyapunov matching of $W(z)$ through a single-rate strategy would induce internal instability due to critical m sampling zeroes. To bi-pass this difficulty, the single-rate controller proposed in section III is designed on the overall control Lyapunov function V with relative degree 1.

The difficulty of critical stability up to instability of the sampling zeroes dynamics is well documented in the linear literature [22] and has been more recently treated in the nonlinear context [23], [37]. In [10], with reference to feedback linearization, multi-rate strategies of order equal to the relative degree of the output map have been proposed to preserve input-output performance and internal stability. A similar reasoning brings in the present context to consider a multi-rate controller with a number of rates equal to $(m+1)$ (the number of connected dynamics), to match at the sampling instants V and the other m Lyapunov functions involved in the continuous design process. More precisely, one requires one-step matching of each $V_j(z^j)$; i.e. denoting by $z^0 = z$ and $F_{z^j}^{(m+1)\bar{\delta}}(\cdot)$ the z^j -component of $F^{(m+1)\bar{\delta}}(\cdot)$, find $u_{ik}^{\bar{\delta}}$ for $i = 1, \dots, m+1$ so that:

$$V_j(F_{z^j}^{(m+1)\bar{\delta}}(z^j, u_1^{\bar{\delta}}, \dots, u_{m+1}^{\bar{\delta}})) - V_j(z^j) = \int_{k\delta}^{(k+1)\delta} \dot{V}_j(z^j(\tau)) d\tau \quad (48)$$

for $j = 0, \dots, m$ when $z_k^m = z^m|_{t=k\delta}$. Arguing so, the minimum phase property verified by each V_j is maintained under digital control and the mismatch between the state evolutions under continuous and digital control is reduced, so improving internal stability. In the sequel, the result is firstly detailed for the 1-connection case with a double-rate controller and then extended to the m -connections case with a multi-rate of order $(m+1)$.

A. Double-rate controller for the 1-connection case

Consider first the single-cascade connection (6-7) and its compact representation (3). The single-rate controller shaped to match the evolution of the function V unavoidably generates a matching error in δ^3 on $W(z)$; i.e.

$$W(z_{k+1}) = W(z|_{t=(k+1)\delta}) + \frac{\delta^3}{2!} \dot{u}_c L_{g_c} L_{f_c} W z|_{t=k\delta} + O(\delta^4)$$

with $L_{g_c} L_{f_c} W(z)|_{t=k\delta} \neq 0$ as $W(z)$ has relative degree 2. As a consequence, a double-rate control (u_{1k}, u_{2k}) defined as:

- u_{1k} active and constant for $t \in [k\delta, k\delta + \delta/2[$,
- u_{2k} active and constant for $t \in [k\delta + \delta/2, (k+1)\delta[$

which generates a two-input equivalent SD dynamics:

$$z_{k+1}^1 = F^{2\bar{\delta}}(z_k^1, u_{1k}, u_{2k})$$

is proposed to achieve one-step matching of both V and W . According to (48), the double-rate LME problem results in computing u_{1k}, u_{2k} to satisfy $\forall z_k^1 = z^1|_{t=k\delta}$:

$$\begin{aligned} V(F^{2\bar{\delta}}(z_k^1, u_{1k}, u_{2k})) - V(z_k^1) &= \int_{k\delta}^{(k+1)\delta} \dot{V}(z^1(\tau)) d\tau \\ W(F_z^{2\bar{\delta}}(z_k^1, u_{1k}, u_{2k})) - W(z_k) &= \int_{k\delta}^{(k+1)\delta} \dot{W}(z(\tau)) d\tau \end{aligned}$$

Theorem 16 below improves the results in Theorem 3.

Theorem 16. *Consider a strict-feedback dynamics (6-7) under the assumptions of Theorem 2, then there exist $T^* > 0$ and for each δ in $]0, T^*]$, a double-rate digital feedback $(u_1^{\bar{\delta}}, u_2^{\bar{\delta}})$ of the form:*

$$u_i^{\bar{\delta}} = u_i^0 + \sum_{j \geq 1} \frac{\bar{\delta}^j}{(j+1)!} u_i^j, \quad i = 1, 2 \quad (49)$$

with $\bar{\delta} = \delta/2$ and $u_i^0 = u_c|_{t=k\delta}$, $i = 1, 2$, which ensures SD-ILM of V and W and guarantees SD-GAS.

Proof: The proof extends that of Theorem 3. Let $\mathbf{Q}(z^1, \delta, u_1^{\bar{\delta}}, u_2^{\bar{\delta}})$ be the map defined as (51) so that one-step Lyapunov matching condition reduces to satisfy $\mathbf{Q}(z^1, \delta, u_1^{\bar{\delta}}, u_2^{\bar{\delta}}) = 0$. For $\delta = 0$, one gets (52) which is satisfied by the choice $u_1^0 = u_2^0 = u_c(z^1)$.

$$\mathbf{Q}(z^1, \delta, u_1^{\bar{\delta}}, u_2^{\bar{\delta}}) = \begin{bmatrix} \frac{1}{\delta} \left(e^{\mathcal{BCH}^2(\bar{\delta}(f_c + u_1^{\bar{\delta}} g_c), \bar{\delta}(f_c + u_2^{\bar{\delta}} g_c))} V(z^1) - e^{\delta(f_c + u_c g_c)} V(z^1) \right) \\ \frac{1}{\delta^2} \left(e^{\mathcal{BCH}^2(\bar{\delta}(f_c + u_1^{\bar{\delta}} g_c), \bar{\delta}(f_c + u_2^{\bar{\delta}} g_c))} W(z) - e^{\delta(f_c + u_c g_c)} W(z) \right) \end{bmatrix} \quad (51)$$

$$\mathbf{Q}(z^1, 0, u_1^0, u_2^0) = \begin{bmatrix} (L_{f_c} + \frac{u_1^0 + u_2^0}{2} L_{g_c}) V(z^1) - (L_{f_c} + u_c L_{g_c}) V(z^1) \\ (3u_1^0 + u_2^0) L_{g_c} L_{f_c} W(z) - 4u_c L_{g_c} L_{f_c} W(z) \end{bmatrix} \quad (52)$$

Then, verifying the non singularity of the Jacobian of \mathbf{Q} at $(\delta = 0, u^0 = u_c)$; i.e.

$$\left[\frac{\partial \mathbf{Q}_i(z^1, \cdot)}{\partial u_j} \Big|_{(0, u_c)} \right]_{ij} = \begin{bmatrix} L_{g_c} V(z^1) & L_{g_c} V(z^1) \\ 3L_{g_c} L_{f_c} W(z) & L_{g_c} L_{f_c} W(z) \end{bmatrix} \neq 0$$

because of the respective relative degrees of $V(z^1)$ and $W(z)$

$$L_{g_c} L_{f_c} W(z) = \frac{\partial W(z)}{\partial z} (\tilde{f}(z) + g(z)y) \tilde{y} b(z, y) \neq 0$$

for $z^1 \neq 0$, one deduces the existence of a pair $(u_1^{\bar{\delta}}, u_2^{\bar{\delta}})$ of the form (49) in the neighborhood of (u_1^0, u_2^0) satisfying $\mathbf{Q}(z^1, \delta, u_1^{\bar{\delta}}, u_2^{\bar{\delta}}) = 0$. ■

The following result which extends Proposition 8 specifies how a double-rate strategy improves the stabilizing performance with respect to the internal z -dynamics.

Proposition 17 (double-rate one-step consistency). *The pair $(u_1^{\bar{\delta}}, u_2^{\bar{\delta}}, F^{2\bar{\delta}})$ computed in Theorem 16 is one-step consistent with the continuous-time evolution $(u_c, e^{f_c + u_c g_c} z^1)$; i.e. there exist $T^* > 0$ and \mathcal{K}_∞ functions θ_1 and θ_2 such that for each δ in $]0, T^*]$, $\forall z_k^1 \in R^{n+1}$ with $z_k^1 = z^1(t)|_{t=k\delta}$, one has:*

$$\begin{aligned} |F_z^{2\bar{\delta}}(z_k^1, u_{1k}^{\bar{\delta}}, u_{2k}^{\bar{\delta}}) - e^{\delta(f_c + u_c g_c)} z(t)|_{t=k\delta} &\leq \delta^3 \theta_2(\delta) \\ |F_{z^1}^{2\bar{\delta}}(z_k^1, u_{1k}^{\bar{\delta}}, u_{2k}^{\bar{\delta}}) - e^{\delta(f_c + u_c g_c)} z^1(t)|_{t=k\delta} &\leq \delta^2 \theta_1(\delta) \end{aligned}$$

with $z^1 = (z', y)'$. The same inequalities hold under approximate controller $(u_1^{[p]}, u_2^{[p]})$ with $p \geq 1$.

Proof: For, it is sufficient to verify that $(u_1^{\bar{\delta}}, u_2^{\bar{\delta}}, F^{2\bar{\delta}})$ guarantees one-step state matching of the closed loop continuous-time dynamics in $O(\delta^4)$ in the z -component because the function $W(z)$ has relative degree 2 and in $O(\delta^3)$ in the y -component because the function $V(z^1)$ has relative degree 1. ■

Formally, the controller $(u_1^{\bar{\delta}}, u_2^{\bar{\delta}}) := \Gamma(z^1, \delta)$ is the reverse series satisfying $\mathbf{Q}(z^1, \delta, \Gamma(z^1, \delta)) = 0$ with $(u_1^0, u_2^0) = \Gamma(z^1, 0)$. As in the single-rate case, the successive parts of the controller can be computed through an executable algorithm so getting for the first terms :

$$\begin{aligned} (u_1^0, u_2^0) &= (u_c, u_c)|_{t=k\delta} \\ (u_1^1, u_2^1) &= \left(\frac{2}{3}, \frac{10}{3}\right) \dot{u}_c|_{t=k\delta} \\ u_1^2 &= -6\dot{u}_c u_c + \frac{4\dot{u}_c (4u_c L_{g_c}^2 L_{f_c} + L_{g_c} L_{f_c}^2) W}{3L_{g_c} L_{f_c} W} \Big|_{t=k\delta} \\ u_2^2 &= \frac{14}{3} \dot{u}_c|_{t=k\delta} + 12\dot{u}_c u_c|_{t=k\delta}. \end{aligned} \quad (53)$$

B. $m+1$ -rate digital backstepping of m -cascade connections

According to the *Multi-Rate Input Lyapunov Matching - MR-ILM* - requirement (48), Theorem 18 below reformulates Theorem 16 in the case of m -connected dynamics.

Theorem 18. *Consider a dynamics in the strict-feedback m -cascade connection form (2) and assume the existence of the continuous-time controller u_c ensuring GAS of the equilibrium with a family of control Lyapunov functions V_j ; $j = 0, \dots, m$ as in (16), then there exist $T^* > 0$ and for each δ in $]0, T^*]$, a $(m+1)$ -rate sampled-data feedback $(u_{ik}^{\bar{\delta}}; i = 1, \dots, m+1)$ with $\bar{\delta} = \frac{\delta}{m+1}$ which ensures MR-ILM of the V_j ; $j = 0, \dots, m$ and guarantees MR-GAS.*

Proof: The proof is similar to the one provided for Theorem 16 rewriting the MR-ILM vector equality as $\mathbf{Q}(z^m, \delta, u_1^{\bar{\delta}}, \dots, u_{m+1}^{\bar{\delta}}) = 0$ with $\mathbf{Q}(z^m, \dots)$ given in (51).

Noting that the MR-digital controller $u_i^0 = u_c(z^1)$ solves the equality MR-ILM (48) for $\delta = 0$, one sets the $(m+1)$ solutions in the form of asymptotic series around u_c

$$u_i^{\bar{\delta}} = u_i^0 + \sum_{j \geq 1} \frac{\bar{\delta}^j}{(j+1)!} u_i^j, \quad i = 1, \dots, m+1. \quad (53)$$

Computing the Jacobian of $\mathbf{Q}(\cdot)$ at $(\delta = 0, u_{i=1, \dots, m+1}^0)$ one gets (52) where the entries of the matrix \mathbf{Q} are defined as $\mathbf{Q}_{ij} = j^i - (j-1)^i$. More in detail, one gets

$$\mathbf{Q}(z^m, 0, u_1^{\bar{\delta}}, \dots, u_{m+1}^{\bar{\delta}}) =$$

$$\begin{bmatrix} \left(\sum_{i=0}^m u_{m+1-i}^{\bar{\delta}} - (m+1)u_c \right) L_{g_c} V_m(z^m) \\ \vdots \\ \left(\sum_{i=0}^m \sum_{j=1}^{m+1} \mathbf{Q}_{ij} u_{m+1-i}^{\bar{\delta}} - (m+1)^{m+1} u_c \right) L_{g_c} L_{f_c}^m V_0(z) \end{bmatrix}$$

so verifying the non singularity of its jacobian; i.e. $\prod_{i=0}^m L_{g_c} L_{f_c}^{m-i} V_i(z^i) \neq 0$ and the existence of a MR-control solution in the form (53) follows. ■

Again, approximate solutions can be computed through an executable algorithm working out the reverse series.

Regarding the stabilizing performance of approximated multi-rate controllers, let us analogously define through truncations at the order p , the p -th approximate multi-rate digital feedback $u_{i=1, \dots, m+1}^{[p]}$ and denote by $\mathbf{O}(\delta^p) := [O(\delta^{p+m}), \dots, O(\delta^p)]^T$ the extended error vector. Under $u_{i=1, \dots, m+1}^{[p]}$, Proposition 11 remains true. Moreover, since by construction MR-ILM of each V_j is satisfied in $\mathbf{O}(\delta^{p+2})$; i.e. $E_{V_j}(z^j, \delta) \in O(\delta^{p+2+m-j})$ for $j = 0, \dots, m$; Proposition 19 below can

$$\mathbf{Q}(z^m, \dots) = \begin{bmatrix} \frac{1}{\delta} \left(e^{\mathcal{BCH}^{m+1}}(\bar{\delta}(f_c + u_1^{\bar{\delta}} g_c), \dots, \bar{\delta}(f_c + u_{m+1}^{\bar{\delta}} g_c)) V_m(z^m) - e^{\delta(f_c + u_c g_c)} V_m(z^m) \right) \\ \vdots \\ \frac{1}{\delta^m} \left(e^{\mathcal{BCH}^{m+1}}(\bar{\delta}(f_c + u_1^{\bar{\delta}} g_c), \dots, \bar{\delta}(f_c + u_{m+1}^{\bar{\delta}} g_c)) V_1(z^1) - e^{\delta(f_c + u_c g_c)} V_1(z^1) \right) \\ \frac{1}{\delta^{m+1}} \left(e^{\mathcal{BCH}^{m+1}}(\bar{\delta}(f_c + u_1^{\bar{\delta}} g_c), \dots, \bar{\delta}(f_c + u_{m+1}^{\bar{\delta}} g_c)) V_0(z) - e^{\delta(f_c + u_c g_c)} V_0(z) \right) \end{bmatrix} \quad (51)$$

$$\left[\frac{\partial \mathbf{Q}_i(\cdot)}{\partial u_j^{\bar{\delta}}} \Big|_{(0, u_c)} \right] = [\mathbf{Q}_{ij}] \begin{bmatrix} L_{g_c} V_m(z^m) & 0 & \cdots & 0 & 0 \\ 0 & L_{g_c} L_{f_c} V_{m-1}(z^{m-1}) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & L_{g_c} L_{f_c}^m V_1(z^1) & 0 \\ 0 & 0 & \cdots & 0 & L_{g_c} L_{f_c}^m V_0(z) \end{bmatrix} \quad (52)$$

be proven so specifying how the multi-rate controller significantly reduces the mismatch between the sampled-data and the continuous-time state evolutions as soon as approximated controllers of order $p \geq 1$ are implemented.

Proposition 19 (MR one-step consistency). *The multi-rate controller computed in Theorem 18 is one-step consistent with the continuous-time evolution $(u_c, e^{f_c + u_c g_c} z^m)$; i.e. there exist $T^* > 0$ and \mathcal{K}_∞ functions θ_j such that for each δ in $]0, T^*]$, $z^j \in \mathcal{R}^{n+j}$, $j = 0, \dots, m$*

$$|F_{z^{m-j}}^{(m+1)\bar{\delta}}(z_k^m, u_{i=1/m+1}^{\bar{\delta}}) - e^{\delta(f_c + u_c g_c)} z^{m-j}|_{t=k\delta} \leq \delta^{j+2} \theta_{j+1}(\delta).$$

The same inequality holds under approximated controller $u_{i=1, \dots, m+1}^{[p]}$ with $p \geq 1$.

Proof: For, it is sufficient to verify that $u_{i=1/m+1}^{\bar{\delta}}$ guarantees one-step state matching of the closed loop continuous-time dynamics in $O(\delta^{j+3})$ with respect to each z^{m-j} -component (equivalently in $\mathbf{O}(\delta^3)$) because each function $V_j(z^j)$ has relative degree $1 + m - j$. ■

It is interesting to note the impact of the cascade strict-feedback form in a multi-rate design. The mismatch in the state evolutions is reduced to $O(\delta^{p+j+2})$ on the z^j -component for $j = 0, \dots, m$, which corresponds to faster convergency to the virtual control variables, typical of the backstepping strategy. Moreover, multi-rate one-step consistency in $\mathbf{O}(\delta^3)$ can be interpreted as multi-step consistency (see [31]) over $m + 1$ steps of length δ in $O(\delta^3)$ with respect to the full state z^m .

C. 1st-approximate MR-controller

1st-approximate multi-rate solution is finally discussed in view of its performance and computational simplicity.

Proposition 20 (1st-approximate MR-controller). *The 1st-approximate multi-rate feedback, $(u_{i=1, \dots, m+1}^{[1]})$:*

$$\begin{bmatrix} u_{m+1}^{[1]} \\ \vdots \\ u_1^{[1]} \end{bmatrix}^T = u_c|_{k\delta} + \frac{2\delta}{m+1} [\mathbf{Q}_{ij}]^{-1} \begin{bmatrix} \frac{(m+1)^2}{2} \\ \vdots \\ \frac{(m+1)^{m+2}}{m+2} \end{bmatrix} \dot{u}_c|_{k\delta}$$

ensures MR-ILM of the family of functions $V_{i=0, \dots, m}$ and one-step state-matching of the closed loop continuous dynamics in $\mathbf{O}(\delta^3) = [O(\delta^{3+m}), \dots, O(\delta^3)]^T$.

Proof: For, it is sufficient to note that the solution of the equality $\mathbf{Q}(z^1 \delta, u_1^{\bar{\delta}}, \dots, u_{m+1}^{\bar{\delta}}) = 0$ in $\mathbf{O}(\delta^2)$ reduces to the solution of the linear equation below:

$$[\mathbf{Q}_{ij}] \begin{bmatrix} u_{m+1}^1 \\ \vdots \\ u_1^1 \end{bmatrix} = 2 \begin{bmatrix} \frac{(m+1)^2}{2} \\ \vdots \\ \frac{(m+1)^{m+2}}{m+2} \end{bmatrix} \dot{u}_c|_{k\delta}. \quad (55)$$

For $m = 2$ and $m = 3$, one gets respectively

$$(u_1^1, u_2^1, u_3^1)|_{z_k^2} = \left(\frac{3}{4}, 3, \frac{21}{4} \right) \dot{u}_c|_{k\delta} \quad (56)$$

$$(u_1^1, u_2^1, u_3^1, u_4^1)|_{z_k^3} = \left(\frac{332}{45}, \frac{68}{15}, \frac{52}{15}, \frac{28}{45} \right) \dot{u}_c|_{k\delta}. \quad (57)$$

■ The efficiency of 1st-approximate multi-rate controllers is illustrated through simulated examples.

D. Example 1 (continued)

The single-rate controller SR2 is given in (46) while the double-rate controller DR2, described in (53), takes the form:

$$\begin{aligned} u_1^0 &= u_2^0 = u_c|_{t=k\delta} \\ u_1^1 &= \frac{4}{3}(-y^2 - z^2 + y + z + 4yz) \\ u_2^1 &= \frac{20}{3}(-y^2 - z^2 + y + z + 4yz) \\ u_1^2 &= 0 \\ u_2^2 &= 32(3y^2 - 4yz - z^2 - z). \end{aligned}$$

E. Example 2 (continued)

To highlight the performance of multi-rate controller, let the 3rd-order example given in section III-G which exhibits a relative degree equal to 3 with respect to z . A 3-rate controller,

denoted by MR2, is compared with the SR2 given in (46). The MR2 takes the form $u_{ai}^{[2]} = u_i^0 + \frac{\delta}{6}u_i^1 + \frac{\delta^2}{54}u_i^2; i = 1, 2, 3$ with:

$$u_1^0 = u_2^0 = u_3^0 = u_c|_{t=k\delta} \quad (58)$$

$$(u_1^1, u_2^1, u_3^1) = \left(\frac{3}{4}, 3, \frac{21}{4} \right) \dot{u}_c|_{t=k\delta}. \quad (59)$$

$$u_1^2 = \frac{1}{10} \left(\frac{360(y_1^2 + z^2)}{y_2} - 9(z + 11y_1) - 162y_1z + y_2(432y_1 - 306z + 27y_1z - 54z^2 + 360) + 63y_1z^2 - 54y_1^2z - 288y_1^2 - 54y_2^2 + \frac{423z^2}{2} + 18z^3 \right) \Big|_{t=k\delta}.$$

$$u_2^2 = \frac{1}{10} \left(108y_1^2z - \frac{99z}{2} - \frac{1260(y_1^2 + z^2)}{y_2} - 756y_1z - y_2(1548z - 2376y_1 + 54y_1z - 108z^2 + 1260) - 126y_1z^2 - \frac{1089y_1}{2} - 1584y_1^2 - 297y_2^2 + 1062z^2 - 36z^3 \right) \Big|_{t=(k+\frac{1}{3})\delta}.$$

$$u_3^2 = \frac{1}{10} \left(\frac{1980(y_1^2 + z^2)}{y_2} - \frac{423z}{2} - \frac{4653y_1}{2} - 3402y_1z + y_2(10152y_1 - 6786z + 27y_1z - 54z^2 + 1980) + 63y_1z^2 - 54y_1^2z - 6768y_1^2 - 1269y_2^2 + \frac{9333z^2}{2} + 18z^3 \right) \Big|_{t=(k+\frac{2}{3})\delta}.$$

V. SIMULATION RESULTS

To better quantify the performance of the proposed controllers we define the *Input-Lyapunov-Index* w. r. to V over a simulation period $T_s = N\delta$, as:

$$I_V = \frac{1}{V(0)} \sqrt{\sum_{k=0}^N (V(z_k^m) - V(z^m(t))|_{k\delta})^2} \quad (60)$$

i.e. the root of the sum of the squares of the mismatch over N steps between the Lyapunov function under continuous-time and sampled-data controllers, weighted by $V(0)$.

In the sequel SR i , DR i and MR i will denote the single-rate, double-rate, and multi-rate, i -th approximate controllers respectively. Different controllers are compared in the sequel; continuous lines will denote the evolutions of the variable under the CT controller, circle-lines under the emulated (EM) one, right triangle and plus-lines under SR1 and SR2 respectively. Finally, the evolutions under DR1 and DR2, will be denoted by star and triangle-lines respectively.

A. Example 1 - Simulation results

Simulation results are given for Example 1 under CT, EM, SR and DR controllers described in sections III-F and IV-D. In Fig. 1, the state evolutions under the single-rate and double-rate controllers are plotted for a sampling period equal to 0.4s. Lyapunov matching is achieved in at most 5s by each controller. Comparing the values of I_V at 0.4s in Fig. 2, it can be noted that better performance is obtained by increasing the rate rather than the approximation order (DR1 performs better than SR2). From a deeper analysis of the results in Fig. 2, where the

errors on V and W are plotted for sampling periods between 0.01s. and 0.5s, it can be observed that I_V corresponding to DR1 at 0.4s is smaller than the one corresponding to SR1 at 0.2s, which implicitly corresponds to a comparison between the single-rate and double-rate controller acting over a double period. It can be noted that for smaller sampling periods, all the proposed strategies are performing better than emulation. The best performance is obtained under DR1 and DR2 which exhibit the maximum admissible sampling period - MASP. The 1st-approximate double-rate controller DR1 represents a good choice with a good rate of simplicity over performance.

B. Example 1 - Comparison with other approaches

We now discuss Example 1 in a comparative perspective with the results obtained in [9], [13]. The simulations are worked out for $\delta = 0.4s$, $x_0 = (0.5, 0.5)$. The results depicted in Fig. 3 compare DR1 and DR2 with the controllers proposed in [9], [12] and [13]: the 1-st approximate controller in [9] (left triangle, "SN1"), the 2nd- approximate controller in [12] (cross, "SN2"), the first (square, "SB1") and 2nd- approximate (diamond, "SB2") controllers in [13]. The 1-st approximate controller described in [12] which coincides with our SR1 solution is not reproduced. Fig. 4 details the total error for various δ 's. The performance obtained under multi-rate control is clearly evidenced.

The result of a last test for a very large sampling period $\delta = 0.9s$ is depicted in Fig. 5. Lyapunov matching is achieved with performance index I_V at 2.85 under DR1 only, as shown by the phase portrait and the control trajectory.

To better evidence the mismatch between the proposed solution and those selected from the literature, we compare Fig. 2a), 4a) and subfigure b). The EM solution is plotted in both figures to make easier the comparison. It can be noticed that with respect to Lyapunov matching, the proposed algorithm gives better performance for small sampling periods (in case of all proposed controllers) and also for larger sampling periods (in case of DR controllers).

C. Example 2 - Simulation results

Simulation results are given for Example 2 under CT, EM, SR and MR controllers described in sections III-G and IV-E. The same initial conditions as for Example 1 are considered. In Fig. 6, the state evolutions under 1st and 2nd-approximate controllers are depicted for a sampling period equal to 0.3s. Lyapunov matching is achieved in 4s. The 3-rate controller gives, as expected, lower errors and higher values of the MASP. Moreover, by comparing these simulations with the ones obtained for Example 1, it follows that the addition of an integrator reduces the rise time, which should bring to increase the gains and consequently to reduce the admissible sampling period length under emulated control. Since we choose the same gains for both simulations, a significant reduction of the MASP can be noticed with respect to the emulated solution as depicted in Fig. 7. This degradation, which is not present if a 2nd-approximate multi-rate controller is used, confirms the intuition that a multi-rate digital controller could be profitably used to control dynamics which include integrators chains.

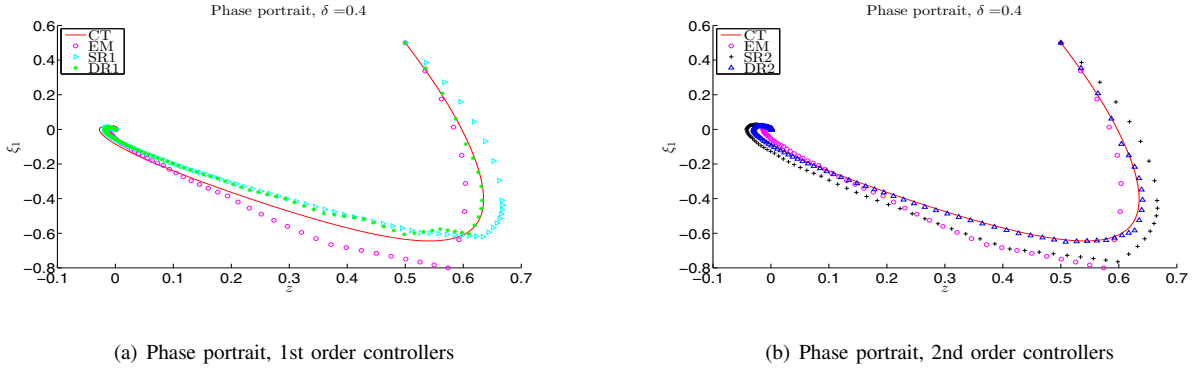


Fig. 1. Simulation results for the proposed algorithms, $\delta = 0.4s$, Example 1

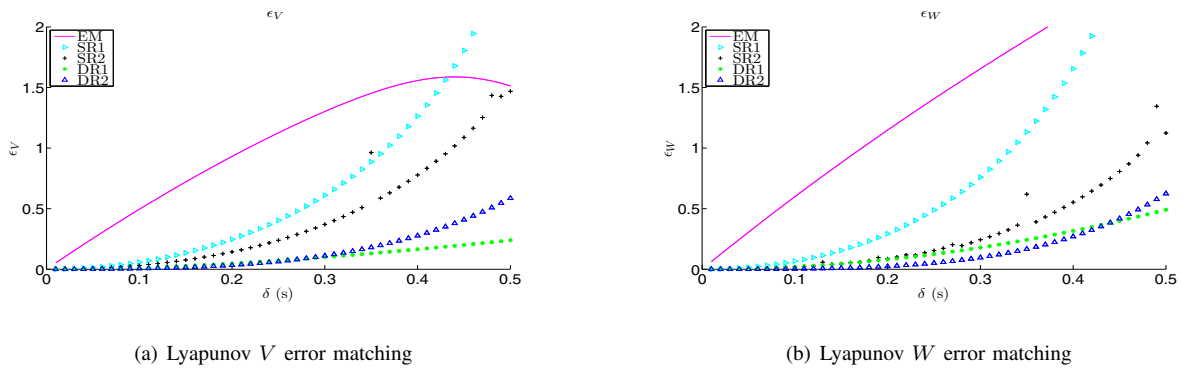


Fig. 2. Lyapunov error matching results for the proposed algorithms - Example 1, $\delta = [0.01 - 0.5]$

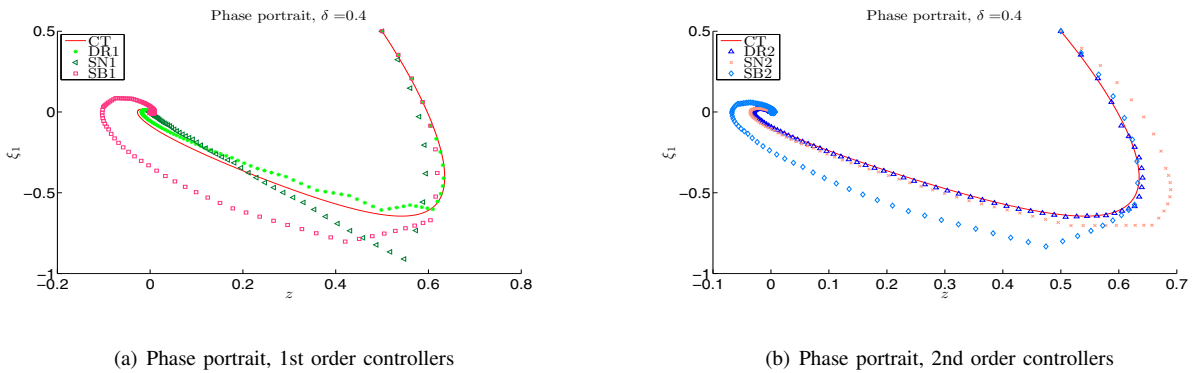


Fig. 3. Simulation results for the compared algorithms, $\delta = 0.4s$, Example 1

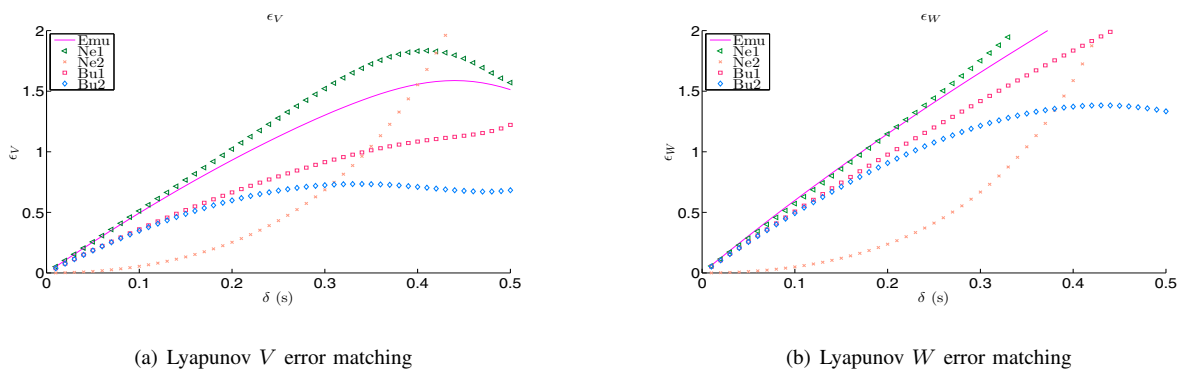


Fig. 4. Lyapunov error matching criteria for the compared algorithms - Example 1, $\delta = [0.01 - 0.5]$

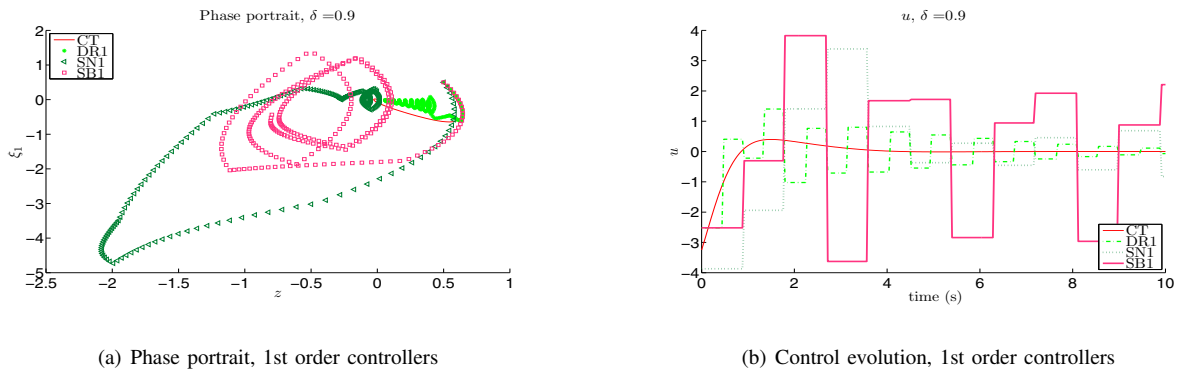


Fig. 5. Simulation results for the compared algorithms of order 1, $\delta = 0.9\text{ms}$, Example 1

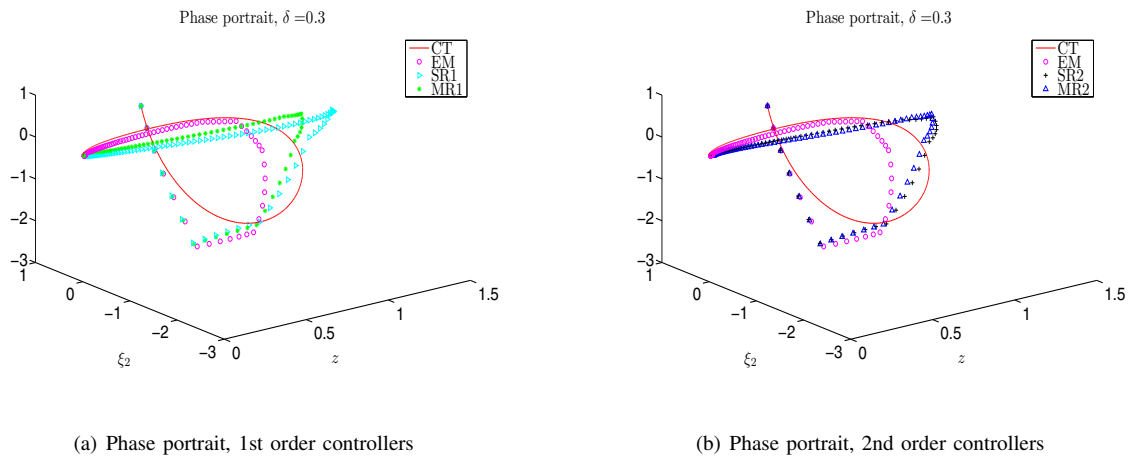


Fig. 6. Simulation results for the proposed algorithms, $\delta = 0.3\text{s}$, Example 2

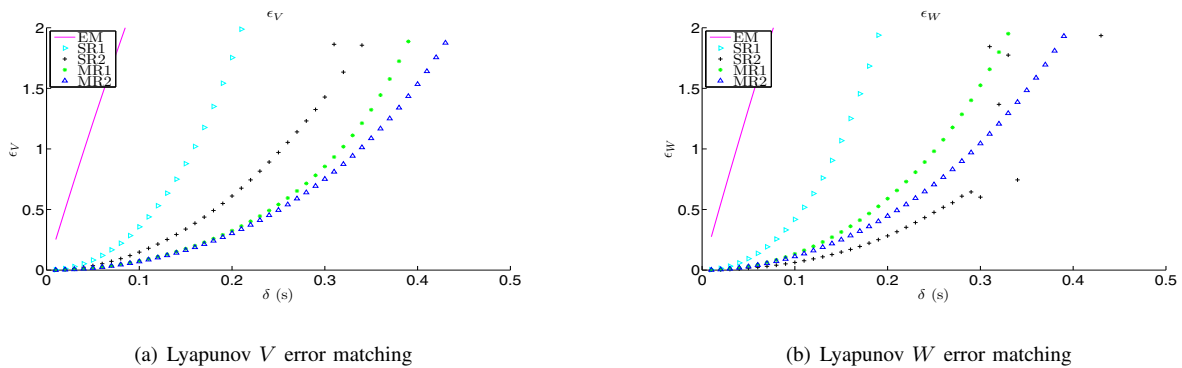


Fig. 7. Lyapunov error matching results for the proposed algorithms - Example $2\delta = [0.01 - 0.5]$

Finally, always with reference to Fig. 7, it can be observed that the same values of I_V and I_W correspond to more than double values of the sampling time under multi-rate controllers, which implicitly corresponds to a comparison between the single-rate and double-rate controller acting over a double period.

VI. CONCLUSIONS

Two sampled-data control schemes have been proposed for stabilizing nonlinear control dynamics which admit continuous backstepping controllers. The solution relies on the idea of using multiple rates on the control to match, at the sampling times, the evolutions of each control Lyapunov function involved in the design of the continuous control law. The performance of each proposed approach is commented and compared with other similar design procedures. Multi-rate approximate controllers at the first order represent a promising alternative to emulated controllers in view of their computational facilities and higher stabilizing performance. The results can be generalized directly to connected structures with connected parts of dimension greater than one. Work is progressing to investigate and quantify the performance of approximate controllers in the ISS formalism.

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