LQ non-Gaussian Regulator with Markovian Control

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***Abstract*—The paper concerns the Linear Quadratic non-Gaussian (LQnG) sub-optimal control problem when the input signal travels through an unreliable network, namely a Gilbert-Elliot channel. In particular, the control input packet losses are modeled by a two-state Markov chain with known transition probability matrix, and we assume that the moments of the non-Gaussian noise sequences up to the fourth order are known. By mean of a suitable rewriting of the system through an output injection term, and by considering an augmented system with the second-order Kronecker power of the measurements, a simple solution is provided by substituting the Kalman predictor of the LQG control law with a quadratic optimal predictor. Numerical simulations show the effective ness of the proposed method.**

***Index Terms*—Optimal Control, LQG Regulator, Kalman filtering, Non-Gaussian Systems.**

I. Introduction

In recent years, an increasing attention has been given to remote control of plants over unreliable networks, *e.g.* [25], [3], [18], [23], [6], [22], [30]. In practical applications, temporary failures are an important issue, due to power constraints, communication delay, multipath fading, data loss, background noise time synchronization or external attacks. Besides, in many relevant technical fields, the widely used Gaussian assumption cannot be accepted as a realistic statistical description of the random quantities involved. Consequently, increasing attention has been paid to non-Gaussian systems in control engineering [17], [2], [27], [33], [5], [7], [9]. In particular, non-Gaussian problems often arise in digital communications when the noise interference includes noise components that are essentially non-Gaussian [31], in problems concerning fault estimation, sensor or actuator faults [26], multiplicative noises and bilinear systems [12]. In monopulse radars, heavy tailed non-Gaussian behavior is present in the angle tracking signals because of target glint ([28], [8]), and an exploratory data analysis on BQM-34A glint signature measurements verify the non-Gaussian character of glint ([19]). Also, under some conditions, Gaussian systems with nonlinear measurements could be transformed, through a suitable rewriting of the output map, into systems with linear measurements and non-Gaussian output noise (see [14] and [13]), and thus, filtering techniques that deal with non-Gaussian noise can be of help even in this context. In this non-Gaussian setting, approximate optimal estimation and control solutions usually have high computational burden and are the results of infinite-dimensional problems [34]. For these reasons, sub-optimal estimation is usually essential to design practical computable control law. Monte-Carlo methods [2], multivariate extremes framework [15], sum of Gaussian densities [1], and weighted sigma points [21] are some of the approaches employed to approximate the conditional distribution of the underlying non-Gaussian process, and they generally have high computational cost.

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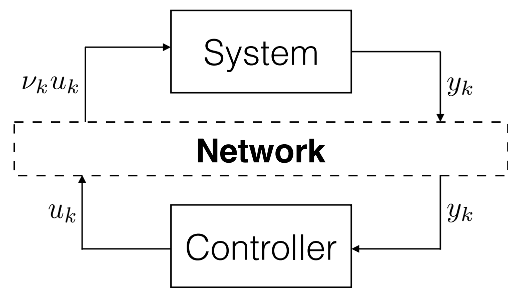

Fig. 1. Network & System diagram.

In this paper we will focus on the Linear Quadratic non-Gaussian (LQnG) sub-optimal control problem when the control input packets can be randomly lost according to a $\{0, 1\}$-valued Markov chain. In this framework, the hypothesis of perfect acknowledgment of packet drops is done, *i.e.* the so called TCP-like case, where the channel provides ordered and error-checked delivery of a stream of packets [20]. Our aim is to extend the work [29], where the TCP-like case is considered, by removing the Gaussian assumption of the noise sequences. In order to cope with these non-Gaussian noise sequences, an effective alternative solution to the aforementioned methods, in the minimum variance sense, is to look for predictors that make use of quadratic (or generally polynomial) transformations of the measurements in order to enhance the estimation accuracy, maintaining simple computability and recursion. We note that the prediction provided by quadratic or polynomial predictors has been exploited in the LQnG regulator problem in [5] and [7] where, however, control input packet losses, that cause a different design of the control law, are not included. We finally remark that when either no acknowledgment or only imperfect acknowledgment occurs, then the separation principle does not hold true, and the joint design of estimator and controller becomes a non-convex problem, as shown in [32]. [24] investigates the latter case. Conversely, since we assume that a perfect acknowledgment mechanism is available, we shall see that the separation result, even in the non-Gaussian framework, is still valid. The resulting quadratic optimal controller yields better performance in terms of the standard quadratic cost function with respect to the standard linear optimal controller, namely the one of [29], and it is simply obtained by replacing the linear optimal prediction provided by the Kalman predictor, with the quadratic optimal one, in virtue of the proved separation principle.

Notation. The Kronecker product of two matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$ is denoted by $A\otimes B$. The i-th Kronecker power of A is $A^{[i]}$, where $A^{[i]} = A \otimes A^{[i-1]}$, with $A^{[1]} = A$. The vectorization (or stack) function is denoted by $\mathrm{st}\{A\}$, and $\mathrm{st}^{-1}\{\cdot\}$ is its inverse function (we omit to specify the column size when it is clear from the context). The trace of a square matrix A is $\mathrm{tr}\{A\}$ and $v = \mathrm{col}(v_1, \dots, v_n)$ denotes the column vector $v = [v_1, \dots, v_n]^\top$, where $v_1, \dots, v_n$ are the entries of the vector v. The Moore-Penrose pseudoinverse of a matrix A is denoted by $A^\dagger$. Moreover, given a vector $v \in \mathbb{R}^n$, then $v^{1:m}$, with $m < n$, denotes the vector of the first m entries of v.

II. Problem Formulation

The control problem we aim to solve concerns the class of linear, detectable and stabilizable systems driven by non-Gaussian additive noise described by the following equations:

$$x_{k+1} = Ax_k + B\bar{u}_k + f_k, \quad (1)$$

$$y_k = Cx_k + g_k, \quad (2)$$

with the associated cost functional

$$J_\mathsf{N} = \mathbb{E}\left[x_\mathsf{N}^\top W_\mathsf{N} x_\mathsf{N} + \sum_{k=0}^{\mathsf{N}-1} x_k^\top W_k x_k + \bar{u}_k^\top U_k \bar{u}_k\right] \quad (3)$$

where $\mathsf{N} \in \mathbb{N}$ is the time-horizon. For $k \geq 0$, $x_k \in \mathbb{R}^n$ is the state, $f_k \in \mathbb{R}^n$ and $g_k \in \mathbb{R}^q$ are non-Gaussian noise sequences with strictly positive covariances, $\bar{u}_k \in \mathbb{R}^p$ is the input signal, $y_k \in \mathbb{R}^q$ is the measurement output and the matrices A, B, C, W_k, U_k are of appropriate dimensions. As usual, the matrices W_k, and U_k are symmetric non-negative definite (strictly positive definite in the case of U_k).

Denoting $\bar{x}_0 = \mathbb{E}[x_0]$, the initial state x_0 and the random sequences $\{f_k\}$ and $\{g_k\}$ satisfy the following conditions for $k \geq 0$:

(i) $\{f_k\}$ and $\{g_k\}$ are a zero mean i.i.d. sequence;
(ii) $\{f_k\}$, $\{g_k\}$ and x_0 have uncorrelated moments up to the fourth order;
(iii) for $i = 1, 2, 3, 4$ there exist finite and known vectors $\Psi_{f,i} \doteq \mathbb{E}\left[f_k^{[i]}\right]$, $\Psi_{g,i} \doteq \mathbb{E}\left[g_k^{[i]}\right]$, and $\Psi_{x_0,i} \doteq \mathbb{E}\left[(x_0 - \bar{x}_0)^{[i]}\right]$.

Clearly $\Psi_{x_0,1} = 0$ and (i) implies $\Psi_{f,1} = \Psi_{g,1} = 0$. Note that, when the sequences $\{f_k\}$, $\{g_k\}$ and x_0 are mutually independent, then assumption (ii) is satisfied. We set $\Psi_{x_0} = \mathrm{st}^{-1}\{\Psi_{x_0,2}\}$, $\Psi_f = \mathrm{st}^{-1}\{\Psi_{f,2}\}$, $\Psi_g = \mathrm{st}^{-1}\{\Psi_{g,2}\}$ the covariance matrices of the initial state, state noise and output noise, respectively. It is clear that in the non-Gaussian framework, the knowledge of the first four moments of the state and output noise sequences is weaker than assuming the knowledge of the whole probability distributions.

Furthermore, we consider the case when the control input signal $u_k \in \mathbb{R}^p$ is sent by the controller through an unreliable channel, namely a Gilbert-Elliot channel model, which is modeled by a two-state discrete Markov chain. Thus, let $\{\nu_k\}$ be a Markov chain taking values in the set $\{0, 1\}$, and modeling the presence of packet losses in the actuators. The sequence $\{\nu_k\}$ is characterized by the known probability transition matrix

$$\Pi = \begin{bmatrix} \mathrm{P}(\nu_{k+1} = 0|\nu_k = 0) & \mathrm{P}(\nu_{k+1} = 1|\nu_k = 0) \\ \mathrm{P}(\nu_{k+1} = 0|\nu_k = 1) & \mathrm{P}(\nu_{k+1} = 1|\nu_k = 1) \end{bmatrix} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \quad (4)$$

We note that if $\nu_k = 1$, then the actuators receive and apply the input u_k, namely no failure or attack have occurred, $\nu_k = 0$ otherwise. For, we have

$$\bar{u}_k = \nu_k u_k. \quad (5)$$

Moreover, we assume the Markov chain described above is independent of $\{f_k\}$, $\{g_k\}$ and x_0, and it is *irreducible*, *i.e.* $\alpha, \beta \in (0, 1]$, and *stationary*, *i.e.* for all $k \geq 0$, we have $\mathrm{P}(\nu_k = 0) = \mathrm{P}(\nu_k = 1) + \frac{\beta-\alpha}{\alpha+\beta} = \frac{\beta}{\alpha+\beta}$. Finally, we shall consider control sequences $\{\bar{u}_k\}_k$ measurable with respect to the σ-Algebra $\mathcal{F}_k = \sigma(y_j, \nu_j, \quad j \leq k-1)$. Thus, we note that the quantity ν_{k-1} is available at time $k \geq 0$, which means that a reliable acknowledgment protocol is implemented (the so-called TCP-like case [20]). As pointed out in [16], this assumption is reasonable in several practical applications.

In this framework, we consider the finite-horizon sub-optimal control problem for non-Gaussian discrete-time linear systems with partial state information and Markovian packet loss in the control. More precisely, our aim is to compute the control law in the class of recursively computable quadratic output feedback which minimizes (3).

We recall the important result of [29].

Proposition 1 ([29]): For the finite-horizon LQG regulator problem with Markovian control packet loss (5), the $\mathcal{F}_k$-measurable optimal output feedback control u_k is given by

$$u_k = -M_k \check{x}_{k|k-1}, \quad (6)$$

with

$$M_k = (U_k + B^\top R_{k+1} B)^{-1} B^\top R_{k+1} A, \quad (7)$$

where R_k is the solution of the backward Riccati equations

$$R_k = W_k + \beta A^\top S_{k+1} A + (1-\beta) A^\top R_{k+1} A - (1-\beta) \cdot \cdot A^\top R_{k+1} B (U_k + B^\top R_{k+1} B)^{-1} B^\top R_{k+1} A, \quad (8)$$

$$S_k = W_k + (1-\alpha) A^\top S_{k+1} A + \alpha A^\top R_{k+1} A + - \alpha A^\top R_{k+1} B (U_k + B^\top R_{k+1} B)^{-1} B^\top R_{k+1} A \quad (9)$$

with final conditions

$$R_\mathsf{N} = S_\mathsf{N} = W_\mathsf{N}, \quad (10)$$

and $\check{x}_{k|k-1}$ is the optimal prediction of x_k provided by the Kalman predictor (see [4]).

The next corollary is a straightforward consequence of the separation principle proved by the previous theorem.

Corollary 1: For the finite-horizon LQ non-Gaussian regulator problem (1)-(2)-(3) with Markovian control packet loss (5), the $\mathcal{F}_k$-measurable output feedback *linear* optimal control u_k is given by (6).

In other words, Corollary 1 states that the control input provided by (6) remains optimal in the class of *linear* transformations of the output: a direct consequence of the fact that, if the state and output noise sequences are non-Gaussian, then the KF (KP) is the optimal estimator (predictor) in the class of *linear* transformations of the output.

III. Quadratic Filtering and Prediction

A. The Geometric Approach

In this section we shall briefly recall some notions about quadratic filtering using the geometric approach (for a more detailed treatment see [11]). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $L^2(\mathcal{F}, n)$ the Hilbert space of the n-dimensional, $\mathcal{F}$ measurable random variables with finite second order moment. We write $L^2(X, n)$ to denote $L^2(\sigma(X), n)$. We recall that $\Pi[\cdot|\ \mathcal{M}]$ is the orthogonal projection onto a given Hilbert space $\mathcal{M}$. Let $Y_k = \mathrm{col}(y_0, y_1, \cdots, y_k)$ be the aggregate vector of the measurements until time $k \geq 0$. The minimum variance estimate of the state x_k of system (1)–(2), can be defined as the orthogonal projection of x_k onto the Hilbert space $L^2(Y_k, n)$:

$$\hat{x}_k = \mathbb{E}\left[x_k \mid \sigma(Y_k)\right] = \Pi\left[x_k \mid L^2(Y_k, n)\right].$$

Defining the auxiliary vector $Y'_k = \mathrm{col}(1, Y_k) \in \mathbb{R}^{\ell+1}$, with $\ell = (k+1)q$ and the space of linear transformations of the output

$$\mathcal{L}_y^k = \{z : \Omega \to \mathbb{R}^n : \exists T \in \mathbb{R}^{n\times(\ell+1)} : z = T\,Y'_k\}, \qquad (11)$$

it is known that the KF recursively computes the projection $\Pi[x_k|\mathcal{L}_y^k]$ which is the best linear estimate of x_k in the minimum variance sense. Let us define now the following space of quadratic transformations of the output

$$\overline{\mathcal{Q}}_y^k = \{z : \Omega \to \mathbb{R}^n : \exists T \in \mathbb{R}^{n\times\bar{\ell}} : z = T\,\overline{Y}_k^{(2)}\}, \qquad (12)$$

where $\overline{Y}_k^{(2)} = \mathrm{col}(Y'_k, y_0^{[2]}, \ldots, y_k^{[2]}) \in \mathbb{R}^{\bar{\ell}}$, with $\bar{\ell} = 1 + \ell + \ell q$. Note that $\overline{\mathcal{Q}}_y^k$ is not the space of *all* quadratic transformations of the output since terms of the form $y_{k_1}(i)y_{k_2}(j)$, with $i, j \leq q$ and $k_1, k_2 \leq k$, are missing. However, this is essential to ensure that the proposed filter shall be recursively implemented. Finally, since the inclusion $\mathcal{L}_y^k \subset \overline{\mathcal{Q}}_y^k$ holds true, the estimate of the state obtained by projecting x_k onto the larger subspace $\overline{\mathcal{Q}}_y^k$ has smaller variance of the estimation error than the one computed by the standard KF.

B. Output Injection

In this section we rewrite the system by mean of an output injection term, which is crucial to ensure some important properties of the quadratic predictor we shall see in Section IV. As in [10], the state equation (1) is transformed using the output equation (2):

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + f_k \\ &= A_k + B\bar{u}_k + f_k + Ly_k - LCx_k - Lg_k \\ &= \tilde{A}x_k + B\bar{u}_k + Ly_k + h_k \end{aligned}$$

where $\tilde{A} = A - LC$, $L \in \mathbb{R}^{n\times q}$ is chosen such that the eigenvalues of $\tilde{A}$ are in the unit circle, and $h_k = f_k - Lg_k$. Moreover, $\psi_h^{(i)} = \mathbb{E}\left[h_k^{[i]}\right]$, $i = 2, 3, 4$, can be computed as functions of $\psi_f^{(i)}$ and $\psi_g^{(i)}$, thus they are known. We split the state process into two sequences: a predictable sequence $\{x_k^p\}$ and a stochastic sequence $\{x_k^s\}$. The predictable component x_k^p satisfies

$$x_{k+1}^p = \tilde{A}x_k^p + B\bar{u}_k + Ly_k, \quad x_0^p = \bar{x}_0, \qquad (13)$$

while the stochastic component x_k^s is the solution of

$$x_{k+1}^s = \tilde{A}x_k^s + h_k, \quad x_0^s = x_0 - \bar{x}_0, \qquad (14)$$

and therefore $\psi_{x_0^s}^{(i)} := E\left[x_0^{s\,[i]}\right] = \psi_{x_0}^{(i)}$. From (13) and (14), for any $k \geq 0$, it follows $x_k = x_k^p + x_k^s$. Moreover, note that at time $k \geq 0$, since the quantities y_{k-1} and $\bar{u}_{k-1}$ are available, the predictable component x_k^p is known. Subsequently, we can define the output map of the stochastic component (14) as

$$y_k^s = y_k - Cx_k^d = Cx_k^s + g_k,$$

where y_k^s is an available quantity at time $k \geq 0$.

By setting the corresponding sequence vector $Y_{s,k} := \mathrm{col}(y_0^s, y_1^s, \ldots, y_k^s$ and the auxiliary vector $Y'_{s,k} = \mathrm{col}(1, Y_{s,k})$, it is possible to define according to (11)-(12) and the definition of $\overline{Y}_k^{(2)}$, the spaces $\mathcal{L}_{y^s}^k$ and $\overline{\mathcal{Q}}_{y^s}^k$, and the vector $\overline{Y}_{s,k}^{(2)}$ by using $Y'_{s,k}$ instead of Y'_k. Since x_k^p is an affine transformation of Y_{k-1} and the sequence $\{\nu_{k-1}\}$, the projection of x_k^p onto $\overline{\mathcal{Q}}_{y^s}^k$ trivially corresponds to itself, and we refer to $\Pi[x_k^s|\overline{\mathcal{Q}}_y^k]$ (respectively $\Pi[x_k^s|\overline{\mathcal{Q}}_y^{k-1}]$) as the *recursive quadratic estimate* (respectively *prediction*) of x_k^s.

Finally, we point out that in [10] it is proved that $Y_{s,k}$ is an affine transformation of the original sequence vector Y_k and *viceversa*, thus $\mathcal{L}_y^k \equiv \mathcal{L}_{y^s}^k$. However this is not true for the recursive quadratic transformations, *i.e.* $\overline{\mathcal{Q}}_y^k \not\equiv \overline{\mathcal{Q}}_{y^s}^k$. In other words, the space $\overline{\mathcal{Q}}_{y^s}^k$ depends on the choice of the output injection gain L.

We shall see in the next section how to compute such optimal recursive quadratic estimate $\hat{x}_k^s = \Pi[x_k^s|\overline{\mathcal{Q}}_{y^s}^k]$.

IV. Separation Principle and Optimal Recursive Quadratic Control with Packet Losses

We prove in this section that the structure of the optimal recursive quadratic controller for the LQ non-Gaussian regulator problem with packet losses in the input remains unchanged, namely the gain (7) with the coupled backward Riccati equations (8)–(9). For, we define the following vectors

$$\mathcal{X}_k = \mathrm{col}\left(x_k^p, x_k^s, x_k^{s\,[2]}\right), \qquad (15)$$

$$\mathcal{Y}_k = \mathrm{col}\left(y_k^s, y_k^{s\,[2]}\right), \qquad (16)$$

where x_k^p and x_k^s are defined in (13)–(14).

Lemma 1: The augmented state and output sequences $\{\mathcal{X}_k\}$ and $\{\mathcal{Y}_k\}$ defined in (15)–(16) obey to the following equations

$$\mathcal{X}_{k+1} = \mathcal{A}\mathcal{X}_k + \mathcal{B}\bar{u}_k + \phi_h + \mathcal{V}_k \tag{17}$$
$$\mathcal{Y}_k = \mathcal{C}\mathcal{X}_k + \varphi_g + \mathcal{G}_k, \tag{18}$$

with

$$\mathcal{A} = \begin{bmatrix} A & LC & 0 \\ 0 & \tilde{A} & 0 \\ 0 & 0 & \tilde{A}^{[2]} \end{bmatrix} \; \mathcal{B} = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} \; \mathcal{C} = \begin{bmatrix} 0 & C & 0 \\ 0 & 0 & C^{[2]} \end{bmatrix},$$

$$\phi_h = \begin{bmatrix} 0 \\ 0 \\ \psi_h^{(2)} \end{bmatrix} \; \mathcal{V}_k = \begin{bmatrix} Lg_k \\ h_k \\ h_k^{(2)} \end{bmatrix} \; \varphi_g = \begin{bmatrix} 0 \\ \psi_g^{(2)} \end{bmatrix} \; \mathcal{G}_k = \begin{bmatrix} g_k \\ g_k^{(2)} \end{bmatrix},$$

where

$$h_k^{(2)} = \tilde{A}x_k^s \otimes h_k + h_k \otimes \tilde{A}x_k^s + h_k^{[2]} - \psi_h^{(2)}, \tag{19}$$
$$g_k^{(2)} = Cx_k^s \otimes g_k + g_k \otimes Cx_k^s + g_k^{[2]} - \psi_g^{(2)}. \tag{20}$$

Proof. By noticing that ${x_k^s}^{[2]}$ and ${y_k^s}^{[2]}$ satisfy

$${x_{k+1}^s}^{[2]} = \tilde{A}^{[2]}{x_k^s}^{[2]} + \psi_h^{(2)} + h_k^{(2)},$$
$${y_k^s}^{[2]} = C^{[2]}{x_k^s}^{[2]} + \psi_g^{(2)} + g_k^{(2)},$$

where $h_k^{(2)}$ and $g_k^{(2)}$ are defined in (19)–(20), it is straightforward to obtain (17)–(18). □

Remark 1: We note that the vector ϕ_h and φ_g are constant and known, whilst the stochastic sequences $\{\mathcal{V}_k\}$ and $\{\mathcal{G}_k\}$ are zero-mean, mutually correlated, white, and uncorrelated with the initial state x_0.

The following lemma provides the algorithm to compute the recursive quadratic estimate and prediction.

Lemma 2: The recursive quadratic estimate $\hat{x}_k = \Pi[x_k|\overline{\mathcal{Q}}_{y^s}^k]$ and the recursive quadratic prediction $\hat{x}_{k|k-1} = \Pi[x_k|\overline{\mathcal{Q}}_{y^s}^{k-1}]$ are given by the following algorithm

$$x_0^p = \bar{x}_0, \quad \widehat{\mathcal{X}}_{0|-1}^s = \mathrm{col}(\Psi_{x_0,1}, \Psi_{x_0,2}) \tag{21}$$
$$P_{0|-1} = \begin{bmatrix} \mathrm{st}^{-1}\{\Psi_{x_0,2}\} & \mathrm{st}^{-1}\{\Psi_{x_0,3}\} \\ \mathrm{st}^{-1}\{\Psi_{x_0,3}\}^\top & \mathrm{st}^{-1}\{\Psi_{x_0,4}\} - \Psi_{x_0,2}\Psi_{x_0,2}^\top \end{bmatrix}$$
$$\hat{x}_{k|k-1} = x_k^p + \hat{x}_{k|k-1}^s, \quad \hat{x}_{k|k-1}^s = \widehat{\mathcal{X}}_{k|k-1}^{s,1:n} \tag{22}$$
$$K_k = P_{k|k-1}\overline{\mathcal{C}}^\top \left(\overline{\mathcal{C}}P_{k|k-1}\overline{\mathcal{C}}^\top + \Psi_k\right)^\dagger \tag{23}$$
$$P_k = P_{k|k-1} - K_k\overline{\mathcal{C}}P_{k|k-1} \tag{24}$$
$$\widehat{\mathcal{X}}_k^s = \widehat{\mathcal{X}}_{k|k-1}^s + K_k\left(\mathcal{Y}_k - \overline{\mathcal{C}}\widehat{\mathcal{X}}_{k|k-1}^s - \varphi_g\right) \tag{25}$$
$$\hat{x}_k = x_k^p + \hat{x}_k^s, \quad \hat{x}_k^s = \widehat{\mathcal{X}}_k^{s,1:n} \tag{26}$$
$$x_{k+1}^p = \tilde{A}x_k^p + B\bar{u}_k + Ly_k \tag{27}$$
$$\Gamma_k = \Upsilon_k\Psi_k^\dagger \tag{28}$$
$$\widehat{\mathcal{X}}_{k+1|k}^s = \bar{\mathcal{A}}\widehat{\mathcal{X}}_k^s + \Gamma_k\left(\mathcal{Y}_k - \overline{\mathcal{C}}\widehat{\mathcal{X}}_k^s\right) + \varphi_h \tag{29}$$
$$P_{k+1|k} = \left(\bar{\mathcal{A}} - \Gamma_k\overline{\mathcal{C}}\right)P_k\left(\bar{\mathcal{A}} - \Gamma_k\overline{\mathcal{C}}\right)^\top + \Xi_k - \Gamma_k\Upsilon_k \tag{30}$$

where $\mathcal{X}_k^s = \mathrm{col}(x_k^s, {x_k^s}^{[2]})$, $\varphi_h = \mathrm{col}(0, \psi_h^{(2)})$,

$$\bar{\mathcal{A}} = \begin{bmatrix} \tilde{A} & 0 \\ 0 & \tilde{A}^{[2]} \end{bmatrix} \quad \overline{\mathcal{C}} = \begin{bmatrix} C & 0 \\ 0 & C^{[2]} \end{bmatrix},$$

and the covariance matrices $\Xi_k = \mathbb{E}\left[\mathcal{H}_k\mathcal{H}_k^\top\right]$ with $\mathcal{H}_k = \mathrm{col}\left(h_k, h_k^{(2)}\right)$, $\Psi_k = \mathbb{E}\left[\mathcal{G}_k\mathcal{G}_k^\top\right]$ and $\Upsilon_k = \mathbb{E}\left[\mathcal{H}_k\mathcal{G}_k^\top\right]$.

Proof. By Lemma 1, it is possible to consider the stochastic augmented sub-system

$$\mathcal{X}_{k+1}^s = \bar{\mathcal{A}}\mathcal{X}_k^s + \varphi_h + \mathcal{H}_k \tag{31}$$
$$\mathcal{Y}_k = \overline{\mathcal{C}}\mathcal{X}_k^s + \varphi_g + \mathcal{G}_k. \tag{32}$$

Note that the noise sequences $\{\mathcal{H}_k\}$ and $\{\mathcal{G}_k\}$ are zero-mean, mutually correlated, white, and uncorrelated with the initial state x_0. Thus, the KF algorithm for mutually correlated state and output noise sequences ([4]) provides the optimal estimate and prediction in the class of linear functions of $\{\mathcal{Y}_k\}$. Moreover, it is clear that the optimal filter in the class of linear functions of $\{\mathcal{Y}_k\}$ corresponds to the optimal filter in the class $\overline{\mathcal{Q}}_{y^s}^k$, and the proof is completed. □

Remark 2: As pointed out in [10], we note that the choice of the output injection gain L, such that the matrix $\tilde{A}$ has eigenvalues in the unit circle, renders the augmented sub-system (31)–(32) detectable and stabilizable since the matrix $\bar{\mathcal{A}}$ is asymptotically stable.

We are now able to state the main theorem of this section.

Theorem 1: For the finite-horizon LQ non-Gaussian regulator problem (1)-(2)-(3) with Markovian control packet loss (5), the $\mathcal{F}_k$-measurable output feedback *quadratic* optimal control u_k is given by

$$u_k = -M_k\hat{x}_{k|k-1}, \tag{33}$$

where M_k is defined in (7) with the coupled backward Riccati equations (8)–(9), and $\hat{x}_{k|k-1} = x_k^p + \hat{x}_{k|k-1}^s$, with the recursive quadratic prediction $\hat{x}_{k|k-1}^s = \Pi[x_k^s|\overline{\mathcal{Q}}_{y_s}^{k-1}]$ given by the algorithm (21)–(30).

Proof. Firstly, we can rewrite the cost index J_N as follows

$$J_\mathsf{N} = \mathbb{E}\left[\mathcal{X}_\mathsf{N}^\top\mathcal{W}_\mathsf{N}\mathcal{X}_\mathsf{N} + \sum_{k=0}^{\mathsf{N}-1}\mathcal{X}_k^\top\mathcal{W}_k\mathcal{X}_k + \bar{u}_k^\top U_k\bar{u}_k\right], \tag{34}$$

where $\mathcal{X}_k$ is the extended state vector defined in (15) and

$$\mathcal{W}_k := \begin{bmatrix} W_k & W_k & 0 \\ W_k & W_k & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By Corollary 1, the LQG solution applied to the extended system (17)–(18) with the cost index (34), yields the optimal linear controller, namely the optimal recursive quadratic control of the original system. In particular, by noticing that the contribute of the known forcing term ϕ_h in (17) vanishes because of its zero-block structure, by Proposition 1, the above sub-optimal control input is given by

$$u_k = -\mathcal{M}_k\widehat{\mathcal{X}}_{k|k-1}, \tag{35}$$

with

$$\mathcal{M}_k = (U_k + \mathcal{B}^\top\mathcal{R}_{k+1}\mathcal{B})^{-1}\mathcal{B}^\top\mathcal{R}_{k+1}\mathcal{A},$$

where $\mathcal{R}_k$ is the solution of the backward Riccati equations

$$\mathcal{R}_k = \mathcal{W}_k + \beta\mathcal{A}^\top\mathcal{S}_{k+1}\mathcal{A} + (1-\beta)\mathcal{A}^\top\mathcal{R}_{k+1}\mathcal{A} - (1-\beta)\cdot \\ \cdot\mathcal{A}^\top\mathcal{R}_{k+1}\mathcal{B}(U_k + \mathcal{B}^\top\mathcal{R}_{k+1}\mathcal{B})^{-1}\mathcal{B}^\top\mathcal{R}_{k+1}\mathcal{A}, \quad (36)$$

$$\mathcal{S}_k = \mathcal{W}_k + (1-\alpha)\mathcal{A}^\top\mathcal{S}_{k+1}\mathcal{A} + \alpha\mathcal{A}^\top\mathcal{R}_{k+1}\mathcal{A} + \\ -\alpha\mathcal{A}^\top\mathcal{R}_{k+1}\mathcal{B}(U_k + \mathcal{B}^\top\mathcal{R}_{k+1}\mathcal{B})^{-1}\mathcal{B}^\top\mathcal{R}_{k+1}\mathcal{A} \quad (37)$$

with final conditions $\mathcal{R}_\mathsf{N} = \mathcal{S}_\mathsf{N} = \mathcal{W}_\mathsf{N}$. It is easy to see that, by backward induction, the matrices

$$\mathcal{R}_k = \begin{bmatrix} R_k & R_k & 0 \\ R_k & R_k & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathcal{S}_k = \begin{bmatrix} S_k & S_k & 0 \\ S_k & S_k & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (38)$$

with R_k and S_k given by (8)–(9) with final conditions (10), are the solutions to (36)–(37). Thus, by equations (38), the control law (35) simplifies in

$$u_k = -M_k\left(\widehat{\mathcal{X}}^{1:n}_{k|k-1} + \widehat{\mathcal{X}}^{s,1:n}_{k|k-1}\right),$$

where M_k is defined in (7), $\widehat{\mathcal{X}}^{1:n}_{k|k-1} = x^p_k$ is known and, by Lemma 2, $\widehat{\mathcal{X}}^{s,1:n}_{k|k-1} = \hat{x}^s_{k|k-1}$ is the optimal recursive quadratic prediction provided by (22). □

Remark 3: The above theorem shows that the optimal controller in the class of quadratic transformation of the output is a linear function of the quadratic prediction. Thus, the separation principle continues to hold even in the non-Gaussian case with *measurable* packet loss in the control input, since estimation and control can be designed separately.

The following proposition is a direct consequence of Theorem 1 of [29] and the Theorem 1 above.

Proposition 2: For the finite-horizon LQ non-Gaussian regulator problem (1)-(2)-(3) with Markovian control packet loss (5) and control input (33), the cost J_N is given by[1]

$$J_\mathsf{N} = \frac{1}{\alpha+\beta}\,\mathrm{tr}\left\{\beta\Psi_{x_0}S_0 + \alpha\Psi_{x_0}R_0 + \sum_{k=0}^{\mathsf{N}-1}\left(\beta S_{k+1}Q + \right.\right. \\ \left.\left. + \alpha R_{k+1}Q + \alpha A^\top R_{k+1}BM_k\bar{P}_{k|k-1}\right)\right\}, \quad (39)$$

where M_k, R_k and S_k are defined in (7)-(8)-(9), and $\bar{P}_{k|k-1}$ is the first $n \times n$ block of the matrix $P_{k|k-1}$ given by (30), namely the covariance of the prediction error.

Remark 4: As pointed out at the end of Section III-B, we note that, since the projective subspace $\overline{\mathcal{Q}}^k_{y^s}$ depends on L, so it is for the matrix $\bar{P}_{k|k-1}$. Therefore, one can enhance the performance by choosing the gain L stabilizing $\tilde{A}$ and minimizing (39), namely if the couple (A, C) is observable

$$L_{\text{opt}} = \underset{\substack{\lambda\in\sigma(\tilde{A}) \\ |\lambda|\le 1}}{\arg\min}\ \mathrm{tr}\left\{\sum_{k=0}^{\mathsf{N}-1}\left(A^\top R_{k+1}BM_k\bar{P}_{k|k-1}\right)\right\}. \quad (40)$$

[1]in the case $\mathbb{E}[x_0] = 0$. Also, we note that a transpose on the matrix A is missing in [29].

V. Simulation Example

In this section we show the effectiveness of the proposed approach. We compare the linear optimal solution of [29] (Corollary 1), namely the control law (6) (we call *Kalman predictor (KP) controller*), with the one proposed in this paper, *i.e.* the control law (33) (we call *Quadratic predictor (QP) controller*). We consider a planar positioning problem where the system in the form (1)–(2) is characterized by

$$A = \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \delta \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

with $\delta = 0.25$ a discretization step, and $x_0 \sim \mathcal{N}(0, I_2)$ with I_2 the identity matrix of dimension 2. Moreover, the input signal is transmitted through an unreliable channel characterized by the transition probability matrix (4) with two different scenarios: Π_1 has $\mathrm{P}(\nu_{k+1} = 1|\nu_k = 0) = \alpha = 0.9$ and $\mathrm{P}(\nu_{k+1} = 0|\nu_k = 1) = \beta = 0.2$, whilst Π_2 has $\mathrm{P}(\nu_{k+1} = 1|\nu_k = 0) = \alpha = 0.6$ and $\mathrm{P}(\nu_{k+1} = 0|\nu_k = 1) = \beta = 0.3$, namely

$$\Pi_1 = \begin{bmatrix} 0.1 & 0.9 \\ 0.2 & 0.8 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 0.4 & 0.6 \\ 0.3 & 0.7 \end{bmatrix}.$$

The cost index (3) to be minimized is defined by $W_k = I_2$, and $U_k = 1$ for all $k \ge 0$. Furthermore, the system is driven by the zero-mean i.i.d. non-Gaussian noise sequences $f_k = \mathrm{col}(f_{1,k}, f_{2,k})$ and g_k, where for any $k \ge 0$ we have $\mathrm{P}(f_{1,k} = 0.05) = 1 - \mathrm{P}(f_{1,k} = -0.2) = 0.8$, $\mathrm{P}(f_{2,k} = -0.01) = 1 - \mathrm{P}(f_{2,k} = 0.09) = 0.9$, and $\mathrm{P}(g_k = 0.01) = 1 - \mathrm{P}(g_k = 0.0025) = 0.2$. Finally, the output injection gain L is chosen according to (40).

Figure 2 shows the empirical cost across 70 Monte Carlo runs of the KP controller and QP controller, *i.e.* the cost $x_\mathsf{N}^\top W_\mathsf{N} x_\mathsf{N} + \sum_{k=0}^{\mathsf{N}-1} x_k^\top W_k x_k + \bar{u}_k^\top U_k \bar{u}_k$ obtained for a single realization. In particular the time horizon of each realization is $\mathsf{N} = 10^3$. In the scenario with Π_2, we see that the averaged cost $\overline{J}^{\text{KP}}_\mathsf{N}$ of the KP controller of [29], *i.e.* the cost obtained with the control law (6), is $\overline{J}^{\text{KP}}_\mathsf{N} = 160$, whilst the averaged cost $\overline{J}^{\text{QP}}_\mathsf{N}$ of the proposed QP controller, *i.e.* the cost obtained with the control law (33), is $\overline{J}^{\text{QP}}_\mathsf{N} = 114$. We define the performance index as the percentage improvement of the proposed solution with respect to the KP controller, namely $\alpha_J = 10^2 \cdot (\overline{J}^{\text{KP}}_\mathsf{N} - \overline{J}^{\text{QP}}_\mathsf{N})/\overline{J}^{\text{KP}}_\mathsf{N}$. Table I summarizes the aforementioned results showing both the *a priori* optimal cost J^{KP}_N and J^{QP}_N, given by (39) for the proposed QP controller, and the one obtained through the numerical simulations, namely $\overline{J}^{\text{KP}}_\mathsf{N}$ and $\overline{J}^{\text{QP}}_\mathsf{N}$. As described in Section III, the superiority of the proposed method descends from the fact that the proposed sub-optimal solution is optimal in the larger class of quadratic transformation output feedback controller, whilst the solution of [29] is the optimal linear solution. Finally, the coherence of the numerical results between the *a priori* and the empirical cost validates the proposed approach that outperforms the linear optimal solution of [29].

VI. Conclusions

In this paper we propose a sub-optimal solution for the Linear Quadratic non-Gaussian (LQnG) regulator problem in

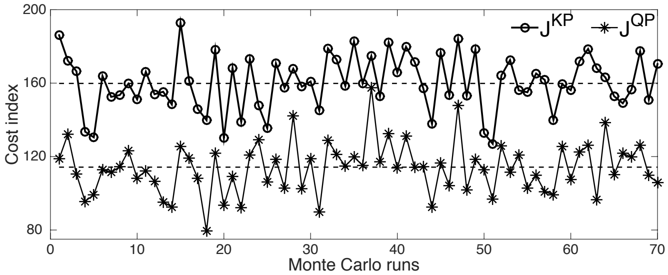

Fig. 2. Cost index across realizations of the KP controller of [29] and the proposed QP controller in the scenario with Π_2.

Scenario Π_1	J^{KP}	$\overline{J}^{\mathrm{KP}}$	α_J
KP controller [29]	146.45	147.46	-
QP controller	105.25	105.60	28.6%

Scenario Π_2	J^{KP}	$\overline{J}^{\mathrm{KP}}$	α_J
KP controller [29]	159.84	160.20	-
QP controller	114.24	113.78	29.0%

TABLE I
A PRIORI COST J^{KP} AND EMPIRICAL COST $\overline{J}^{\mathrm{KP}}$ OF THE LINEAR OPTIMAL SOLUTION OF [29], *i.e.* THE KALMAN PREDICTOR (KP) CONTROLLER, AND THE PROPOSED QUADRATIC OPTIMAL SOLUTION, *i.e.* THE QUADRATIC PREDICTOR (QP) CONTROLLER.

the presence of measurable packet losses in the control input. We show that the optimal controller in the class of quadratic transformation of the output is a linear map of the quadratic prediction of the state, *i.e.* the optimal prediction in the class of quadratic transformation of the output. As a consequence, the separation principle continue to be true even in this case since estimation and control can be designed separately. Numerical results validate the proposed approach that outperforms the linear optimal solution. Further developments can include: extension to the polynomial filtering, intermittent observations, partial packet losses, Semi-Markov packet losses, intermittent and/or probabilistic acknowledgment of packet drops.

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