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Bounded solutions to the 1–Laplacian equation with a total variation term

A. Dall'Aglio · S. Segura de León

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Abstract In this paper we study the Dirichlet problem for two related equations involving the 1–Laplacian and a total variation term as reaction, namely:

$$g(u) - \operatorname{div}\left(\frac{Du}{|Du|}\right) = |Du| + f(x)\,, \tag{1}$$

$$-\operatorname{div}\left(\frac{Du}{|Du|}\right) = |Du| + f(x)\,, \tag{2}$$

with homogeneous Dirichlet boundary conditions on $\partial\Omega$, where Ω is a regular, bounded domain in $\mathbb{R}^N$. Here f is a measurable function belonging to some suitable Lebesgue space, while $g(u)$ is a continuous function having the same sign as u and such that $g(\pm\infty) = \pm\infty$. As far as equation (1) is concerned, we show that a bounded solution exists if the datum f belongs to $L^N(\Omega)$. When the absorption term $g(u)$ is missing, i.e. in the case of equation (2), we show that if $f \in L^N(\Omega)$, and its norm is small, then the only solution of (2) is $u \equiv 0$. In the case where the norm of f is not small, several cases may happen. Depending on Ω and f, we show examples where no solution of (2) exists, other examples where $u \equiv 0$ is still a solution, and finally examples with nontrivial solutions. Some of these results can be viewed as a translation to the 1–Laplacian operator of known results by Ferone and Murat (see [14], [15] and [16]).

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1 Introduction and Statement of Main Results

In this paper we study two related Dirichlet problems involving the 1-Laplacian and a total variation term. The first one is

$$\begin{cases} g(u) - \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = |Du| + f(x) & \text{in } \Omega\,; \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3}$$

Here Ω is an open, bounded, regular subset of $\mathbb{R}^N$, $f(x)$ is a function which belongs to $L^N(\Omega)$, while $g(s) \; : \; \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(s)\, s \geq 0$ and $g(\pm\infty) = \pm\infty$. The second problem is similar, but without the zero-order term $g(u)$:

$$\begin{cases} -\operatorname{div}\left(\dfrac{Du}{|Du|}\right) = |Du| + f(x) & \text{in } \Omega\,; \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{4}$$

We are interested in existence, regularity and nonexistence results for a solution to these two problems.

The concept of solution to equations involving the 1–Laplacian was developed by Andreu, Ballester, Caselles and Mazón (see [3] and the book [4]). The natural energy space for this operator is the space $BV(\Omega)$ of functions having bounded variation. Using the theory by Anzellotti [6], they introduce a bounded vector field $\mathbf{z}$ which plays the role of the ratio $\frac{Du}{|Du|}$. The boundary condition must not be understood in terms of the trace of BV–functions, but in a weaker sense involving the vector field $\mathbf{z}$ (see Section 3).

In the case where $g(u) = u$, problem (3) was studied by Andreu, Dall'Aglio and Segura de León in [5]. In that paper, they proved that there exists a bounded solution u when $f \in L^m(\Omega)$, with $m > N$. Moreover, that solution is unique under the stronger assumption $0 \leq f(x) \leq \alpha < 2$.

Later on, in [1], Abdellaoui, Dall'Aglio and Segura de León studied the existence of infinitely many unbounded solutions of problems (3) and (4). This solutions may have prescribed singularities and are related to some elliptic problems involving singular Radon measures.

In the present paper, we prove that problem (3) admits a bounded solution even in the limit case $f \in L^N(\Omega)$. This result is somewhat surprising, because, for the similar problems for the p-laplacian, with $p > 1$, i.e.

$$\begin{cases} g(u) - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |\nabla u|^p + f(x) & \text{in } \Omega\,; \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{5}$$

one can prove the existence of bounded solutions for $f \in L^m(\Omega)$, $m > N/p$, but solutions are usually unbounded in the limit case $f \in L^{N/p}(\Omega)$ (see Boccardo, Murat and Puel [8], Ferone and Murat [14], [15] and [16], Ferone, Posteraro and Rakotoson [17], Dall'Aglio, Giachetti and Puel [12]). Therefore, while boundedness should be expected (see [5]) for problem (3) when $f \in L^m(\Omega)$ and $m > N$, it is unexpected in the limit case $f \in L^N(\Omega)$.

Subsequently, we study problem (4), where the term $g(u)$ is absent. In the case $1 < p < N$, that is, for problem (5) when $g \equiv 0$, it was proved by Ferone and Murat in [14] and [15] that a solution might not exist, unless an appropriate smallness assumption is made on the datum f. This smallness condition reads as follows:

$$\Big(\frac{1}{p-1}\Big)^{p-1} \|f\|_{N/p} < S_{N,p}^{-1}\,,$$

where $S_{N,p}$ denotes the best constant in Sobolev's imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^{\frac{Np}{N-p}}(\Omega)$, that is,

$$\Big(\int_\Omega |u|^{p^*}\Big)^{p/p^*} \le S_{N,p} \int_\Omega |\nabla u|^p.$$

This condition is related to the regularity enjoyed by the solution, namely,

$$e^{\frac{\delta}{p-1}|u|} - 1 \in W_0^{1,p}(\Omega)$$

for every $\delta > 0$ satisfying

$$\Big(\frac{\delta}{p-1}\Big)^{p-1} \|f\|_{N/p} < S_{N,p}^{-1}\,.$$

We analyze the limit problem (4), for $p = 1$, and prove that:

- if the datum $f(x)$ is "small", more precisely if $f \in L^N(\Omega)$ and $\|f\|_N < S_N^{-1}$ (where $S_N = S_{N,1}$ is the Sobolev constant appearing in the embedding of $W_0^{1,1}(\Omega)$ into $L^{N/(N-1)}(\Omega)$), then $u \equiv 0$ is the only bounded solution $u \in BV(\Omega)$ of problem (4). More precisely, $u \equiv 0$ is the only solution such that $e^{\lambda u} \in BV(\Omega)$ for every $\lambda > 0$.
- if $\|f\|_N > S_N^{-1}$, several situations may happen, depending on the actual form of f. We show that if f is a multiple of the characteristic function of a ball $B_r \subset \Omega$, then problem (4) admits no solutions as soon as $\|f\|_N > S_N^{-1}$. On the other hand, we show that $u \equiv 0$ may well be a solution when f is a (large) multiple of a characteristic of a "thin" set, like a strip or an annulus. We also show some cases where nonzero solutions appear.
- in the limit case $\|f\|_N = S_N^{-1}$, we show that $u \equiv 0$ is always a solution and non trivial solutions exist if Ω is a ball and f is constant.

The paper is organized as follows. In Section 2, we introduce our notation and state the main features of functions of bounded variation and of L^∞–divergence–measure vector fields. Section 3 is devoted to study problem (3), while problem (4) is considered in Section 4. The final section is devoted to explicit examples of existence and nonexistence for problem (4).

2 Preliminaries

From now on, we fix an integer $N \geq 2$. The symbol $\mathcal{H}^{N-1}(E)$ stands for the $(N-1)$–dimensional Hausdorff measure of a set $E \subset \mathbb{R}^N$ and $|E|$ for its Lebesgue measure. We will denote by ω_N the measure of the unit ball in $\mathbb{R}^N$. Moreover, Ω will always denote an open, bounded subset of $\mathbb{R}^N$ with Lipschitz boundary. Thus, an outward normal unit vector $\nu(x)$ is defined for $\mathcal{H}^{N-1}$–almost every $x \in \partial\Omega$.

The truncation function will be used throughout this paper. Given $k > 0$, it is defined by

$$T_k(s) = \min\{|s|, k\} \operatorname{sign}(s)\,, \tag{6}$$

for all $s \in \mathbb{R}$. Moreover we will denote by $G_k(s)$ the function defined by

$$G_k(s) = s - T_k(s)\,.$$

The space of all C^∞–functions having compact support in Ω is denoted by $C_0^\infty(\Omega)$. The symbol $L^q(\Omega)$, with $1 \leq q \leq \infty$, denotes the usual Lebesgue space with respect to Lebesgue measure and q' is the conjugate of q: $q' = \dfrac{q}{q-1}$. We will denote by $W_0^{1,q}(\Omega)$ the usual Sobolev space of measurable functions having weak gradient in $L^q(\Omega;\mathbb{R}^N)$ and zero trace on $\partial\Omega$. Finally, if $1 \leq p < N$, we will denote by $p^* = Np/(N-p)$ its Sobolev conjugate exponent and by $S_{N,p}$ the best constant in the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, that is,

$$\left(\int_\Omega |u|^{p^*}\right)^{p/p^*} \leq S_{N,p} \int_\Omega |\nabla u|^p \qquad \text{for all } u \in W_0^{1,p}(\Omega)\,.$$

We will also write S_N instead of $S_{N,1}$.

The natural energy space to study equations involving the 1–Laplacian is the space $BV(\Omega)$ of functions of bounded variation. It is defined as the space of functions $u \in L^1(\Omega)$ whose distributional gradient Du is a vector–valued Radon measure on Ω with finite total variation. This space is a Banach space with the norm defined by

$$\|u\|_{BV} = \int_\Omega |u|\,dx + |Du|(\Omega)\,.$$

We recall that the notion of trace can be extended to any $u \in BV(\Omega)$ and this fact allows us to interpret it as the boundary values of u and to write $u\big|_{\partial\Omega}$. Moreover, it holds that the trace is a linear bounded operator $BV(\Omega) \to L^1(\partial\Omega)$ which is onto. Using the trace, an equivalent norm in $BV(\Omega)$ can be defined by

$$\|u\| = \int_{\partial\Omega} |u|\,d\mathcal{H}^{N-1} + |Du|(\Omega)\,.$$

The Sobolev embedding $W_0^{1,1}(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$ extends to BV–Functions; it yields

$$\left(\int_\Omega |u|^{\frac{N}{N-1}}\right)^{(N-1)/N} \leq S_N \left[|Du|(\Omega) + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1}\right],$$

for all $u \in BV(\Omega)$. We point out that the same constant can be taken.

For every $u \in BV(\Omega)$, the Radon measure Du can be decomposed into three parts: $Du = D^a u + D^c u + D^j u$, where $D^a u$ is its absolutely continuous part (we mean absolutely continuous with respect to Lebesgue measure), $D^c u$ its Cantor part and $D^j u$ its jump part. This decomposition is defined as follows. We denote by S_u the set of all $x \in \Omega$ at which the approximate limit of u does not exist: if $x \in \Omega\backslash S_u$, we denote by $\tilde{u}(x)$ the approximate limit of u at x. On the other hand, we denote by $J_u \subset S_u$ the set of approximate jump points of u, that is, those points where there exist "one side" limits of u: $u_+(x)$ and $u_-(x)$. Then $D^c u = D^s u \llcorner (\Omega\backslash S_u)$ and $D^j u = D^s u \llcorner J_u$, where $D^s u = D^c u + D^j u$ stands for the singular part of Du with respect to the Lebesgue measure. The precise representative $u^* : \Omega\backslash(S_u\backslash J_u) \to \mathbb{R}$ of u is defined as equal to $\tilde{u}$ on $\Omega\backslash S_u$ and equal to $\frac{u_+ + u_-}{2}$ on J_u. Since $\mathcal{H}^{N-1}(S_u\backslash J_u) = 0$, due to the Federer–Vol'pert Theorem, it follows that u^* is defined $\mathcal{H}^{N-1}$–a.e. in Ω.

A compactness result in $BV(\Omega)$ will be used several times in what follows. It states that every sequence that is bounded in $BV(\Omega)$ has a subsequence which converges strongly in $L^1(\Omega)$ to certain $u \in BV(\Omega)$.

To pass to the limit we will often apply that some functionals defined on $BV(\Omega)$ are lower semicontinuous with respect to the convergence in $L^1(\Omega)$. We recall that the functional defined by

$$u \mapsto |Du|(\Omega) + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} \tag{7}$$

is lower semicontinuous with respect to the convergence in $L^1(\Omega)$. Similarly, if we fix $\varphi \in C_0^1(\Omega)$, with $\varphi \geq 0$, the functional defined by

$$u \mapsto \int_\Omega \varphi\, d|Du|,$$

is lower semicontinuous in $L^1(\Omega)$.

For further information concerning functions of bounded variation we refer to [2] or [22].

In our definition of solution we will need some features of L^∞–divergence–measure vector fields and functions of bounded variation (see [6] and [10]). Basically, a type of dot product of a vector field and the gradient of a bounded variation function is used to give sense to $\mathbf{z} = \frac{Du}{|Du|}$, namely, $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ satisfies $\|\mathbf{z}\|_\infty \leq 1$ and $(\mathbf{z}, Du) = |Du|$.

From now on, we denote by $\mathcal{DM}^\infty(\Omega)$ the space of all vector fields $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ whose divergence in the sense of distributions is a Radon measure with finite total variation.

To define $(\mathbf{z}, Dv)$ in Anzellotti's theory, we need some compatibility conditions, for instance $\operatorname{div}\mathbf{z}$ is a Radon measure with finite total variation and $v \in BV(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$. In this case, we define the dot product as a distribution: for every $\varphi \in C_0^\infty(\Omega)$, we write

$$\langle(\mathbf{z}, Du), \varphi\rangle = -\int_\Omega u^*\,\varphi\,d\mu - \int_\Omega u\mathbf{z}\cdot\nabla\varphi\,,$$

where $\mu = \operatorname{div}\mathbf{z}$. This distribution $(\mathbf{z}, Dv)$ is actually a Radon measure. Moreover, the following basic inequality holds: $|(\mathbf{z}, Dv)| \le \|\mathbf{z}\|_\infty |Dv|$. On the other hand, for every $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$, a weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}$ is defined in [6] and denoted by $[\mathbf{z}, \nu]$. Anzellotti's definition of $(\mathbf{z}, Dv)$ can be extended to the case where $\operatorname{div}\mathbf{z}$ is a Radon measure with finite total variation and $v \in BV(\Omega) \cap L^\infty(\Omega)$ (see [20, Appendix A] and [9, Section 5]). A further extension can be found in [1, Section 3] for bounded vector fields $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ satisfying $-\operatorname{div}\mathbf{z} \ge f \in L^N(\Omega)$ and a general $v \in BV(\Omega)$. (We also refer to [11] for additional information.) These extensions will be used throughout this paper.

Under these extended assumptions, a Green formula holds.

Proposition 1 *(see [6], [20], [9], [1]). Let $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ be such that $-\operatorname{div}\mathbf{z} = \beta + f$, where β is a nonnegative Radon measure on Ω, and $f \in L^N(\Omega)$. Let $u \in BV(\Omega)$. Then, with the above definitions, $u^* \in L^1(\Omega, d\mu)$, and the following Green formula holds*

$$\int_\Omega u^*\,d\mu + \int_\Omega d(\mathbf{z}, Du) = \int_{\partial\Omega} [\mathbf{z}, \nu]u\ d\mathcal{H}^{N-1}\,, \tag{8}$$

where $\mu = \operatorname{div}\mathbf{z}$.

3 Existence of a solution to (3)

This Section is devoted to obtain an existence result for problem (3), where $f \in L^N(\Omega)$.

Definition 1 A solution of problem (3) is a function $u \in BV(\Omega)$, with an associated vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ satisfying

$$\|\mathbf{z}\|_\infty \le 1\,; \tag{9}$$

$$g(u) - \operatorname{div}\mathbf{z} = |Du| + f(x) \quad \text{in the sense of distributions;} \tag{10}$$

$$(\mathbf{z}, Du) = |Du| \quad \text{as measures;} \tag{11}$$

$$[\mathbf{z}, \nu] \in \operatorname{sign}(-u) \quad \mathcal{H}^{N-1}\text{–a.e. on } \partial\Omega; \tag{12}$$

$$D^j u = 0. \tag{13}$$

Theorem 1 *For every $f \in L^N(\Omega)$ there exists a bounded solution to problem (3).*

Proof The proof will be divided into several steps.

Step 1: Approximating problems: $p > 1$. Assume for the moment that $f \in L^\infty(\Omega)$. We consider the following problems, for $p > 1$:

$$\begin{cases} g(u_p) - \operatorname{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right) = |\nabla u_p|^p + f(x) \text{ in } \Omega\,; \\ \qquad\qquad u_p \in W_0^{1,p}(\Omega)\,. \end{cases} \tag{14}$$

By the results proved by Ferone and Murat in [16], for every $p > 1$ there exists a solution $u_p \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ of (14). We wish to show that the L^∞–estimate does not depend on p. Indeed, we can multiply the equation in (14) by

$$v = \left(e^{\lambda|G_k(u_p)|} - 1\right) \operatorname{sign} u_p\,,$$

for $\lambda > 1$ and for some positive k. Since $g(s)$ has the same sign as s, we easily obtain

$$\begin{aligned} &\int_\Omega |g(u_p)| \left(e^{\lambda|G_k(u_p)|} - 1\right) + \lambda \int_\Omega |\nabla G_k(u_p)|^p\, e^{\lambda|G_k(u_p)|} \\ &\quad \le \int_\Omega |\nabla G_k(u_p)|^p \left(e^{\lambda|G_k(u_p)|} - 1\right) + \|f\|_\infty \int_\Omega \left(e^{\lambda|G_k(u_p)|} - 1\right)\,. \end{aligned} \tag{15}$$

By the assumptions on g, there exists $k > 0$ such that $|g(s)| > \|f\|_\infty$ for all s such that $|s| > k$. With this choice of k, one has

$$\int_\Omega |g(u_p)| \left(e^{\lambda|G_k(u_p)|} - 1\right) \ge \|f\|_\infty \int_\Omega \left(e^{\lambda|G_k(u_p)|} - 1\right)\,,$$

therefore the two integrals in (15) cancel out, and one can conclude that

$$(\lambda - 1) \int_\Omega |\nabla G_k(u_p)|^p \left(e^{\lambda|G_k(u_p)|} - 1\right) \le 0\,,$$

which gives

$$\|u_p\|_\infty \le k = k(g, \|f\|_\infty)\,.$$

We emphasize that this estimate is independent on p.

Once this estimate is proved, one can follow the same steps as in [5], pass to the limit for $p \downarrow 1$ and conclude that, when $f \in L^\infty(\Omega)$, there exists a bounded solution of problem (3).

Step 2: L^∞–estimate for unbounded f. Assume now that $f \in L^N(\Omega)$. Then, for $n \in \mathbb{N}$, let u_n be a solution of problem

$$\begin{cases} g(u_n) - \operatorname{div}\left(\dfrac{Du_n}{|Du_n|}\right) = |Du_n| + T_n(f(x))\,, \text{ in } \Omega\,; \\ \qquad\qquad u_n = 0\,, \qquad\qquad \text{on } \partial\Omega\,. \end{cases} \tag{16}$$

Such a solution exists by Step 1. Let $\mathbf{z}_n$ be the associated vector field according to Definition 1. For every $k > 0$, by taking $(e^{2|G_k(u_n)|} - 1)\operatorname{sign} u_n$ as test function in (16) (see (10) for the meaning), we get

$$\int_\Omega |g(u_n)|\Big(e^{2|G_k(u_n)|} - 1\Big) \\ +\int_\Omega \Big(\mathbf{z}_n, D\Big[\Big(e^{2|G_k(u_n)|}-1\Big)\operatorname{sign} u_n\Big]\Big) - \int_{\partial\Omega} \Big(e^{2|G_k(u_n)|}-1\Big)\operatorname{sign} u_n\,[\mathbf{z}_n,\nu]\,d\mathcal{H}^{N-1} \\ \leq \int_\Omega \Big(e^{2|G_k(u_n)|} - 1\Big)^*|Du_n| + \int_\Omega |T_n(f)|\Big(e^{2|G_k(u_n)|} - 1\Big). \quad (17)$$

Taking into account $D^j u_n = 0$, we may apply [18, Proposition 2.7] to the Lipschitz-continuous function $s \mapsto \Big(e^{2|G_k(s)|} - 1\Big)\operatorname{sign} s$ and deduce from (11) that

$$\Big(\mathbf{z}_n, D\Big[\Big(e^{2|G_k(u_n)|} - 1\Big)\operatorname{sign} u_n\Big]\Big) = \Big|D\Big[\Big(e^{2|G_k(u_n)|} - 1\Big)\operatorname{sign} u_n\Big]\Big|\,.$$

Now the chain rule (for a BV–function without jump part) yields

$$\Big(\mathbf{z}_n, D\Big[\Big(e^{2|G_k(u_n)|} - 1\Big)\operatorname{sign} u_n\Big]\Big) = 2\Big(e^{2|G_k(u_n)|}\Big)^*|DG_k(u_n)|\,.$$

On the other hand, $[\mathbf{z}_n,\nu] \in \operatorname{sign}(-u_n)$ on $\partial\Omega$. Hence, inequality (17) becomes

$$\int_\Omega |g(u_n)|\Big(e^{2|G_k(u_n)|} - 1\Big) \\ + 2\int_\Omega \Big(e^{2|G_k(u_n)|}\Big)^*|DG_k(u_n)| + \int_{\partial\Omega} \Big(e^{2|G_k(u_n)|} - 1\Big)\,d\mathcal{H}^{N-1} \\ \leq \int_\Omega \Big(e^{2|G_k(u_n)|} - 1\Big)^*|Du_n| + \int_\Omega |T_n(f)|\Big(e^{2|G_k(u_n)|} - 1\Big).$$

Since $e^{2|G_k(u_n)|} - 1$ vanishes on $\{|u_n| > k\}$, simplifying and dropping nonnegative terms, it yields

$$\int_\Omega |g(u_n)|\Big(e^{2|G_k(u_n)|} - 1\Big) \\ + \int_\Omega \Big(e^{2|G_k(u_n)|}\Big)^*|DG_k(u_n)| + \int_{\partial\Omega} \Big(e^{2|G_k(u_n)|} - 1\Big)\,d\mathcal{H}^{N-1} \\ \leq \int_\Omega |T_n(f)|\Big(e^{2|G_k(u_n)|} - 1\Big) \\ \leq h\int_{\{|T_n(f)|\leq h\}} \Big(e^{2|G_k(u_n)|} - 1\Big) + \int_{\{|T_n(f)|>h\}} |T_n(f)|\Big(e^{2|G_k(u_n)|} - 1\Big),$$

for some $h > 0$ to be chosen hereafter. As before, we can take $k = k(h)$ such that $|g(s)| \geq h$ for $|s| > k$. With this choice of k, one has

$$
\begin{aligned}
h \int_\Omega \Big(e^{2|G_k(u_n)|} - 1 \Big) + \frac{1}{2} \int_\Omega \Big| D\Big(e^{2|G_k(u_n)|} - 1 \Big) \Big| + \int_{\partial\Omega} \Big(e^{2|G_k(u_n)|} - 1 \Big) \, d\mathcal{H}^{N-1} \\
\leq h \int_{\{|T_n(f)| \leq h\}} \Big(e^{2|G_k(u_n)|} - 1 \Big) + \int_{\{|T_n(f)| > h\}} |T_n(f)| \Big(e^{2|G_k(u_n)|} - 1 \Big) ,
\end{aligned}
$$

so that the first integral of each side cancels, and Hölder's inequality implies

$$
\begin{aligned}
\int_\Omega \Big| D\Big(e^{2|G_k(u_n)|} - 1 \Big) \Big| + \int_{\partial\Omega} \Big(e^{2|G_k(u_n)|} - 1 \Big) \, d\mathcal{H}^{N-1} \\
\leq 2 \int_{\{|T_n(f)| > h\}} |T_n(f)| \Big(e^{2|G_k(u_n)|} - 1 \Big) \\
\leq 2 \| T_n(f) \chi_{\{|T_n(f)| > h\}} \|_N \, \| e^{2|G_k(u_n)|} - 1 \|_{N/(N-1)} .
\end{aligned}
$$

Applying Sobolev's inequality, we can write

$$
\begin{aligned}
\| e^{2|G_k(u_n)|} - 1 \|_{N/(N-1)} \\
\leq S_N \Big[\int_\Omega \Big| D\Big(e^{2|G_k(u_n)|} - 1 \Big) \Big| + \int_{\partial\Omega} \Big(e^{2|G_k(u_n)|} - 1 \Big) \, d\mathcal{H}^{N-1} \Big] \\
\leq 2 S_N \| T_n(f) \chi_{\{|T_n(f)| > h\}} \|_N \| e^{2|G_k(u_n)|} - 1 \|_{N/(N-1)} \\
\leq 2 S_N \| f \chi_{\{|f| > h\}} \|_N \, \| e^{2|G_k(u_n)|} - 1 \|_{N/(N-1)} ,
\end{aligned}
$$

wherewith the right hand side can be absorbed if $\| f \chi_{\{|f|>h\}} \|_N$ is small enough, that is, if we choose h large enough. Thus, for $k = k(h)$, this leads to $\| e^{2|G_k(u_n)|} - 1 \|_{N/(N-1)} = 0$ for all $n \in \mathbb{N}$, and as a consequence $\| u_n \|_\infty \leq k = k(f, g)$ for all $n \in \mathbb{N}$.

Step 3: BV–estimate. We take $\big(e^{2|u_n|} - 1 \big)$ sign u_n as test function in (16), obtaining

$$
\begin{aligned}
\int_\Omega |g(u_n)| \big(e^{2|u_n|} - 1 \big) \\
+ \int_\Omega \Big(\mathbf{z}_n \,,\, D\big[\big(e^{2|u_n|} - 1 \big) \text{ sign } u_n \big] \Big) - \int_{\partial\Omega} \big(e^{2|u_n|} - 1 \big) \text{ sign } u_n \, [\mathbf{z}_n, \nu] \, d\mathcal{H}^{N-1} \\
\leq \int_\Omega \big(e^{2|u_n|} - 1 \big)^* |Du_n| + \int_\Omega |f| \big(e^{2|u_n|} - 1 \big) .
\end{aligned}
$$

Having in mind (11) and (12), applying the chain rule and disregarding non-negative terms, it yields

$$
\int_\Omega \big(e^{2|u_n|} \big)^* |Du_n| + \int_\Omega |Du_n| + \int_{\partial\Omega} \big(e^{2|u_n|} - 1 \big) \, d\mathcal{H}^{N-1} \leq \int_\Omega |f| \big(e^{2|u_n|} - 1 \big) .
$$

Observe that the right hand side is bounded due to the L^∞–estimate. Hence, in addition to be bounded in $L^\infty(\Omega)$, we have that the sequence $\left(e^{2|u_n|}-1\right)_n$ is bounded in $BV(\Omega)$, so that (up to subsequences) there exists $u \in BV(\Omega) \cap L^\infty(\Omega)$ satisfying $e^{2|u|}-1 \in BV(\Omega)$ and

$$\begin{gathered}
De^{2|u_n|} \rightharpoonup De^{2|u|} \quad \text{*–weakly as measures} \\
u_n \to u \quad \text{pointwise a.e in } \Omega \\
u_n \to u \quad \text{strongly in } L^r(\Omega)\,, \quad 1 \le r < \infty
\end{gathered}$$

Step 4: Convergence of $(\mathbf{z}_n)_n$*.* It follows from (9) that there exists $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$ such that (up subsequences) $\mathbf{z}_n \rightharpoonup \mathbf{z}$ *–weakly in $L^\infty(\Omega;\mathbb{R}^N)$. Obviously, $\|\mathbf{z}\|_\infty \le 1$ holds.

Now, we may take $e^{u_n}\varphi$, where $\varphi \in C_0^\infty(\Omega)$, as test function in (16), simplify and pass to the limit which leads to $-\mathrm{div}\,(e^u\mathbf{z}) = e^u\,(f-g(u))$ in the sense of distributions. Thus, $\mathrm{div}\,(e^u\mathbf{z}) \in L^N(\Omega)$.

In a similar way, choosing $\varphi \in C_0^\infty(\Omega)$ with $\varphi \ge 0$ as test function in (10), the lower–semicontinuity of the total variation implies that

$$\int_\Omega g(u)\varphi + \int_\Omega \mathbf{z}\cdot\nabla\varphi \ge \int_\Omega \varphi|Du| + \int_\Omega f\varphi\,.$$

Therefore, the inequality $g(u) - \mathrm{div}\,\mathbf{z} \ge |Du| + f(x)$ holds in the sense of distributions. As a consequence, $\mathrm{div}\,\mathbf{z}$ is a Radon measure. Furthermore, it follows from (10) that the sequence of measures $(\mathrm{div}\,\mathbf{z}_n)_n$ is bounded, and so (up to subsequences) it converges *–weakly in the sense of measures. Since its limit must be $\mathrm{div}\,\mathbf{z}$, it follows that $\mathrm{div}\,\mathbf{z}$ is a Radon measure with finite total variation.

It remains to prove the points:

$$\begin{gathered}
D^j u = 0\,; \\
g(u) - \mathrm{div}\,\mathbf{z} = |Du| + f(x) \quad \text{in the sense of distributions;} \\
(\mathbf{z}, Du) = |Du| \quad \text{as measures;} \\
[\mathbf{z},\nu] \in \ \mathrm{sign}\,(-u) \quad \mathcal{H}^{N-1}\text{–a.e. on } \partial\Omega;
\end{gathered}$$

To see them, we may follow the steps 7–10 of the proof of [5, Theorem 1].

Remark 1 It is worth observing that the same proof works for any increasing real function g such that $g(\pm\infty) = \pm\infty$. We just have to replace $g(s)$ with $g(s) - g(0)$ and the datum $f(x)$ with $f(x) - g(0)$.

Remark 2 We point out that a similar argument to that used in the proof of Theorem 1 leads to the boundedness of the solutions to the Dirichlet problem for

$$g(u) - \mathrm{div}\left(\frac{Du}{|Du|}\right) = f(x)\,,$$

with $f \in L^N(\Omega)$.

4 Existence of a solution to (4)

In this Section we will study existence and non existence for the Dirichlet problem (4).

Theorem 2 *If $f \in L^N(\Omega)$ satisfies $\|f\|_N < S_N^{-1}$, then $u \equiv 0$ is the only solution to problem (4) satisfying $e^{\lambda u} \in BV(\Omega)$ for all $\lambda > 1$.*

Proof Step 1: Existence. Since a solution of (4) is actually the pair $(u, \mathbf{z})$, we still have to get the vector field $\mathbf{z}$. To this end, apply a duality argument to obtain the embedding $L^N(\Omega) \hookrightarrow W^{-1,\infty}(\Omega)$. Thus, given $f \in L^N(\Omega)$, we find $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ satisfying $f = -\operatorname{div} \mathbf{z}$ and $\|\mathbf{z}\|_\infty = \|\operatorname{div} \mathbf{z}\|_{W^{-1,\infty}(\Omega)}$. Moreover,

$$\begin{aligned}
\|\operatorname{div} \mathbf{z}\|_{W^{-1,\infty}(\Omega)} &= \sup\left\{\int_\Omega \mathbf{z}\cdot\nabla u \;:\; u \in W_0^{1,1}(\Omega)\,, \quad \int_\Omega |\nabla u| \le 1\right\} \\
&= \sup\left\{\int_\Omega f u \;:\; u \in W_0^{1,1}(\Omega)\,, \quad \int_\Omega |\nabla u| \le 1\right\} \\
&\le \sup\left\{\int_\Omega f u \;:\; u \in L^{\frac{N}{N-1}}(\Omega)\,, \quad \|u\|_{\frac{N}{N-1}} \le S_N\right\} \le \|f\|_N S_N < 1\,.
\end{aligned}$$

Hence, $\|\mathbf{z}\|_\infty < 1$.

Finally, taking $u \equiv 0$, we have seen that

$$\begin{aligned}
-\operatorname{div} \mathbf{z} &= |Du| + f\,, \quad \text{in } \mathcal{D}'(\Omega) \\
(\mathbf{z}, Du) &= |Du|\,, \quad \text{as measures} \\
u\big|_{\partial\Omega} &= 0 \quad \text{in the sense of traces, which imply condition (12).}
\end{aligned}$$

Therefore, $u \equiv 0$ is a solution to problem (4).

Step 2: Uniqueness. Assume that $u \in BV(\Omega)$ is a solution to problem (4) satisfying $e^{\lambda u} \in BV(\Omega)$ for all $\lambda > 1$. Then $D^j u = 0$ and there exists $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ such that

i) $-\operatorname{div} \mathbf{z} = |Du| + f$ in $\mathcal{D}'(\Omega)$
ii) $(\mathbf{z}, Du) = |Du|$ as measures
iii) $[\mathbf{z}, \nu] \in$ sign $(-u)$ $\mathcal{H}^{N-1}$–a.e. on $\partial\Omega$

Fix λ such that $\frac{\lambda-1}{\lambda} > \|f\|_N S_N$ and apply Green's formula to get

$$\int_\Omega (\mathbf{z}, D(e^{\lambda u} - 1)) - \int_{\partial\Omega} (e^{\lambda u} - 1)[\mathbf{z}, \nu]\, d\mathcal{H}^{N-1} = \int_\Omega (e^{\lambda u} - 1)^* |Du| + \int_\Omega f(e^{\lambda u} - 1)\,. \tag{18}$$

We now consider each term appearing in (18). By Lemma 1 below,

$$\int_\Omega (\mathbf{z}, D(e^{\lambda u} - 1)) = \int_\Omega \left|D(e^{\lambda u} - 1)\right|\,. \tag{19}$$

As far as the right hand side of (18) is concerned, the chain rule leads to

$$\int_\Omega (e^{\lambda u}-1)^*|Du| = \frac{1}{\lambda}\int_\Omega |D(e^{\lambda u}-1)| - \int_\Omega |Du|\,. \tag{20}$$

Finally, it follows from condition iii) that

$$\int_{\partial\Omega} (e^{\lambda u}-1)[\mathbf{z},\nu]\,d\mathcal{H}^{N-1} = -\int_{\partial\Omega} \big|e^{\lambda u}-1\big|\,d\mathcal{H}^{N-1}\,. \tag{21}$$

Having in mind (19), (20) and (21), equation (18) becomes

$$\frac{\lambda-1}{\lambda}\int_\Omega \big|D(e^{\lambda u}-1)\big| + \int_{\partial\Omega}\big|e^{\lambda u}-1\big|\,d\mathcal{H}^{N-1} + \int_\Omega |Du| = \int_\Omega f(e^{\lambda u}-1)\,.$$

Applying Hölder's and Sobolev's inequalities on the right hand side, it yields

$$\int_\Omega f(e^{\lambda u}-1) \le \|f\|_N S_N \left[\int_\Omega \big|D(e^{\lambda u}-1)\big| + \int_{\partial\Omega}\big|e^{\lambda u}-1\big|\,d\mathcal{H}^{N-1}\right],$$

wherewith

$$\Big(\frac{\lambda-1}{\lambda} - \|f\|_N S_N\Big)\left[\int_\Omega \big|D(e^{\lambda u}-1)\big| + \int_{\partial\Omega}\big|e^{\lambda u}-1\big|\,d\mathcal{H}^{N-1}\right] \le 0\,.$$

Since $\frac{\lambda-1}{\lambda} - \|f\|_N S_N$ is positive, it follows that

$$\int_\Omega \big|D(e^{\lambda u}-1)\big| + \int_{\partial\Omega}\big|e^{\lambda u}-1\big|\,d\mathcal{H}^{N-1} = 0\,,$$

and so $e^{\lambda u}-1\equiv 0$. Therefore, $u\equiv 0$.

Remark 3 As far as existence of trivial solutions is concerned, we point out that the same proof leads to the following result:

If $f\in L^N(\Omega)$ satisfies $\|f\|_N \le S_N^{-1}$, then $u\equiv 0$ is a solution to problem (4).

Lemma 1 *Let $\mathbf{z}\in L^\infty(\Omega;\mathbb{R}^N)$ be such that $-\operatorname{div}\mathbf{z} = \beta + f$, where β is a nonnegative Radon measure on Ω, and $f\in L^N(\Omega)$. Assume that u is a function in $BV(\Omega)$, with $D^j(u)=0$, and that $\psi(s) : \mathbb{R}\to\mathbb{R}$ is an increasing locally Lipschitz function, such that $w=\psi(u)\in BV(\Omega)$.*

If $(z,Du)=|Du|$ as measures, then

$$(\mathbf{z}\,,\,Dw) = |Dw| \quad \textit{as measures.} \tag{22}$$

Proof We first remark that

$$\begin{aligned}|Du| = (\mathbf{z},Du) = (\mathbf{z},DT_k(u)) + (\mathbf{z},DG_k(u))&\\ \le |DT_k(u)| + |DG_k(u)| = |Du|&\end{aligned}$$

and consequently the inequality is actually an equality. Thus, as measures, $(\mathbf{z},DT_k(u)) = |DT_k(u)|$ for every $k>0$. We also write $w_k=\psi(T_k(u))$ for all

$k > 0$. We remark that $|Dw_k| = |Dw|\chi_{\{|u|<k\}}$ holds for all $k > 0$. Indeed, since $T_k(u)$ is bounded, and ψ is Lipschitz-continuous in the interval $[-k, k]$, it is enough to apply the chain rule [2, Theorem 99] to deduce from $D^j u = 0$ that $D^j w_k = 0$ for all $k > 0$. Hence, fixed $k > 0$, we obtain from the chain rule that $|Dw_k| = |Dw_h|\chi_{\{|u|<k\}}$ for all $h > k$, so that $|Dw_k| = |Dw|\chi_{\{|u|<k\}}$ holds.

On the other hand, having in mind again that $T_k(u)$ is bounded, and ψ is Lipschitz-continuous in the interval $[-k, k]$, and applying [18, Proposition 2.7], we deduce that the Radon–Nikodým derivative of $(\mathbf{z}, Dw_k)$ with respect to $|Dw_k|$ coincides with the Radon–Nikodým derivative of $(\mathbf{z}, DT_k(u))$ with respect to $|DT_k(u)|$. Thus, $(\mathbf{z}, DT_k(u)) = |DT_k(u)|$ implies $(\mathbf{z}, Dw_k) = |Dw_k|$ for every $k > 0$. Then, for each nonnegative $\varphi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}\int_\Omega \varphi\,|Dw_k| = \int_\Omega \varphi\,(\mathbf{z}, Dw_k) &= -\int_\Omega w_k\varphi \operatorname{div}\mathbf{z} - \int_\Omega w_k\mathbf{z}\cdot\nabla\varphi \\ &= \int_\Omega w_k\varphi\,d\beta + \int_\Omega w_k\varphi f - \int_\Omega w_k\mathbf{z}\cdot\nabla\varphi\,. \end{aligned} \tag{23}$$

In order to let k go to infinity, we apply Levi's monotone convergence theorem to deal with the left hand side:

$$\lim_{k\to\infty}\int_\Omega \varphi\,|Dw_k| = \lim_{k\to\infty}\int_{\{|u|<k\}} \varphi\,|Dw| = \int_\Omega \varphi\,|Dw|\,.$$

Moreover, by Proposition 1, w is summable with respect to the measure β. It follows from Lebesgue's dominated convergence theorem that

$$\lim_{k\to\infty}\int_\Omega w_k\varphi\,d\beta = \int_\Omega w\varphi\,d\beta\,.$$

Applying again Lebesgue's dominated convergence theorem, we obtain

$$\lim_{k\to\infty}\left(\int_\Omega w_k\varphi f - \int_\Omega w_k\mathbf{z}\cdot\nabla\varphi\right) = \int_\Omega w\varphi f - \int_\Omega w\mathbf{z}\cdot\nabla\varphi\,,$$

due to $w \in BV(\Omega) \subset L^{\frac{N}{N-1}}(\Omega)$, $f \in L^N(\Omega)$ and $\mathbf{z} \in L^\infty(\Omega;\mathbb{R}^N)$. Hence, we may take the limit in (23) to conclude that

$$\int_\Omega \varphi\,|Dw| = -\int_\Omega w\varphi\operatorname{div}\mathbf{z} - \int_\Omega w\mathbf{z}\cdot\nabla\varphi = \int_\Omega \varphi\,(\mathbf{z}, Dw)$$

for every nonnegative $\varphi \in C_0^\infty(\Omega)$. Therefore, (22) holds true.

5 Examples of existence and non existence for (4)

This final section is devoted to several examples of existence and non existence of solutions for problem (4).

We will use several times the well-known fact that the Sobolev constant S_N is indeed the isoperimetric constant, that is,

$$S_N = \frac{|B_R(x_0)|^{(N-1)/N}}{\mathcal{H}^{N-1}(\partial B_R(x_0))} = \frac{1}{N\,\omega_N^{1/N}}\,. \tag{24}$$

The first two examples will show that, even when $\|f\|_N$ is larger than S_N^{-1}, there are cases where $u \equiv 0$ is still a solution of problem (4).

Example 1 Let Ω be a bounded domain with Lipschitz boundary. We will see that for every $t > S_N^{-1}$ there exists f such that $\|f\|_N = t$ and problem (4) admits the trivial solution. There is no loss of generality in assuming $0 \in \Omega$.

Fix $R > 0$ such that $B_R(0) \subset \Omega$, choose ϵ such that $0 < \epsilon < 1$ and take $\rho > 0$ satisfying $\rho^N = (1-\varepsilon)R^N$. Consider $\lambda = \frac{N}{R\varepsilon}$ and the datum $f = \lambda\chi_{B_R(0)\setminus B_\rho(0)}$. Then

$$\|f\|_N^N = \lambda^N\omega_N(R^N - \rho^N) = \lambda^N\omega_N R^N\varepsilon = \frac{\omega_N N^N}{\varepsilon^{N-1}}\,.$$

Note that, using (24), the norm $\|f\|_N$ takes all values larger than S_N^{-1} when ε varies between 0 and 1.

Now it is easy to check that $u \equiv 0$ is a solution with a vector field given by

$$\mathbf{z}(x) = \begin{cases} 0\,, & \text{if } |x| \le \rho\,; \\ -x\xi(|x|)\,, & \text{if } \rho < |x| < R\,; \\ -R^N\xi(R)\frac{x}{|x|^N}\,, & \text{if } |x| \ge R\,; \end{cases}$$

where $\xi(r) = \dfrac{\lambda}{N}\Big(1 - \Big(\dfrac{\rho}{r}\Big)^N\Big)$. Indeed, $\|\mathbf{z}\|_\infty \le 1$ since $R\xi(R) = 1$ as a consequence of our choice of λ. On the other hand, $\xi(\rho) = 0$ and

$$-\operatorname{div}\mathbf{z}(x) = \begin{cases} 0\,, & \text{if } |x| \le \rho\,; \\ N\xi(|x|) + |x|\xi'(|x|)\,, & \text{if } \rho < |x| < R\,; \\ 0\,, & \text{if } |x| \ge R\,. \end{cases}$$

The result follows from the identity $N\xi(r) + r\xi'(r) = \lambda$, for $\rho < r < R$.

Example 2 We now consider a two dimensional example, though it can easily be generalized to a higher dimension. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and containing the origin, and choose $\ell > 0$. Denote by

$$\Omega_\ell = \Omega \cap \{(x,y) \in \mathbb{R}^2 \;:\; |y| \le 1/\ell\}\,.$$

For the sake of simplicity, assume that Ω_ℓ is a rectangle of sides $2/\ell$ and L, centered in the origin. Taking $f = \ell\chi_{\Omega_\ell}$, its L^2-norm is

$$\|f\|_2 = \ell\sqrt{|\Omega_\ell|} = \ell\sqrt{\frac{2L}{\ell}} = \sqrt{2L\ell}\,,$$

which can be made as large or as small as we wish, by choosing ℓ accordingly.

Now, considering f as a datum, it is easy to check that $u \equiv 0$ is a solution to problem (4) with vector field defined by

$$\mathbf{z}(x) = \begin{cases} (0, -1)\,, & \text{if } y > 1/\ell\,; \\ (0, -y\ell)\,, & \text{if } |y| < 1/\ell\,; \\ (0, 1)\,, & \text{if } y < -1/\ell\,. \end{cases}$$

The following example shows that the threshold S_N^{-1} is sharp for the existence of a solution solution to problem (4).

Example 3 Fix $x_0 \in \Omega$ and let $R > 0$ satisfy $B_R(x_0) \subset \Omega$. Let t be any number larger than S_N^{-1}. Consider $f = \lambda \chi_{B_R(x_0)}$, where $\lambda = \frac{t}{|B_R(x_0)|^{1/N}}$ is chosen such that $\|f\|_N = t$.

Assume, by contradiction, that there exists a solution $u \in BV(\Omega)$ to problem (4). Hence, we can find a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ satisfying $\|\mathbf{z}\|_\infty \leq 1$ and $-\operatorname{div} \mathbf{z} = |Du| + f$ in the sense of distributions. Integrating in the ball $B_R(x_0)$ and applying the Green formula, we get

$$-\int_{\partial B_R(x_0)} [\mathbf{z}, \nu]\, d\mathcal{H}^{N-1} = \int_{B_R(x_0)} |Du| + \int_{B_R(x_0)} f(x)\, dx \geq \lambda |B_R(x_0)|\,.$$

Observe that the left-hand side is smaller than $\mathcal{H}^{N-1}(\partial B_R(x_0))$. Therefore $\mathcal{H}^{N-1}(\partial B_R(x_0)) \geq \lambda |B_R(x_0)|$, which gives

$$t = \lambda |B_R(x_0)|^{1/N} \leq \frac{\mathcal{H}^{N-1}(\partial B_R(x_0))}{|B_R(x_0)|^{(N-1)/N}} = S_N^{-1}\,,$$

a contradiction.

The previous examples show that, in the case where f is constant on some set, the isoperimetric inequality plays an important role for existence and nonexistence of solutions. For instance, in the Examples 1 and 2, f is different from zero on some very "thin" sets, for which the isoperimetric ratio is far from optimal. On the contrary, in the example given in Example 3 f is a multiple of a characteristic function of a ball.

We summarize the previous examples in the following result:

Proposition 2 *Assume that Ω is a bounded domain with Lipschitz boundary. Then, for every $t > S_N^{-1}$, there exists $f \in L^N(\Omega)$ such that $\|f\|_N = t$ and problem (4) has no solution. On the other hand, there exists $f \in L^N(\Omega)$ such that $\|f\|_N = t$ and problem (4) has the trivial solution.*

One may wonder whether there exist nontrivial solutions to problem (4). This is indeed the case, as we will show in the following three examples, which examine different cases, according to the size of the datum.

Example 4 (*The case of f small and regular enough*) More precisely, $f \in L^m(\Omega)$ with $m > N$ and

$$\|f\|_m < \Big(\frac{m-N}{N(m-1)}\Big)^{\frac{m-1}{m}} \frac{|\Omega|^{\frac{1}{m}-\frac{1}{N}}}{S_N}\,.$$

In [1] non regular solutions to problem (4) have been studied. The main result states, under the above smallness condition on the datum, the existence of unbounded solutions to (4). These solutions $u \in BV(\Omega)$ do not contradict Theorem 2, since $e^{\lambda u} \notin BV(\Omega)$ for $\lambda \ge 1$.

Example 5 (*The critical case* $\|f\|_N = S_N^{-1}$) Let Ω is a ball, say B_R, and f is a constant datum. Let $f(x) = \lambda$, it follows from $\|f\|_N = S_N^{-1}$ that

$$\lambda = S_N^{-1}|B_R|^{-1/N} = \frac{\mathcal{H}^{N-1}(\partial B_R)}{|B_R|} = \frac{N}{R}\,.$$

Then any positive constant is a solution to (4) with vector field given by $\mathbf{z}(x) = -\dfrac{x}{R}$, since $\|\mathbf{z}\|_\infty = 1$, $-\operatorname{div}\mathbf{z} = \lambda = |Du| + \lambda$ and $[\mathbf{z}(x), \nu(x)] = -\frac{x}{R}\cdot\frac{x}{|x|} = -1$ on ∂B_R.

Example 6 (*The case* $\|f\|_N > S_N^{-1}$) Let Ω be a bounded domain with Lipschitz boundary; for $t > S_N^{-1}$ we will find a nonnegative datum f such that $\|f\|_N = t$ and problem (4) has a nontrivial solution. As above, there is no loss of generality in assuming $0 \in \Omega$. Fix $R > 0$ such that $B_R(0) \subset \Omega$ and take $\rho \in (0, R)$, to be determined later.

We define

$$f(x) = \frac{N}{\rho}\chi_{B_\rho(0)} + \frac{\mu}{|x|}\chi_{B_R(0)\setminus B_\rho(0)}$$

for some $0 < \mu < N-1$. It is straightforward that

$$\|f\|_N^N = N^N\omega_N + \mu^N N\omega_N \log\Big(\frac{R}{\rho}\Big) = S_N^{-N} + \mu^N N\omega_N \log\Big(\frac{R}{\rho}\Big)\,,$$

which (by suitably choosing ρ) can take any value larger than S_N^{-N}.

Now consider the real function given by

$$g(r) = (N-1-\mu)\log\Big(\frac{R}{r}\Big)$$

and define

$$u(x) = \begin{cases} g(\rho)\,, & \text{if } |x| \le \rho\,;\\ g(|x|)\,, & \text{if } \rho < |x| < R\,;\\ 0\,, & \text{if } |x| \ge R\,. \end{cases}$$

Then it is easy to check that u is a solution to problem (4) with an auxiliary vector field defined by

$$\mathbf{z}(x) = \begin{cases} -\frac{x}{\rho}\,, & \text{if } |x| \le \rho\,; \\ -\frac{x}{|x|}\,, & \text{if } \rho < |x| < R\,; \\ -R^{N-1}\frac{x}{|x|^N}\,, & \text{if } |x| \ge R\,. \end{cases}$$

Finally, we point out that, when $\Omega = B_R(0)$, then $v(x) = u(x) + C$, where $C > 0$, is also a solution to this problem, with the same choice of $\mathbf{z}$.

References

1. B. Abdellaoui, A. Dall'Aglio and S. Segura de León: "Multiplicity of Solutions to Elliptic Problems Involving the 1–Laplacian with a Critical Gradient Term", *Adv. Nonlinear Stud.* **17** (2017), no. 2, 333–353.
2. L. Ambrosio, N. Fusco and D. Pallara: *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, 2000.
3. F. Andreu, C. Ballester, V. Caselles and J.M. Mazón: "The Dirichlet problem for the total variation flow", *J. Funct. Anal.* **180** (2001), no. 2, 347–403.
4. F. Andreu-Vaillo, V. Caselles and J.M. Mazón: *Parabolic quasilinear equations minimizing linear growth functionals*, Progress in Mathematics, 223. Birkhäuser Verlag, Basel, 2004.
5. F. Andreu, A. Dall'Aglio and S. Segura de León: "Bounded solutions to the 1-Laplacian equation with a critical gradient term", *Asymptot. Anal.* **80** (2012), no. 1–2, 21–43.
6. G. Anzellotti: "Pairings between measures and bounded functions and compensated compactness", *Ann. Mat. Pura Appl. (4)* **135** (1983), 293–318.
7. G. Bellettini, V. Caselles and M. Novaga: "The Total Variation Flow in $\mathbb{R}^N$", *J. Differ. Equations* **184** (2002), 475–525.
8. L. Boccardo, F. Murat and J.-P. Puel: "L^∞-estimate for some nonlinear elliptic partial differential equation and application to an existence result", *SIAM J. Math. Analysis,* **213** (1992), 326–333.
9. V. Caselles: "On the entropy conditions for some flux limited diffusion equations", *J. Differ. Equations* **250** (2011), 3311–3348.
10. G.–Q. Chen and H. Frid: "Divergence-measure fields and hyperbolic conservation laws", *Arch. Ration. Mech. Anal.* **147** (1999), no. 2, 89–118.
11. G. Crasta and V. De Cicco: "Anzellotti's pairing theory and the Gauss-Green theorem", preprint.
12. A. Dall'Aglio, D. Giachetti and J.-P. Puel: "Nonlinear elliptic equations with natural growth in general domains", *Ann. Mat. Pura Appl. (4)* **181** (2002), no. 4, 407–426.
13. F. Demengel: "Functions locally almost 1-harmonic", *Appl. Anal.* **83** (2004), 865–896.
14. V. Ferone and F. Murat: "Quasilinear problems having quadratic growth in the gradient: an existence result when the source term is small", *Équations aux dérivées partielles et applications*, 497–515, Gauthier-Villars, Éd. Sci. Méd. Elsevier, Paris, 1998.
15. V. Ferone and F. Murat: "Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small", *Nonlinear Anal.* **42** (2000), no. 7, Ser. A: Theory Methods, 1309–1326.
16. V. Ferone and F. Murat: "Nonlinear elliptic equations with natural growth in the gradient and source terms in Lorentz spaces", *J. Differ. Equations* **256** (2014), no. 2, 577–608.
17. V. Ferone, M.R. Posteraro and and J.M. Rakotoson: "L^∞-estimates for nonlinear elliptic problems with p-growth in the gradient", *J. of Inequal. & Appl.* **3** (1999), 109–125.

18. M. Latorre and S. Segura de León: "Existence and comparison results for an elliptic equation ing the 1–Laplacian and L^1–data", *J. Evol. Equ.* **18** (2018), no. 1, 1–28.
19. A. Mercaldo, S. Segura de León and C. Trombetti: "On the behaviour of the solutions to p–Laplacian equations as p goes to 1" *Publ. Mat.* **52** (2008), no. 2, 377–411.
20. A. Mercaldo, S. Segura de León and C. Trombetti: "On the solutions to 1–Laplacian equation with L^1 data", *J. Funct. Anal.* **256** (2009), no. 8, 2387–2416.
21. A. Mercaldo, J.D. Rossi, S. Segura de León and C. Trombetti: "Behaviour of p–Laplacian problems with Neumann boundary conditions when p goes to 1", *Commun. Pure Appl. Anal.* **12** (2013), no. 1, 253–267.
22. W.P. Ziemer: *Weakly differentiable functions. Sobolev spaces and functions of bounded variation.* Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989.