An extended–observer approach to robust stabilization of linear differential algebraic systems

ARTICLE HISTORY
Compiled October 8, 2018

ABSTRACT
In this paper we address the problem of robustly stabilizing a class of linear differential algebraic systems characterized by autonomous and asymptotically stable zero dynamics, in spite of parameter uncertainties ranging over *a priori* fixed bounded sets. We take advantage of some recent results related to the structural properties and normal forms of this class of systems and propose a robust control that asymptotically recovers, in practical terms, the performance of a nominal, though non implementable, stabilizing control. More specifically, the proposed control combines a partial output feedback control, aimed at letting the system behave as a regular system, and a robust control, based on an extended observer, using which the dynamic of the closed loop system is rendered arbitrarily close to the one of a properly selected stable system. The extended observer, originally conceived in the context of standard differential systems, is here shown to be the key ingredient for robustly stabilizing the targeted class of differential algebraic systems, provided that the gain of the partial output feedback control used to make the system regular is chosen sufficiently high.

KEYWORDS
Linear differential algebraic systems; robust stabilization; extended observer

1. Introduction

It is a pleasure to contribute an essay to this special issue dedicated to Alexander Fradkov on the occasion of his 70-th birthday. Professor Fradkov's research achievements have influenced the field of systems and control in many aspects, notably in the areas of passive systems, adaptive control and control of chaotic systems. As an academic leader, he has favored cross-fertilization between control theory, information theory and physics, inspiring in this way the work of many young talents. In this article, we present some advances in a subject that has been central in Alexander's interests: the synthesis of observers for robust feedback design. The target of our study is the use of robust observers in the stabilization of systems modeled by differential-algebraic equations (DAEs).

Representations of physical processes in the form of differential algebraic equations characterize a large variety of systems in which the dynamics of state variables are subject to algebraic constraints. Among the others, examples of systems motivating the theoretical investigation of this class of representations arise in the context of power systems (Hill and Mareels (1990), Venkatasubramanian, Schattler, and Zaborsky (1995)), electronics (Riaza (2008, 2013)), chemical processes (Kumar and Daoutidis (1998, 1999)) and mechanics (Eich-Soellner and Fuhrer (1998)).

Well-established methods for the analysis and control of physical processes modeled by DAEs can be found in the book of Dai (1989). Contributions in this field cover classical control problems, such as stabilization (Varga (1995), Xu, Dooren, Stefan, and Lam (2002), Liu and Ho (2004), Benner (2011), Berger (2014, 2016)), output regulation (Lin and Dai (1996), Pang, Huang, and Bai (2005)), observer design (Darouach and Boutayeb (1995), Zimmer and Meier (1997), Hou and Müller (1999), Osorio-Gordillo, Darouach, Astorga-Zaragoza, and Boutat-Baddas (2016), Berger and Reis (2017),) and disturbance decoupling (Banaszuk, Kociecki, and Przy Iuski (1990), Lebret (1994), Duan, Liu, and Thompson (2000), Y. Wang, Zhu, and Cheng (2004), Berger (2017)). In the interesting work of Berger (2013), the notions of zero dynamics, normal forms and invertibility - whose relevance in analysis and design of ordinary differential systems is well-known - are developed thoroughly and analyzed for DAEs.

In this paper we consider a system modeled by a DAE, whose parameters are affected by uncertainties ranging over *a priori* fixed bounded sets and we address the problem of designing a feedback law that *robustly* stabilizes such a system. The point of departure of our work is the normal form introduced by Berger (2013) for systems possessing an autonomous zero dynamics. In particular, we reconsider the stabilization problem addressed in Berger (2016) under a different perspective and different assumptions on the system representation. As in Berger (2016) a partial *high-gain output feedback control* is used in order to let the DAE behave as a regular system with uncertain parameters. In addition to that, we show how the influence of the uncertainties can be dominated by means of an *extended observer*. In this way, we can prove that the performance that would have been obtained if the uncertain parameters were known can be asymptotically recovered by means of a dynamic output feedback driven by the available measurements. The method of the extended observer reposes on some fundamental results originally presented in Han (1995) and Jiang and Praly (1998), further developed by Freidovich and Khalil (2008) in the context of single-input single-output (SISO) systems and later extended in L. Wang, Isidori, and Su (2015) to the case of multi-input multi-output (MIMO) systems. In these works a *high-gain extended observer* is employed to the purpose of obtaining a robust *proxy* of a nominal control that would stabilize the system in case of exact knowledge of the parameters. The resulting robust stabilizing control is here shown to be effective also in presence of algebraic constraints, provided that the gain used to make the system regular is chosen sufficiently high.

The paper is organized as follows. Section 2 summarizes some fundamental results developed in Berger (2016), which represents the basis of our work. Section 3 introduces the nominal stabilizing control. Section 4 discusses the proposed robust stabilizing control, the main result of the paper and the related proof, also providing a numerical example. Finally Section 5 is dedicated to the concluding remarks.

2. A summary of some relevant prior results

In this paper we consider linear constant coefficient DAEs of the form

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

in which $E, A \in \mathbb{R}^{\ell \times n}$, $B \in \mathbb{R}^{\ell \times m}$, $C \in \mathbb{R}^{p \times n}$ and $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$. Following Berger (2016), we do not assume that $sE - A$ is *regular*, that is $\ell = n$

and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$. [1] The matrices E, A, B, C in (1) depend on a vector of uncertain parameters, not explicitly indicated for notational convenience, whose values range over a *fixed* compact set. The basic problem that we address is that of finding a (possibly dynamic) feedback law that robustly stabilizes such system in spite of parameter uncertainties. The problem in question has been addressed, under suitable assumptions, in Berger (2013) and Berger (2016) where it is shown that *high-gain output feedback* and *funnel control* can be profitably used to this purpose. In our paper we still take advantage of the general framework of analysis and design developed in Berger (2016), but we propose a somewhat different and alternative design strategy, that requires different, perhaps milder, assumptions.

Cornerstones of Berger's apparatus are the concepts of *autonomous zero dynamics* and the developments of *normal forms* for DAEs possessing an autonomous zero dynamics.[2] Specifically, Berger (2016), Proposition 3.5, proves that the zero dynamics of (1) are autonomous if and only if

$$\operatorname{rank}_{\mathbb{R}[s]} \begin{pmatrix} sE - A & -B \\ -C & 0 \end{pmatrix} = n + m\,. \tag{2}$$

If such assumption holds, Berger (2016), Theorem 3.6, derives a change of variables under which the equations of (1) are changed into a *normal form*, that can be seen as an extension, to the case of DAEs, of the classical normal form considered in the analysis of systems described by ODE. Under the additional assumption that $\operatorname{rank}(C) = p$, Berger (2016), Theorem 4.3, proposes a further refinement of such normal form, that he refers to as *system inversion form*.

Specifically, under such assumptions, it is proven by Berger (2016) that there exist nonsingular matrices S, T such that the matrices

$$\hat{E} = SET\,, \quad \hat{A} = SAT\,, \quad \hat{B} = SB\,, \quad \hat{C} = CT \tag{3}$$

have the following form

$$\hat{E} = \begin{pmatrix} I_k & 0 & 0 \\ 0 & E_{22} & E_{23} \\ 0 & E_{32} & N \\ 0 & E_{42} & E_{43} \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & I_{n-k-p} \\ 0 & A_{42} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ I_m \\ 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & I_p & 0 \end{pmatrix} \tag{4}$$

in which N is nilpotent (with $N^\nu = 0$ and $N^{\nu-1} \neq 0$). The number n_4 of rows of the lower strings of blocks in $\hat{E}, \hat{A}, \hat{B}$ is equal to [3]

$$n_4 = \ell - n + p - m\,. \tag{5}$$

In what follows, we consider the special case in which the integer n_4 in (5) is zero. For convenience we summarize as follows all the hypotheses considered so far.

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[1] Throughout most of the paper, we follow the notation used in Berger (2016). Specifically, here by $\mathbb{R}[s]$ we denote the ring of polynomials with coefficients in $\mathbb{R}$.

[2] See Berger (2016) for a precise definition of the concept of an *autonomous zero dynamics*. Roughly speaking, the zero dynamics are said to be autonomous if the choice of the input $u(t)$ cannot have an influence on the (forced) internal motions that are consistent with the constraint $y(t) \equiv 0$. In the case of a system modeled by ordinary differential equations, the zero dynamics are autonomous whenever $\mathcal{R}^*$, the largest controllability subspace contained in $\ker(C)$, is $\{0\}$.

[3] Note, in this respect, that assumption (2) implies $\ell + p \geq n + m$

Assumption 1. System (1) satisfies (2), $\ell + p = n + m$ and $\text{rank}(C) = p$, for all values of the uncertain parameters.

It is clear from all of the above that, if such Assumption holds, there exists nonsingular matrices S and T such that system (1) is equivalent to the system

$$\begin{array}{rcl} \hat{E}\dot{\hat{x}} & = & \hat{A}\hat{x} + \hat{B}u \\ y & = & \hat{C}\hat{x} \end{array}$$

in which $\hat{x} = T^{-1}x$ and $\hat{E}, \hat{A}, \hat{B}$ have the form (4), where in particular the lower strings of blocks are missing. Splitting $\hat{x}$ as $\text{col}(x_1, x_2, x_3)$ it is seen that the system is modeled by equations of the form

$$\begin{array}{rcl} \dot{x}_1 & = & A_{11}x_1 + A_{12}x_2 \\ E_{22}\dot{x}_2 + E_{23}\dot{x}_3 & = & A_{21}x_1 + A_{22}x_2 + u \\ E_{32}\dot{x}_2 + N\dot{x}_3 & = & x_3 \\ y & = & x_2\,, \end{array} \tag{6}$$

in which $x_1 \in \mathbb{R}^k$, $x_2 \in \mathbb{R}^p$, $x_3 \in \mathbb{R}^{n-k-p}$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. Note that, since the matrix N is nilpotent, the third equation can explicitly solved for in x_2, by recursion, yielding

$$x_3(t) = \sum_{k=0}^{\nu-1} N^k E_{32} x_2^{(k+1)}(t)\,.$$

Hence, system (6) can be rewritten as [4]

$$\begin{array}{rcl} \dot{x}_1 & = & A_{11}x_1 + A_{12}x_2 \\ E_{22}\dot{x}_2 + \sum_{k=0}^{\nu-1} E_{32}N^k E_{23} x_2^{(k+2)} & = & A_{21}x_1 + A_{22}x_2 + u \\ x_3 & = & \sum_{k=0}^{\nu-1} N^k E_{32} x_2^{(k+1)} \\ y & = & x_2\,. \end{array} \tag{7}$$

As anticipated at the beginning, in this paper we strongly rely upon the apparatus developed by Berger (2016). Thus, we proceed by describing some additional assumptions considered in that paper when dealing with the design of a stabilizing feedback. Suppose Assumption 1 holds and let $L(s)$ be a left-inverse of the matrix in (2) over $\mathbb{R}(s)$.[5] Then, it can be shown [6] that the matrix

$$\begin{pmatrix} 0 & I_m \end{pmatrix} L(s) \begin{pmatrix} 0 \\ I_p \end{pmatrix}$$

is *independent* of the choice of $L(s)$. This being the case, we consider, as in Berger (2016), the following hypothesis.

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[4]Nota also that the internal dynamics consistent with the constraint $y(t) \equiv 0$ are those of $\dot{x}_1 = A_{11}x_1$.

[5]Here $\mathbb{R}(s)$ denotes the quotient field of $\mathbb{R}[s]$. The matrix $L(s)$ is any matrix with entries in $\mathbb{R}(s)$ satisfying

$$L(s)\begin{pmatrix} sE - A & -B \\ -C & 0 \end{pmatrix} = I_{n+m}.$$

[6]See Berger (2016), Lemma A.1.

Assumption 2. The matrix

$$\bar{E} = -s^{-1} \lim_{s \to \infty} \begin{pmatrix} 0 & I_m \end{pmatrix} L(s) \begin{pmatrix} 0 \\ I_p \end{pmatrix} \tag{8}$$

exists.

Under such additional hypothesis, Berger (2016), Lemma A.2, shows that

$$\begin{aligned} \bar{E} &= E_{22} \\ E_{23} N^k E_{32} &= 0, \qquad \forall k = 0, 1, \ldots, \nu - 1. \end{aligned}$$

Therefore, if Assumptions 1 and 2 hold, system (1) can be rewritten in equivalent form as

$$\begin{aligned} \dot{x}_1 &= A_{11} x_1 + A_{12} x_2 \\ \bar{E} \dot{x}_2 &= A_{21} x_1 + A_{22} x_2 + u \\ x_3 &= \textstyle\sum_{k=0}^{\nu-1} N^k E_{32} x_2^{(k+1)} \\ y &= x_2 \,. \end{aligned} \tag{9}$$

Finally, we assume the following.

Assumption 3. The zero dynamics of (1) are asymptotically stable, i.e. $\sigma(A_{11}) \in \mathbb{C}^-$.

All of the above summarizes, to some extent, the technical apparatus developed in Berger (2016) to the purpose of designing a stabilizing feedback law. In a nutshell, if Assumptions 1,2,3 holds, then system (1) can be brought to the equivalent form (9), with A_{11} a Hurwitz matrix. Then, Berger (2016) proceeds by considering the case of systems having the same number of inputs and outputs (i.e. $m = p$) and, further, *assumes* that the (square) matrix $\bar{E}$ in (8) satisfies

$$\bar{E} = \bar{E}^{\mathrm{T}} \geq 0 \,. \tag{10}$$

Taking advantage of such assumption, Berger (2016) shows how robust stabilization can be obtained, via output feedback, by means either pure *high-gain* $u = -ky$ or *funnel control*. In the present paper, we discuss an alternative stabilization scheme that reposes on a different, perhaps milder, assumption.

3. The nominal stabilizing control

To be consistent with a good part of the literature on ordinary differential systems, and also to avoid possible conflicts of notations induced by subsequent changes of variables, we rewrite system (9) in the form

$$\begin{aligned} \dot{z} &= Fz + Gx \\ \bar{E} \dot{x} &= Ax + Hz + u \\ y &= x \end{aligned} \tag{11}$$

where, by hypothesis, F is a Hurwitz matrix. As mentioned at the end of the last section, we replace the assumption (10) with a different assumption.

Assumption 4. The system has the same number of inputs and outputs (i.e. $m = p$). The $m \times m$ matrix $\bar{E}$ has a constant rank $r \leq m$ for all values of the uncertain parameters. Two permutation matrices P_ℓ and P_r *exist and are known* such that

$$P_\ell \bar{E} P_r = \begin{pmatrix} \bar{E}_{11} & \bar{E}_{12} \\ \bar{E}_{21} & \bar{E}_{22} \end{pmatrix} \tag{12}$$

in which $\bar{E}_{11} \in \mathbb{R}^{r \times r}$ is nonsingular. Moreover, a nonsingular $r \times r$ matrix B_0 *is known* such that, for some $0 < \delta_0 < 1$,

$$\|[\bar{E}_{11}^{-1} - B_0]\Lambda B_0^{-1}\| \leq \delta_0 \tag{13}$$

for all diagonal matrices Λ such that $\|\Lambda\| \leq 1$, and for all values of the uncertain parameters. [7] [8]

Since the matrices P_ℓ and P_r in (12) are by hypothesis known, we can assume that, after a permutation of state variables and equations, the matrix $\bar{E}$ is partitioned as in (12), with nonsingular $\bar{E}_{11}$. In fact, to change x into $P_r^{-1}x$ is equivalent to reordering the components of y, while left-multiplication of the DAE by P_ℓ entails a reordering of the components of u. Let $x = y$ and u be partitioned accordingly, as

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

in which $x_1, y_1, u_1 \in \mathbb{R}^r$,

It is straightforward to see that two matrices S_0 and T_0 exist such that

$$S = \begin{pmatrix} I_r & 0 \\ S_0 & I_{m-r} \end{pmatrix} \qquad T = \begin{pmatrix} I_r & T_0 \\ 0 & I_{m-r} \end{pmatrix}$$

satisfy

$$S\bar{E}T = \begin{pmatrix} \bar{E}_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

If SAT is partitioned accordingly and x_1 is replaced by $\xi = x_1 - T_0 x_2$, the system (11) can be written in the form

$$\begin{array}{rcl} \dot{z} & = & Fz + G_1\xi + G_2 x_2 \\ \dot{\xi} & = & A_{11}\xi + A_{12}x_2 + H_1 z + Bu_1 \\ 0 & = & A_{21}\xi + A_{22}x_2 + H_2 z + S_0 u_1 + u_2 \end{array} \tag{14}$$

where $B = \bar{E}_{11}^{-1}$.

To develop a nominal stabilizing controller for (14) we proceed in two steps, detailed in the reminder of this Section: in the first one, feedback from y_2 is used to regularize

[7] The procedure described below can be extended, without difficulties, to the case $m > p$.

[8] It is worth observing that, in case $m = 1$, (13) reduces to $\left|\frac{E_{11}^{-1} - B_0}{B_0}\right| \leq \delta_0 < 1$ which, in turn, holds if there exists two numbers $b_{\min}, b_{\max}$ such that $0 < b_{\min} \leq |E_{11}^{-1}| \leq b_{\max}$. The condition in (13) can be regarded as a multivariable version of such assumption.

the DAE; in the second one the resulting regular DAE is stabilized. To make the system regular it suffices to pick

$$u_2 = hy_2 \,. \tag{15}$$

Recalling that $y_2 = x_2$, the following equations are obtained

$$\begin{array}{rcl} \dot{z} & = & Fz + G_1\xi + G_2x_2 \\ \dot{\xi} & = & A_{11}\xi + A_{12}x_2 + H_1z + Bu_1 \\ 0 & = & A_{21}\xi + [A_{22} + hI]x_2 + H_2z + S_0u_1 \end{array} \tag{16}$$

As pointed out in Berger (2013), if h is large the system is regular, since

$$\det(A_{22} + hI) \neq 0\,. \tag{17}$$

In fact, equation (17) guarantees that x_2 is uniquely determined by the algebraic equation as

$$x_2 = -(A_{22} + hI)^{-1}[A_{21}\xi + H_2z + S_0u_1]\,. \tag{18}$$

The second step consists in using the residual control input u_1 to stabilize the *nominal* system (the system (16) in which we assume that no parameter is uncertain). Since the matrix F is by assumption a Hurwitz matrix, a natural choice to obtain stability would be to seek a control u_1 for which

$$Bu_1 = -\xi - A_{11}\xi - A_{12}x_2 - H_1z. \tag{19}$$

If this were the case, in fact, the second equation of (16) would become

$$\dot{\xi} = -\xi\,.$$

It is easy to see that, if h is large, the equation (19) has a unique solution u_1. To this end, recall that – if h is large – x_2 can be expressed as in (18). Hence, the equation (19) can be written as

$$Bu_1 = -[I + A_{11}]\xi + A_{12}(A_{22} + hI)^{-1}[A_{21}\xi + H_2z + S_0u_1] - H_1z.$$

If h is large, the matrix $(B - A_{12}(A_{22} + hI)^{-1}S_0)$ is nonsingular and u_1 is seen to be a well-defined, linear, function of z, ξ

$$u_1 = u_{\text{id}}(z, \xi)\,. \tag{20}$$

In conclusion, under the control input

$$u = \begin{pmatrix} u_{\text{id}}(z, \xi) \\ hy_2 \end{pmatrix}$$

the system becomes

$$\begin{aligned}
\dot z &= Fz + G_1\xi + G_2x_2\\
\dot\xi &= -\xi\\
0 &= A_{21}\xi + [A_{22}+hI]x_2 + H_2z + S_0u_{\rm id}(z,\xi)\,.
\end{aligned}$$

Such system is a system of the form

$$\begin{pmatrix}\dot z\\ \dot\xi\end{pmatrix} = \begin{pmatrix}F & G_1\\ 0 & -I_r\end{pmatrix}\begin{pmatrix}z\\ \xi\end{pmatrix} + \begin{pmatrix}G_2\\ 0\end{pmatrix}x_2$$

in which x_2 has the form (18), a linear function of (z,ξ). This system, in turn, can be seen as a stable linear system (recall that F is Hurwitz) subject to a memoryless linear state feedback $x_2 = x_2(z,\xi)$. The gain matrices that characterize such feedback can be made arbitrarily small by increasing h. Thus, from the small-gain theorem it is concluded that if h is sufficiently large the equilibrium $(z,\xi) = (0,0)$ of this system is stable.

The stabilizing controller thus found, though, can only be implemented if the "exact cancelation" entailed by the choice of $u_{\rm id}(z,\xi)$ can take place and this requires – in turn - that all involved parameters are accurately known. Hence, the control in question is not implementable in practice when the parameters that characterize the model are uncertain. However, a robust "proxy" of such control can be designed, as it will be shown in the next section.

Remark 1. It should be noticed that the approach described above requires a sufficiently large value of the gain parameter h, the "minimal" value of which clearly depends on the (compact) set where the uncertain parameters are allowed to range. This is not the case if the robust stabilization problem is addressed via funnel control, as in Berger (2016), where however an assumption like (10) is present.

4. The robust stabilizing control

4.1. Main result

We develop now a robust version of the control (20). In this framework, we take specific advantage of the second part of Assumption 4, namely the availability of a fixed matrix B_0 that makes condition (13) fulfilled, assumption that has not been used in the development of the nominal control. The stabilization result that we obtain is a kind of "semiglobal stabilization" result, i.e. a fixed – but possibly large – arbitrary *compact* set $\mathcal{C}$ of initial condition is given and a control law is found with the property that, in the resulting closed loop system, the prescribed equilibrium is asymptotically stable with a domain of attraction that contains the set $\mathcal{C}$.

A basic ingredient of the proposed control is a vector-valued saturation function, defined as follows. Let $g_L : \mathbb{R} \to \mathbb{R}$ be a smooth function (henceforth referred to as a "saturation function") characterized by the following properties:

- $g_L(s) = s$ if $|s| \le L$,
- $g_L(s)$ is odd and monotonically increasing, with $0 < g'_L(s) \le 1$,
- $\lim_{s\to\infty} g_L(s) = L(1+c)$ with $0 < c \ll 1$.

and define $G_L : \mathbb{R}^r \to \mathbb{R}^r$ as

$$G_L(s) = \mathrm{col}(g_L(s_1), g_L(s_2), \ldots, g_L(s_r)). \tag{21}$$

Following the "extended observer" paradigm (see e.g. Han (1995), Jiang and Praly (1998), Freidovich and Khalil (2008)), we propose the control

$$u_1 = u_{\mathrm{act}} = G_L(\psi(\hat{\xi}, \sigma))$$

in which

$$\psi(\xi, \sigma) = B_0^{-1}[-\xi - \sigma] \tag{22}$$

and $(\hat{\xi}, \sigma)$ are the states of the "extended observer"

$$\begin{array}{rcl} \dot{\hat{\xi}} &=& \sigma + B_0 G_L(\psi(\hat{\xi}, \sigma)) + \kappa c_1(y_1 - \hat{\xi}) \\ \dot{\sigma} &=& \kappa^2 c_0(y_1 - \hat{\xi})\,. \end{array} \tag{23}$$

In these equations, the parameter L, the coefficient κ and c_0, c_1 are *design parameters.*

The following Theorem describes the main result of the paper.

Theorem 4.1. *Consider system (11). Suppose F is a Hurwitz matrix and suppose Assumptions 4 holds. Let the order of the components of y and u be changed so as to bring the matrix $\bar{E}$ in the form (12), with $\bar{E}_{11}$ a nonsingular matrix. Let the system be controlled by*

$$u = \begin{pmatrix} G_L(\psi(\hat{\xi}, \sigma)) \\ hy_2 \end{pmatrix}$$

where $G_L(\cdot)$ is defined as in (21), $\psi(\xi, \sigma)$ is defined as in (22), with B_0 chosen so as to satisfy the condition (13), and $(\hat{\xi}, \sigma)$ are states of the extended observer (23). For every choice of a compact set $\mathcal{C}$ there is a choice of the design parameters L and c_0, c_1, a number κ^* and, for all $\kappa > \kappa^*$ a number h^*_κ, such that, if $\kappa > \kappa^*$ and $h > h^*_\kappa$, then the equilibrium $(z, \xi, \hat{\xi}, \sigma) = (0, 0, 0, 0)$ is asymptotically stable, with a domain of attraction $\mathcal{A}$ that contains the set $\mathcal{C}$.

4.2. Proof of the main result

Recall that the system under consideration is a system of the form (14). The proof of Theorem 4.1 is split in 7 steps.

Step 1. We introduce new variables defined as

$$\begin{array}{rcl} e_1 &=& \kappa(\xi - \hat{\xi}) \\ e_2 &=& Q(z, \xi, x_2) + [B - B_0]G_L(\psi(\xi, \sigma)) - \sigma\,. \end{array} \tag{24}$$

in which

$$Q(z, \xi, x_2) = A_{11}\xi + A_{12}x_2 + H_1 z\,,$$

and we use such variables to replace $\hat{\xi}$ and σ in the previous equations. To this end, we first check that such change of variables is well-defined. In this respect, observe that the first one of such relations can be trivially solved for $\hat{\xi}$ yielding

$$\hat{\xi} = \xi - \kappa^{-1} e_1 \,. \tag{25}$$

The second relation, on the other hand, can be solved for σ thanks to the following result.

Lemma 4.2. *If (13) holds, the map $f : \mathbb{R}^r \to \mathbb{R}^r$ defined as*

$$s \quad \mapsto \quad f(s) = B_0^{-1}[B - B_0]G_L(s) + s \tag{26}$$

is globally invertible. As a consequence

$$\sigma = -\xi - B_0 f^{-1}(B_0^{-1}[-\xi - Q(z, \xi, x_2) + e_2]) \,. \tag{27}$$

Proof. Since $G_L(s)$ is bounded, the map $f(s)$ is proper (that is, $\|f(s)\| \to \infty$ as $\|s\| \to \infty$). Thus, according to Hadamard's Theorem, the map $f(s)$ has a globally defined inverse if the Jacobian of $f(s)$ is nowhere singular. The Jacobian of $f(s)$ has the following expression

$$\frac{\partial f}{\partial s} = B_0^{-1}[B - B_0]G'_L(s) + I = B_0^{-1}\Big[[B - B_0]G'_L(s)B_0^{-1} + I\Big]B_0 \tag{28}$$

in which $G'_L(s)$ denotes the matrix

$$G'_L(s) = \mathrm{diag}(g'_L(s_1), g'_L(s_2), \ldots, g'_L(s_m)).$$

Thus, the jacobian of $f(s)$ is nonsingular if and only if

$$\det\Big[[B - B_0]G'_L(s)B_0^{-1} + I\Big] \neq 0 \qquad \forall s \in \mathbb{R}^r.$$

Clearly, a sufficient condition for this to be true is that $\|[B - B_0]G'_L(s)B_0^{-1}\| < 1$, which is guaranteed by assumption (13).

Having shown this, observe – using the definition of $\psi(\xi, \sigma)$ – that

$$\begin{aligned} e_2 &= Q(z, \xi, x_2) + [B - B_0]G_L(\psi(\xi, \sigma)) + \xi - \xi - \sigma \\ &= Q(z, \xi, x_2) + [B - B_0]G_L(\psi(\xi, \sigma)) + \xi + B_0\psi(\xi, \sigma) \,. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} B_0^{-1}[e_2 - Q(z, \xi, x_2) - \xi] &= B_0^{-1}[B - B_0]G_L(\psi(\xi, \sigma)) + \psi(\xi, \sigma) \\ &= f(\psi(\xi, \sigma)) \end{aligned} \tag{29}$$

Thus,

$$\psi(\xi, \sigma) = f^{-1}(B_0^{-1}[e_2 - Q(z, \xi, x_2) - \xi])$$

from which (27) follows. □

In view of this, in what follows we can think of $\hat{\xi}$ and σ as functions of (e_1, e_2) and of $\bar{x} := (z, \xi, x_2)$.

Step 2. We compute now the dynamics of $e = \text{col}(e_1, e_2)$. So long as e_1 is concerned, we have (recall that $y_1 = \xi + T_0 x_2$)

$$\begin{aligned}
\dot{e}_1 &= \kappa[Q(z,\xi,x_2) + [B - B_0]G_L(\psi(\hat{\xi},\sigma)) - \sigma - c_1 e_1] - \kappa^2 c_1 T_0 x_2 \\
&= \kappa[e_2 - c_1 e_1] + \kappa[B - B_0][G_L(\psi(\hat{\xi},\sigma)) - G_L(\psi(\xi,\sigma))] - \kappa^2 c_1 T_0 x_2 \\
&:= \kappa[e_2 - c_1 e_1] + \Delta_1(\bar{x}, e) - \kappa^2 c_1 T_0 x_2 \,,
\end{aligned}$$

in which [9]

$$\Delta_1(\bar{x}, e) = \kappa[B - B_0][G_L(\psi(\hat{\xi},\sigma)) - G_L(\psi(\xi,\sigma))] \,.$$

So long as e_2 is concerned, we have

$$\dot{e}_2 = \dot{Q}(z,\xi,x_2) + [B - B_0]G'_L(\psi(\xi,\sigma))B_0^{-1}[-\dot{\xi} - \dot{\sigma}] - \dot{\sigma} \,.$$

Tedious, but elementary, calculations show that, if one sets

$$\begin{aligned}
\Delta_2(\bar{x}, e) = \; & [A_{11} - A_{12}(A_{22} + hI)^{-1}A_{21} \\
& - [B - B_0]G'_L(\psi(\xi,\sigma))B_0^{-1}][A_{11}\xi + A_{12}x_2 + H_1 z + BG_L(\psi(\hat{\xi},\sigma))] \\
& + [H_1 - A_{12}(A_{22} + hI)^{-1}H_2][Fz + G_1\xi + G_2 x_2] \\
& + A_{12}(A_{22} + hI)^{-1}S_0 G'_L(\psi(\hat{\xi},\sigma))B_0^{-1}[\sigma + B_0 G_L(\psi(\hat{\xi},\sigma))]
\end{aligned}$$

and

$$\Delta_0(\bar{x}, e) = [B - B_0]G'_L(\psi(\xi,\sigma))B_0^{-1} - A_{12}(A_{22} + hI)^{-1}S_0 G'_L(\psi(\hat{\xi},\sigma))B_0^{-1}[1 + c_1(\kappa c_0)^{-1}]$$

an equation of the form

$$\dot{e}_2 = \Delta_2(\bar{x}, e) - \kappa[\Delta_0(\bar{x}, e) + I]c_0 e_1 - \kappa^2 c_0[\Delta_0(\bar{x}, e) + I]T_0 x_2$$

is obtained.

In summary, the dynamics of e have the form

$$\dot{e} = \kappa[\mathbf{A} - \mathbf{B}_2\Delta_0(\bar{x}, e)\mathbf{C}]e + \mathbf{B}_1\Delta_1(\bar{x}, e) + \mathbf{B}_2\Delta_2(\bar{x}, e) + \kappa^2\mathbf{B}_3(\bar{x}, e)x_2 \tag{30}$$

in which

$$\mathbf{A} = \begin{pmatrix} -c_1 I_r & I_r \\ -c_0 I_r & 0 \end{pmatrix}, \quad \mathbf{B_1} = \begin{pmatrix} I_r \\ 0 \end{pmatrix}, \quad \mathbf{B_2} = \begin{pmatrix} 0 \\ I_r \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} c_0 I_r & 0 \end{pmatrix}, \quad \mathbf{B}_3(\bar{x}, e) = \begin{pmatrix} c_1 I_r \\ c_0[\Delta_0(\bar{x}, e) + I_r] \end{pmatrix} T_0 \,.$$

[9] Knowing that the states $(\hat{\xi}, \sigma)$ of the observer can be uniquely expressed as functions of $\bar{x}$ and e, we denote by $(\bar{x}, e)$ the arguments of the functions defined below.

Step 3. We highlight some useful features of the equation (30). [10]

Lemma 4.3. *Suppose assumption (13) holds and* $\kappa \geq 1$*. There exists a number* δ_1 *such that*

$$\|\Delta_1(\bar{x}, e)\| \leq \delta_1 \|e\| \qquad \text{for all } (\bar{x}, e) \text{ and all } \kappa \tag{31}$$

There exist numbers $0 < \delta'_0 < 1$ *and* h_0 *such that, if* $h > h_0$*, then*

$$\|\Delta_0(\bar{x}, e)\| \leq \delta'_0 < 1 \qquad \text{for all } (\bar{x}, e) \text{ and all } \kappa \tag{32}$$

Moreover, for each $R > 0$ *there is a number* M_R *and a strictly decreasing continuous function* $\varsigma : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\lim_{h\to\infty} \varsigma(h) = 0$ *such that*

$$\bar{x} \in R \quad \Rightarrow \quad \|\Delta_2(\bar{x}, e)\| \leq M_R + \varsigma(h)\|e\| \qquad \text{for all } e \in \mathbb{R}^{2r} \text{ and all } \kappa. \tag{33}$$

Proof. The proof of (31) is identical to the proof of a similar result in Isidori (2016), page 304. So is the proof of (32), if one bears in mind the fact that $G'_L(\cdot)$ is a diagonal matrix whose norm does not exceed 1 and hence, by Assumption, there exists a number $0 < \delta_0 < 1$ such that

$$\|[B - B_0]G'_L(\psi(\xi, \sigma))B_0^{-1}\| \leq \delta_0 \,.$$

This together with the fact that $\lim_{h\to\infty} \|(A_{22} + hI)^{-1}\| = 0$ proves that (32) holds for some δ'_0 satisfying $\delta_0 < \delta'_0 < 1$ if h is large enough. Finally, observe that the variable σ appearing in $\Delta_2(\bar{x}, e)$, thought as a function of $(\bar{x}, e)$, has the expression (27). Since the function $f^{-1}(\cdot)$ is globally Lypschitz, it is seen that $\|\sigma\|$ can be bounded as

$$\|\sigma\| \leq M_0\|\bar{x}\| + M_1\|e\| \,.$$

Using this, the fact that all other terms appearing in the expression of $\Delta_2(\bar{x}, e)$ remain bounded so long as $\bar{x} \in B_R$, regardless of how κ is chosen, and the property $\lim_{h\to\infty} \|(A_{22} + hI)^{-1}\| = 0$, the bound (33) follows. □

Lemma 4.4. *Let be* h *large enough so that the bound (32) holds. If* c_1, c_0 *are such that the polynomial* $d(\lambda) = \lambda^2 + c_1\lambda + c_0$ *has two negative real roots, there exists a positive definite and symmetric* $2r \times 2r$ *matrix* P *and a number* $\lambda > 0$ *such that*

$$P[\mathbf{A} - \mathbf{B}_2\Delta_0(\bar{x}, e)\mathbf{C}] + [\mathbf{A} - \mathbf{B}_2\Delta_0(\bar{x}, e)\mathbf{C}]^\top P \leq -\lambda I. \tag{34}$$

The proof of this Lemma uses arguments identical to those used in the proof of a similar Lemma in Isidori (2016), pages 309, and is not repeated here.

Step 4. We consider the overall system, that we express in the form

$$\begin{array}{rcl}
\dot{z} &=& Fz + G_1\xi + G_2x_2 \\
\dot{\xi} &=& A_{11}\xi + A_{12}x_2 + H_1z + BG_L(\psi(\hat{\xi}, \sigma)) \\
\dot{e} &=& \kappa[\mathbf{A} - \mathbf{B}_2\Delta_0(\bar{x}, e)\mathbf{C}]e + \mathbf{B}_1\Delta_1(\bar{x}, e) + \mathbf{B}_2\Delta_2(\bar{x}, e) + \kappa^2\mathbf{B}_3(\bar{x}, e)x_2 \\
0 &=& A_{21}\xi + [A_{22} + hI]x_2 + H_2z + S_0G_L(\psi(\hat{\xi}, \sigma))
\end{array} \tag{35}$$

[10] Here and in the remaining portion of the paper, we denote by B_R the closed ball of radius R, with the tacit understanding that the space in which the ball is considered is specified by the context.

in which the arguments $(\hat{\xi}, \sigma)$ of $\psi(\hat{\xi}, \sigma)$ are meant to be replaced by the expressions (25)–(27).

As expected, if h is large, this (augmented) system is regular. We express this result and a related property as follows.

Lemma 4.5. *There is a number h_0 such that, if $h > h_0$, the algebraic constraint in (35) has a unique solution $x_2 = X_{2,h}(z, \xi, e)$. Moreover, given any pair of positive numbers R, δ, there is a value $h^* > h_0$ such that, if $h > h^*$, then*

$$(z, \xi) \in B_R \times B_R \quad \Rightarrow \quad \|X_{2,h}(z, \xi, e)\| \leq \delta \qquad \text{for all } e \in \mathbb{R}^{2r} \text{ and all } \kappa\,.$$

Proof. The proof of the first part of the statement uses arguments similar to those used in Lemma 4.2. Observe that

$$S_0 G_L(\psi(\hat{\xi}, \sigma)) = S_0 G_L\Big(B_0^{-1}\Big[\kappa^{-1} e_1 + B_0 f^{-1}(B_0^{-1}[-\xi - Q(z, \xi, x_2) + e_2])\Big]\Big)$$

The equation that determines x_2 is an equation of the form

$$[A_{22}+hI]x_2+S_0 G_L\Big(B_0^{-1}\kappa^{-1}e_1+f^{-1}(B_0^{-1}[-\xi-A_{11}\xi-A_{12}x_2-H_1 z+e_2])\Big) = -A_{21}\xi-H_2 z$$

If h is large, the map

$$\tilde{f}(s) = [A_{22} + hI]s + S_0 G_L\Big(v_1 + f^{-1}(B_0^{-1}[v_2 - A_{12}s])\Big)$$

is globally invertible (here $v_1 = B_0^{-1}\kappa^{-1}e_1$ and $v_2 = -\xi - A_{11}\xi - H_1 z + e_2$) and hence x_2 can be expressed uniquely in terms of z, ξ, e. The second part of the statement is a consequence of the fact that $G_L(\cdot)$ is a bounded function. $\square$

This Lemma is instrumental in determining solutions of (35).[11] Assuming that $h > h_0$, we rewrite the top two equations of system (35) in more compact form as

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}\Delta_3(\bar{x}, e) + \mathbf{L}x_2\,, \tag{37}$$

in which

$$\mathbf{x} = \begin{pmatrix} z \\ \xi \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} F & G_1 \\ 0 & -I_r \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} 0 \\ I_r \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} G_2 \\ 0 \end{pmatrix},$$

$$\Delta_3(\bar{x}, e) = [I_r + A_{11}]\xi + A_{12}x_2 + H_1 z + B G_L(\psi(\hat{\xi}, \sigma)).$$

[11] Equation (35) is a DAE of the form

$$E\dot{\mathcal{X}}(t) = \mathcal{F}(\mathcal{X}(t)) \tag{36}$$

in which $\mathcal{F} : \mathbb{R}^\nu \to \mathbb{R}^\nu$ is locally Lypschitz and $\mathcal{X}(t) = \mathrm{col}(z(t), \xi(t), e(t), x_2(t))$. Let I be an open interval of $\mathbb{R}$, containing the origin. Given $\mathcal{X}_0 \in \mathbb{R}^\nu$, a continuously differentiable function $\mathcal{X} : I \to \mathbb{R}^\nu$ is a solution of the initial value problem $\mathcal{X}(0) = \mathcal{X}_0$ if $\mathcal{X}(t)$ satisfies (36) for all $t \in I$ and $\mathcal{X}(0) = \mathcal{X}_0$. The initial value $\mathcal{X}(0)$ is consistent if at least one solution exists to the initial value problem $\mathcal{X}(0) = \mathcal{X}_0$. It is seen from all of the above that, if h is large, $x_2(t) = X_{2h}(z(t), \xi(t), e(t))$. Thus the initial value $\mathcal{X}_0 = \mathrm{col}(z_0, \xi_0, e_0, x_{2,0})$ is consistent if and only if $x_{2,0} = X_{2h}(z_0, \xi_0, e_0)$. For all such consistent initial values, a solution of equation (35) can be determined inserting $x_2 = X_{2h}(z, \xi, e)$ in the top three equations and then solving the resulting ODE. In what follows, we will show that – if the design parameters are appropriately chosen – a unique solution of such ODE exists, defined over an interval $I \supset [0, \infty)$.

(where, of course, the arguments $\hat{\xi}$ and σ are meant to be replaced by the expressions (25)–(27)).

With $\mathcal{C}$ being the given compact set of initial conditions of the system, pick a number $R > 0$ such that

$$(\mathbf{x}, \hat{\xi}, \sigma) \in \mathcal{C} \qquad \Rightarrow \qquad (\mathbf{x}, \hat{\xi}, \sigma) \in B_R \times B_R \times B_R \,.$$

The first design parameter which is being fixed is the "saturation" level L that characterizes the function $g_L(\cdot)$. To this end, observe that the matrix $\mathbf{F}$ in (37) is a Hurwitz matrix, and hence there exists a matrix $\mathbf{P} = \mathbf{P}^{\mathrm{T}} > 0$ satisfying

$$\mathbf{PF} + \mathbf{F}^{\mathrm{T}}\mathbf{P} = -I \,.$$

Set $V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{x}$ and pick a number $c > 0$ such that

$$\Omega_c := \{\mathbf{x} : V(\mathbf{x}) \le c\} \supset B_R \,.$$

Then, choose for L the value

$$L = \max_{\mathbf{x} \in \Omega_{c+1}, \|x_2\| \le 1} \Big[B^{-1}(-Q(z, \xi, x_2) - \xi)\Big] + 1 \,.$$

Moreover, with Lemma 4.5 in mind, let h_1^* be such that if $h \ge h_1^*$

$$\|x_2\| = \|X_{2,h}(\mathbf{x}, e)\| \le 1$$

for all $\mathbf{x} \in \Omega_{c+1}$ and assume $h \ge h_1^*$.

With L fixed in this way, we examine the behavior of $\mathbf{x}(t)$ for small values of $t \ge 0$. To this end observe that, since $G_L(\cdot)$ is a bounded function, there exists a number δ_3, such that

$$\|\Delta_3(\bar{x}, e)\| \le \delta_3 \qquad \text{for all } \mathbf{x} \in \Omega_{c+1} \text{ and all } e \in \mathbb{R}^{2r} \,.$$

Such bound δ_3 does not depend on κ nor on h, so long as the latter satisfies $h \ge h_1^*$.

Consider now a trajectory with initial condition $\mathbf{x}(0) \in B_R \subset \Omega_c$. In view of the above bound for $\Delta_3(\bar{x}, e)$ and of the fact that $\|x_2\| \le 1$, it can be claimed that, so long as $\mathbf{x}(t) \in \Omega_{c+1}$,

$$\dot{V}(\mathbf{x}) \le -\|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{P}\|\Big[\delta_3 + \|G_2\|\Big] \,.$$

Letting $M := \max_{\mathbf{x} \in \Omega_{c+1}} 2\|\mathbf{x}\|\|\mathbf{P}\|\Big[\delta_3 + \|G_2\|\Big]$ the previous estimate yields, in particular,

$$\dot{V}(\mathbf{x}(t)) \le M$$

from which it is seen that (recall that $\mathbf{x}(0) \in \Omega_c$)

$$V(\mathbf{x}(t)) \le V(\mathbf{x}(0)) + Mt \le c + Mt \,.$$

This inequality shows that $\mathbf{x}(t)$ remains in Ω_{c+1} at least until time $\overline{T} = 1/M$. This time might be very small but it is independent of κ and h, because so is M.

During the time interval $[0, \overline{T}]$ the state $e(t)$ remains bounded. This is seen from the third equation of (35), using the bounds determined for $\Delta_0(\bar{x}, e)$, $\Delta_1(\bar{x}, e)$, $\Delta_2(\bar{x}, e)$ in Lemma 4.3 and the fact that $\mathbf{x}(t) \in \Omega_{c+1}$ for all $t \in [0, \overline{T}]$. The bound on $e(t)$, though, is affected by the value of κ. In fact, looking at the definitions of the various components of e, it is seen that $\|e(0)\|$ grows with κ (despite of the fact that, by assumption, $\mathbf{x}(0) \in B_R$ and $(\hat{\xi}, \sigma) \in B_R \times B_R$).

Step 5. Having established that trajectories of the system exist on the time interval $[0, \overline{T}]$, we analyze next the behavior of $e(t)$ and $\mathbf{x}(t)$ for times larger than $\overline{T}$. Knowing that $\mathbf{x}(t) \in \Omega_{c+1}$ for all $t \leq \overline{T}$, let $T_{\max} \geq \overline{T}$ be any number such that $\mathbf{x}(t) \in \Omega_{c+1}$ for all $t \in [0, T_{\max})$. Then the properties indicated in the following Lemma hold.

Lemma 4.6. *Let the c_0, c_1 be such that (34) holds. Suppose $\mathbf{x}(t) \in \Omega_{c+1}$ for all $t \in [0, T_{\max})$ and suppose that $(\hat{\xi}(0), \sigma(0)) \in B_R \times B_R$. Then, for every $0 < T < T_{\max}$ and every $\varepsilon > 0$, there is a number κ^* and, for all $\kappa > \kappa^*$ a number h^*_κ, such that, if $\kappa > \kappa^*$ and $h > h^*_\kappa$, then*

$$\|e(t)\| \leq 2\varepsilon \quad \textit{for all } t \in [T, T_{\max}).$$

Proof. Set $U(e) = e^\top P e$ and recall that

$$a_1 \|e\|^2 \leq U(e) \leq a_2 \|e\|^2 \,, \tag{38}$$

for some pair a_1, a_2. Using the result of Lemma 4.3 and the bound (31), we have

$$\begin{aligned} \dot{U}(e(t)) &= 2e^\top P\Big[\kappa[\mathbf{A} - \mathbf{B}_2\Delta_0(\bar{x}, e)\mathbf{C}]e + \mathbf{B}_1\Delta_1(\bar{x}, e) + \mathbf{B}_2\Delta_2(\bar{x}, e) + \kappa^2 \mathbf{B}_3(\bar{x}, e)x_2\Big] \\ &\leq -\kappa\lambda\|e\|^2 + 2e^\top P\mathbf{B}_1\Delta_1(\bar{x}, e) + 2e^\top P[\mathbf{B}_2\Delta_2(\bar{x}, e) + \kappa^2\mathbf{B}_3(\bar{x}, e)x_2] \\ &\leq -(\kappa\lambda - 2\delta_1\|P\|)\|e\|^2 + 2\|e\|\|P\|[\|\Delta_2(\bar{x}, e)\| + \kappa^2\|\mathbf{B}_3(\bar{x}, e)\| \, \|x_2\|] \end{aligned}$$

Let now R' be a number such that $\Omega_{c+1} \subset B_{R'}$. Using the bound (33) we see that there is a number N and a continuous function $\varsigma(h)$, with $\lim_{h\to\infty} \varsigma(h) = 0$, such that

$$\|\Delta_2(\bar{x}, e)\| \leq \frac{1}{2}N + \varsigma(h)\|e\|$$

so long as $\mathbf{x}(t) \in \Omega_{c+1}$. Observe also that, by construction,

$$\|\mathbf{B}_3(\bar{x}, e)\| \leq \|T_0\|(c_1^2 + 4c_0^2)^{\frac{1}{2}} \,.$$

With Lemma 4.5 in mind, we know that for any N' and for any κ we can find number h^*_κ, that without loss of generality we assume larger than h^*_1, such that, if $h > h^*_\kappa$, then

$$\|x_2\| \leq \frac{1}{\kappa^2} N' \,.$$

Choosing N' so that

$$N'\|T_0\|(c_1^2 + 4c_0^2)^{\frac{1}{2}} = \frac{1}{2}N$$

we deduce that

$$\|\Delta_2(\bar{x}, e)\| + \kappa^2 \|\mathbf{B}_3(\bar{x}, e)\| \, \|x_2\| \leq N + \varsigma(h)\|e\| \,.$$

Let h be large enough so that $\varsigma(h) < \delta_1$. Then

$$\begin{aligned}
\dot{U}(e(t)) &\leq -(\kappa\lambda - 2\delta_1\|P\|)\|e\|^2 + 2\|e\|\|P\|[N + \varsigma(h)\|e\|] \\
&\leq -(\kappa\lambda - 4\delta_1\|P\|)\|e\|^2 + 2\|e\|\|P\|N \\
&\leq -(\kappa\lambda - 4\delta_1\|P\| - \mu\|P\|)\|e\|^2 + \tfrac{\|P\|}{\mu}N^2 \,,
\end{aligned}$$

in which μ is any arbitrary (positive) number. Choose now μ as

$$\mu = \frac{N^2\|P\|}{\varepsilon^2}$$

in which case we arrive at

$$\dot{U}(e(t)) \leq -(\kappa\lambda - 4\delta_1\|P\| - \mu\|P\|)\|e\|^2 + \varepsilon^2 \,.$$

Bearing in mind the estimates in (38), set

$$\alpha_\kappa = \frac{\kappa\lambda - (4\delta_1 + \mu)\|P\|}{2a_2} \tag{39}$$

and suppose that κ is large enough so as to make

$$2\alpha_\kappa a_1 > 1 \,,$$

which implies $\alpha_\kappa > 0$. Then, the inequality

$$\dot{U}(e(t)) \leq -2\alpha_\kappa U(e(t)) + \varepsilon^2,$$

holds, for any $t \in [0, T_{\max})$. From this, by means of standard arguments, it can be concluded that [12]

$$\|e(t)\| \leq Ae^{-\alpha_\kappa t}\|e(0)\| + \varepsilon \,, \qquad \text{in which} \quad A = \sqrt{\tfrac{a_2}{a_1}}.$$

At this point, it is necessary to obtain a bound for $\|e(0)\|$. Bearing in mind the definition of e, the fact that $\mathbf{x}(0) \in B_R$, $\hat{\xi}(0) \in B_R$, $\sigma(0) \in B_R$, and assuming $\kappa \geq 1$, it is seen that

$$\|e(0)\| \leq \hat{R}\kappa$$

[12]Use the comparison Lemma to get

$$U(t) \leq e^{-2\alpha_\kappa t}U(0) + \frac{\varepsilon^2}{2\alpha_\kappa}$$

from which, using the estimates (38) and the fact that $2\alpha_\kappa a_1 > 1$, the claimed inequality follows.

in which $\hat{R}$ is a number only depending on R. Therefore, $\|e(t)\|$ is bounded as

$$\|e(t)\| \leq A\hat{R}e^{-\alpha_\kappa t}\kappa + \varepsilon$$

for all $t \in [0, T_{\max})$. Let now T be any time satisfying $0 < T < T_{\max}$, and – using (39) – observe that

$$A\hat{R}e^{-\alpha_\kappa T}\kappa = A\hat{R}e^{\frac{(2\delta_1+\mu)\|P\|T}{2a_2}} e^{-\frac{\lambda T}{2a_2}\kappa}\kappa\,.$$

Clearly, the function $\vartheta : \kappa \mapsto \vartheta(\kappa) = e^{-\frac{\lambda T}{2a_2}\kappa}\kappa$ decays to 0 as $\kappa \to \infty$. Thus, there is a number κ^* such that

$$A\hat{R}e^{-\alpha_\kappa T}\kappa = A\hat{R}e^{\frac{(2\delta_1+\mu)\|P\|T}{2a_2}} \vartheta(\kappa) \leq \varepsilon\,.$$

for all $\kappa \geq \kappa^*$. [13] From this, it is concluded that, for all $\kappa \geq \kappa^*$,

$$\|e(T)\| \leq 2\varepsilon.$$

Finally, bearing in mind the fact that $\alpha_\kappa > 0$, we see that for $t \in [T, T_{\max})$

$$\|e(t)\| \leq A\hat{R}e^{-\alpha_\kappa(t-T)}e^{-\alpha_\kappa T}\kappa + \varepsilon \leq e^{-\alpha_\kappa(t-T)}\varepsilon + \varepsilon \leq 2\varepsilon\,,$$

and this concludes the proof. □

This result can be used to prove that, if ε is small enough, none of the components of the control u_{act} is "saturated" on the entire time interval $t \in [T, T_{\max})$.

Lemma 4.7. *Suppose $\mathbf{x}(t) \in \Omega_{c+1}$, $\|x_2(t)\| \leq 1$ and $\|e(t)\| \leq 2\varepsilon(t)$, for all $t \in [T, T_{\max})$. There is a number ε^* such that, if $\varepsilon < \varepsilon^*$, then for all $t \in [T, T_{\max})$,*

$$\begin{array}{rcl} G_L(\psi(\xi,\sigma)) &=& \psi(\xi,\sigma) \\ G_L(\psi(\hat{\xi},\sigma)) &=& \psi(\hat{\xi},\sigma)\,. \end{array} \tag{40}$$

Proof. Let b_0 be such that $\|B^{-1}\| \leq b_0$. Let ε satisfies

$$4\varepsilon b_0 < 1\,. \tag{41}$$

Then, bearing in mind the definition of e_2, we obtain

$$\|B^{-1}e_2\| \leq \|B^{-1}\|\,\|e\| \leq b_0 2\varepsilon < \frac{1}{2}\,,$$

for all $t \in [T, T_{\max})$, which in turn implies, because of the choice of L,

$$\|B^{-1}[-\xi - Q(z,\xi,x_2) + e_2]\| \leq L - \frac{1}{2}\,. \tag{42}$$

[13]Note that μ depends on ε and, actually, increases as ε decreases. This fact, however, does not affect the prior conclusion.

Using (29), observe that

$$B^{-1}[-\xi - Q(z, \xi, x_2) + e_2] = B^{-1}B_0 f(\psi(\xi, \sigma)) \,.$$

Hence, if we consider the function

$$\hat{f}(s) = B^{-1}B_0 f(s) = [I - B^{-1}B_0]G_L(s) + B^{-1}B_0 s \,,$$

the inequality (42) can be rewritten as

$$\|\hat{f}(\psi(\xi, \sigma))\| \leq L - \frac{1}{2}. \tag{43}$$

The function $\hat{f}(s)$ is globally invertible (because so is $f(s)$, as shown earlier in Lemma 4.2). Moreover, it can be easily seen that the function $\hat{f}(s)$ is an identity on the set

$$C_L := \{s \in \mathbb{R}^r : \ |s_i| \leq L, \text{ for all } i = 1, \dots, r\} \,.$$

In fact,

$$s \in C_L \qquad \Rightarrow \qquad G_L(s) = s \,,$$

and this, using the expression shown above for $\hat{f}(s)$, proves that $\hat{f}(s) = s$ for all $s \in C_L$. As a consequence, the pre-image of any point p in the set C_L is the point p itself. Since the inequality (43) implies $\hat{f}(\psi(\xi, \sigma)) \in C_L$, it is concluded that $\hat{f}(\psi(\xi, \sigma)) = \psi(\xi, \sigma)$ and hence

$$\|\psi(\xi, \sigma)\| \leq L - \frac{1}{2} \,, \tag{44}$$

for all $t \in [T, T_{\max})$, which proves the first identity in (40). To prove the second identity, recall that

$$\hat{\xi} = \xi - \begin{pmatrix} \kappa^{-1} I_r & 0 \end{pmatrix} e$$

and, without loss of generality, assume $\kappa \geq 1$. Since, by definition,

$$\psi(\hat{\xi}, \sigma) = \psi(\xi, \sigma) + B_0^{-1} \begin{pmatrix} \kappa^{-1} & 0 \end{pmatrix} e \,,$$

we have

$$\|\psi(\hat{\xi}, \sigma)\| \leq \|\psi(\xi, \sigma)\| + \|B_0^{-1}\| \|e\|.$$

Suppose ε satisfies

$$4\|B_0^{-1}\|\varepsilon < 1 \,. \tag{45}$$

Then, using (44), we see that $\|\psi(\hat{\xi}, \sigma)\| < L$, and this proves that the second identity in (40) holds on the time interval $[T, T_{\max})$. $\square$

Step 6. The properties indicated in the previous Lemma make it possible to obtain simpler expression for $\Delta_3(\bar{x}, e)$ in (37). From the definition of e_2, we get

$$\begin{array}{rcl} e_2 & = & Q(z,\xi,x_2) + [B - B_0]\psi(\xi,\sigma) - \sigma \\ & = & Q(z,\xi,x_2) + [B - B_0]B_0^{-1}(-\xi - \sigma) - \sigma \\ & = & Q(z,\xi,x_2) + BB_0^{-1}(-\xi - \sigma) + \xi \\ & = & Q(z,\xi,x_2) + B\psi(\xi,\sigma) + \xi\,. \end{array}$$

Hence

$$\begin{array}{rcl} B\psi(\xi,\sigma) & = & e_2 - Q(z,\xi,x_2) - \xi \\ B\psi(\hat{\xi},\sigma) & = & e_2 - Q(z,\xi,x_2) - \xi - BB_0^{-1}\kappa^{-1}e_1\,. \end{array} \tag{46}$$

Since $G_L(\psi(\hat{\xi},\sigma)) = \psi(\hat{\xi},\sigma)$ for all $t \in [T, T_{\max})$, we obtain (using, in the last passage, the identities (46))

$$\begin{array}{rcl} \Delta_3(\bar{x},e) & = & \xi + Q(z,\xi,x_2) + BG_L(\psi(\hat{\xi},\sigma)) \\ & = & \xi + Q(z,\xi,x_2) + B\psi(\hat{\xi},\sigma) \\ & = & \xi + Q(z,\xi,x_2) + B\psi(\xi,\sigma) - BB_0^{-1}\kappa^{-1}e_1 \\ & = & e_2 - BB_0^{-1}\kappa^{-1}e_1\,, \end{array}$$

that can be rewritten as

$$\Delta_3(\bar{x},e) = \mathbf{H}_\kappa e$$

having defined

$$\mathbf{H}_\kappa = \begin{pmatrix} -BB_0^{-1}\kappa^{-1} & I_r \end{pmatrix}$$

Returning now to the equation (37), it is seen that

$$\dot{V}(\mathbf{x}) = -\|\mathbf{x}\|^2 + 2\mathbf{x}^{\mathrm{T}}\mathbf{P}[\mathbf{G}\mathbf{H}_\kappa e + \mathbf{L}x_2]$$

Bearing in mind the fact that $\|e(t)\| \leq 2\varepsilon$ and that $\|x_2(t)\|$ can be arbitrarily lowered by increasing h, standard manipulations can be used to show that, for any choice of a number $0 < \delta < 1$ there is a number h_2^* such that, if $h \geq h_2^*$, an estimate of the form

$$\dot{V}(\mathbf{x}) = -(1-\delta)\|\mathbf{x}\|^2 + \varepsilon^2$$

holds. Hence, so long as $\|\mathbf{x}\| > \frac{\varepsilon}{1-\delta}$, the function $\dot{V}(\mathbf{x})$ is decreasing. This shows, that $\mathbf{x}(t) \in \Omega_{c+1}$ for all $t > 0$ and hence $T_{\max} = \infty$.

Step 7. The properties indicated in Lemma 4.7 make also possible to simplify the expressions of the terms $\Delta_0(\bar{x}, e)$, $\Delta_1(\bar{x}, e)$, $\Delta_2(\bar{x}, e)$, and $\mathbf{B}_3(\bar{x}, e)$ in the dynamics of

e. Simple manipulations show that

$$\begin{array}{rcl}\Delta_0(\bar{x},e) & = & [BB_0^{-1}-I] := \Delta_0 \\ \Delta_1(\bar{x},e) & = & [BB_0^{-1}-I]e_1 := \Delta_1 e_1 \\ \mathbf{B}_3(\bar{x},e) & = & \begin{pmatrix} c_1 I \\ c_0 BB_0^{-1}\end{pmatrix} T_0 := \mathbf{B}_3 \\ \Delta_2(\bar{x},e) & = & \dot{Q}(z,\xi,x_2)+[I-BB_0^{-1}]\dot{\xi}\,.\end{array}$$

To find a more explicit expression for $\Delta_2(\bar{x},e)$, it is convenient to solve first the algebraic constraint that defines x_2. The constraint in question has the form (use here (46))

$$0 = A_{21}\xi + [A_{22}+hI]x_2 + H_2 z + S_0 B^{-1}[-A_{11}\xi - A_{12}x_2 - H_1 z - \xi + \mathbf{H}_\kappa e]$$

that is

$$0 = [A_{21}-S_0B^{-1}(A_{11}+I)]\xi + [H_2 - S_0B^{-1}H_1]z + [A_{22}+hI-S_0B^{-1}A_{12}]x_2 + S_0B^{-1}\mathbf{H}_\kappa e$$

which we rewrite, for convenience, as

$$\bar{A}_{21}\xi + \bar{H}_2 z + \overline{\mathbf{H}}_\kappa e = [\bar{A}_{22}+hI]x_2\,.$$

Note that $\bar{A}_{21}, \bar{A}_{22}$ and $\bar{H}_2$ do not depend on the design parameters (κ,h), while $\overline{\mathbf{H}}_\kappa$, that depends on κ because so does $\mathbf{H}_\kappa$, is bounded for $\kappa \geq 1$. Then

$$x_2 = [\bar{A}_{22}+hI]^{-1}[\bar{A}_{21}\xi + \bar{H}_2 z + \overline{\mathbf{H}}_\kappa e] \tag{47}$$

This being the case, we have

$$\begin{array}{rcl}\Delta_2(\bar{x},e) & = & \dot{Q}(z,\xi,x_2)+[I-BB_0^{-1}]\dot{\xi} \\ & = & [A_{11}+I-BB_0^{-1}]\dot{\xi} + A_{12}\dot{x}_2 + H_1\dot{z} \\ & = & [A_{11}+I-BB_0^{-1}+A_{12}[\bar{A}_{22}+hI]^{-1}\bar{A}_{21}]\dot{\xi} + \\ & & +[H_1 + A_{12}[\bar{A}_{22}+hI]^{-1}\bar{H}_2]\dot{z} + A_{12}[\bar{A}_{22}+hI]^{-1}\overline{\mathbf{H}}_\kappa \dot{e}\end{array}$$

that we rewrite as

$$\Delta_2(\bar{x},e) = \mathbf{D}_h\dot{\mathbf{x}} + A_{12}[\bar{A}_{22}+hI]^{-1}\overline{\mathbf{H}}_\kappa\dot{e}$$

in which

$$\mathbf{D}_h = \begin{pmatrix}[A_{11}+I-BB_0^{-1}+A_{12}[\bar{A}_{22}+hI]^{-1}\bar{A}_{21}] \\ [H_1+A_{12}[\bar{A}_{22}+hI]^{-1}\bar{H}_2]\end{pmatrix}\,.$$

With the help of such notations, we obtain

$$\dot{e} = \kappa[\mathbf{A}-\mathbf{B}_2\Delta_0\mathbf{C}]e + \mathbf{B}_1\Delta_1 e + \mathbf{B}_2[\mathbf{D}_h(\mathbf{F}\mathbf{x}+\mathbf{G}\mathbf{H}_\kappa e + \mathbf{L}x_2) + A_{12}[\bar{A}_{22}+hI]^{-1}\overline{\mathbf{H}}_\kappa\dot{e}] + \kappa^2\mathbf{B}_3 x_2$$

that is

$$
\begin{aligned}
&(I - A_{12}[\bar{A}_{22} + hI]^{-1}\overline{\mathbf{H}}_\kappa)\dot{e} = \\
&\qquad = \kappa[\mathbf{A} - \mathbf{B}_2\Delta_0\mathbf{C}]e + [\mathbf{B}_1\Delta_1 + \mathbf{B}_2\mathbf{D}_h\mathbf{G}\mathbf{H}_\kappa]e + \mathbf{B}_2\mathbf{D}_h\mathbf{F}\mathbf{x} + [\mathbf{B}_2\mathbf{D}_h\mathbf{L} + \kappa^2\mathbf{B}_3]x_2
\end{aligned}
$$

In summary, it is seen that, on the time interval $[T, +\infty)$, the system (35) can be expressed in the form

$$
\begin{array}{rcl}
\dot{\mathbf{x}} & = & \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{H}_\kappa e + \mathbf{L}x_2 \\
(I - A_{12}[\bar{A}_{22} + hI]^{-1}\overline{\mathbf{H}}_\kappa)\dot{e} & = & \kappa[\mathbf{A} - \mathbf{B}_2\Delta_0\mathbf{C}]e + [\mathbf{B}_1\Delta_1 + \mathbf{B}_2\mathbf{D}_h\mathbf{G}\mathbf{H}]e + \\
 & & +\mathbf{B}_2\mathbf{D}_h\mathbf{F}\mathbf{x} + [\mathbf{B}_2\mathbf{D}_h\mathbf{L} + \kappa^2\mathbf{B}_3]x_2
\end{array}
\tag{48}
$$

in which x_2 has the expression (47).

The "input" x_2 in (48) can be interpreted as a "memoryless feedback" from $(\mathbf{x}, e)$. Since the gain matrices that characterize such feedback can be made arbitrarily small by increasing h, then – according to well-known principles – to establish stability of the equilibrium $(\mathbf{x}, e) = (0, 0)$ of (48) it suffices to establish stability of the equilibrium $(\mathbf{x}, e) = (0, 0)$ of the system obtained setting $x_2 = 0$, which we rewrite as

$$
\begin{array}{rcl}
\dot{\mathbf{x}} & = & \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{H}_\kappa e \\
\dot{e} & = & \kappa(I + \bar{E}_{\kappa,h})[\mathbf{A} - \mathbf{B}_2\Delta_0\mathbf{C}]e + \mathbf{M}_h e + \mathbf{N}_h\mathbf{x}
\end{array}
\tag{49}
$$

in which

$$
\begin{array}{rcl}
\bar{E}_{\kappa,h} & = & (I - A_{12}[\bar{A}_{22} + hI]^{-1}\overline{\mathbf{H}}_\kappa)^{-1} - I \\
\mathbf{M}_h & = & (I + \bar{E}_{\kappa,h})[\mathbf{B}_1\Delta_1 + \mathbf{B}_2\mathbf{D}_h\mathbf{G}\mathbf{H}] \\
\mathbf{N}_h & = & (I + \bar{E}_{\kappa,h})\mathbf{B}_2\mathbf{D}_h\mathbf{F}\,.
\end{array}
$$

Note that $\bar{E}_{\kappa,h} \to 0$ as $h \to \infty$, while $\mathbf{M}_h$ and $\mathbf{N}_h$ can be bounded by numbers that are independent of h (so long as $\kappa > 1$).

Recall now that the matrix $\mathbf{A} - \mathbf{B}_2\Delta_0\mathbf{C}$ has the property indicated in Lemma 4.4 and consider the Lyapunov function $U(e) = e^{\mathrm{T}}P\mathbf{e}$. Then

$$
\begin{array}{rcl}
\dot{U}(e) & = & 2e^{\mathrm{T}}P(I + \bar{E}_{\kappa,h})[\kappa[\mathbf{A} - \mathbf{B}_2\Delta_0\mathbf{C}]e + \mathbf{M}_h e + \mathbf{N}_h\mathbf{x}] \\
 & \le & -\kappa\lambda\|e\|^2 + [2\kappa\|\bar{E}_{\kappa,h}\|\,\|P\|\,\|(\mathbf{A} - \mathbf{B}_2\Delta_0\mathbf{C}\|)]\|e\|^2 + \\
 & & +2\|P\mathbf{M}_h\|\,\|e\|^2 + 2\|P\mathbf{N}_h\|\,\|e\|\,\|\mathbf{x}\|
\end{array}
$$

Pick a number $\bar{d}$ such that

$$
\max\{2\|P\mathbf{M}_h\|, 2\|P\mathbf{N}_h\|\} \le \bar{d}\,.
$$

Since $\bar{E}_{\kappa,h} \to 0$ as $h \to \infty$, for any choice of κ there is a value h^*_κ such that, if $h > h^*_\kappa$, then

$$
2\kappa\|\bar{E}_{\kappa,h}\|\,\|P\|\,\|(\mathbf{A} - \mathbf{B}_2\Delta_0\mathbf{C}\|) \le \bar{d}\,.
$$

Thus, if $h > h^*_\kappa$, then

$$
\dot{V}(e) \le -\kappa\lambda\|e\|^2 + 2\bar{d}\|e\|^2 + \bar{d}\|e\|\,\|\mathbf{x}\|\,.
$$

From this, using standard arguments, one can conclude that, given any number c^*, there is a choice of κ^* and, for any choice of $\kappa > \kappa^*$ a value of h^*_κ such that, if $\kappa > \kappa^*$ and $h > h^*_\kappa$, then the lower subsystem of (35), viewed as a system with state e and input $\mathbf{x}$ is input-to-state stable, with a linear gain function $\gamma(s) = c^*$. By the small-gain theorem, if c^* is sufficiently small, the equilibrium $(\mathbf{x}, e) = (0, 0)$ of (49) is asymptotically stable. [14]

4.3. Numerical example

We consider the example of a system modeled by a DAE of the form (9), with matrices

$$F = -1, \quad G = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}, \quad \bar{E} = \begin{pmatrix} 1 & 2 & 2 \\ b & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -1 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},$$

where b is an uncertain parameter with values in the range $[3, 5]$. The eigenvalues of F and A are -1 and $\{1, -1, 0\}$, respectively. The rank of the matrix $\bar{E}$ is $r = 2$ and the matrix

$$\bar{E}_{11} = \begin{pmatrix} 1 & 2 \\ b & 1 \end{pmatrix}$$

is nonsingular regardless of the value of b in its range. The gain of the feedback (15) is set as $h = 10$. The parameters of the extended observer (23) are set as

$$B_0 = \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}^{-1}, \quad \kappa = 15, \quad c_0 = 0.02, \quad c_1 = 0.3\,,$$

and the saturation function in (21) is defined as

- $g_L(s) = s$ if $|s| \leq L$,
- $g_L(s) = L + c(1 - e^{\frac{-s+L}{c}})$ if $s > L$,
- $g_L(s) = -L - c(1 - e^{\frac{s+L}{c}})$ if $s < -L$,

with $L = 5$ and $c = 0.01$. It can be verified that, with this choice for B_0, the term appearing in the left-hand side of equation (13) of Assumption 4 is bounded by 0.527 (i.e., there exist values for δ_0 smaller than 1 for which (13) is satisfied).

From what is written above it is then verified that Theorem 4.1 holds and hence the proposed robust control method can be applied. We observe that, on the contrary, the stabilization method proposed in Berger (2016) is not applicable since $\bar{E} \neq \bar{E}^T$.

Figures 1 and 2 present the results of a numerical simulation of the dynamics of the described system and of the error dynamics (24) starting from the initial state $z(0) = 1$, $x_1(0) = \begin{pmatrix} 1 & 2 \end{pmatrix}^T$, $x_2(0) = 0.2$, $\hat{\xi}(0) = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$, $\sigma(0) = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$, with parameter $b = 3$. The figures show that, in the beginning, the control $u_{\text{act}}(t)$ is saturated. Nonetheless, the extended-observer errors decrease and, from about $1s$ on, $u_{\text{act}}(t)$ is not saturated, so that the overall system behaves as a linear one. As expected, the states of the system as well that of the observer converge to 0. Analogous results were obtained for several values of the uncertain parameter $b \in [3, 5]$.

[14] For the notion of input-to-state stability and the associated small-gain theorem, see e.g. Khalil (2002).

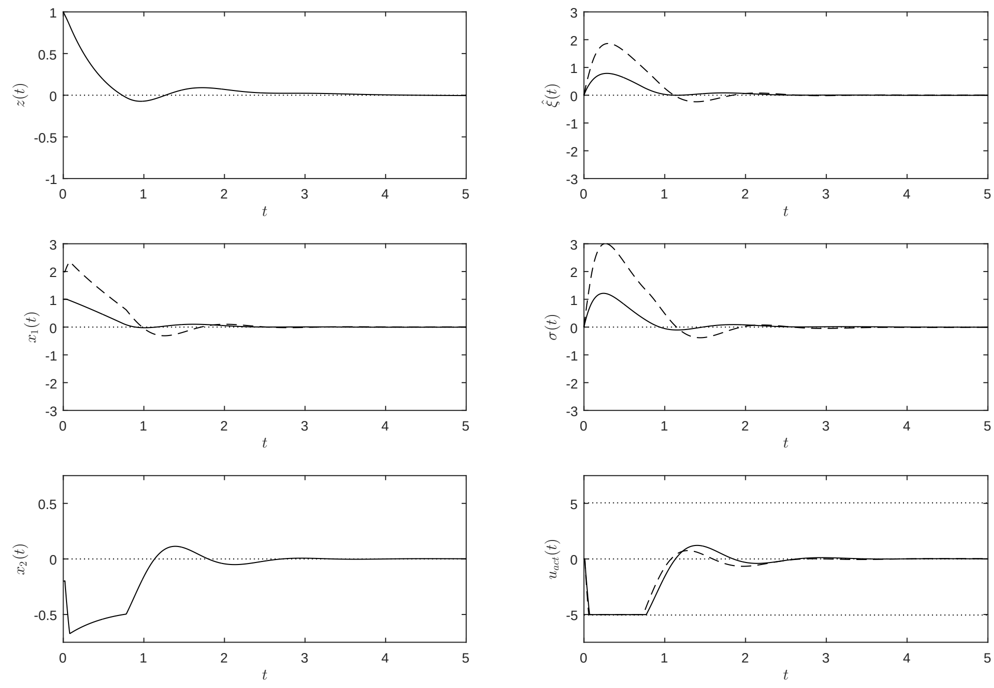

Figure 1. DAE system and extended-observer dynamics (the dotted lines in the plot of $u_{\text{act}}(t)$ indicate the saturation levels)

.

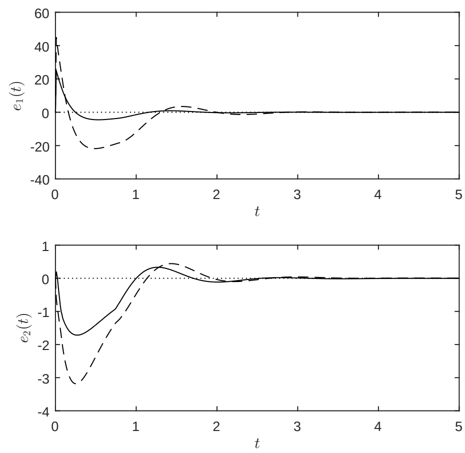

Figure 2. Error dynamics

5. Conclusions

In this paper we presented a robust stabilizing control for a class of linear differential algebraic systems characterized by an autonomous and stable zero dynamics, in spite of parameter uncertainties ranging over *a priori* fixed bounded sets.

We reconsidered the stabilization method described in Berger (2016) under different assumptions on the system representation. The proposed robust stabilizer relies on the idea of using a partial output feedback control to let the DAE behave as a regular system and an extension of the high-gain extended-observer-based stabilizing control presented in Freidovich and Khalil (2008). The proposed robust control framework, originally conceived in the context of standard dynamical systems, has been shown to be effective also in presence of algebraic constraints, provided that the gain used to make the system regular is chosen sufficiently high. Among other possible applications, the proposed stabilization method is helpful in solving also a problem of robust output regulation for systems modeled by DAEs, as it is shown in Di Giorgio, Pietrabissa, Delli Priscoli, and Isidori (2018).

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