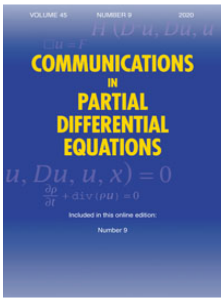

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On the improvement of the Hardy inequality due to singular magnetic fields

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ABSTRACT
We establish magnetic improvements upon the classical Hardy inequality for two specific choices of singular magnetic fields. First, we consider the Aharonov-Bohm field in all dimensions and establish a sharp Hardy-type inequality that takes into account both the dimensional as well as the magnetic flux contributions. Second, in the three-dimensional Euclidean space, we derive a non-trivial magnetic Hardy inequality for a magnetic field that vanishes at infinity and diverges along a plane.

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1. Introduction

The subcriticality of the Laplacian in $\mathbb{R}^d$ for $d \geq 3$ can be quantified by means of the classical Hardy inequality

$$-\Delta \geq \left(\frac{d-2}{2}\right)^2 \frac{1}{r^2} \tag{1}$$

valid in the sense of quadratic forms in $L^2(\mathbb{R}^d)$, where $-\Delta$ is the standard self-adjoint realisation of the Laplacian in $L^2(\mathbb{R}^d)$ and r is the distance to the origin of $\mathbb{R}^d$. On the other hand, the Laplacian is critical in $\mathbb{R}$ and $\mathbb{R}^2$ in the sense that the spectrum of the shifted operator $-\Delta + V$ starts below zero whenever the operator of multiplication V is bounded, non-positive and non-trivial. In quantum mechanics, interpreting $-\Delta$ as the Hamiltonian of a free electron, the Hardy inequality (1) can be interpreted as the uncertainty principle with important consequences for the stability of atoms and molecules.

Inequality (1) goes back to 1920 [1] and it is well known that it is optimal in the sense that the dimensional constant is the best possible and no other non-negative term could be added on the right-hand side of (1). A much more recent observation is that adding any magnetic field leads to an improved Hardy inequality, including dimension $d=2$. A variant of this statement is the magnetic Hardy inequality

$$(-i\nabla + A)^2 - \left(\frac{d-2}{2}\right)^2 \frac{1}{r^2} \geq \frac{c_{d,B}}{1 + r^2 \log^2(r)}, \tag{2}$$

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valid in the sense of quadratic forms in $L^2(\mathbb{R}^d)$ for $d \geq 2$. Here $A : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth vector potential and $c_{d,B}$ is a non-negative constant that depends only on the dimension d and the magnetic field $B = \mathrm{d}A$; the constant $c_{d,B}$ is positive if, and only if, the field B is not identically equal to zero. This inequality was first proved by Laptev and Weidl in 1999 [2] for $d = 2$ under an extra flux condition, in which case the lower bound holds with a better weight (without the logarithm) on the right-hand side of (2). A general version of (2) is due to Cazacu and Krejčiřík [3], but we also refer to [4, 5, 7; 6, Sec. 6] and [8] for previous related works.

Inequality (2) has important physical consequences. In the absence of magnetic field, the operator appearing on the left-hand side is critical for every $d \geq 2$. Adding the magnetic field, however, it becomes subcritical, in the sense that additive (*e.g.* electric) perturbations smaller comparing to the right-hand side of (2) will not produce negative (discrete) spectrum. Hence, magnetic field stabilises quantum transport in the sense of preventing the creation of bound states due to attractive electric potentials. In the time-dependent setting, this phenomenon has been recently quantified in [9–11]. Similar repulsive effects of the magnetic field exist in the context of the heat flow, see [3, 12, 13]. For applications, it is thus important to understand the structure of the magnetic improvement on the right-hand side of (2). In this paper we quantify it in two models.

Our first result deals with the Aharonov-Bohm potential

$$A_\alpha(x, y, z_1, ..., z_{d-2}) := \alpha\left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0, ..., 0\right), \qquad \alpha \in \mathbb{R}, \tag{3}$$

where $(x, y, z_1, ..., z_{d-2}) \in \mathbb{R}^d$ with $d \geq 2$. We abbreviate $z := (z_1, ..., z_{d-2}) \in \mathbb{R}^{d-2}$ and denote by $\rho(x, y, z) := \sqrt{x^2 + y^2}$ the distance of a point $(x, y, z) \in \mathbb{R}^d$ to the subspace $\{x = y = 0\} \subset \mathbb{R}^d$ of dimension $d-2$. Let us also recall that $r(x, y, z) := \sqrt{x^2 + y^2 + |z|^2}$ denotes the distance of $(x, y, z) \in \mathbb{R}^d$ to the origin of $\mathbb{R}^d$. Because of the singularity of A_α at the origin, it is important to specify the self-adjoint realisation of the associated magnetic Laplacian; we customarily understand $(-i\nabla + A_\alpha)^2$ as the Friedrichs extension of this operator initially defined on $C_0^\infty(\mathbb{R}^d \setminus \{\rho = 0\})$.

Theorem 1. *Let A_α be given by (3). For every $\alpha \in \mathbb{R}$, one has*

$$(-i\nabla + A_\alpha)^2 - \left(\frac{d-2}{2}\right)^2 \frac{1}{r^2} \geq \frac{\operatorname{dist}(\alpha, \mathbb{Z})^2}{\rho^2} \tag{4}$$

in the sense of quadratic forms in $L^2(\mathbb{R}^d)$ with $d \geq 2$.

This theorem in dimension $d = 2$ is due to Laptev and Weidl [2]. The novelty of Theorem 1 consists in the present extension to the higher dimensions, $d \geq 3$. The result is optimal in the sense that the constants appearing in (4) are the best possible and no other non-negative term could be added on the right-hand side of the inequality. In other words, subtracting the right-hand side from the left-hand side, the obtained operator would be critical. At the same time, the equality sign in the inequality (4) is never achieved, in the sense that the equality of the values of the quadratic forms corresponding to the left- and right-hand sides of the inequality does not occur for any non-trivial function.

Notice also that the flux-type condition $\alpha \notin \mathbb{Z}$ is necessary to have the subcriticality of the operator on the left-hand side of (4). Indeed, $(-i\nabla + A_\alpha)^2$ is unitarily equivalent to the magnetic-free Laplacian whenever $\alpha \in \mathbb{Z}$, so in this case the criticality of the operator on the left-hand side of (4) follows from the optimality of the classical Hardy inequality (1).

Because of the special form of the vector potential (3), the operator $(-i\nabla + A_\alpha)^2$ admits a natural decomposition with respect to the variables $(x, y) \in \mathbb{R}^2$ and $z \in \mathbb{R}^{d-2}$. However, it is important to stress that (4) does not follow as a result of this separation of variables. In fact, while the right-hand side of (4) is the two-dimensional contribution coming from the angular component of the magnetic Laplacian in the $(x,\ y)$-plane, the second (dimensional) term on the left-hand side of the inequality is a contribution coming from both the radial component of the magnetic Laplacian in the $(x,\ y)$-plane as well as the Laplacian in the z-space.

While the structure of the Aharonov-Bohm potential (3) might seem very special, it is in fact a canonical example, for it is a magnetic analogue of the electric Dirac delta potential. The feature of this vector potential is that its singularity is supported on a manifold of codimension two. Our next interest lies in a vector potential with a singularity supported on a plane. In this case, we restrict our attention to the three-dimensional toy model

$$A_\beta(x, y, z) := \beta\left(\frac{y}{z^2}, 0, 0\right), \qquad \beta \in \mathbb{R}, \tag{5}$$

where $(x, y, z) \in \mathbb{R}^3$. The seemingly simple choice (5) for the vector potential is of course very special, but on the other hand the model is intrinsically three-dimensional in the sense that no reduction to lower dimensions via a separation of variables is available.

We understand the magnetic Laplacian $-\Delta_\beta := (-i\nabla + A_\beta)^2$ corresponding to (5) with $\beta \in \mathbb{R} \setminus \{0\}$ as the Friedrichs extension of this operator initially defined on $C_0^\infty(\mathbb{R}^3 \setminus \{z = 0\})$. Because of the strong singularity of A_β on the plane $\{z = 0\}$, the unperturbed version of $-\Delta_\beta$ is not the Laplacian $-\Delta$ in $\mathbb{R}^3$ but rather the Dirichlet Laplacian in $\mathbb{R}^3 \setminus \{z = 0\}$ that we denote by $-\Delta_0$. More specifically, the singularity of the potential requires that the functions from the operator domain of $-\Delta_\beta$ vanish on $\{z = 0\}$, representing thus certain confinement of the electron to one of the two half-spaces $\{z > 0\}$ or $\{z < 0\}$. The unperturbed operator $-\Delta_0$ satisfies the Hardy inequality

$$-\Delta_0 \geq \frac{1}{4}\frac{1}{\varrho^2}, \tag{6}$$

where $\varrho(x, y, z) := |z|$, which is optimal (in the same way as (1) is optimal for $-\Delta$). Notice that the distance r to the origin in (1) is naturally replaced by the distance ϱ to the plane $\{z = 0\}$ in (6). Our next result shows that there is always a specific improvement whenever $\beta \neq 0$.

Theorem 2. *Let A_β be given by (5). For every $\beta \in \mathbb{R}$, one has*

$$-\Delta_\beta - \frac{1}{4}\frac{1}{\varrho^2} \geq \frac{|\beta|}{\varrho^2} \tag{7}$$

in the sense of quadratic forms in $L^2(\mathbb{R}^3)$.

Contrary to Theorem 1, we do not know whether the inequality of Theorem 2 is optimal if $\beta \neq 0$.

The rest of the paper naturally splits into two independent sections. In Section 2 we quickly prove Theorem 1, while Theorem 2 is established in a longer Section 3.

2. The Aharonov-Bohm field

In this section we exclusively consider the vector potential A_α from (3) with any $d \geq 2$.

2.1. Preliminaries

For every $\alpha \in \mathbb{R}$, we introduce the magnetic Laplacian $(-i\nabla + A_\alpha)^2$ as the self-adjoint non-negative operator in $L^2(\mathbb{R}^d)$ associated with the quadratic form

$$Q_\alpha[\psi] := \|(\nabla + iA_\alpha)\psi\|^2, \qquad \mathsf{D}(Q_\alpha) := \overline{C_0^\infty(\mathbb{R}^d \setminus \{\rho = 0\})}^{|||\cdot|||},$$

where $\|\cdot\|$ denotes the usual norm of $L^2(\mathbb{R}^d)$ and

$$|||\psi||| := \sqrt{\|(\nabla + iA_\alpha)\psi\|^2 + \|\psi\|^2}. \tag{8}$$

By the diamagnetic inequality, we have that if $\psi \in \mathsf{D}(Q_\alpha)$ then $|\psi| \in W_0^{1,2}(\mathbb{R}^d \setminus \{\rho = 0\}) = W^{1,2}(\mathbb{R}^d)$, where the equality follows from the fact that the subset $\{\rho = 0\} \subset \mathbb{R}^d$ is a polar set (*cf.* [14, Sec. VIII.6]). Using the special structure (5) of the potential A_β, we have

$$Q_\alpha[\psi] = \int_{\mathbb{R}^d} \left(\left| \left(-i\partial_x - \frac{\alpha y}{x^2 + y^2} \right) \psi \right|^2 + \left| \left(-i\partial_y + \frac{\alpha x}{x^2 + y^2} \right) \psi \right|^2 + |\nabla_z \psi|^2 \right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$$

where $z = (z_1, \dots, z_{d-2})$ is a $(d-2)$-dimensional coordinate. In the sense of distributions,

$$(-i\nabla + A_\alpha)^2 = \left(-i\partial_x - \frac{\alpha y}{x^2 + y^2} \right)^2 + \left(-i\partial_y + \frac{\alpha x}{x^2 + y^2} \right)^2 - \Delta_z,$$

where $-\Delta_z$ is the usual (distributional) Laplacian in the z variables.

If $\alpha = 0$, then $\mathsf{D}(Q_0) = W^{1,2}(\mathbb{R}^d)$ and the operator associated with Q_0 is just the standard self-adjoint realisation of the Laplacian $-\Delta$ in $L^2(\mathbb{R}^d)$. More generally, if $\alpha \in \mathbb{Z}$ then $(-i\nabla + A_\alpha)^2$ is unitarily equivalent to the (magnetic-free) Laplacian $-\Delta$. This can be seen as follows. Passing to the polar coordinates in the (x, y)-plane, *i.e.* writing $(x, y) = (\rho \cos\varphi, \rho \sin\varphi)$ with $\rho \in (0, \infty)$ and $\varphi \in (0, 2\pi]$, we have the obvious unitary equivalences

$$\begin{aligned} (-i\nabla + A_\alpha)^2 &\cong -\rho^{-1}\, \partial_\rho\, \rho\, \partial_\rho + \frac{(-i\partial_\varphi + \alpha)^2}{\rho^2} - \Delta_z \\ &\cong \bigoplus_{m \in \mathbb{Z}} \left(-\rho^{-1}\, \partial_\rho\, \rho\, \partial_\rho + \frac{\nu_m^2}{\rho^2} \right) - \Delta_z. \end{aligned} \tag{9}$$

Here $\nu_m := m + \alpha$ are the eigenvalues of the one-dimensional operator $-i\partial_\varphi + \alpha$ in $L^2([0, 2\pi))$, subject to periodic boundary conditions. The corresponding set of eigenfunctions read $\{e^{im\varphi}\}_{m\in\mathbb{Z}}$. If α is an integer, then the direct sum is indistinguishable from the usual partial-wave decomposition of the Laplacian $-\Delta$.

Finally, let us notice that the spectrum of $(-i\nabla + A_\alpha)^2$ equals the semiaxis $[0,\infty)$ for every real α.

2.2. The improved hardy inequality

Let $\psi \in C_0^\infty(\mathbb{R}^d \setminus \{\rho = 0\})$, a core of Q_α (recall that a subspace of $\mathsf{D}(Q_\alpha)$ is a core of Q_α if, and only if, it is dense in the topology (8) generated by the form, *cf.* [15, Thm. VI.1.21]). Employing the polar coordinates in the (x, y)-plane as in (9) and writing $\phi(\rho,\varphi,z) =: \psi(\rho\cos\varphi, \rho\sin\varphi, z)$, we have

$$
\begin{aligned}
Q_\alpha[\psi] &= \int_{\mathbb{R}^{d-2}} \int_0^{2\pi} \int_0^\infty \left(|\partial_\rho \phi|^2 + \frac{|\partial_\varphi \phi + i\alpha\phi|^2}{\rho^2} + |\nabla_z \phi|^2 \right) \rho \, \mathrm{d}\rho \, \mathrm{d}\varphi \, \mathrm{d}z \\
&\geq \int_{\mathbb{R}^{d-2}} \int_0^{2\pi} \int_0^\infty \left(|\partial_\rho \phi|^2 + \frac{\operatorname{dist}(\alpha,\mathbb{Z})^2}{\rho^2} |\phi|^2 + |\nabla_z \phi|^2 \right) \rho \, \mathrm{d}\rho \, \mathrm{d}\varphi \, \mathrm{d}z,
\end{aligned}
\tag{10}
$$

where we omit to specify the arguments of ϕ and abuse a bit the notation for ρ. This inequality explains the quantity on the right-hand side of (4). To obtain the dimensional term on the left-hand side of (4), we write

$$
\phi(\rho,\varphi,z) = f(\rho,\varphi,z)\,(\rho^2 + |z|^2)^{-(d-2)/4},
$$

which is in fact the definition of the new test function f. Notice that $f(0,\varphi,z) = 0$ for all $\varphi \in [0, 2\pi)$ and $z \in \mathbb{R}^{d-2}$. A straightforward computation employing an integration by parts yields

$$
\begin{aligned}
\int_{\mathbb{R}^{d-2}} \int_0^\infty \left(|\partial_\rho \phi|^2 + |\nabla_z \phi|^2 \right) \rho \, \mathrm{d}\rho \, \mathrm{d}z &= \left(\frac{d-2}{2} \right)^2 \int_{\mathbb{R}^{d-2}} \int_0^\infty \frac{|\phi|^2}{\rho^2 + |z|^2} \, \rho \, \mathrm{d}\rho \, \mathrm{d}z \\
&\quad + \int_{\mathbb{R}^{d-2}} \int_0^\infty \left(|\partial_\rho f|^2 + |\nabla_z f|^2 \right) (\rho^2 + |z|^2)^{-(d-2)/2} \, \rho \, \mathrm{d}\rho \, \mathrm{d}z \\
&\geq \left(\frac{d-2}{2} \right)^2 \int_{\mathbb{R}^{d-2}} \int_0^\infty \frac{|\phi|^2}{\rho^2 + |z|^2} \, \rho \, \mathrm{d}\rho \, \mathrm{d}z.
\end{aligned}
\tag{11}
$$

Estimates (10) and (11) yield (4), after coming back to the Cartesian coordinates and noticing that $\rho^2 + |z|^2 = r^2$. This concludes the proof of Theorem 1.

The present proof also explains why the inequality (4) is optimal. Indeed, the inequality (10) is sharp in the sense that it is achieved by any function of the form $\phi(\rho,\varphi,z) = g(\rho,z)e^{im\varphi}$, where $m \in \mathbb{Z}$ is chosen in such a way that it minimises the distance $\operatorname{dist}(\alpha,\mathbb{Z})$ (so that $e^{im\varphi}$ is an eigenfunction corresponding to the lowest eigenvalue $\operatorname{dist}(\alpha,\mathbb{Z})^2$ of the operator $(-i\partial_\varphi + \alpha)^2$). The other inequality (11) is not achieved by a non-trivial ϕ, but it is also sharp in the following sense. For any function f depending only on r, we have

$$\int_{\mathbb{R}^{d-2}}\int_0^\infty \left(|\partial_\rho f|^2+|\nabla_z f|^2\right)(\rho^2+|z|^2)^{-(d-2)/2}\,\rho\,\mathrm{d}\rho\,\mathrm{d}z=\int_{\mathbb{S}_+^{d-2}}\int_0^\infty |\partial_r f|^2\,\rho\,\mathrm{d}r\,\mathrm{d}\sigma$$
$$\le |\mathbb{S}_+^{d-2}|\int_0^\infty |\partial_r f|^2\,r\,\mathrm{d}r,$$

where $\mathbb{S}_+^{d-2} := \mathbb{S}^{d-2}\cap\{\rho>0\}$ and $\mathbb{S}^{d-2}$ is the unit sphere in the (ρ,z)-half-space. It is well known that there exists a sequence of functions $\{f_n\}_{n=1}^\infty\subset C_0^\infty((0,\infty))$ such that

$$f_n(r)\underset{n\to\infty}{\longrightarrow}1\ \text{pointwise}\qquad\text{and}\qquad\int_0^\infty|\partial_r f_n(r)|^2\,r\,\mathrm{d}r\underset{n\to\infty}{\longrightarrow}0,$$

for the integral corresponds to the radial part of the two-dimensional Laplacian.

3. The confining field

The organisation of this section dealing with the vector potential (5) is as follows. In Section 3.1 we rigorously introduce the corresponding magnetic Laplacian $-\Delta_\beta$ as a special case in a large class of magnetic Schrödinger operators in $L^2(\mathbb{R}^d)$ and state their basic spectral properties. In Section 3.2, we establish an improved Hardy inequality for the general operators (*cf.* Theorem 3). The complete proof of Theorem 2 follows as a corollary of this more universal result. Finally, in Section 3.3, we present a more quantitative version of the proof of Theorem 3.

3.1. Preliminaries

Let $d\ge 3$. Given any function $b:\mathbb{R}^{d-2}\to\mathbb{R}$ such that b is locally bounded in $\mathbb{R}^{d-2}\setminus\{0\}$, let us consider the vector potential

$$A_b(x,y,z) := b(z)(y,0,0,\dots,0), \tag{12}$$

where $x,y\in\mathbb{R}$ and $z=(z_1,\dots,z_{d-2})\in\mathbb{R}^{d-2}$ as in (3). We introduce the corresponding magnetic Laplacian $-\Delta_b$ as the self-adjoint non-negative operator in $L^2(\mathbb{R}^d)$ associated with the quadratic form

$$Q_b[\psi] := \|(\nabla+iA_b)\psi\|^2,\qquad \mathsf{D}(Q_b) := \overline{C_0^\infty(\mathbb{R}^d\setminus\{z=0\})}^{|||\cdot|||},$$

where $\|\cdot\|$ denotes the usual norm of $L^2(\mathbb{R}^d)$ and

$$|||\psi||| := \sqrt{\|(\nabla+iA_b)\psi\|^2+\|\psi\|^2}.$$

By the diamagnetic inequality, we have that if $\psi\in\mathsf{D}(Q_\alpha)$ then $|\psi|\in W_0^{1,2}(\mathbb{R}^d\setminus\{z=0\})$. Using the special structure (12) of the potential A_b, we have

$$Q_b[\psi]=\int_{\mathbb{R}^d}\left(\left|\left(-i\partial_x+b(z)\,y\right)\psi\right|^2+|\partial_y\psi|^2+|\nabla_z\psi|^2\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

and, in the sense of distributions,

$$-\Delta_b=\left(-i\partial_x+b(z)\,y\right)^2-\partial_y^2-\Delta_z=-\Delta-2i\,b(z)\,y\,\partial_x+b(z)^2\,y^2, \tag{13}$$

where $-\Delta$ is the usual (distributional) Laplacian in the (x, y, z) variables. Obviously, $-\Delta_b$ reduces to $-\Delta_\beta$ for the special choice

$$d = 3 \qquad \text{and} \qquad b(z) := \frac{\beta}{z^2}. \tag{14}$$

Notice that $-\Delta_0$ (*i.e.* $b=0$) is just the Dirichlet Laplacian in $\mathbb{R}^d \setminus \{z=0\}$ for which the form domain equality $\mathsf{D}(Q_0) = W_0^{1,2}(\mathbb{R}^d \setminus \{z=0\})$ holds. One has $\mathsf{D}(Q_0) = W^{1,2}(\mathbb{R}^d)$ if $d \geq 4$, because then $\{z=0\}$ is a polar set in $\mathbb{R}^d$. Hence, unless $d=3$, $-\Delta_0$ is just the standard self-adjoint realisation of the Laplacian in $L^2(\mathbb{R}^d)$. In any dimension, the spectrum of $-\Delta_0$ is well known, $\sigma(-\Delta_0) = [0, \infty)$. Moreover, $-\Delta_0$ is subcritical in the sense that the Hardy inequality

$$-\Delta_0 \geq \left(\frac{d-2}{2}\right)^2 \frac{1}{r^2}$$

holds as a consequence of the classical Hardy (1) and the inclusion $\mathsf{D}(Q_0) \subset W^{1,2}(\mathbb{R}^d)$. If $d \neq 4$, we also have another Hardy inequality

$$-\Delta_0 \geq \left(\frac{d-4}{2}\right)^2 \frac{1}{\varrho^2}, \tag{15}$$

where we use the same notation $\varrho(x,y,z) = |z|^2$ as below (6). Equivalently, $\forall \psi \in W_0^{1,2}(\mathbb{R}^d \setminus \{z=0\})$,

$$\int_{\mathbb{R}^d} |\nabla \psi(x,y,z)|^2 \, dx \, dy \, dz \geq \left(\frac{d-4}{2}\right)^2 \int_{\mathbb{R}^d} \frac{|\psi(x,y,z)|^2}{|z|^2} \, dx \, dy \, dz.$$

For $d \geq 4$ this inequality follows from (1) when applied to the Laplacian in z-coordinates only (it becomes trivial if $d=4$), while the three-dimensional situation employs the classical one-dimensional Hardy inequality

$$\forall u \in W_0^{1,2}(\mathbb{R} \setminus \{0\}), \qquad \int_{\mathbb{R}} |u'(s)|^2 \, ds \geq \frac{1}{4} \int_{\mathbb{R}} \frac{|u(s)|^2}{s^2} \, ds. \tag{16}$$

Obviously, (15) reduces to (6) if $d=3$.

It is not difficult to establish a sufficient condition about b to guarantee that the spectrum of $-\Delta_b$ coincides with the spectrum of the unperturbed operator $-\Delta_0$ (as well as of the usual Laplacian without the extra Dirichlet condition).

Proposition 1. *Let $b \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{d-2} \setminus \{0\})$ with $d \geq 3$. If $b(z)|z| \to 0$ as $|z| \to \infty$, then*

$$\sigma(-\Delta_b) = [0, \infty).$$

Proof. The inclusion $\sigma(-\Delta_b) \subset [0, \infty)$ follows trivially because of the non-negativity of $-\Delta_b$. The opposite inclusion $\sigma(-\Delta_b) \supset [0, \infty)$ can be established by the Weyl criterion, by choosing the singular sequence localised at the infinity of the conical domain $\{|z| > y > 1\}$, where the terms in (13) containing b can be made arbitrarily small. We omit the details and refer the reader to a fairly general case treated in [16]. □

Obviously, the sufficient condition is satisfied for the special case (14).

3.2. The improved hardy inequality

By the diamagnetic inequality, (15) remains valid for $-\Delta_b$ instead of $-\Delta_0$. Now we show that there is always an improvement whenever $b \neq 0$.

Theorem 3. *Let $b \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{d-2} \setminus \{0\})$ with $d \geq 3$. Then $\forall \psi \in \mathsf{D}(Q_b)$,*

$$Q_b[\psi] - \left(\frac{d-4}{2}\right)^2 \int_{\mathbb{R}^d} \frac{|\psi(x,y,z)|^2}{|z|^2} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \geq \int_{\mathbb{R}^d} |b(z)| \, |\psi(x,y,z)|^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \tag{17}$$

Proof. Since the result will be re-proved in the following subsection, here we provide just a sketch proof.

Writing

$$-\Delta_b = \underbrace{\left(-i\partial_x + b(z)\ y\right)^2 - \partial_y^2}_{-\Delta'_b} - \Delta_z, \tag{18}$$

we notice that $-\Delta'_b$ is the magnetic Laplacian in $L^2(\mathbb{R}^2)$ corresponding to the two-dimensional vector potential

$$A'_b(x,y) := b(z)\ (y, 0),$$

which depends parametrically on z. The corresponding two-dimensional magnetic field is constant

$$B'_b(x,y) := \operatorname{rot} A'_b(x,y) = -b(z). \tag{19}$$

The operator $-\Delta'_b$ is the celebrated Landau Hamiltonian.

The spectral problem for $-\Delta'_b$ is explicitly solvable. The easiest way how to see it is to perform a partial Fourier transform with respect to the x-variable, which yields a unitary equivalence

$$-\Delta'_b \cong \left(\xi + b(z)\ y\right)^2 - \partial_y^2, \tag{20}$$

where $\xi \in \mathbb{R}$ is the dual variable to x. Noticing that the right-hand side of (20) is the Hamiltonian of a shifted harmonic oscillator (the shift can be handled as yet another unitary transform), we get the familiar formula (the natural numbers $\mathbb{N}$ contain zero in our convention)

$$\sigma(-\Delta'_b) = 2\ |b(z)| \left(\mathbb{N} + \frac{1}{2}\right) \qquad \text{if} \qquad b(z) \neq 0.$$

Each point in the spectrum is an eigenvalue of infinite multiplicity (Landau levels). (If $b(z) = 0$, then $\sigma(-\Delta'_b) = [0, \infty)$.) In particular,

$$\inf \sigma(\Delta'_b) = |b(z)|$$

(which is trivially valid also for $b(z) = 0$).

Using the last result in (18), we get

$$-\Delta_b \geq -\Delta_z + |b(z)| \geq \left(\frac{d-4}{2}\right)^2 \frac{1}{|z|^2} + |b(z)|, \tag{21}$$

which is the desired result (17). The second estimate in (21) follows from the classical Hardy inequality (1) if $d \geq 4$ (the result is trivial if $d=4$) or (16) if $d=1$, by noticing in the latter case that the form core consists of functions that vanish on the plane $\{z=0\}$. □

Theorem 2 follows as a corollary of Theorem 3 in the special case (14).

Remark 1. In some situations, Theorem 3 can be alternatively proved also by a standard commutator trick (see, *e.g.*, [5, Sec. 2.4]). Let us denote by $\Pi_j := -i\partial_j + (A_b)_j$ with $j \in \{1,2,3,\ldots,d\} \cong \{x,y,z_1,\ldots,z_{d-2}\}$ the j^{th} component of the magnetic gradient. Let us assume for a moment that b is boundedly differentiable outside the plane $\{z=0\}$. Then one has the identity

$$\forall C_0^\infty(\mathbb{R}^d \setminus \{z=0\}), \qquad \|\Pi_j\psi\|^2 + \|\Pi_k\psi\|^2 = \|(\Pi_j \pm i\Pi_k)\psi\|^2 \pm \langle \psi, (B_b)_{jk}\psi\rangle, \tag{22}$$

for *any* pair $j,k \in \{1,2,3,\ldots,d\}$, where $(B_b)_{jk} := \partial_j(A_b)_k - \partial_k(A_b)_j$ are the coefficients of the magnetic tensor $B_b := \mathrm{d}A_b$ and $\langle\cdot,\cdot\rangle$ denotes the inner product of $L^2(\mathbb{R}^d)$. In our case (5), we have

$$B_b = \begin{pmatrix} 0 & -b(z) & -\partial_1 b(z)\ y & \ldots & -\partial_{d-2} b(z)\ y \\ b(z) & 0 & 0 & \ldots & 0 \\ \partial_1 b(z)\ y & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \partial_{d-2} b(z)\ y & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Using the formula (22) with the special choice $(j,k) := (1,2)$, which does not require any differentiability of b, one therefore obtains, for every $\psi \in C_0^\infty(\mathbb{R}^d \setminus \{z=0\})$,

$$\begin{aligned} Q_b[\psi] = \|\Pi_1\psi\|^2 + \|\Pi_2\psi\|^2 + \sum_{l=2}^{d-2} \|\Pi_l\psi\|^2 &\geq \int_{\mathbb{R}^d} \mp b(z)\, |\psi(x,y,z)|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \|\nabla_z\psi\|^2 \\ &\geq \int_{\mathbb{R}^d} \left[\mp b(z) + \left(\frac{d-4}{2}\right)^2 \frac{1}{|z|^2}\right] |\psi(x,y,z)|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \end{aligned}$$

where the last inequality is due to the classical Hardy inequality (1) or (16). Since the obtained result holds with either the plus or minus sign, we arrive at (17) for every $\psi \in C_0^\infty(\mathbb{R}^d \setminus \{z=0\})$, provided that b does not change sign (this is certainly true for the special choice (14)). By density, the result extends to all $\psi \in \mathsf{D}(Q_\beta)$.

3.3. Quantification of the improved hardy inequality

Our final goal is to present a more quantitative proof of Theorem 3. To do so, we have to employ the terms we neglected in the crude estimates (21).

Let $\psi \in C_0^\infty(\mathbb{R}^d \setminus \{z=0\})$, a core of Q_β. The function is implicitly assumed to depend on the space variables $(x, y, z) \in \mathbb{R}^d$ and for brevity we omit to specify the arguments in the integrals below.

First of all, let us perform the partial Fourier transform with respect to the x-variable as in (20):

$$Q_b[\psi] = \int_{\mathbb{R}^d} \left(\left| (\xi + b(z)\,y)\hat{\psi} \right|^2 + |\partial_y \hat{\psi}|^2 + |\nabla_z \hat{\psi}|^2 \right) \mathrm{d}\xi\, \mathrm{d}y\, \mathrm{d}z. \tag{23}$$

Notice that the transformed function $\hat{\psi} = \hat{\psi}(\xi, y, z)$ still vanishes in a neighbourhood of $\{z = 0\}$.

In the second step, we make the change of test function

$$\hat{\psi}(\xi, y, z) = |z|^{-(d-4)/2}\ \phi(\xi, y, z) \tag{24}$$

to single out the second term on the left-hand side of (17). Putting (24) into (23) and integrating by parts with respect to z, we arrive at

$$Q_b[\psi] - \left(\frac{d-4}{2}\right)^2 \left\| \frac{\psi}{|z|} \right\|^2 = \int_{\mathbb{R}^d} \left(\left| (\xi + b(z)\,y)\phi \right|^2 + |\partial_y \phi|^2 + |\nabla_z \phi|^2 \right) |z|^{-(d-4)}\, \mathrm{d}\xi\, \mathrm{d}y\, \mathrm{d}z. \tag{25}$$

In the third step, we introduce the function

$$\eta(\xi, y, z) := \begin{cases} \exp\left(-\frac{1}{2}\ |b(z)|\ (y - y_0)^2 \right) \quad \text{with} \quad y_0 := -\dfrac{\xi}{b(z)} & \text{if} \quad b(z) \neq 0, \\ \exp(-\xi y) & \text{if} \quad b(z) = 0. \end{cases}$$

If $b(z) \neq 0$ (respectively, $b(z) = 0$), η is an eigenfunction (respectively, generalised eigenfunction) of the operator on the right-hand side of (20) corresponding to the lowest eigenvalue $|b(z)|$ (respectively, to 0, the lowest point in the continuous spectrum); in this two-dimensional context, the variable z is understood as a parameter and ξ gives rise to the degeneracies. In the third step, we make the change of test function

$$\phi(\xi, y, z) = \eta(\xi, y, z)\ \varphi(\xi, y, z) \tag{26}$$

to single out the second term on the right-hand side of (17). Putting (26) into (25) and integrating by parts with respect to y, we arrive at

$$\begin{aligned} &Q_b[\psi] - \left(\frac{d-4}{2}\right)^2 \left\| \frac{\psi}{|z|} \right\|^2 - \left\| |b(z)|^{1/2} \psi \right\|^2 \\ &\quad = \int_{\mathbb{R}^d} \left(|\partial_y \varphi|^2\, \eta^2(\xi, y, z) + |\nabla_z \phi|^2 \right) |z|^{-(d-4)}\, \mathrm{d}\xi\, \mathrm{d}y\, \mathrm{d}z. \end{aligned} \tag{27}$$

Since the right-hand side of (27) is non-negative, we have just re-proved (17).

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