REGULARITY OF RENORMALIZED SOLUTIONS TO NONLINEAR ELLIPTIC EQUATIONS AWAY FROM THE SUPPORT OF MEASURE DATA

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Abstract. We prove boundedness and continuity for solutions to the Dirichlet problem for the equation

$$-\mathrm{div}\,(a(x,\nabla u)) = h(x,u) + \mu\,, \quad \text{in } \Omega \subset \mathbb{R}^N\,,$$

where the left–hand side is a Leray–Lions operator from $W_0^{1,p}$ into $W^{-1,p'}(\Omega)$, with $1 < p < N$, $h(x,s)$ is a Carathéodory function which grows like $|s|^{p-1}$ and μ is a finite Radon measure. We prove that renormalized solutions, though not globally bounded, are Hölder-continuous far from the support of μ.

Keywords: bounded solutions, p-Laplacian, renormalized solutions, measure data

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1. Introduction

In this note, we prove that solutions to the Dirichlet problem for nonlinear elliptic equations having measure datum are locally Hölder–continuous. For the sake of concreteness, consider the following simple linear problem:

$$\begin{cases} -\lambda\Delta u = f(x)(1+u) + \mu\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$ is an open and bounded set, $f \in L^m(\Omega)$, with $m > \frac{N}{2}$, is small enough (in a sense to be determined) and μ is a Radon measure having finite total variation and whose support does not include the whole domain. Our motivation to study

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these kind of problems comes from searching non–regular solutions to equations with a gradient term having "natural" growth which, by means of the Cole–Hopf change of unknown, are reduced to (1.1) (see Abdellaoui, Dall'Aglio and Peral [2] and Abdel-Hamid and Bidaut-Veron [1]). In fact, we apply the results of this paper to obtain non–regular solutions to an equation involving the 1–Laplacian and a total variation term in Abdellaoui, Dall'Aglio and Segura de León [3].

Two features of problem (1.1) deserve a comment. The first point to note is that we have to restrict the size of f in order to get existence of a solution, even if $\mu = 0$ and $f(x) \equiv f_0$ is a constant. Indeed, taking u as test function and applying Poincaré's inequality, we may perform the following calculation:

$$\lambda \int_\Omega |\nabla u|^2 = f_0 \int_\Omega |u|^2 + f_0 \int_\Omega u \leq \frac{f_0}{\lambda_1} \int_\Omega |\nabla u|^2 + f_0 \int_\Omega u \,,$$

where λ_1 is the first eigenvalue of the Laplacian. So, an estimate in the energy space is only possible if $f_0 < \lambda\lambda_1$. Observe that this bound on f_0 depends on the coercivity of the principal part.

In order to deal with a general Radon measure μ, we consider the notion of renormalized solution introduced by Dal Maso, Murat, Orsina and Prignet in [8]. We point out the related concept of entropy solution introduced by Benilan et al. in [4] for proving an existence and uniqueness result for L^1–data and extended by Boccardo, Gallouët and Orsina in [5] to measures which do not charge the sets of zero capacity. The existence of a renormalized solution to problem (1.1) under a smallness assumption on f has been proved in Grenon [10] and Abdel-Hamid and Bidaut-Veron [1].

A classical result by Stampacchia (see [12]) shows that if μ is actually a function belonging to $L^m(\Omega)$, for some $m > \frac{N}{2}$, then the solution is bounded and continuous but, if not, is unbounded in general. Actually, the simple case where $h(x, s) \equiv 0$ shows that we cannot hope to prove global boundedness of solutions, but nevertheless we prove boundedness and continuity of the solution in a zone far away from the support of μ. Heuristically, the idea is that local boundedness of the solution only depends on the local summability of the datum μ. In the special case where $p = 2$ and $f(x) = 0$, a similar result was proved by Boccardo and Leonori in [6].

Our setting is more general than problem (1.1) and includes nonlinear operators of p–Laplacian type, so that it is similar to the one studied by Grenon in [10]. The only change is our restriction on the growth of function $h(x, s)$, we always assume the critical exponent $p - 1$ because of our interest in equations which appear as a consequence of the Cole–Hopf transformation. In her paper, Grenon proves existence of a renormalized solution under a hypothesis of smallness of f. Our main result is that every renormalized solution is Hölder–continuous outside of the support of the measure μ. We point out that we only analyze the case $1 < p < N$ since for $p > N$ every renormalized solution is actually a weak solution and so it is globally bounded and Hölder–continuous.

This paper is organized as follows. The next section is devoted to introducing our notation and precise hypotheses. Section 3 deals with the definition of renormalized solutions, while Section 4 contains the results on regularity.

2. Preliminaries and assumptions

We begin by introducing our notation. From now on, Ω is an open bounded set in $\mathbb{R}^N$, with $N \geq 2$, and $|E|$ denotes the Lebesgue measure of $E \subset \Omega$. The symbol $L^q(\Omega)$ stands for the usual Lebesgue space and q' denotes the conjugate of q: $q' = \dfrac{q}{q-1}$.

We will denote by $W_0^{1,q}(\Omega)$ the usual Sobolev space, of measurable functions having weak derivative in $L^q(\Omega)$ and zero trace on $\partial\Omega$. Finally, if $1 \leq q < N$, we will denote by $q^* = Nq/(N-q)$ its Sobolev conjugate exponent.

Let us state our hypotheses more precisely. We will consider the following problem

$$\begin{cases} -\mathrm{div}\,(a(x, \nabla u)) = h(x, u) + \mu\,, & \text{in } \Omega\,; \\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{2.1}$$

The function

$$a(x, \xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$$

satisfies the Carathéodory conditions and there exist some constants $\lambda > 0$ and $\nu > 0$ such that

$$a(x, \xi) \cdot \xi \geq \lambda |\xi|^p, \tag{2.2}$$

$$|a(x, \xi)| \leq \nu |\xi|^{p-1}, \tag{2.3}$$

$$(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) > 0\,; \tag{2.4}$$

for all $\xi, \eta \in \mathbb{R}^N$, with $\xi \neq \eta$ and for almost all $x \in \Omega$.

The function

$$h(x, s) : \Omega \times \mathbb{R} \to \mathbb{R}$$

also satisfies the Carathéodory conditions and there exists a nonnegative function $f \in L^m(\Omega)$, for some $m > N/p$, such that

$$|h(x, s)| \leq f(x)(1 + |s|^{p-1}); \tag{2.5}$$

for all $s \in \mathbb{R}$ and for almost all $x \in \Omega$. As far as the datum μ is concerned, we assume that

(2.6) μ is a Radon measure with bounded total variation.

Throughout this paper, we will use two auxiliary real functions: given $k > 0$, we define

$$T_k(s) = \begin{cases} s\,, & \text{if } |s| \le k\,; \\ k\frac{s}{|s|}\,, & \text{if } |s| > k\,; \end{cases} \qquad G_k(s) = s - T_k(s)\,.$$

3. Renormalized solutions

In this Section, we define renormalized solution to problem (2.1); we refer to [8] for a detailed study of renormalized solutions and several equivalent definitions.

Definition 3.1. Given the measure μ, we decompose it as $\mu = \mu_0 + \mu_s^+ - \mu_s^-$, where μ_0 is absolutely continuous with respect to the p–capacity, while μ_s^+ and μ_s^- are two nonnegative measures which are concentrated on two disjoint subsets of zero p–capacity.

A measurable function $u : \Omega \to \mathbb{R}$ is a renormalized solution to problem (2.1) if the following conditions hold:

(1) The function u is finite almost everywhere and $T_k(u) \in W_0^{1,p}(\Omega)$ for all $k > 0$. (As a consequence, a generalized gradient ∇u can be defined, see [4, Lemma 2.1].)
(2) The gradient satisfies $|\nabla u|^{p-1} \in L^q(\Omega)$ for every $q < \frac{N}{N-1}$.
(3) $|u|^{p-1} \in L^s(\Omega)$ for every $s < \frac{N}{N-p}$. In particular, by assumption (2.5), this implies that the function $h(x,u)$ belongs to $L^1(\Omega)$.
(4) For every $S \in W^{1,\infty}(\mathbb{R})$ such that S' has compact support in $\mathbb{R}$ (consequently S is constant for $|s|$ large and so the limits $S(+\infty) = \lim_{s\to+\infty} S(s)$ and $S(-\infty) = \lim_{s\to-\infty} S(s)$ exist), we have

$$\int_\Omega S'(u)\varphi a(x,\nabla u)\cdot\nabla u + \int_\Omega S(u) a(x,\nabla u)\cdot\nabla\varphi$$
$$= \int_\Omega h(x,u)S(u)\varphi + \int_\Omega S(u)\varphi\, d\mu_0 + S(+\infty)\int_\Omega \varphi\, d\mu_s^+ - S(-\infty)\int_\Omega \varphi\, d\mu_s^-$$

for all $\varphi \in W^{1,r}(\Omega)\cap L^\infty(\Omega)$, with $r > N$, such that $S(u)\varphi \in W_0^{1,p}(\Omega)$.

In [10] and [1] it was proved that, under a smallness assumption on f, there exists a renormalized solution for problem (2.1). In particular, the next theorem can be found in [1]:

Theorem 3.2. *Assume that hypotheses* (2.2)–(2.6) *are true, and that*

$$\frac{1}{\lambda} < \lambda_1(f) := \inf\left\{ \frac{\displaystyle\int_\Omega |\nabla w|^p}{\displaystyle\int_\Omega f\,|w|^p} \;:\; w \in W_0^{1,p}(\Omega)\,,\ \int_\Omega f\,|w|^p \ne 0 \right\}$$

Then there exists a renormalized solution u to problem (2.1)*.*

4. Regularity away from the support of μ.

In the next results we aim to prove that a renormalized solution of problem (2.1) is "regular enough" (in particular, it is bounded) far from the set where the measure μ is concentrated. We start by proving that it belongs to all L^q spaces.

We recall that a measure μ is said to be concentrated on a set A if $\mu(E) = \mu(E \cap A)$ for every measurable set E.

Proposition 4.1. *Assume that hypotheses* (2.2)–(2.6) *are true, and that the measure μ is concentrated on a set $A \subset \Omega$.*

Then, for every open set $U \subset \Omega$ having positive distance from A, and for every $q < \infty$, $|u|^q \in W^{1,p}(U)$. It follows that $u \in L^q(U)$, for every $q < \infty$.

In order to prove Proposition 4.1, we need an iteration lemma, inspired in the Brezis–Kato approach (see [7]), which allows to improve the summability of a renormalized solution of problem (2.1):

Lemma 4.2. *Under the same hypotheses of Proposition 4.1, assume that U and V are two open bounded sets in $\mathbb{R}^N$ such that $\overline{U} \subset V \subset \mathbb{R}^N \setminus A$. Define*

$$M = \frac{N}{(N-p)m'} > 1\,,$$

and assume that there exist $\theta > \dfrac{p-1}{Mp}$ and $k \in \mathbb{N}$ such that

$$(1+|u|)^{\theta M^k} \in W^{1,p}(V \cap \Omega)\,.$$

Then

$$(1+|u|)^{\theta M^{k+1}} \in W^{1,p}(U \cap \Omega)\,.$$

Proof. First of all, by standard inclusions between Lebesgue spaces, we can always assume that f satisfies the assumption

$$f \in L^m(\Omega)\,, \quad \text{with} \quad \frac{N}{p} < m \leq \frac{N}{p-1}\,.$$

Let φ be a function in $C_0^\infty(V)$ such that $0 \leq \varphi \leq 1$ in V, $\varphi \equiv 1$ in $\overline{U}$. For $\alpha > 0$ (to be chosen later) let us take $w = \varphi^p \big[(1+|T_L u|)^{\alpha(p-1)} - 1\big]$ sign u as test function in the definition of renormalized solution. We obtain

$$\alpha(p-1)\lambda\int_{V\cap\Omega}(1+|T_L(u)|)^{\alpha(p-1)-1}|\nabla T_L(u)|^p\varphi^p \leq p\nu\int_{V\cap\Omega}\varphi^{p-1}|\nabla u|^{p-1}|\nabla\varphi|\,(1+|T_L(u)|)^{\alpha(p-1)} + \int_{V\cap\Omega} f\,\,(1+|u|)^{p-1}\,(1+|T_L(u)|)^{\alpha(p-1)}\,\varphi^p\,. \tag{4.1}$$

Note that the measure μ disappears due to the presence of φ. By the monotone convergence theorem, it is easy to pass to the limit for $L\to\infty$ in all the integrals in (4.1), thus obtaining:

$$\alpha(p-1)\lambda\int_{V\cap\Omega}(1+|u|)^{\alpha(p-1)-1}|\nabla u|^p\varphi^p \leq \leq p\nu\int_{V\cap\Omega}\varphi^{p-1}|\nabla u|^{p-1}|\nabla\varphi|\,(1+|u|)^{\alpha(p-1)} + \int_{V\cap\Omega} f\,\,(1+|u|)^{(p-1)(\alpha+1)}\,\,\varphi^p\,. \tag{4.2}$$

We only have to check that the last two integrals in (4.2) are finite. Let us start with the last one, which is finite if $\int_{V\cap\Omega} f\,\,(1+|u|)^{(p-1)(\alpha+1)}$ is finite. By the assumptions, we know that $f\in L^m(\Omega)$ and, using Sobolev's inequality, that $1+|u|\in L^{p^*\theta M^k}(V\cap\Omega)$. Therefore, the integral is finite if we choose α such that

$$\frac{1}{m}+\frac{(\alpha+1)(p-1)}{p^*\theta M^k}=1\,,$$

that is,

$$\alpha=\frac{p}{p-1}\theta M^{k+1}-1\,. \tag{4.3}$$

We point out that $\alpha>0$ due to our assumption $\theta>\dfrac{p-1}{Mp}>\dfrac{p-1}{M^{k+1}p}$. We only have to check that, with this choice of α, the second integral in (4.2) is finite, that is, we have to make sure that $\int_{V\cap\Omega}|\nabla u|^{p-1}\,(1+|u|)^{\alpha(p-1)}<\infty$. Indeed, using Young's inequality,

$$\begin{aligned}\int_{V\cap\Omega}|\nabla u|^{p-1}\,(1+|u|)^{\alpha(p-1)} &= \int_{V\cap\Omega}|\nabla u|^{p-1}\,(1+|u|)^{p\theta M^{k+1}-p+1}\\ &= \int_{V\cap\Omega}|\nabla u|^{p-1}\,(1+|u|)^{(\theta M^k-1)(p-1)}(1+|u|)^{\theta M^k\,(pM-p+1)}\\ &\leq \int_{V\cap\Omega}|\nabla u|^p\,(1+|u|)^{(\theta M^k-1)p}+\int_{V\cap\Omega}(1+|u|)^{\theta pM^k\,(pM-p+1)}\\ &= \frac{1}{(\theta M^k)^p}\int_{V\cap\Omega}|\nabla(1+|u|)^{\theta M^k}|^p+\int_{V\cap\Omega}(1+|u|)^{\theta pM^k\,(pM-p+1)}\,.\end{aligned}$$

Using the assumption on u, Sobolev's inequality and the fact that

$$\theta p M^k (pM - p + 1) \le \theta p^* M^k$$

due to the assumption $m \le \frac{N}{p-1}$, we obtain that the last two integrals are finite. The Lemma is thus proved. $\square$

PROOF OF PROPOSITION 4.1. In order to apply Lemma 4.2, we need a startpoint, that is, we need to verify that the assumption of Lemma 4.2 is valid for $k = 0$. In other words, we need to show that there exists a number $\theta > (p-1)/Mp$ such that $(1+|u|)^\theta$ is in $W^{1,p}$ far from the support of μ. To this aim, assume that V is an open set such that $\overline{U} \subset V \subset \mathbb{R}^N \setminus A$. Let us again consider a cut-off function φ which vanishes outside V and is 1 on U.

Multiplying the equation by $\varphi^p \left[(1 + |T_L(u)|)^{\alpha(p-1)} - 1\right]$ sign u, and letting L go to infinity, we obtain

$$\begin{aligned}(4.4)\quad & \alpha(p-1)\lambda \int_{V\cap\Omega} (1+|u|)^{\alpha(p-1)-1}|\nabla u|^p \varphi^p \le \\ & \le p\nu \int_{V\cap\Omega} \varphi^{p-1}|\nabla u|^{p-1}|\nabla\varphi|\,(1+|u|)^{\alpha(p-1)} + \int_{V\cap\Omega} f\ (1+|u|)^{(\alpha+1)(p-1)}\ \varphi^p\,.\end{aligned}$$

We choose α such that

$$0 < \alpha < \frac{N}{m'(N-p)} - 1\,. \qquad (4.5)$$

With this choice, the last integral in (4.4) is finite since $u^{p-1} \in L^s(\Omega)$ for every $s < \frac{N}{N-p}$. As far as the second integral is concerned, by the definition of renormalized solution we know that $|\nabla u|^{p-1} \in L^r(\Omega)$ for every $r < \frac{N}{N-1}$. Therefore,

$$\int_{V\cap\Omega} \varphi^{p-1}|\nabla u|^{p-1}|\nabla\varphi|\,(1+|u|)^{\alpha(p-1)} < \infty$$

as soon as

$$\frac{N-1}{N} + \frac{\alpha(N-p)}{N} < 1\,,$$

which corresponds to

$$\alpha < \frac{1}{N-p}\,.$$

it is easy to see that, under the condition $m \le \frac{N}{p-1}$, which can always be assumed without loss of generality, one has

$$\frac{N}{m'(N-p)} - 1 \le \frac{1}{N-p}\,,$$

therefore all the integrals in (4.4) are finite for every α as in (4.5). From there, it easily follows that $(1+|u|)^\theta \in W^{1,p}(U)$ for every θ such that

$$\theta < \frac{N(p-1)}{m'p(N-p)} = \frac{M(p-1)}{p}\,.$$

In order to apply Lemma 4.2 it is enough to choose θ satisfying

$$\frac{p-1}{Mp} < \theta < \frac{N(p-1)}{m'p(N-p)} = \frac{M(p-1)}{p}\,,$$

which is posssible since $M > 1$. So, we have completed the proof of Proposition 4.1. ■

In order to prove the following two results, we need to use some Caccioppoli estimate techniques. In order to obtain the estimates up to the boundary of Ω, it is convenient to extend the renormalized solution u to be zero outside of Ω. We therefore define

$$\tilde{u}(x) = \begin{cases} u(x)\,, & \text{if } x \in \Omega\,; \\ 0\,, & \text{if } x \in \mathbb{R}^N \setminus \Omega\,. \end{cases}$$

Theorem 4.3. *Assume that hypotheses* (2.2)–(2.6) *are true, and that μ is concentrated on a set $A \subset \Omega$. Let u be a renormalized solution of problem* (2.1).

Then, for every open set $U \subset \Omega$ having positive distance from A,

$$u \in L^\infty(U)\,.$$

Proof. Our aim is to see that for every x_0 far from the support of μ there exists a ball centered at x_0 in which $\tilde{u}$ is bounded. To this end, we get a Caccioppoli type inequality that allows us to deduce a L^∞_{loc}–estimate far from the support of μ.

From now on, $B_\rho \subset B_R$ stand for concentric open balls, we will always assume that B_R has positive distance from A. In addition, if u is the renormalized solution and $k \geq 0$, we will write

$$A(k,\rho) = \{x \in B_\rho : |\tilde{u}(x)| \geq k\}.$$

Fix $R > 0$ such that $|B_R|$ is small enough (it will be determined later). Let $\varphi \in C_0^\infty(B_R)$ satisfy $0 \leq \varphi \leq 1$. Given $k > 0$, since we know that $u \in W^{1,p}(B_R \cap \Omega)$, we may choose $T_L\big(G_k(u)\big)\varphi^p$ as test function in (2.1) and then let L go to ∞. We obtain

$$\begin{aligned} \text{(4.6)} \quad \lambda \int_\Omega \varphi^p |\nabla G_k(u)|^p & \\ \leq p\nu \int_\Omega \varphi^{p-1} |G_k(u)|\, |\nabla G_k(u)|^{p-1} |\nabla \varphi| &+ \int_\Omega f(1+|u|)^{p-1} |G_k(u)| \varphi^p\,. \end{aligned}$$

We analyze the right hand side of (4.6). By applying Young's inequality, we get

$$(4.7)\quad p\nu\int_\Omega \varphi^{p-1}|G_k(u)|\,|\nabla G_k(u)|^{p-1}|\nabla\varphi| \le \epsilon\int_\Omega \varphi^p|\nabla G_k(u)|^p + C(\epsilon)\int_\Omega |G_k(u)|^p|\nabla\varphi|^p\,.$$

Observe that, choosing $\epsilon=\lambda/2$, the first term can be absorbed by the left hand side of (4.6).

The last term in the right hand side of (4.6) is estimated as follows.

$$\begin{aligned}(4.8)\quad \int_\Omega f(1+|u|)^{p-1}|G_k(u)|\varphi^p &\le \int_\Omega f\left[2^{p-1}\big((1+k)^{p-1}+|G_k(u)|^{p-1}\big)\right]|G_k(u)|\varphi^p\\ &= 2^{p-1}\int_\Omega f(1+k)^{p-1}|G_k(u)|\varphi^p + 2^{p-1}\int_\Omega f|G_k(u)|^p\varphi^p\\ &\le 2^{p-1}\frac{p-1}{p}\int_{A(k,R)} f(1+k)^p\varphi^p + 2^{p-1}\Big(1+\frac{1}{p}\Big)\int_\Omega f|G_k(u)|^p\varphi^p\,,\end{aligned}$$

where in the last step we have applied Young's inequality. Hence, on account of (4.7) and (4.8), it follows from (4.6) that

$$(4.9)\quad \int_\Omega \varphi^p|\nabla G_k(u)|^p \le C\int_\Omega |G_k(u)|^p|\nabla\varphi|^p + C\int_{A(k,R)} f(1+k)^p\varphi^p + C\int_\Omega f|G_k(u)|^p\varphi^p\,,$$

for some constant $C>0$. The next step is to estimate the last term on the right hand side. We apply Hölder's and Sobolev's inequalities to deduce

$$\begin{aligned}\int_\Omega f|G_k(u)|^p\varphi^p &\le \|f\|_m|A(k,R)|^{\frac{1}{m'}-\frac{p}{p^*}}\Big[\int_\Omega |G_k(u)|^{p^*}\varphi^{p^*}\Big]^{p/p^*}\\ &\le \|f\|_m|B_R|^{\frac{1}{m'}-\frac{p}{p^*}}S_{N,p}\int_\Omega |\nabla(G_k(u)\varphi)|^p\\ &= \|f\|_m|B_R|^{\frac{1}{m'}-\frac{p}{p^*}}S_{N,p}2^{p-1}\Big[\int_\Omega \varphi^p|\nabla G_k(u)|^p + \int_\Omega |G_k(u)|^p\,|\nabla\varphi|^p\Big]\,.\end{aligned}$$

Choosing R such that $C\|f\|_m|B_R|^{\frac{1}{m'}-\frac{p}{p^*}}S_{N,p}2^{p-1}\le\frac{1}{2}$, where C is the same constant occurring in (4.9), this first term can be absorbed by the left hand side of (4.9). Thus (4.9) becomes

$$(4.10)\quad \int_\Omega \varphi^p|\nabla G_k(u)|^p \le C\int_\Omega |G_k(u)|^p|\nabla\varphi|^p + C\int_{A(k,R)} f(1+k)^p\varphi^p\,,$$

for a different constant $C > 0$.

Now, take $0 < \rho < R$, consider B_ρ a ball centered at the same point as B_R and choose $\varphi \in C_0^\infty(B_R)$ satisfying $\varphi(x) = 1$ for all $x \in B_\rho$ and $|\nabla\varphi| \le \frac{2}{R-\rho}$. Then we deduce from (4.10) that

$$\begin{aligned}(4.11)\qquad \int_{B_\rho} |\nabla G_k(\tilde u)|^p &\le \frac{C}{(R-\rho)^p}\int_{B_R} |G_k(\tilde u)|^p + C(1+k)^p \int_{A(k,R)} f \\ &\le \frac{C}{(R-\rho)^p}\int_{B_R} |G_k(\tilde u)|^p + C(1+k)^p \|f\|_m |A(k,R)|^{1/m'}\,,\end{aligned}$$

which is the desired Caccioppoli type inequality. Since $m > \frac{N}{p}$, it yields

$$\frac{1}{m'} > 1 - \frac{p}{N}\,,$$

so that (4.11) is similar to the estimate found in [9, Theorem 7.1]. Now we may follow the same arguments of [9, Chapter 7] to infer that $\tilde u \in L^\infty(B_{R/2})$.

Let U be an open set $U \subset \Omega$ having positive distance from A. We have seen that, for every $x \in \overline{U}$, there exists $r > 0$ (depending on x) such that $\tilde u \in L^\infty(B_r(x))$. The compactness of $\overline{U}$ implies the desired conclusion. □

Remark 4.4. If we assume $h(x,s) = f(x)s^{p-1}$ (where f is a nonnegative function belonging to $L^m(\Omega)$ for some $m > N/p$) and $\mu \ge 0$, then we may apply [11, Theorem 2.4] to obtain a pointwise estimate of the solution. It is easy to check that this estimate is bounded far from the support of μ. Therefore, in this case, we may deduce Theorem 4.3 above from [11, Theorem 2.4]. The authors thank the referee for bringing [11] to our attention.

It is now straightforward to prove that the renormalized solution is actually continuous outside the support of μ.

Theorem 4.5. *Assume that hypotheses* (2.2)–(2.6) *are true, and that μ is concentrated on a closed set $A \subset \Omega$. Let u be a renormalized solution of problem* (2.1).

Then u is Hölder–continuous in $\overline{\Omega} \setminus A$.

Proof. Let $B_\rho \subset\subset B_R$ be a pair of concentric balls having positive distance from A. We have proved in Theorem 4.3 that u is bounded in $\Omega \cap B_R$. It follows that (4.11) can be rewritten as

$$\int_{B_\rho} |\nabla G_k(\tilde u)|^p \le \frac{C}{(R-\rho)^p}\int_{B_R} |G_k(\tilde u)|^p + C\big(1 + \|\tilde u\|_{L^\infty(B_R)}\big)^p \|f\|_m |A(k,R)|^{1/m'}\,,$$

for all $k < \|\tilde u\|_{L^\infty(B_R)}$. Therefore it is possible to apply Theorem 7.6 of [9] to obtain that $\tilde u$ is Hölder–continuous far from A. □

Remark 4.6. We point out that the smallness assumption on f, which is needed in the existence result, is not used in the proof of the summability/boundedness/continuity of the renormalized solution.

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