# Robust Output Regulation for a Class of Linear Differential-Algebraic Systems

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***Abstract*—This paper addresses the problem of robust output regulation of systems described by linear differential-algebraic equations (DAEs). Taking advantage of some recent relevant results concerned with the development of normal forms for general DAEs, it is shown how an internal model can be designed in such a way that the problem in question is reduced to the problem of robustly stabilizing an augmented system modeled by DAEs. Then, a suitable enhancement - within the context of systems modeled by DAEs - of the theory of extended observers is proposed, by means of which such robust stabilization problem is solved.**

***Index Terms*—Output regulation, differential-algebraic systems, robust control.**

## I. INTRODUCTION

### A. Preliminaries

SYSTEMS modeled by differential-algebraic equations (DAE) have been the subject of a lot of intense investigation over the years. Classical design problems such as feedback stabilization, disturbance decoupling, non-interacting control have been thoroughly investigated, initially for linear systems (see, in this respect, the classical book of L. Dai [1]) and subsequently also for classes of nonlinear systems. More recently, a very general apparatus for analysis and control of systems modeled by linear DAEs, in which most of the earlier methods are unified and extended, has been proposed by T. Berger in the monograph [2]. We will summarize and stress some of the outstanding features of the approach of [2] in the subsequent sections of the present paper.

One of the design problems that have been addressed for systems modeled by DAEs is the so-called problem of *output regulation*, namely the problem in which the purpose of the design is to guarantee asymptotic tracking/rejection of exogenous commands/disturbances, under the assumption that a model of such exogenous inputs is given. For systems modeled by linear DAEs, the problem of output regulation was initially studied in [3], where the so-called *regulator equations for singular systems* where introduced for the first time. Subsequently, the problem in question was thoroughly investigated also for nonlinear systems, in the series of papers [6], [7], [8], [9]. All such papers elaborate on the pioneering approach of the seminal paper [4] in which the systematic design of robust nonlinear regulators was initiated. See, in this respect, the excellent monograph of J. Huang [5]. In these papers it was shown how the problem can be solved, at least in a neighborhood of a fixed operating point, under appropriate “stabilizability/detectability” hypotheses on the linear approximation of the equations at the given operating point. As a special case, these works address the problem of output regulation for systems modeled by linear DAEs and, as such, extend the work of [3]. Further results on the problem of output regulation for systems modeled by linear DAEs can be found in [10] and [11, Ch4].

As it is well-known, key ingredients in the design of a controller that solves the problem of output regulation are: (i) an *internal model* of the exogenous input, and (ii) a *stabilizer* for the composite system obtained when the plant to be controlled is *augmented* with the internal model. This problem becomes a challenging one when the parameters of the controlled plant are affected by uncertainties that range over fixed tolerance bounds (and not simply “small” uncertainties). In such cases, the assumptions of “stabilzability/detectability” of the underlying DAEs, that have been considered in the prior literature, are no longer sufficient to guarantee the existence of a robust stabilizer (if robustness is sought with respect to uncertainties ranging of prescribed, *a priori fixed*, bounds). Thus, alternative design methods are to be sought. In the respect, the recent approach of Berger to the design of robust stabilizers for systems modeled by DAEs proves particularly promising. The purpose of this paper is to show how the apparatus developed by Berger for analysis and robust stabilization of DAEs can be fruitfully used in the context of the design of internal-model-based regulators and how some of the assumptions considered by Berger can be to some extent weakened.

### B. Problem statement

In the present paper we study a problem of output regulation for a system modeled by a differential-algebraic equation (DAE) of the form

$$\begin{aligned} E\dot{x} &= Ax + Bu + Pw \\ e &= Cx + Qw \end{aligned} \tag{1}$$

in which $u$ denotes the input to be used for *control*, $w$ denotes an *exogenous* input, that includes disturbances/commands to be rejected/tracked, and $e$ denotes a *regulated* output, to be asymptotically steered to zero. In these equations, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $e(t) \in \mathbb{R}^p$, $w(t) \in \mathbb{R}^{n_w}$ and $E \in \mathbb{R}^{\ell \times n}$. The exogenous input $w$ is modeled by the linear autonomous system

$$\dot{w} = A_e w\,, \tag{2}$$

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known as the *exosystem*, in which it is assumed, without loss of generality, that all eigenvalues of $A_\mathrm{e}$ have non-negative real part. The matrices $E,A,B,C,P,Q$ in (1) may depend on a vector $\mu$ of constant *uncertain parameters*, which is not explicitly indicated for the sake of notational simplicity, that ranges over a fixed compact set. On the contrary, the matrix $A_\mathrm{e}$ is assumed to be known.

The problem of output regulation consists in finding a controller, driven by the measured output $e$, that generates a control input $u$ yielding an asymptotically stable closed-loop system in which, for every $w(0)$ and for every admissible $x(0)$, the regulated variable $e(t)$ asymptotically decays to 0 as $t \to \infty$.

*Remark.* As kindly suggested by an anonymous reviewer, the potentially larger class of exosystems modeled by a DAE $E_\mathrm{e}\dot{w} = A_\mathrm{e}w$ could be considered. Using the Kroeneker form of the matrix pencil $sE_\mathrm{e} - A_\mathrm{e}$ it would be possible reduce the problem to a problem modeled as in (1) – (2), after suitable redefinition of the matrices $B,P,Q$ and of the input $u$ (the free variables of $E_\mathrm{e}\dot{w} = A_\mathrm{e}w$ being viewed as additional inputs).

### C. Background material on DAEs

In the present paper, we base our study on a set of relevant results initially presented in the excellent monograph [2] and subsequently published in a series of related papers, out of which we specifically refer to the paper [12]. In such paper, Berger develops a very detailed study of *normal forms* for system modeled by DAEs and of a series of related concepts such as those of *zero dynamics* and of *invertibility*. As Berger rightfully stresses in the Introduction of [2], "exploiting the zero dynamics proved fruitful in a lot of control theoretic topics such as output regulation". The contribution of the present paper is precisely that of showing how the apparatus developed by Berger can be profitably used in the design of controllers that solve a problem of output regulation for systems modeled by DAEs.

We begin by a short summary of some major findings of [2], to which the reader is referred for further details. Unless otherwise noted, we adopt the notations used in [2].

*Assumption 1:* The quadruplet $[E,A,B,C]$ satisfies[1]

$$\mathrm{rank}_{\mathbb{R}[s]} \begin{pmatrix} sE-A & -B \\ -C & 0 \end{pmatrix} = n+m \tag{3}$$

Proposition 3.5 of [12] proves that such Assumption is necessary and sufficient for the *zero dynamics* of the system

$$\begin{array}{rcl} E\dot{x} & = & Ax+Bu \\ y & = & Cx \end{array} \tag{4}$$

to be *autonomous*.[2] Since we are dealing with a problem of output regulation, in which the output $y$ of a system like (4) is required to track/reject exogenous signals, it is reasonable to assume that the quadruplet $[E,A,B,C]$ be *right-invertible*.[3]

[1]Note that this condtion implies $\ell + p \geq n+m$, i.e. $\ell - n \geq m-p$.

[2]See [12] for a precise definition of the concept of an *autonomous zero dynamics* for a system modeled by a DAE.

[3]See [12] for a precise definition of *right-invertibility* for a system modeled by a DAE.

As shown in Theorem 4.3 and Proposition 4.6 of [12], this is the case if, in addition to Assumption 1, also the following assumption holds.

*Assumption 2:* The matrix $C$ has rank $p$ and $\ell - n = m - p$.

As shown in [12], if Assumptions 1 and 2 hold, there exist nonsingular matrices $S,T$ such that the matrices

$$\hat{E} = SET, \quad \hat{A} = SAT, \quad \hat{B} = SB, \quad \hat{C} = CT \tag{5}$$

have the following form (referred to, in [12], as "system inversion form")

$$\hat{E} = \begin{pmatrix} I_k & 0 & 0 \\ 0 & E_{22} & E_{23} \\ 0 & E_{32} & N \end{pmatrix}, \; \hat{A} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & I_{n-k-p} \end{pmatrix}, \; \hat{B} = \begin{pmatrix} 0 \\ I_m \\ 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 0 & I_p & 0 \end{pmatrix}$$

in which $N$ is nilpotent (with $N^\nu = 0$ and $N^{\nu-1} \neq 0$).

Since $S$ and $T$ are nonsingular, the system (4) is equivalent to the system

$$\begin{array}{rcl} \hat{E}\dot{\hat{x}} & = & \hat{A}\hat{x} + \hat{B}u \\ y & = & \hat{C}\hat{x} \end{array}$$

in which $\hat{x} = T^{-1}x$. Splitting $\hat{x}$ as $\mathrm{col}(x_1,x_2,x_3)$ we obtain the equations

$$\begin{array}{rcl} \dot{x}_1 & = & A_{11}x_1 + A_{12}x_2 \\ E_{22}\dot{x}_2 + E_{23}\dot{x}_3 & = & A_{21}x_1 + A_{22}x_2 + u \\ E_{32}\dot{x}_2 + N\dot{x}_3 & = & x_3 \\ y & = & x_2\,. \end{array} \tag{6}$$

The third equation can be made explicit in $x_3$. In fact, exploiting the fact that $N$ is nilpotent, its explicit solution can be easily determined, by recursion, as

$$x_3(t) = \sum_{k=0}^{\nu-1} N^k E_{32} x_2^{(k+1)}(t)$$

and hence system (6) can be rewritten as

$$\begin{array}{rcl} \dot{x}_1 & = & A_{11}x_1 + A_{12}x_2 \\ E_{22}\dot{x}_2 + \sum_{k=0}^{\nu-1} E_{23}N^k E_{32} x_2^{(k+2)} & = & A_{21}x_1 + A_{22}x_2 + u \\ x_3 & = & \sum_{k=0}^{\nu-1} N^k E_{32} x_2^{(k+1)} \\ y & = & x_2 \end{array} \tag{7}$$

Finally, as in [12], the following assumptions are considered.

*Assumption 3:* The zero dynamics of $[E,A,B,C]$ are *asymptotically stable*.

*Assumption 4:* The matrix

$$\bar{E} = -\lim_{s\to\infty} s^{-1} \begin{pmatrix} 0 & I_m \end{pmatrix} L(s) \begin{pmatrix} 0 \\ I_p \end{pmatrix} \tag{8}$$

exists, in which $L(s)$ denotes a left-inverse of $\begin{pmatrix} sE-A & -B \\ -C & 0 \end{pmatrix}$ over $\mathbb{R}(s)$.

The zero dynamics of $[E,A,B,C]$ are asymptotically stable if and only if $\sigma(A_{11}) \in \mathbb{C}^-$. Lemma A.1 of [12] shows that, under Assumptions 1 and 2, the left-inverse $L(s)$ exists and the matrix $\bar{E}$, if it exists, is independent of the choice of $L(s)$. Moreover, Lemma A.2 of [12] shows that, under the said assumptions

$$\begin{array}{rcl} \bar{E} & = & E_{22} \\ E_{23}N^k E_{32} & = & 0, \qquad \forall k = 0,1,\ldots,\nu-1, \end{array} \tag{9}$$

in which case (7) becomes

$$\begin{array}{rcl} \dot{x}_1 & = & A_{11}x_1 + A_{12}x_2 \\ \bar{E}\dot{x}_2 & = & A_{21}x_1 + A_{22}x_2 + u \\ x_3 & = & \sum_{k=0}^{\nu-1} N^k E_{32} x_2^{(k+1)} \\ y & = & x_2\,. \end{array} \tag{10}$$

## II. THE STRUCTURE OF THE REGULATOR

### A. The regulator equations

We now look at system (1) extended with (2), that we express as

$$\begin{array}{rcl} \begin{pmatrix} I_{n_w} & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \dot{w} \\ \dot{x} \end{pmatrix} & = & \begin{pmatrix} A_{\rm e} & 0 \\ P & A \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix} + \begin{pmatrix} 0 \\ B \end{pmatrix} u \\ e & = & \begin{pmatrix} Q & C \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix} \end{array} \tag{11}$$

and we retain the hypotheses that the quadruplet $[E,A,B,C]$ have the properties indicated in Assumptions 1-4.

It is easy to check that the zero dynamics of such extended system are autonomous. Using Assumptions 1 and 2, a normal form equivalent to (6) is determined very easily. To this end, we rewrite (11) in the equivalent form

$$\begin{array}{rcl} \dot{w} & = & A_{\rm e}w \\ \hat{E}\dot{\hat{x}} & = & \hat{A}\hat{x} + \hat{B}u + \hat{P}w \\ e & = & \hat{C}\hat{x} + \hat{Q}w \end{array} \tag{12}$$

in which $\hat{E},\hat{A},\hat{B},\hat{C}$ are as in (5), $\hat{P} = SP$ and $\hat{Q} = Q$.

Splitting $\hat{x}$ as before we obtain a set of equations of the form

$$\begin{array}{rcl} \dot{w} & = & A_{\rm e}w \\ \dot{x}_1 & = & A_{11}x_1 + A_{12}x_2 + P_1 w \\ E_{22}\dot{x}_2 + E_{23}\dot{x}_3 & = & A_{21}x_1 + A_{22}x_2 + P_2 w + u \\ E_{32}\dot{x}_2 + N\dot{x}_3 & = & x_3 + P_3 w \\ e & = & x_2 + Qw \end{array} \tag{13}$$

in which $\mathrm{col}(P_1,P_2,P_3)$ is a partition of $\hat{P}$.

Recalling that we have assumed $\sigma(A_{\rm e}) \in \overline{\mathbb{C}}^+$ we look now for the existence of a control (if any of such control exists) that secures $e(t) \equiv 0$.

From the fourth equation, we see that if $e(t) \equiv 0$, then necessarily $x_2(t) = -Qw(t)$ and we rewrite the latter as $x_2(t) = \Pi_2 w(t)$. Bearing in mind that (see previous section)

$$x_3(t) = \sum_{k=0}^{\nu-1} N^k [E_{32}\dot{x}_2(t) - P_3 w(t)]^{(k)}$$

we observe that necessarily

$$x_3(t) = \sum_{k=0}^{\nu-1} N^k [E_{32}\Pi_2 A_{\rm e}^{k+1} - P_3 A_{\rm e}^k] w(t)$$

and rewrite the latter as $x_3(t) = \Pi_3 w(t)$. So long as $x_1(t)$ is concerned, we see that, if $e(t) \equiv 0$, necessarily $x_1(t)$ obeys

$$\begin{array}{rcl} \dot{w} & = & A_{\rm e}w \\ \dot{x}_1 & = & A_{11}x_1 + [A_{12}\Pi_2 + P_1]w\,. \end{array} \tag{14}$$

By Assumption 3, $\sigma(A_{11}) \in \mathbb{C}^-$. Hence $A_{11}$ and $A_{\rm e}$ have disjoint spectra and the Sylvester equation

$$\Pi_1 A_{\rm e} = A_{11}\Pi_1 + [A_{12}\Pi_2 + P_1]$$

has a unique solution $\Pi_1$. As a consequence, (14) yields

$$\dot{x}_1 = A_{11}x_1 + (\Pi_1 A_{\rm e} - A_{11}\Pi_1)w = A_{11}x_1 - A_{11}\Pi_1 w + \Pi\dot{w}$$

which implies $\frac{\mathrm{d}}{\mathrm{d}t}(x_1 - \Pi_1 w) = A_{11}(x_1 - \Pi_1 w)$. From this, it is seen that $x_1(t) = \Pi_1 w(t) + e^{A_{11}t}[x_1(0) - \Pi_1 w(0)]$. We have in this way identified all solutions compatible with the constraint $e(t) \equiv 0$. In particular, in $x_1(t)$ we identify a *steady state component* $\Pi_1 w(t)$ and a *transient* component $e^{A_{11}t}[x_1(0) - \Pi_1 w(0)]$.

Finally, observe that, if $x_i(t) = \Pi_i w(t)$ for $i = 1,2,3$, then $u(t)$ is necessarily

$$\begin{array}{rcl} u(t) & = & E_{22}\Pi_2 A_{\rm e}w(t) + E_{23}\Pi_3 A_{\rm e}w(t) - A_{21}\Pi_1 w(t) \\ & & -A_{22}\Pi_2 w(t) - P_2 w(t) \end{array}$$

that we rewrite as

$$u(t) = \Psi w(t)\,. \tag{15}$$

It follows from all of the above that $\hat{\Pi} = \mathrm{col}(\Pi_1,\Pi_2,\Pi_3)$ and $\Psi$ satisfy

$$\begin{array}{rcl} \hat{E}\hat{\Pi}A_{\rm e} & = & \hat{A}\hat{\Pi} + \hat{B}\Psi + \hat{P} \\ 0 & = & \hat{C}\hat{\Pi} + \hat{Q} \end{array}$$

i.e. that $\Pi = T\hat{\Pi}$ satisfies

$$\begin{array}{rcl} E\Pi A_{\rm e} & = & A\Pi + B\Psi + P \\ 0 & = & C\Pi + Q \end{array}$$

which are the *regulator equations* of [3]. The subspace $x = \Pi w$ is rendered invariant by $u = \Psi w$ and on such subspace the regulated variable $e$ is zero.

Observe also that, if the $x_i$'s, $i = 1,2,3$ are scaled as

$$\tilde{x}_i = x_i - \Pi_i w \qquad i = 1,2,3,$$

the equations (13) can be rewritten as

$$\begin{array}{rcl} \dot{w} & = & A_{\rm e}w \\ \dot{\tilde{x}}_1 & = & A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2 \\ E_{22}\dot{\tilde{x}}_2 + E_{23}\dot{\tilde{x}}_3 & = & A_{21}\tilde{x}_1 + A_{22}\tilde{x}_2 + u - \Psi w \\ E_{32}\dot{\tilde{x}}_2 + N\dot{\tilde{x}}_3 & = & \tilde{x}_3 \\ e & = & \tilde{x}_2\,. \end{array} \tag{16}$$

In the analysis so far, we have only used Assumptions 1-3. If also Assumption 4 is invoked, then a simpler set of equations is obtained, because (9) holds. Under such additional assumption (compare (7) with (10)), instead of (16) one obtains a system of the form

$$\begin{array}{rcl} \dot{w} & = & A_{\rm e}w \\ \dot{\tilde{x}}_1 & = & A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2 \\ \bar{E}\dot{\tilde{x}}_2 & = & A_{21}\tilde{x}_1 + A_{22}\tilde{x}_2 + u - \Psi w \\ \tilde{x}_3 & = & \sum_{k=0}^{\nu-1} N^k E_{32}\tilde{x}_2^{(k+1)} \\ e & = & \tilde{x}_2 \end{array} \tag{17}$$

in which $\bar{E}$ is the matrix (8), and the matrix $A_{11}$ is Hurwitz.

### B. Adding an internal model

Following a well-established paradigm, we choose now the control as

$$\begin{array}{rcl} \dot{\eta} & = & (F+G\Gamma)\eta + G\bar{u} \\ u & = & \Gamma\eta + \bar{u} \end{array} \tag{18}$$

with

$$F = \begin{pmatrix} 0 & I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I \\ -a_0 I & -a_1 I & \cdots & -a_{d-1} I \end{pmatrix} \quad G = \begin{pmatrix} 0 \\ \cdots \\ 0 \\ I \end{pmatrix}$$

in which all blocks are $m \times m$, the integer $d$ is the degree of the minimal polynomial of $A_{\mathrm{e}}$, the coefficients $a_0, \ldots, a_{d-1}$ are such that the polynomial $p(\lambda) = \lambda^d + a_{d-1}\lambda^{d-1} + \cdots + a_1\lambda + a_0$ is Hurwitz, and $\Gamma$ is such that the minimal polynomial of $(F+G\Gamma)$ coincides with the minimal polynomial of $A_{\mathrm{e}}$. Since $(F,G)$ is a controllable pair, this is always possible.

According to Lemma 4.8 of [13, p. 110], there always exists a matrix $\Sigma$ such that

$$\Sigma A_{\mathrm{e}} = (F+G\Gamma)\Sigma \qquad\qquad \Psi = \Gamma\Sigma,$$

where $\Psi$ is the matrix defined in (15). Thus, scaling also $\eta$ as $\tilde{\eta} = \eta - \Sigma w$, system (16) with control (18) becomes

$$\begin{array}{rcl} \dot{w} & = & A_{\mathrm{e}} w \\ \dot{\tilde{\eta}} & = & (F+G\Gamma)\tilde{\eta} + G\bar{u} \\ \dot{\tilde{x}}_1 & = & A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2 \\ E_{22}\dot{\tilde{x}}_2 + E_{23}\dot{\tilde{x}}_3 & = & A_{21}\tilde{x}_1 + A_{22}\tilde{x}_2 + \Gamma\tilde{\eta} + \bar{u} \\ E_{32}\dot{\tilde{x}}_2 + N\dot{\tilde{x}}_3 & = & \tilde{x}_3 \\ e & = & \tilde{x}_2 \end{array} \tag{19}$$

Note the fact that $w$, the state of the upper equation, is totally decoupled from the state $(\tilde{\eta}, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ of the lower set of equations. If an input $\bar{u}$ can be found that steers $\tilde{x}_2$ to zero, the problem of output regulation is solved.

The lower set of equations in (19) can be brought to a form similar to that of (16) by means of the additional change of variables

$$\zeta = \tilde{\eta} - G[E_{22}\tilde{x}_2 + E_{23}\tilde{x}_3]\,.$$

In fact, in this way, the following system is obtained

$$\begin{array}{l} \dot{\zeta} = F\zeta - GA_{21}\tilde{x}_1 + (FGE_{22} - GA_{22})\tilde{x}_2 + FGE_{23}\tilde{x}_3 \\ \dot{\tilde{x}}_1 = A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2 \\ E_{22}\dot{\tilde{x}}_2 + E_{23}\dot{\tilde{x}}_3 = A_{21}\tilde{x}_1 + (A_{22} + \Gamma GE_{22})\tilde{x}_2 + \Gamma GE_{23}\tilde{x}_3 + \Gamma\zeta + \bar{u} \\ E_{32}\dot{\tilde{x}}_2 + N\dot{\tilde{x}}_3 = \tilde{x}_3 \\ e = \tilde{x}_2\,. \end{array} \tag{20}$$

Finally, note that, by virtue of Assumption 4, the identities (9) hold and then the system above simplifies to

$$\begin{array}{rcl} \dot{\zeta} & = & F\zeta - GA_{21}\tilde{x}_1 + (FGE_{22} - GA_{22})\tilde{x}_2 \\ \dot{\tilde{x}}_1 & = & A_{11}\tilde{x}_1 + A_{12}\tilde{x}_2 \\ \bar{E}\dot{\tilde{x}}_2 & = & A_{21}\tilde{x}_1 + (A_{22} + \Gamma GE_{22})\tilde{x}_2 + \Gamma\zeta + \bar{u} \\ \tilde{x}_3 & = & \sum_{k=0}^{\nu-1} N^k E_{32}\tilde{x}_2^{(k+1)} \\ e & = & \tilde{x}_2 \end{array} \tag{21}$$

This system has a structure *identical* to that of system (10), as it can be highlighted by putting together the variables $\tilde{x}_1$ and $\zeta$

$$\begin{array}{rcl} \begin{pmatrix} \dot{\zeta} \\ \dot{\tilde{x}}_1 \end{pmatrix} & = & \begin{pmatrix} F & -GA_{21} \\ 0 & A_{11} \end{pmatrix} \begin{pmatrix} \zeta \\ \tilde{x}_1 \end{pmatrix} + \begin{pmatrix} FGE_{22} - GA_{22} \\ A_{12} \end{pmatrix} \tilde{x}_2 \\ \bar{E}\dot{\tilde{x}}_2 & = & \begin{pmatrix} \Gamma & A_{21} \end{pmatrix} \begin{pmatrix} \zeta \\ \tilde{x}_1 \end{pmatrix} + (A_{22} + \Gamma GE_{22})\tilde{x}_2 + \bar{u} \\ \tilde{x}_3 & = & \sum_{k=0}^{\nu-1} N^k E_{32}\tilde{x}_2^{(k+1)} \\ e & = & \tilde{x}_2 \end{array} \tag{22}$$

Note, in particular, that since under the said assumptions both $F$ and $A_{11}$ are Hurwitz matrices, so is the matrix

$$\begin{pmatrix} F & -GA_{21} \\ 0 & A_{11} \end{pmatrix}.$$

We conclude that the design of a regulator can be satisfactorily completed if, for a system of the form (10), under the assumption that the matrix $A_{11}$ is Hurwitz, one is able to find a control $u$ that steers to 0 the state $x_2$.

## III. STABILIZING THE AUGMENTED SYSTEM

### A. Basic assumptions

In view of the results established so far, we focus now on the problem of stabilizing a system of the form (10), in which $A_{11}$ is a Hurwitz matrix. The literature on stabilization of systems modeled by DAEs is vast: the interested reader may consult [2] and the references therein, [14], [15] and also [16], where in particular the problem of achieving robust stability in spite of parameter uncertainties is addressed by means of an LMI-based approach. In this section, exploiting the assumption that the zero dynamics are asymptotically stable, we rather focus – to cope with uncertainties – on methods based on "high-gain" output feedback.

For simplicity, we assume that system (1) has the same number of inputs and outputs and, for notational convenience, we rewrite system (22), or – what is the same – system (10), in the form

$$\begin{array}{rcl} \dot{z} & = & Fz + Gx \\ \bar{E}\dot{x} & = & Ax + Hz + u \\ y & = & x \end{array} \tag{23}$$

in which $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $z(t) \in \mathbb{R}^k$ and, as assumed above, $F$ is a Hurwitz matrix. We recall that we are considering the case in which all coefficient matrices are possibly dependent on a vector of uncertain parameters (not written for notational convenience) that ranges over a fixed compact set. The issue is to find a control $u$ that steers $x$ to zero.

A solution to this specific stabilization problem has been proposed in Chapter 5 of [2], under the *additional assumption* that the matrix $\bar{E}$ satisfies the condition $\bar{E} = \bar{E}^{\mathrm{T}} \geq 0$. In this case, in fact, Theorem 5.1.4 of [2] shows that asymptotic convergence of $x$ to zero can be achieved using a control $u = -hy$ in which $h > 0$ is a sufficiently large gain coefficient.

In what follows, we suggest the use of an alternative method, that does not require the matrix $\bar{E}$ to by symmetric and positive semi-definite, but reposes on different, perhaps milder, hypotheses. We rather assume the following.

*Assumption 5:* The matrix $\bar{E}$ has a constant rank $r \leq n$ for all values of the uncertain parameters. Two permutation matrices $P_\ell$ and $P_r$ *exist and are known* such that

$$P_\ell \bar{E} P_r = \begin{pmatrix} \bar{E}_{11} & \bar{E}_{12} \\ \bar{E}_{21} & \bar{E}_{22} \end{pmatrix} \tag{24}$$

in which $\bar{E}_{11} \in \mathbb{R}^{r\times r}$ is nonsingular. Moreover, a nonsingular matrix $B_0 \in \mathbb{R}^{r\times r}$ *is known* such that, for some $0 < \delta_0 < 1$,

$$\|[\bar{E}_{11}^{-1} - B_0]\Lambda B_0^{-1}\| \leq \delta_0 \tag{25}$$

for all diagonal matrices $\Lambda$ such that $\|\Lambda\| \leq 1$, and for all values of the uncertain parameters.

*Remark.* The first part of this hypothesis is equivalent to assume the knowledge of a selection of rows/columns that identifies an $r \times r$ submatrix of $\bar{E}$ which is nonsingular for all values of the uncertain parameters. The second part of the assumption can be seen as equivalent to assuming "upper and lower bounds" on such submatrix.[4]

With this assumption in mind, it is possible to reorder input and output channels [5] of (23) in such a way that, after such reordering, the matrix $\bar{E}$ is partitioned as in (24), with nonsingular $\bar{E}_{11}$. Let *the reordered* $u$ and $y$ be partitioned accordingly, as

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

in which $u_1 \in \mathbb{R}^r$ and $x_1 \in \mathbb{R}^r$, and pick

$$u_2 = h y_2\,, \qquad h > 0.$$

Clearly, there exists matrices $S_0$ and $T_0$ such that

$$S = \begin{pmatrix} I_r & 0 \\ S_0 & I_{n-r} \end{pmatrix}, \qquad T = \begin{pmatrix} I_r & T_0 \\ 0 & I_{n-r} \end{pmatrix}$$

satisfy

$$S\bar{E}T = \begin{pmatrix} \bar{E}_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

Partition $SAT$ accordingly, and define $\xi = x_1 - T_0 x_2$ so as to obtain equations of the form

$$\begin{array}{rcl} \dot{z} &=& Fz + G_1\xi + G_2 x_2 \\ \dot{\xi} &=& A_{11}\xi + A_{12}x_2 + H_1 z + Bu_1 \\ 0 &=& A_{21}\xi + [A_{22} + hI]x_2 + H_2 z + S_0 u_1 \end{array}$$

in which $B = \bar{E}_{11}^{-1}$. Note that all matrices here (notably $B$) can be uncertain.

Clearly, as pointed out in [2, p.216], if $h$ is large the system is regular, because in this case $\det(A_{22} + hI) \neq 0$ and the algebraic constraint uniquely determines $x_2$.

[4]In fact, in case $r = 1$, the assumption in question is equivalent to the assumption that $\left|\frac{\bar{E}_{11}^{-1} - B_0}{B_0}\right| \leq \delta_0 < 1$ which, in turn, is equivalent to the assumption of the existence of two numbers $b_{\min}, b_{\max}$ such that $0 < b_{\min} \leq |\bar{E}_{11}^{-1}| \leq b_{\max}$. The condition in Assumption 5 can be regarded as a multivariable version of such assumption.

[5]Change of $x$ into $P_r^{-1}x$ corresponds to a reorder of the components of $y$, while left-multiplication of the DAE by $P_\ell$ corresponds to a reorder of the components of $u$.

### B. A naïve control

Consider now the equation for $\xi$ and write it as

$$\dot{\xi} = Q(\xi, x_2, z) + Bu_1$$

where $Q(\xi, x_2, z) = A_{11}\xi + A_{12}x_2 + H_1 z$. Note that, if $h$ is large, the equation

$$Q(\xi, x_2, z) + Bu_1 = -\xi$$

has a unique solution, which we denote as $u_1 = u_{\rm id}(z, \xi)$. In fact, this equation reads as

$$Bu_1 = -[I + A_{11}]\xi + A_{12}(A_{22} + hI)^{-1}[A_{21}\xi + H_2 z + S_0 u_1] - H_1 z.$$

If $h$ is large, the matrix $B_h := B - A_{12}(A_{22} + hI)^{-1}S_0$ is nonsingular and hence $u_1$ is a well-defined, linear function of $(\xi, z)$. Note also that

$$\lim_{h\to\infty} B_h^{-1} = B^{-1} = \bar{E}_{11}\,. \tag{26}$$

Pick $u_1 = u_{\rm id}(z, \xi)$. The resulting system becomes

$$\begin{array}{rcl} \dot{z} &=& Fz + G_1\xi + G_2 x_2 \\ \dot{\xi} &=& -\xi\,, \end{array} \tag{27}$$

in which

$$\begin{array}{rcl} x_2 &=& -(A_{22} + hI)^{-1}[A_{21}\xi + H_2 z + S_0 u_{\rm id}(z, \xi)] \\ &=& -(A_{22} + hI)^{-1}\Big[A_{21}\xi + H_2 z + S_0 B_h^{-1}\Big(-[I + A_{11}]\xi \\ && \quad + A_{12}[A_{22} + hI]^{-1}(A_{21}\xi + H_2 z) - H_1 z\Big)\Big]. \end{array}$$

Recall that $F$ is a Hurwitz matrix. Thus, system (27) with $x_2 = 0$ is a stable system. On the other hand, as seen from the expression above, $x_2$ is an $h$-dependent linear "feedback" from $(z, \xi)$ whose gain matrices can be rendered arbitrarily small by increasing $h$ (consider also (26) in this respect). Thus, by the small-gain Theorem, it can be concluded that for large $h$ system (27) is asymptotically stable. We see in this way that the control $u_{\rm id}(z, \xi)$ solves the problem of steering $x$ to zero, under the said assumptions. However, this control is not implementable as such because it depends on a lot of uncertain quantities.

### C. An extended-observer-based control

The purpose of the control $u_{\rm id}$ is to cancel the term $Q(\xi, x_2, z)$ and to replace it by $-\xi$. However, this cancelation cannot be exactly carried out if $B$ and the parameters in $Q(\xi, x_2, z)$ are not accurately known. In what follows, we show that the lack of accurate knowledge of $B$ and $Q(\xi, x_2, z)$ can be overcome if the "ideal" memoryless control $u_{\rm id}$ is replaced by a *dynamical feedback*, driven by $y_1$, whose internal states are those of an *extended observer*. In fact, the use of extended observers – developed in the works [17], [18], and [19], [13, pp.297-316] – has proven to be effective in addressing uncertainties of this kind. In what follows, we show how the theory of extended observers can be enhanced so as to deal with the present case of a system described by the DAE (23).

Following the "extended observer" paradigm, use – instead of $u_{\rm id}(z, \xi)$ – a control

$$u_{\rm act} = G_L(\psi(\hat{\xi}, \sigma)) \tag{28}$$

where $G_L(s) = \mathrm{col}(g_L(s_1), g_L(s_2), \ldots, g_L(s_r))$, in which $g_L(\cdot)$ is a fixed saturation function, defined immediately below, and

$$\psi(\hat{\xi}, \sigma) = B_0^{-1}[-\hat{\xi} - \sigma] \tag{29}$$

in which $(\hat{\xi}, \sigma)$ are the states of the "extended observer"

$$\begin{array}{rcl} \dot{\hat{\xi}} & = & \sigma + B_0 G_L(\psi(\hat{\xi}, \sigma)) + \kappa c_1(y_1 - \hat{\xi}) \\ \dot{\sigma} & = & \kappa^2 c_0(y_1 - \hat{\xi}) \end{array} \tag{30}$$

In these equations, the saturation function $g_L : \mathbb{R} \to \mathbb{R}$ is a $C^1$ function having the following properties: (i) $g_L(s) = s$ if $|s| \leq L$, (ii) $g_L(s)$ is odd and monotonically increasing, with $0 < g_L'(s) \leq 1$, (iii) $\lim_{s\to\infty} g_L(s) = L(1+c)$ with $0 < c \ll 1$. $B_0$ is any matrix for which (25) holds, and the quantities $L$, $\kappa$ and $c_0, c_1$ are design parameters.

If the control (28)–(29)–(30) is used, the following result can be established.

*Proposition 1:* Consider system (23). Suppose $F$ is a Hurwitz matrix and suppose Assumption 5 holds. Let the order of the components of $y$ and $u$ be changed so as to bring the matrix $\bar{E}$ in the form (24), with $\bar{E}_{11}$ a nonsingular matrix. Let the system be controlled by

$$u = \begin{pmatrix} G_L(\psi(\hat{\xi}, \sigma)) \\ h y_2 \end{pmatrix}$$

where $G_L(s) = \mathrm{col}(g_L(s_1), g_L(s_2), \ldots, g_L(s_r))$, with $g_L(\cdot)$ a saturation function, $\psi(\hat{\xi}, \sigma)$ is defined as in (29), with $B_0$ chosen so as to satisfy the condition (25), and $(\hat{\xi}, \sigma)$ are states of the extended observer (30). For every choice of a compact set $\mathscr{C}$ there is a choice of the design parameters $L$ and $c_0, c_1$, a number $\kappa^*$ and, for all $\kappa > \kappa^*$ a number $h_\kappa^*$, such that, if $\kappa > \kappa^*$ and $h > h_\kappa^*$, then the equilibrium $(z, \xi, \hat{\xi}, \sigma) = (0,0,0,0)$ of the resulting closed-loop system is asymptotically stable, with a domain of attraction $\mathscr{A}$ that contains the set $\mathscr{C}$ .

A proof of such Proposition can be obtained by adapting, to the current setup, the arguments of the proof developed, in the context of MIMO systems modeled by ODEs, in [19] (see also [20, pp.176-183]). The adaptation involves a number of subtleties, which depend on the fact that, in the current setting, there are *two* design parameters that need to be tuned: the "gain" $\kappa$ of the observer (30) and the "gain" $h$ of the control $u_2 = hy_2$. The value of $\kappa$ has to be sufficiently large so as to dominate the uncertainties while the value of $h$ has to be sufficiently large so as to render the algebraic constraint solvable in $x_2$ and, at the same time, to facilitate the fulfillment of the conditions of the small-gain Theorem. The two choices are not independent, because the "minimal" admissible value of $h$ turns out to depend on the choice of $\kappa$, as the statement of the Proposition says. A detailed proof of Proposition 1 can be found in [21].

One might be puzzled by the remark that a design problem for a linear system is solved by means of a saturated controller, i.e. a nonlinear controller. The major consequence of this is that, while the design parameters can always be chosen in such a way that the desired equilibrium is asymptotically stable with a region of attraction that contains any *a priori* fixed compact set, global stability is not necessarily ensured. The use of a saturated control, though, is helpful in proving that – by increasing $\kappa$ – the "observer error" decays to an arbitrary small value in an arbitrarily small time *regardless of how large h is*. In other words, the use of a saturation makes it possible to prove that $\kappa$ can be tuned independently of $h$.

## IV. CONCLUSIONS

In this paper, we have considered the problem of output regulation for a system modeled by linear DAEs. Within the general framework for analysis and design recently developed in [12], we have shown how the regulator equations can be solved and how the theory of extended observers can be fruitfully used, so as to obtain robustness in spite of parameter uncertainties.

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