HOMOGENIZATION RESULTS FOR A CLASS OF PARABOLIC EQUATIONS WITH A NON-LOCAL INTERFACE CONDITION VIA TIME-PERIODIC UNFOLDING

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Abstract. We study the thermal properties of a composite material in which a periodic array of finely mixed perfect thermal conductors is inserted. The suitable model describing the behaviour of such physical materials leads to the so-called equivalued surface boundary value problem. To analyze the overall conductivity of the composite medium (when the size of the inclusions tends to zero), we make use of the homogenization theory, employing the unfolding technique.
The peculiarity of the problem under investigation asks for a particular care in developing the unfolding procedure, giving rise to a non-standard two-scale problem.

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1. Introduction

We study, via homogenization techniques, the thermal properties of a composite material made up of a hosting medium in which a periodic array of conductive fillers is inserted. The microscopic inclusions are assumed to be perfect heat conductors

(i.e. they have infinite thermal conductivity). This last assumption is motivated by the fact that, in applications, the heat conductivity of the inclusions is much larger than the one of the hosting medium.
These models are drawing increasing attention in last years, due to the appearance on the market of new composite materials, produced with the purpose of increasing the overall thermal conductivity. For example, this is the case in the packaging of electronic devices, in which rubber is used as an encapsulating medium. Then, an efficient heat dispersion device is needed, justifying the insertion of highly conductive inclusions into the rubber itself. The purpose of this investigation is to give a theoretical justification of some heuristic models used by engineers in applications (see, for instance, [21, 23, 27, 28, 29, 30]).
From a mathematical point of view, the problem reduces to a heat equation satisfied by the temperature u_ε in the hosting medium, while on the boundary of the inclusions (i.e. on the interface between the two different conductive phases) u_ε is assumed to be constant with respect to the space variable and determined only by a heat balance, in which the total flux entering the inclusions is taken into account. This corresponds to assuming a perfect thermal contact between the two conductive phases of the medium. More precisely, on each interface Γ^ε_ξ, the temperature satisfies the non-standard boundary condition (see (2.5))

$$\lambda^\xi_\varepsilon u_{\varepsilon t} = \frac{1}{\varepsilon^N} \int\limits_{\Gamma^\varepsilon_\xi} \kappa_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \,\mathrm{d}\sigma \,,$$

where λ^ξ_ε is proportional to the specific heat capacity, κ_ε takes into account the diffusion properties of the hosting material and ε represents the characteristic length of the inclusions. As already mentioned, the previous boundary condition is not classical. Well-posedness of evolutive problems involving such a condition was studied in [8], for fixed $\varepsilon = 1$. However, it seems that no homogenization results are known for problems of this type. For a different mathematical modelling of connected physical problems see, for instance, [9, 10].
It is worthwhile noting that non-local boundary conditions have a wide area of possible applications ranging from heat diffusion (as in the case treated in this paper) to electric conduction, to petroleum exploitation, to wave equations or to the elastic behaviour of perforated materials (we refer to [11, 12, 13, 17, 24, 25, 26] for a more extensive description of these models).
Parabolic problems in the presence of spatial inhomogeneities, coupled with sharp time oscillations, have been discussed in [6, 7], in connection with intracellular diffusion, with the aim of investigating local accumulation effects and their interplay with boundary flux conditions.
In the present paper, we consider the case where the heat capacities of the hosting medium and of the inclusions are assumed to oscillate in time. More precisely, we consider a family of possible time scalings of the type $s = \varepsilon^{-\alpha} t$, with $\alpha \geq 1$. The presence of these time-oscillations makes the mathematical approach much harder technically. We stress again that the problem we are addressing here is, up to our knowledge, new in the literature, because of its non-standard evolutive character.

The case of constant coefficients, which is considerably simpler, has been treated in [5], for more general initial conditions.
Our proofs are quite complex and are based on the time-periodic unfolding technique (see [3]), recently developed as a generalization of the one introduced in [16]. Indeed, the homogenization procedure calls for the creation of non-standard test functions for the weak formulation of our problem, which are inspired from the construction in [13, 17], for the elliptic case. Nevertheless, our case is more complicated, due to the presence of the aforementioned time-dependence. In particular, the presence of oscillations in time implies that there is no variational formulation for the limiting two-scale problems, contrarily to what happens in [17]. For this reason, it is not possible to directly get uniqueness for the macroscopic two-scale system (4.13)–(4.14) and for the two-scale problem (4.59).
In the case of the system (4.13)–(4.14), which corresponds to $\alpha = 1$, we are forced to provide a proof based on a highly non-standard factorization procedure which, however, leads to a standard parabolic problem, whence uniqueness can ultimately be recovered (see Subsection 4.1 and, in particular, formula (4.44)). On the contrary, the problem (4.59), which corresponds to the case $\alpha > 1$, cannot be treated in its full generality, so that we are led to consider a special *factorized* case (see Remark 4.16).

The paper is organized as follows. In Section 2, we introduce the problem and its geometrical setting. In Section 3, we recall the definition and the main properties of the time-periodic unfolding operator. In Section 4, we state and prove our main homogenization results.

2. Preliminaries

2.1. **Geometrical setting.** The typical periodic geometrical setting is displayed in Figure 1. Here, we give, for the sake of clarity, its detailed formal definition.

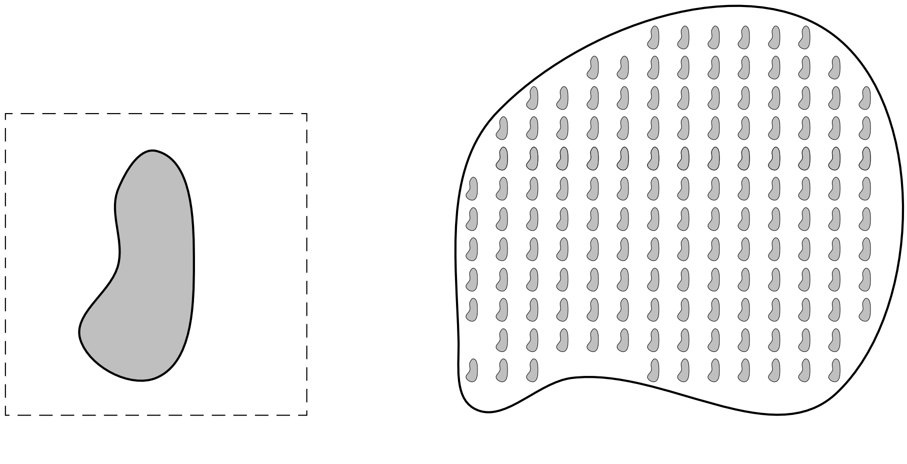

Figure 1. *Left*: the periodic cell Y. E_{v} is the shaded region and E_{s} is the white region. *Right*: the region Ω.

Let us introduce a periodic open subset E of $\mathbb{R}^N$, so that $E+z=E$ for all $z\in\mathbb{Z}^N$. We employ the notation $Y=(0,1)^N$ and $E_{\rm v}=E\cap Y$, $E_{\rm s}=Y\setminus\overline{E}$, $\mathit{\Gamma}=\partial E\cap Y$, so that $E_{\rm v}$ denotes the inclusion (which we assume to be a connected set) in the unit reference cell, while $E_{\rm s}$ is the solid part in the unit reference cell. Further, we stipulate that $\partial E_{\rm v}\cap\partial Y=\emptyset$, so that $\partial E_{\rm v}=\mathit{\Gamma}$.
Let $\mathit{\Omega}$ be an open connected bounded subset of $\mathbb{R}^N$ and $T>0$. We set

$$\Sigma=(0,1)\,,\qquad Q=Y\times\Sigma\,,\qquad Q_{\rm s}=E_{\rm s}\times\Sigma\,,\qquad Q_{\rm v}=E_{\rm v}\times\Sigma\,,\qquad \mathit{\Omega}_T=\mathit{\Omega}\times(0,T)\,,$$

and

$$\Xi_\varepsilon=\left\{\xi\in\mathbb{Z}^N\,,\quad \varepsilon(\xi+Y)\subset\mathit{\Omega}\right\},$$

where $\varepsilon\in(0,1)$ is a small positive parameter, related to the characteristic dimension of the microstructure and which takes values in a sequence of strictly positive numbers tending to zero. For $\xi\in\Xi_\varepsilon$, we define

$$T^\varepsilon_\xi:=\varepsilon(E_{\rm v}+\xi)\,,\qquad \mathit{\Gamma}^\varepsilon_\xi:=\partial T^\varepsilon_\xi\,,\qquad\text{and}\qquad T^\varepsilon=\bigcup_{\xi\in\Xi_\varepsilon}T^\varepsilon_\xi\,;$$

moreover, we set

$$\mathit{\Gamma}^\varepsilon=\partial T^\varepsilon\qquad\text{and}\qquad \mathit{\Omega}_\varepsilon=\mathit{\Omega}\setminus\overline{T^\varepsilon}\,.$$

We assume that $\mathit{\Omega}$ and E have regular boundary. We remark also that $\mathit{\Omega}_\varepsilon$ is connected, while T^ε is disconnected. Finally, let ν denote the normal unit vector to $\mathit{\Gamma}$ pointing into $E_{\rm s}$, extended by periodicity to the whole $\mathbb{R}^N$, so that $\nu_\varepsilon(x)=\nu(x/\varepsilon)$ denotes the normal unit vector to $\mathit{\Gamma}^\varepsilon$ pointing into $\mathit{\Omega}_\varepsilon$.
In the following, by γ we shall denote a strictly positive constant, independent of ε, which may vary from line to line.

2.2. **Position of the problem.** For every $\xi\in\Xi_\varepsilon$, let $\lambda^\xi\in L^\infty(0,T;L^\infty_\#(\Sigma))$ and $A\in L^\infty(\mathit{\Omega}_T;L^\infty_\#(Q))$ be such that

$$\lambda^\xi(t,s)>\overline{\gamma}\,,\quad A(x,t,y,s)\geq\overline{\gamma}\,,\qquad\text{for a.e. }(x,t,y,s)\in\mathit{\Omega}_T\times Q\,,\tag{2.1}$$

with $\overline{\gamma}>0$. Let $K=[\kappa_{ij}]$ be a symmetric matrix such that $\kappa_{ij}\in L^\infty(\mathit{\Omega};L^\infty_\#(Y))$ and there exist $\gamma_0,\widetilde{\gamma}_0>0$ with

$$\gamma_0|\zeta|^2\leq K(x,y)\zeta\cdot\zeta\leq\widetilde{\gamma}_0|\zeta|^2,\qquad\text{for every }\zeta\in\mathbb{R}^N\text{ and a.e. }(x,y)\in\mathit{\Omega}\times Y.\tag{2.2}$$

Moreover, for $\alpha\geq1$, set $\lambda^\xi_\varepsilon(t)=\lambda^\xi(t,\varepsilon^{-\alpha}t)$, $a_\varepsilon(x,t)=A(x,t,\varepsilon^{-1}x,\varepsilon^{-\alpha}t)$ and $\kappa_\varepsilon(x)=K(x,\varepsilon^{-1}x)$ for a.e. $(x,t)\in\mathit{\Omega}_T$, and assume that all these functions are measurable. We give here a complete formulation of the problem described in the Introduction (the operators div and ∇ act only with respect to the space variable x).

Assume that $f\in L^2(\mathit{\Omega}_T)$ and, for every $\varepsilon>0$, let $\overline{u}_{0\varepsilon}\in H^1_0(\mathit{\Omega})$ be such that $\overline{u}_{0\varepsilon}$ is constant (with possibly different values) on each inclusion T^ε_ξ, $\xi\in\Xi_\varepsilon$, and

$$\int_{\mathit{\Omega}}|\nabla\overline{u}_{0\varepsilon}|^2\,{\rm d}x\leq\gamma\,.\tag{2.3}$$

Remark 2.1. Initial data of this type can be obtained following the construction in [13, Proposition 2.1] (see, also, [17, Lemma 4.1]). It is worth pointing out that, in fact, starting from any given $\overline{u}_0 \in H^1_0(\Omega)$, the sequence $\{\overline{u}_{0\varepsilon}\}$ can be chosen in such a way that $\overline{u}_{0\varepsilon} \to \overline{u}_0$ strongly in $L^2(\Omega)$ and also weakly in $H^1_0(\Omega)$. □

Let us consider the problem for $u_\varepsilon(x,t)$ given by

$$a_\varepsilon u_{\varepsilon t} - \operatorname{div}(\kappa_\varepsilon \nabla u_\varepsilon) = f\,, \qquad \text{in } \Omega_\varepsilon \times (0,T); \tag{2.4}$$

$$\lambda^\xi_\varepsilon u_{\varepsilon t} = \frac{1}{\varepsilon^N} \int\limits_{\Gamma^\varepsilon_\xi} \kappa_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \,\mathrm{d}\sigma\,, \qquad \text{on } \Gamma^\varepsilon_\xi \times (0,T),\ \xi \in \Xi_\varepsilon; \tag{2.5}$$

$$u_\varepsilon = 0\,, \qquad \text{on } \partial\Omega \times (0,T); \tag{2.6}$$

$$u_\varepsilon(x,0) = \overline{u}_{0\varepsilon}(x)\,, \qquad \text{on } \Omega. \tag{2.7}$$

Notice that u_ε is spatially constant (with possibly different values) a.e. on each Γ^ε_ξ, $\xi \in \Xi_\varepsilon$; hence, we can extend it inside T^ε_ξ by means of these constant values and, for the sake of simplicity, we will denote by u_ε both the original function and its extension to the whole of Ω.
Let us denote by H^ε the space

$$H^\varepsilon := \{u \in \mathcal{C}\left([0,T]; L_\varepsilon\right) \cap L^2\left(0,T; W^\varepsilon_0\right) \ :\ u_t \in L^2\big(0,T; (W^\varepsilon_0)'\big)\}\,, \tag{2.8}$$

where (as in [17])

$$L_\varepsilon = \{u \in L^2(\Omega) \ :\ u\,|_{T^\varepsilon_\xi},\ \text{with } \xi \in \Xi_\varepsilon, \text{ is a constant function with the constant depending on } \xi\}$$

and

$$W^\varepsilon_0 = \{u \in H^1_0(\Omega) \ :\ u\,|_{T^\varepsilon_\xi},\ \text{with } \xi \in \Xi_\varepsilon, \text{ is a constant function with the constant depending on } \xi\}.$$

We remark that, if $u_\varepsilon \in H^\varepsilon$ is solution of problem (2.4)–(2.7), it satisfies in a suitable sense

$$\int\limits_0^T\!\!\int\limits_{\Omega_\varepsilon} a_\varepsilon u_{\varepsilon t} \phi \,\mathrm{d}x\,\mathrm{d}t + \int\limits_0^T\!\!\int\limits_{\Omega_\varepsilon} \kappa_\varepsilon \nabla u_\varepsilon \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t + \sum_{\xi\in\Xi_\varepsilon} \int\limits_0^T\!\!\int\limits_{\Gamma^\varepsilon_\xi} \kappa_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \phi \,\mathrm{d}\sigma\,\mathrm{d}t = \int\limits_0^T\!\!\int\limits_{\Omega_\varepsilon} f\phi \,\mathrm{d}x\,\mathrm{d}t\,, \tag{2.9}$$

for every test function $\phi \in \mathcal{C}^\infty(\Omega_T)$ such that ϕ has compact support in Ω for every $t \in (0,T)$.
In order to take into account the full strong formulation of problem (2.4)–(2.7), we need to restrict the class of admissible test functions, introducing the set

$$\mathcal{X}^\varepsilon := \{\phi_\varepsilon \in \mathcal{C}^\infty(\overline{\Omega}_T) \ :\ \phi_\varepsilon \text{ has compact support in } \Omega \text{ for every } t \in (0,T),\ \phi_\varepsilon \text{ is spatially constant on each } T^\varepsilon_\xi,\ \xi \in \Xi_\varepsilon\}. \tag{2.10}$$

Then, we can write the weak formulation of problem (2.4)–(2.7) in the following way:

$$\int_0^T\int_{\Omega_\varepsilon} a_\varepsilon u_{\varepsilon t}\phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_\varepsilon} \kappa_\varepsilon \nabla u_\varepsilon \cdot \nabla\phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{|E_{\mathrm{v}}|}\int_0^T\int_{T^\varepsilon} \lambda_\varepsilon\, u_{\varepsilon t}\phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_0^T\int_{\Omega_\varepsilon} f\phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t\,, \quad (2.11)$$

where $\lambda_\varepsilon = \lambda_\varepsilon(t, [\varepsilon^{-1}x], \varepsilon^{-\alpha}t)$ is such that, for a.e. $t \in (0,T)$, $\lambda_\varepsilon(t, [\varepsilon^{-1}x]_Y, \varepsilon^{-\alpha}t) = \lambda^\xi_\varepsilon(t, \varepsilon^{-\alpha}t)$, when $[\varepsilon^{-1}x]_Y = \xi$, $\xi \in \Xi_\varepsilon$.
Here and below, $[r]$ denotes the integer part of $r \in \mathbb{R}$. For $x \in \mathbb{R}^N$, we define

$$\Big[\frac{x}{\varepsilon}\Big]_Y = \Big(\Big[\frac{x_1}{\varepsilon}\Big], \dots, \Big[\frac{x_N}{\varepsilon}\Big] \Big)\,.$$

Existence for the problem (2.4)–(2.7) for each fixed $\varepsilon > 0$ follows from the approach of [8] (see also Remark 1.2 there), at least for bounded data f and $\overline{u}_{0\varepsilon}$, for non-vanishing λ_ε, even when $\lambda_\varepsilon < 0$. As a difference with [8], we deal with a finite number of well-stirred inclusions rather than with just one, but this point can be easily circumvented by localization. In the case of $\lambda_\varepsilon > 0$, an alternative proof of existence for $\overline{u}_{0\varepsilon} \in H^1_0(\Omega)$ and $f \in L^2(\Omega_T)$ can be based on the energy inequality and on approximating the differential equations with a strictly parabolic equation set in the whole spatial domain, by defining $\kappa_\varepsilon = 1/\delta$ and $a_\varepsilon = \lambda^\xi_\varepsilon/|E_{\mathrm{v}}|$ in each inclusion and then letting δ go to 0.
Taking into account that u_ε is constant on each T^ε_i, up to a standard regularization procedure, we may test (2.4)–(2.5) directly with $u_{\varepsilon t}$ obtaining

$$\int_0^T\int_{\Omega_\varepsilon} a_\varepsilon u^2_{\varepsilon t} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_\varepsilon} \kappa_\varepsilon \nabla u_\varepsilon \cdot \nabla u_{\varepsilon t} \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{|E_{\mathrm{v}}|}\int_0^T\int_{T^\varepsilon} \lambda_\varepsilon u^2_{\varepsilon t} \,\mathrm{d}\sigma\,\mathrm{d}t$$

$$= \int_0^T\int_{\Omega_\varepsilon} f u_{\varepsilon t} \,\mathrm{d}x\,\mathrm{d}t\,. \quad (2.12)$$

Using Gronwall inequality, (2.12) leads to the following energy estimate:

$$\int_0^T\int_{\Omega_\varepsilon} u^2_{\varepsilon t} \,\mathrm{d}x\,\mathrm{d}t + \sup_{t\in(0,T)} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \,\mathrm{d}x \le \gamma\,, \quad (2.13)$$

where γ depends on $\overline{\gamma}, \gamma_0, \widetilde{\gamma}_0, |E_{\mathrm{v}}|, \|\overline{u}_{0\varepsilon}\|^2_{H^1_0(\Omega)}, \|f\|^2_{L^2(\Omega_T)}$, but it is independent of ε.

3. Definition and main properties of the time-periodic unfolding operators

In this section, we define and collect some properties of a time-periodic version (as in [2, 3]) of the space-unfolding operator introduced and developed in [19, 15, 16, 18].

We define

$$\widehat{\Omega}_\varepsilon = \text{interior}\left\{ \bigcup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi+\overline{Y}) \right\} ;$$

$$\widehat{\mathbb{T}}_\varepsilon = \left\{ t\in(0,T) \,\middle|\, \varepsilon^\alpha\left(\left[\frac{t}{\varepsilon^\alpha}\right]+1\right)\le T \right\} , \qquad \Lambda^\varepsilon_T = \widehat{\Omega}_\varepsilon \times \widehat{\mathbb{T}}_\varepsilon .$$

Then, we introduce the space and the space-time cell containing (x,t) as being

$$Y_\varepsilon(x) = \varepsilon\Big(\left[\frac{x}{\varepsilon}\right]_Y + Y\Big) , \qquad Q_\varepsilon(x,t) = \varepsilon\Big(\left[\frac{x}{\varepsilon}\right]_Y + Y\Big) \times \varepsilon^\alpha\Big(\left[\frac{t}{\varepsilon^\alpha}\right]+\Sigma\Big) .$$

We also define

$$\left\{\frac{x}{\varepsilon}\right\}_Y := \frac{x}{\varepsilon} - \left[\frac{x}{\varepsilon}\right]_Y \qquad \text{and} \qquad \left\{\frac{t}{\varepsilon^\alpha}\right\} := \frac{t}{\varepsilon^\alpha} - \left[\frac{t}{\varepsilon^\alpha}\right] ,$$

so that we can write

$$x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right) \qquad \text{and} \qquad t = \varepsilon^\alpha\left(\left[\frac{t}{\varepsilon^\alpha}\right] + \left\{\frac{t}{\varepsilon^\alpha}\right\}\right) .$$

Definition 3.1. For w Lebesgue-measurable on Ω_T, the *time-periodic unfolding operator* $\mathcal{T}_\varepsilon$ is defined as

$$\mathcal{T}_\varepsilon(w)(x,t,y,s) = \begin{cases} w\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, \varepsilon^\alpha\left[\frac{t}{\varepsilon^\alpha}\right] + \varepsilon^\alpha s\right) , & (x,t,y,s)\in\Lambda^\varepsilon_T\times Q , \\ 0 , & \text{otherwise.} \end{cases}$$

□

Clearly, for w_1, w_2 as in Definition 3.1,

$$\mathcal{T}_\varepsilon(w_1 w_2) = \mathcal{T}_\varepsilon(w_1)\mathcal{T}_\varepsilon(w_2) . \tag{3.1}$$

Notice that the operator $\mathcal{T}_\varepsilon$ introduced in Definition 3.1 coincides with the usual unfolding operator defined in [15], when w does not depend on time, and, respectively, with the pure time unfolding operator, when w does not depend on space. We will use the same notation for all these operators, when no confusion arises.
We need also an average operator in space-time.

Definition 3.2. Let w be integrable in Ω_T. The space-time average operator is defined by

$$\mathcal{M}_\varepsilon(w)(x,t) = \begin{cases} \dfrac{1}{\varepsilon^N\varepsilon^\alpha} \displaystyle\int_{Q_\varepsilon(x,t)} w(y,s)\,\mathrm{d}y\,\mathrm{d}s , & \text{if } (x,t)\in\Lambda^\varepsilon_T , \\ 0 , & \text{otherwise.} \end{cases} \tag{3.2}$$

□

Remark 3.3. From our definitions, it follows that

$$\mathcal{M}_\varepsilon(w)(x,t) = \int_Q \mathcal{T}_\varepsilon(w)(x,t,y,s)\,\mathrm{d}y\,\mathrm{d}s = \mathcal{M}_Q(\mathcal{T}_\varepsilon(w))(x,t) , \tag{3.3}$$

where in general $\mathcal{M}_I$ denotes the integral average on the set I. □

In practice, the average operator will be mostly used in connection with the oscillation operator which we define presently.

Definition 3.4. Let w be integrable in Ω_T. The space-time oscillation operator is defined as

$$\mathcal{Z}_\varepsilon(w)(x,y,t,s) = \mathcal{T}_\varepsilon(w)(x,y,t,s) - \mathcal{M}_\varepsilon(w)(x,t)\,. \tag{3.4}$$

□

We collect some properties of the operators defined above.

Proposition 3.5. *The operator $\mathcal{T}_\varepsilon : L^2(\Omega_T) \to L^2(\Omega_T \times Q)$ is linear and continuous. In addition, for all $w \in L^2(\Omega_T)$, we have*

$$\|\mathcal{T}_\varepsilon(w)\|_{L^2(\Omega_T\times Q)} \le \|w\|_{L^2(\Omega_T)} \tag{3.5}$$

and

$$\left| \int_{\Omega_T} w \,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T}\int_{Q} \mathcal{T}_\varepsilon(w)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \right| \le \int_{\Omega_T\setminus\Lambda^\varepsilon_T} |w|\,\mathrm{d}x\,\mathrm{d}t\,. \tag{3.6}$$

Remark 3.6. Notice that, by (3.6), it follows that, for $w \in H^1(\Omega_T)$, we have

$$\mathcal{T}_\varepsilon(w) \to w\,, \qquad \text{strongly in } L^2(\Omega_T \times Q); \tag{3.7}$$

$$\mathcal{T}_\varepsilon(\nabla w) \to \nabla w\,, \qquad \text{strongly in } L^2(\Omega_T \times Q); \tag{3.8}$$

$$\mathcal{T}_\varepsilon(w_t) \to w_t\,, \qquad \text{strongly in } L^2(\Omega_T \times Q). \tag{3.9}$$

□

Lemma 3.7. *Let $\phi \in H^1(\Omega_T \times Q)$ and define*

$$\phi^\varepsilon(x,t) = \phi\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right)\,, \qquad (x,t) \in \Omega_T\,, \tag{3.10}$$

where ϕ has been extended by Q-periodicity to $\Omega_T \times \mathbb{R}^{N+1}$. Then, in $\Omega_T \times Q$,

$$\frac{\partial}{\partial s}\mathcal{T}_\varepsilon(\phi^\varepsilon) = \varepsilon^\alpha \mathcal{T}_\varepsilon\left(\frac{\partial \phi}{\partial t}\right) + \mathcal{T}_\varepsilon\left(\frac{\partial \phi}{\partial s}\right) \tag{3.11}$$

and

$$\nabla_y \mathcal{T}_\varepsilon(\phi^\varepsilon) = \varepsilon \mathcal{T}_\varepsilon\left(\nabla_x \phi\right) + \mathcal{T}_\varepsilon\left(\nabla_y \phi\right)\,. \tag{3.12}$$

Proposition 3.8. *For ϕ measurable on Q, extended by Q-periodicity to the whole of $\mathbb{R}^N \times \mathbb{R}$, define the sequence*

$$\phi^\varepsilon(x,t) = \phi\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right)\,, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}\,.$$

Then,

$$\mathcal{T}_\varepsilon(\phi^\varepsilon)(x,y,t,s) = \begin{cases} \phi(y,s)\,, & (x,y,t,s) \in \Lambda^\varepsilon_T\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{3.13}$$

Moreover, if $\phi \in L^2(Q)$, *as* $\varepsilon \to 0$,

$$\mathcal{T}_\varepsilon(\phi^\varepsilon) \to \phi\,, \qquad \textit{strongly in}\, L^2(\Omega_T \times Q)\,. \tag{3.14}$$

If there exist $\nabla_y \phi\,, \frac{\partial \phi}{\partial s} \in L^2(Q)$, *then*

$$\nabla_y(\mathcal{T}_\varepsilon(\phi^\varepsilon)) \to \nabla_y \phi\,, \quad \textit{strongly in}\, L^2(\Omega_T \times Q)\,, \tag{3.15}$$

$$\frac{\partial}{\partial s}(\mathcal{T}_\varepsilon(\phi^\varepsilon)) \to \frac{\partial \phi}{\partial s}\,, \quad \textit{strongly in}\, L^2(\Omega_T \times Q)\,. \tag{3.16}$$

Proposition 3.9. *Let* $\{w_\varepsilon\}$ *be a sequence of functions in* $L^2(\Omega_T)$.
If $w_\varepsilon \to w$ *strongly in* $L^2(\Omega_T)$ *as* $\varepsilon \to 0$, *then*

$$\mathcal{T}_\varepsilon(w_\varepsilon) \to w\,, \quad \textit{strongly in}\, L^2(\Omega_T \times Q)\,. \tag{3.17}$$

If we only assume that (3.17) *holds true and that* $w_\varepsilon \geq \gamma_1 > 0$, *then we have*

$$\mathcal{T}_\varepsilon(w_\varepsilon^{-1}) \to w^{-1}\,, \qquad \textit{strongly in}\ L^2(\Omega_T \times Q). \tag{3.18}$$

If w_ε *is a bounded sequence of functions in* $L^2(\Omega_T)$, *then, up to a subsequence,*

$$\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup \widehat{w}\,, \quad \textit{weakly in}\, L^2\,(\Omega_T \times Q) \tag{3.19}$$

and

$$w_\varepsilon \rightharpoonup \mathcal{M}_Q(\widehat{w})\,, \quad \textit{weakly in}\, L^2(\Omega_T)\,. \tag{3.20}$$

Remark 3.10. Actually, the only classes for which the strong convergence of the unfolding $\mathcal{T}_\varepsilon(w_\varepsilon)$ is known to hold, even without strong convergence of w_ε, are sums of the following cases: $w_\varepsilon(x,t) = f_1(x,t) f_2(\varepsilon^{-1}x, \varepsilon^{-\alpha}t)$, $w_\varepsilon(x,t) = w(x,t,\varepsilon^{-1}x,\varepsilon^{-\alpha}t)$ with $w \in L^2(Y \times \Sigma; C(\Omega_T))$ or $w \in L^2(\Omega_T; C(Y \times \Sigma))$. In all such cases, $\mathcal{T}_\varepsilon(w_\varepsilon) \to w$ strongly in $L^2(\Omega_T \times Q)$ (see [1, 15, 16]). □

Theorem 3.11. *Let* $\{w_\varepsilon\}$ *be a sequence converging strongly to* w *in* $L^2(0,T;H^1(\Omega))$, *as* $\varepsilon \to 0$. *Then,*

$$\mathcal{T}_\varepsilon(\nabla w_\varepsilon) \to \nabla w\,, \quad \textit{strongly in}\, L^2(\Omega_T \times Q)\,. \tag{3.21}$$

Let $\{w_\varepsilon\}$ *be a sequence converging strongly to* w *in* $H^1(\Omega_T)$, *as* $\varepsilon \to 0$. *Then,*

$$\mathcal{T}_\varepsilon\left(\frac{\partial w_\varepsilon}{\partial t}\right) \to \frac{\partial w}{\partial t}\,, \quad \textit{strongly in}\, L^2(\Omega_T \times Q)\,. \tag{3.22}$$

Theorem 3.12. (See [3, Proposition 2.15]) *Let* $\alpha = 1$. *Let* $\{w_\varepsilon\}$ *be a sequence converging strongly to* w *in* $H^1(\Omega_T)$, *as* $\varepsilon \to 0$. *Then,*

$$\frac{1}{\varepsilon}\mathcal{Z}_\varepsilon(w_\varepsilon) \to y^c \cdot \nabla w + (s - 1/2)\frac{\partial w}{\partial t}\,, \quad \textit{strongly in}\, L^2(\Omega_T; H^1(Q))\,, \tag{3.23}$$

where

$$y^c = \left(y_1 - \frac{1}{2}\,, y_2 - \frac{1}{2}\,, \cdots, y_N - \frac{1}{2}\right)\,. \tag{3.24}$$

Theorem 3.13. (See [3, Theorem 2.20]) *Let $\alpha = 1$ and $\{w_\varepsilon\}$ be a sequence converging weakly to w in $H^1(\Omega_T)$. Then, up to a subsequence, there exists $\hat{w} = \hat{w}(x, y, t, s) \in H^1(\Omega_T \times Q)$, periodic in Q and with $\mathcal{M}_Q(\hat{w}) = 0$, such that, as $\varepsilon \to 0$,*

$$\mathcal{T}_\varepsilon\left(\frac{\partial w_\varepsilon}{\partial t}\right) \rightharpoonup \frac{\partial w}{\partial t} + \frac{\partial \hat{w}}{\partial s}\,, \qquad \textit{weakly in}\, L^2(\Omega_T \times Q)\,, \tag{3.25}$$

$$\mathcal{T}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \hat{w}\,, \qquad \textit{weakly in}\, L^2(\Omega_T \times Q)\,, \tag{3.26}$$

$$\frac{1}{\varepsilon}\mathcal{Z}_\varepsilon(w_\varepsilon) \rightharpoonup y^c \cdot \nabla w + (s - 1/2)\frac{\partial w}{\partial t} + \hat{w}\,, \qquad \textit{weakly in}\, L^2(\Omega_T; H^1(Q))\,. \tag{3.27}$$

Theorem 3.14. (See [3, Proposition 2.14]) *Let $\alpha > 1$ and $\{w_\varepsilon\}$ be a sequence converging strongly to w in $L^2(0, T; H^1(\Omega))$, as $\varepsilon \to 0$. Assume also that the condition*

$$\left\| \frac{\partial w_\varepsilon}{\partial t} \right\|_{L^2(\Omega_T)} \le \gamma \tag{3.28}$$

holds. Then, as $\varepsilon \to 0$, we have

$$\frac{1}{\varepsilon}\mathcal{Z}_\varepsilon(w_\varepsilon) \to y^c \cdot \nabla w\,, \quad \textit{strongly in}\, L^2(\Omega_T; H^1(Q))\,, \tag{3.29}$$

where y^c is defined in (3.24).

Theorem 3.15. *Let $\alpha > 1$. Assume that $w_\varepsilon \rightharpoonup w$ weakly in $H^1(\Omega_T)$. Then, up to a subsequence, there exist $\widetilde{w} \in L^2(\Omega_T; H^1_{per}(Q))$, with $\mathcal{M}_Q(\widetilde{w}) = 0$ and $\frac{\partial \widetilde{w}}{\partial s} = 0$, and $\mathring{w} \in L^2(\Omega_T \times Y; H^1(\Sigma))$, with $\mathcal{M}_\Sigma(\mathring{w}) = 0$, such that, as $\varepsilon \to 0$, we have*

$$\mathcal{T}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widetilde{w}\,, \qquad \textit{weakly in}\, L^2(\Omega_T \times Q)\,, \tag{3.30}$$

$$\frac{1}{\varepsilon}\mathcal{Z}_\varepsilon(w_\varepsilon) \rightharpoonup y^c \cdot \nabla w + \widetilde{w}\,, \qquad \textit{weakly in}\, L^2(\Omega_T; H^1(Q))\,, \tag{3.31}$$

$$\mathcal{T}_\varepsilon\left(\frac{\partial w_\varepsilon}{\partial t}\right) \rightharpoonup \frac{\partial w}{\partial t} + \frac{\partial \mathring{w}}{\partial s}\,, \qquad \textit{weakly in}\, L^2(\Omega_T \times Q)\,, \tag{3.32}$$

$$\frac{1}{\varepsilon^\alpha}(\mathcal{T}_\varepsilon(w_\varepsilon) - \mathcal{M}_\Sigma(w_\varepsilon)) \rightharpoonup (s - 1/2)\frac{\partial w}{\partial t} + \mathring{w}\,, \qquad \textit{weakly in}\, L^2(\Omega_T \times Q)\,. \tag{3.33}$$

Proof. Properties (3.30) and (3.31) are proven in [3, Theorem 2.16]. In order to prove (3.32) and (3.33), we first notice that the weak $H^1(\Omega_T)$-convergence of the sequence $\{w_\varepsilon\}$ implies that condition (3.28) is satisfied in this case, as well. Then, we can appeal to Poincaré-Wirtinger inequality in Σ. Indeed, we have

$$\left\| \frac{1}{\varepsilon^\alpha}(\mathcal{T}_\varepsilon(w_\varepsilon) - \mathcal{M}_\Sigma(w_\varepsilon)) - (s - 1/2)\frac{\partial w}{\partial t} \right\|_{L^2(\Omega_T \times Q)} \le$$
$$\gamma \left\| \frac{1}{\varepsilon^\alpha}\frac{\partial}{\partial s}\mathcal{T}_\varepsilon(w_\varepsilon) - \frac{\partial w}{\partial t} \right\|_{L^2(\Omega_T \times Q)} = \gamma \left\| \mathcal{T}_\varepsilon\left(\frac{\partial w_\varepsilon}{\partial t}\right) - \frac{\partial w}{\partial t} \right\|_{L^2(\Omega_T \times Q)} \le \gamma\,,$$

where we used (3.5) and (3.28). Therefore, there exists $\mathring{w} \in L^2(\Omega_T \times Q)$ such that

$$\frac{1}{\varepsilon^\alpha}(\mathcal{T}_\varepsilon(w_\varepsilon) - \mathcal{M}_\Sigma(w_\varepsilon)) - (s - 1/2)\frac{\partial w}{\partial t} \rightharpoonup \mathring{w}\,,$$

which implies

$$\mathcal{T}_\varepsilon\left(\frac{\partial w_\varepsilon}{\partial t}\right) = \frac{1}{\varepsilon^\alpha}\frac{\partial}{\partial s}\mathcal{T}_\varepsilon(w_\varepsilon) = \frac{1}{\varepsilon^\alpha}\frac{\partial}{\partial s}\left(\mathcal{T}_\varepsilon(w_\varepsilon) - \mathcal{M}_\Sigma(w_\varepsilon)\right) \rightharpoonup \frac{\partial w}{\partial t} + \frac{\partial \mathring{w}}{\partial s}\,.$$

Since, by construction, $\mathcal{M}_\Sigma\left(\frac{1}{\varepsilon^\alpha}(\mathcal{T}_\varepsilon(w_\varepsilon) - \mathcal{M}_\Sigma(w_\varepsilon)) - (s-1/2)\frac{\partial w}{\partial t}\right) = 0$, we immediately get $\mathcal{M}_\Sigma(\mathring{w}) = 0$. □

4. Homogenization

Our goal in this section is to describe the asymptotic behavior, as $\varepsilon \to 0$, of the solution u_ε of problem (2.4)–(2.7), for $\alpha \geq 1$. To this end, in the following, we will assume

$$\begin{aligned} \mathcal{T}_\varepsilon(\kappa_\varepsilon) &\to \kappa, && \text{strongly in } L^2(\Omega \times Y); \\ \mathcal{T}_\varepsilon(a_\varepsilon) &\to a, && \text{strongly in } L^2(\Omega_T \times Q); \\ \mathcal{T}_\varepsilon(\lambda_\varepsilon) &\to \lambda, && \text{strongly in } L^2(\Omega_T \times Q). \end{aligned} \tag{4.1}$$

Since the techniques are different for the case $\alpha = 1$ and $\alpha > 1$, we shall split the analysis of our problem in two different subsections.

4.1. The case $\alpha = 1$.

We state the following compactness result.

Lemma 4.1. *Assume that $\|\overline{u}_{0\varepsilon}\|_{H^1_0(\Omega)} \leq \gamma$, with γ independent of ε, and that, for every $\varepsilon > 0$, u_ε is the unique solution of problem* (2.11)*. Then, up to a subsequence, still denoted by ε, there exist $u \in L^2(0,T;H^1_0(\Omega)) \cap H^1(\Omega_T)$, $u_1 \in L^2(\Omega_T; H^1_{per}(Q))$, $\mathcal{M}_Q(u_1) = 0$, such that*

$$u_\varepsilon \to u \qquad \text{strongly in } L^2(\Omega_T), \tag{4.2}$$

$$u_\varepsilon \rightharpoonup u \qquad \text{weakly in } H^1(\Omega_T), \tag{4.3}$$

$$\mathcal{T}_\varepsilon(u_{\varepsilon t}) \rightharpoonup u_t + u_{1s} \qquad \text{weakly in } L^2(\Omega_T \times Q), \tag{4.4}$$

$$\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla u + \nabla_y u_1 \qquad \text{weakly in } L^2(\Omega_T \times Q_{\mathrm{s}}), \tag{4.5}$$

$$\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup 0 \qquad \text{weakly in } L^2(\Omega_T \times Q_{\mathrm{v}}), \tag{4.6}$$

$$\frac{1}{\varepsilon}\mathcal{Z}_\varepsilon(u_\varepsilon) \rightharpoonup y^c \cdot \nabla u + (s-1/2)u_t + u_1 \qquad \text{weakly in } L^2(\Omega_T \times Q), \tag{4.7}$$

$$y^c \cdot \nabla u + (s-1/2)u_t + u_1 \qquad \text{is independent of } y \text{ on } \Omega_T \times Q_{\mathrm{v}}. \tag{4.8}$$

Remark 4.2. Following [14, Remark 1.11], with a slight abuse of notation, in (4.5) $\mathcal{T}_\varepsilon$ stands for the restriction to Q_{s} of the unfolding operator defined above and in (4.6) $\mathcal{T}_\varepsilon$ stands for the restriction to Q_{v}, respectively. The same notation will be used also in the following. □

Remark 4.3. The assertion in (4.8) is a consequence of the total flux condition given in (2.5). However, since the second term in (4.8) is, in fact, independent of y, such a condition is prescribed only for $y^c \cdot \nabla u + u_1$. □

Proof. Assertions (4.2) and (4.3) are direct consequences of the energy estimate (2.13). Assertions (4.4)–(4.7) follow from Theorem 3.13, where we have taken into account that $\nabla u_\varepsilon = 0$ a.e. in $T^\varepsilon \times (0,T)$, since u_ε is spatially constant in each T^ε_ξ, $\xi \in \Xi_\varepsilon$, for a.e. $t \in (0,T)$. Finally, for the same reason, we get that $y^c \cdot \nabla u + (s-1/2)u_t + u_1$ is independent of y on $\Omega_T \times Q_{\rm v}$. Indeed, (4.8) is a direct consequence of (4.6) and (4.7), recalling that

$$\nabla_y \mathcal{Z}_\varepsilon(u_\varepsilon) = \nabla_y \left[\mathcal{T}_\varepsilon(u_\varepsilon) - \mathcal{M}^\varepsilon(u_\varepsilon)\right] = \nabla_y \mathcal{T}_\varepsilon(u_\varepsilon) = \varepsilon \mathcal{T}_\varepsilon(\nabla u_\varepsilon)\,, \tag{4.9}$$

(see also [17, Proof of Proposition 4.1]). □

For later use, we set

$$H^\Gamma_\#(Y) := \{\psi \in H^1_\#(Y) \,:\, \psi \text{ is constant on } E_{\rm v}\} \tag{4.10}$$

and

$$\mathcal{H}^\Gamma(\Omega_T;Q) := L^2((0,T)\times Q; H^1_0(\Omega)) \cap H^1(\Omega_T;L^2(Q)) \cap L^2(\Omega_T;H^\Gamma_{\rm per}(Q))\,, \tag{4.11}$$

where $H^\Gamma_{\rm per}(Q)$ is the space of the $H^1_{\rm per}(Q)$-functions which are independent of y on $E_{\rm v}$ a.e. in $\Omega_T \times \Sigma$. Moreover, we introduce the space

$$\begin{aligned} W(\Omega_T;Q) := \{(w,w^1) \;:\; w \in L^2(0,T;H^1_0(\Omega)) \cap H^1(\Omega_T),\; w^1 \in L^2(\Omega_T;H^1_{\rm per}(Q))\,, \\ \mathcal{M}_Q(w^1) = 0\,,\; y^c\cdot\nabla w + (s-1/2)w_t + w^1 \text{ is independent of } y \text{ on } \Omega_T \times Q_{\rm v}\}. \end{aligned} \tag{4.12}$$

Notice that the pair (u,u^1) given in Lemma 4.1 belongs to the space $W(\Omega_T;Q)$.

Theorem 4.4. *Assume that (4.1) holds. Assume also that $\|\overline{u}_{0\varepsilon}\|_{H^1_0(\Omega)} \le \gamma$, with γ independent of ε, and that there exists a function $\overline{u}_0 \in H^1_0(\Omega)$ such that $\overline{u}_{0\varepsilon} \to \overline{u}_0$, strongly in $L^2(\Omega)$. Then, the pair $(u,u^1) \in W(\Omega_T;Q)$, appearing in the statement of Lemma 4.1, is a weak solution of the following two-scale system:*

$$\int\!\!\int_{\Omega_T Q_{\rm s}} \kappa(\nabla u + \nabla_y u^1)\cdot\nabla_y \Psi \,{\rm d}y\,{\rm d}s\,{\rm d}x\,{\rm d}t = 0\,, \tag{4.13}$$

$$\begin{aligned} &\int\!\!\int_{\Omega_T E_{\rm s}} a(u_t + u^1_s) w \,{\rm d}y\,{\rm d}x\,{\rm d}t \\ &\quad + \int\!\!\int_{\Omega_T E_{\rm s}} \kappa(\nabla u + \nabla_y u^1)\cdot\left[\nabla w - \nabla_y\big((y^c\cdot\nabla w + (s-1/2)w_t)\psi\big)\right]{\rm d}y\,{\rm d}x\,{\rm d}t \\ &+ \frac{1}{|E_{\rm v}|}\int\!\!\int_{\Omega_T E_{\rm v}} \lambda(u_t + u^1_s) w\,{\rm d}y\,{\rm d}x\,{\rm d}t = \int\!\!\int_{\Omega_T E_{\rm s}} f w\,{\rm d}y\,{\rm d}x\,{\rm d}t\,, \qquad \text{for a.e. } s \in \Sigma, \end{aligned} \tag{4.14}$$

for every $\Psi \in \mathcal{H}^\Gamma(\Omega_T;Q)$, $w \in L^2(0,T;H^1_0(\Omega)) \cap H^1(\Omega_T)$ and $\psi \in H^1_0(Y)$, such that $\psi \equiv 1$ in $E_{\rm v}$, with the initial condition $u(x,0) = \overline{u}_0(x)$, a.e. in Ω.

Proof. Similarly to [13, 17], we can take as test function in (2.11) $\phi_\varepsilon(x,t) = \varepsilon\phi(x,t,\varepsilon^{-1}x,\varepsilon^{-1}t)$, where

$$\phi(x,t,y,s) = z(s)[\mathcal{M}^\varepsilon(w)(x,t)\psi(y) + w(x,t)\varphi(y)] \tag{4.15}$$

with $z \in \mathcal{C}^\infty_\#(\Sigma)$, $w \in \mathcal{C}^\infty([0,T];\mathcal{C}^\infty_c(\Omega))$, $\psi \in \mathcal{C}^\infty_c(Y) \cap H^\Gamma_\#(Y)$ and $\varphi \in \mathcal{C}^\infty_\#(\overline{Y})$, with $\varphi\,|_{E_\mathrm{v}} = 0$. This implies

$$
\begin{aligned}
\varepsilon \int_0^T\!\!\int_{\Omega_\varepsilon} & a_\varepsilon u_{\varepsilon t} z[\mathcal{M}^\varepsilon(w)\psi + w\varphi]\,\mathrm{d}x\,\mathrm{d}t \\
& + \int_0^T\!\!\int_{\Omega_\varepsilon} \kappa_\varepsilon \nabla u_\varepsilon \cdot [\mathcal{M}^\varepsilon(w)\nabla_y\psi + \varepsilon\nabla_x w\varphi + w\nabla_y\varphi] z\,\mathrm{d}x\,\mathrm{d}t \\
+ \frac{\varepsilon}{|E_\mathrm{v}|} \int_0^T\!\!\int_{T^\varepsilon} & \lambda_\varepsilon u_{\varepsilon t} z[\mathcal{M}^\varepsilon(w)\psi + w\varphi]\,\mathrm{d}x\,\mathrm{d}t = \varepsilon \int_0^T\!\!\int_{\Omega_\varepsilon} f[\mathcal{M}^\varepsilon(w)\psi + w\varphi] z\,\mathrm{d}x\,\mathrm{d}t\,. \qquad (4.16)
\end{aligned}
$$

Unfolding and then passing to the limit for $\varepsilon \to 0$, we get

$$
\begin{aligned}
& \int_{\Omega_T}\!\!\int_{Q_\mathrm{s}} \kappa(\nabla u + \nabla_y u^1) \cdot \nabla_y(\psi + \varphi) w z\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \longleftarrow \\
& \int_{\Omega_T}\!\!\int_{Q_\mathrm{s}} \mathcal{T}_\varepsilon(\kappa_\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \cdot [\mathcal{T}_\varepsilon(\mathcal{M}^\varepsilon(w)\nabla_y\psi) + \mathcal{T}_\varepsilon(w\nabla_y\varphi)]\mathcal{T}_\varepsilon(z)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t = O(\varepsilon) \to 0\,.
\end{aligned}
\tag{4.17}
$$

Taking into account that a general function in $\mathcal{C}^\infty_\#(\overline{Y}) \cap H^\Gamma_\#(Y)$ can always be split in the form $\psi + \varphi$, with ψ, φ as before, and recalling the density of product functions in $\mathcal{H}^\Gamma(\Omega_T; Q)$, we obtain exactly (4.13).
Now we take as test function in (2.11) $\phi_\varepsilon(x,t) = \phi(x,t,\varepsilon^{-1}x,\varepsilon^{-1}t)$, where

$$
\phi(x,t,y,s) = z(s)[\mathcal{M}^\varepsilon(w)(x,t)\psi(y) + w(x,t)(1 - \psi(y))] \tag{4.18}
$$

with z, w, ψ as in (4.15) and $\psi \equiv 1$ on E_v. Clearly, $\mathcal{T}_\varepsilon(\phi_\varepsilon) \to zw$ strongly in $L^2(\Omega_T \times Q)$. From the weak formulation, it follows

$$
\begin{aligned}
\int_0^T\!\!\int_{\Omega_\varepsilon} & a_\varepsilon u_{\varepsilon t} z[\mathcal{M}^\varepsilon(w)\psi + w(1-\psi)]\,\mathrm{d}x\,\mathrm{d}t \\
& + \int_0^T\!\!\int_{\Omega_\varepsilon} \kappa_\varepsilon \nabla u_\varepsilon \cdot \left[\frac{1}{\varepsilon}(\mathcal{M}^\varepsilon(w) - w)\nabla_y\psi + \nabla w(1-\psi)\right] z\,\mathrm{d}x\,\mathrm{d}t \\
+ \frac{1}{|E_\mathrm{v}|} \int_0^T\!\!\int_{T^\varepsilon} & \lambda_\varepsilon u_{\varepsilon t} z \mathcal{M}^\varepsilon(w)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\int_{\Omega_\varepsilon} f z[\mathcal{M}^\varepsilon(w)\psi + w(1-\psi)]\,\mathrm{d}x\,\mathrm{d}t\,. \qquad (4.19)
\end{aligned}
$$

Unfolding and then passing to the limit for $\varepsilon \to 0$, we get

$$\iint_{\Omega_T Q_s} a(u_t + u^1_s) w z \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_{\Omega_T Q_s} \kappa(\nabla u + \nabla_y u^1) \cdot [\nabla w - \nabla_y\big((y^c \cdot \nabla w + (s-1/2)w_t)\psi\big)] z \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \frac{1}{|E_v|} \iint_{\Omega_T Q_v} \lambda(u_t + u^1_s) z w \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t = \iint_{\Omega_T Q_s} f z w \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\,, \quad (4.20)$$

where we used Lemma 4.1 and we took into account that, by (3.23), $\frac{1}{\varepsilon}(\mathcal{M}^\varepsilon(w) - \mathcal{T}_\varepsilon(w)) \to -y^c \cdot \nabla w - (s-1/2)w_t$ strongly in $L^2(\Omega \times Y)$. Localizing the previous equation with respect to s, we get (4.14).
The initial condition can be easily recovered since $u_\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega_T)$. □

Corollary 4.5. *Under the assumptions of Theorem 4.4, the pair $(u, u^1) \in W(\Omega_T; Q)$ is a weak solution of the two-scale problem*

$$\iint_{\Omega_T E_s} a(u_t + u^1_s) w \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega_T E_s} \kappa(\nabla u + \nabla_y u^1) \cdot (\nabla w + \nabla_y w^1) \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \frac{1}{|E_v|} \iint_{\Omega_T E_v} \lambda(u_t + u^1_s) w \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = |E_s| \int_{\Omega_T} f w \,\mathrm{d}x\,\mathrm{d}t\,, \quad (4.21)$$

for every $(w, w^1) \in W(\Omega_T; Q)$, for a.e. $s \in \Sigma$, with the initial condition $u(x,0) = \overline{u}_0(x)$, a.e. in Ω.

Proof. We take in (4.13) a test function $\Psi(x,t,y,s)z(s)$, where $\Psi \in \mathcal{H}^\Gamma(\Omega_T; Q)$ and $z(s)$ is the same function appearing in (4.20). Thus, we obtain

$$\iint_{\Omega_T Q_s} \kappa(\nabla u + \nabla_y u^1) \cdot \nabla_y \Psi z \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t = 0\,.$$

Summing this last equation with (4.20), it follows

$$\iint_{\Omega_T Q_s} a(u_t + u^1_s) w z \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_{\Omega_T Q_s} \kappa(\nabla u + \nabla_y u^1) \cdot [\nabla w + \nabla_y\big(\Psi - (y^c \cdot \nabla w + (s-1/2)w_t)\psi\big)] z \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \frac{1}{|E_v|} \iint_{\Omega_T Q_v} \lambda(u_t + u^1_s) z w \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t = \iint_{\Omega_T Q_s} f z w \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\,, \quad (4.22)$$

which can be written as

$$\iint_{\Omega_T Q_s} a(u_t + u^1_s) wz \, dy \, ds \, dx \, dt + \iint_{\Omega_T Q_s} \kappa(\nabla u + \nabla_y u^1) \cdot (\nabla w + \nabla_y w^1) z \, dy \, ds \, dx \, dt$$

$$+ \frac{1}{|E_v|} \iint_{\Omega_T Q_v} \lambda(u_t + u^1_s) wz \, dy \, ds \, dx \, dt = \iint_{\Omega_T Q_s} fwz \, dy \, ds \, dx \, dt \,, \quad (4.23)$$

by setting

$$w^1(x,t,y,s) = \Psi(x,t,y,s) - (y^c \cdot \nabla w + (s - 1/2) w_t) \psi(y)$$
$$- \int_Q \big[\Psi(x,t,y,s) - (y^c \cdot \nabla w + (s-1/2) w_t) \psi(y) \big] \, dy \, ds, \quad (4.24)$$

which gives (4.21), after localizing with respect s. □

We remark that, even in the form given in Corollary 4.5, due to the presence of u^1_s, our homogenized two-scale problem is not variational and, therefore, we are not able to prove a direct uniqueness result for it. To overcome this difficulty, we are forced to pass to the single-scale formulation (4.44), which cannot be obtained directly from equation (4.21), as usual. Then, we are led to provide a non-standard factorization procedure, which allows us to remove the residual microscopic term u^1_s appearing in the macroscopic part of the above mentioned equation.

Lemma 4.6. *Assume that κ is as in Theorem 4.4 and that $u \in L^2(0,T;\, H^1_0(\Omega)) \cap H^1(\Omega_T)$ is the function given by Lemma 4.1. Let $v_1, v_2 \in L^2(\Omega_T; H^1_{per}(Q))$, with null mean average over Q, be two solutions of the problem*

$$\begin{aligned} & -\operatorname{div}_y(\kappa(\nabla u + \nabla_y v)) = 0, && \text{in } \Omega_T \times Q_s; \\ & \int_\Gamma \kappa(\nabla u + \nabla_y v) \cdot \nu \, d\sigma = 0 && \text{in } \Omega_T \times \Sigma; \\ & y^c \cdot \nabla u + (s-1/2) u_t + v \quad \text{is independent of } y && \text{on } \Omega_T \times Q_v. \end{aligned} \quad (4.25)$$

Then, there exists a function $\overline{\chi} = \overline{\chi}(x,t,s) \in L^2(\Omega_T; H^1_\#(\Sigma))$, with $\mathcal{M}_\Sigma(\overline{\chi}) = 0$, such that $v_1(x,t,y,s) = v_2(x,t,y,s) + \overline{\chi}(x,t,s)$ a.e. in $\Omega_T \times Q$.

Proof. Set $V = v_1 - v_2$. Clearly, $\mathcal{M}_Q(V) = 0$ and V satisfies

$$-\operatorname{div}_y(\kappa \nabla_y V) = 0, \qquad \text{in } \Omega_T \times Q_s; \quad (4.26)$$

$$\int_\Gamma \kappa \nabla_y V \cdot \nu \, d\sigma = 0, \qquad \text{in } \Omega_T \times \Sigma; \quad (4.27)$$

moreover, V is independent of y on $\Omega_T \times Q_v$. Then, taking V as test function in (4.26) and using (4.27), by the coercivity of κ, it follows that $\nabla_y V = 0$ in $\Omega_T \times Q_s$, which implies that $\nabla_y V = 0$ in the whole $\Omega_T \times Q$. Therefore, there exists $\overline{\chi} \in L^2(\Omega_T; H^1_\#(\Sigma))$ such that $V(x,t,y,s) = \overline{\chi}(x,t,s)$ a.e. in $\Omega_T \times Q$ and $\mathcal{M}_\Sigma(\overline{\chi}) = 0$. □

Notice that, given u as in Lemma 4.1, the corresponding corrector u^1 is a solution of problem (4.25); therefore, it is uniquely determined up to a function $\overline{\chi}$ depending only on (x, t, s).

Lemma 4.7. *Let κ be as in Theorem 4.4. For $j = 1, \dots, N$, let us consider the problem*

$$\int_Y \kappa(x,y)\nabla_y(\chi^j(x,y) - y_j) \cdot \nabla_y \varphi \, \mathrm{d}y = 0 \,, \qquad \forall \varphi \in H^{\Gamma}_{\#}(Y) \,; \tag{4.28}$$

$$\chi^j(x,y) - y_j \qquad \textit{is independent of } y \textit{ on } E_{\mathrm{v}}; \tag{4.29}$$

$$\int_Y \chi^j(x,y) \, \mathrm{d}y = 0 \,, \tag{4.30}$$

where $H^{\Gamma}_{\#}(Y)$ has been defined in (4.10)*. Then, problem* (4.28)–(4.30) *admits a unique solution $\chi^j \in L^\infty(\Omega; H^1_{\#}(Y))$.*

Remark 4.8. We point out that the strong formulation of the problem above is given by

$$-\operatorname{div}_y \Big(\kappa(x,y)\nabla_y(\chi^j(x,y) - y_j) \Big) = 0 \,, \qquad \text{in } E_{\mathrm{s}}; \tag{4.31}$$

$$\int_\Gamma \kappa(x,y)\nabla_y(\chi^j(x,y) - y_j) \cdot \nu \, \mathrm{d}\sigma = 0 \,; \tag{4.32}$$

$$\chi^j(x,y) - y_j \qquad \text{is independent of } y \text{ on } E_{\mathrm{v}}; \tag{4.33}$$

$$\chi^j(x,\cdot) \quad \text{is } Y\text{-periodic}; \tag{4.34}$$

$$\int_Y \chi^j(x,y) \, \mathrm{d}y = 0 \,. \tag{4.35}$$

Notice that condition (4.32) is automatically satisfied, as a consequence of the weak formulation (4.28). □

This cell problem is rather classical and has a long history. It appears, for instance, in [20] or, more recently, in [22], for the case where E_{v} has more than one connected component, with regular boundary. A similar result was proven independently also in [4]. On the other hand, we refer to the recent proof in [17], for the case of multiple holes without regularity assumptions. In this paper, we consider a simpler geometry, where there is only one smooth inclusion inside the elementary cell Y, providing here an alternative direct proof.

Proof. Clearly, problem (4.31)–(4.35) has uniqueness. In order to prove existence, we consider, for $j = 1, \dots, N$, the following auxiliary problem

$$- \operatorname{div}_y \Big(\kappa(x,y) \nabla_y (\widetilde{\chi}^j(x,y) - y_j) \Big) = 0 \,, \quad \text{in } E_{\mathrm{s}}; \tag{4.36}$$

$$\int_{\Gamma} \kappa(x,y) \nabla_y (\widetilde{\chi}^j(x,y) - y_j) \cdot \nu \, \mathrm{d}\sigma = 0 \,; \tag{4.37}$$

$$\widetilde{\chi}^j(x,y) = y_j \qquad \text{on } \Gamma; \tag{4.38}$$

$$\widetilde{\chi}^j(x,\cdot) \quad \text{is } Y\text{-periodic.} \tag{4.39}$$

Existence and uniqueness of a solution $\widetilde{\chi}^j(x,\cdot) \in H^1_{\#}(E_{\mathrm{s}})$ for problem (4.36)–(4.39) is classical, the proof being based on a suitable version of Lax-Milgram Lemma. Therefore, let us extend $\widetilde{\chi}^j$ as y_j in E_{v}, denoting also this extension by $\widetilde{\chi}^j$, so that $\widetilde{\chi}^j(x,\cdot) \in H^1_{\#}(Y)$, and set

$$-c_j(x) = \int_{Y} \widetilde{\chi}^j(x,y) \, \mathrm{d}y \,. \tag{4.40}$$

Then, it is easy to see that $\chi^j = \widetilde{\chi}^j + c_j$ verifies (4.31) and (4.33)–(4.35). Moreover, it satisfies also (4.32), due to (4.31) and the Y-periodicity of the function $\kappa(x,y)\nabla_y(\chi^j(x,y) - y_j)$. Therefore, it is the unique solution of problem (4.28)–(4.30). Since the dependence of χ^j on x is only parametric, it is easy to see that $\chi^j \in L^\infty(\Omega; H^\Gamma_{\#}(Y))$. □

Remark 4.9. From Lemmas 4.6 and 4.7, we can factorize the corrector u^1 as

$$u^1(x,t,y,s) = \widetilde{u}^1(x,t,y) + \overline{\chi}(x,t,s) \,, \tag{4.41}$$

where

$$\widetilde{u}^1(x,t,y) = -\chi^j(x,y) \partial_j u(x,t) \,. \tag{4.42}$$

Indeed, by construction, $\widetilde{u}^1$ is a solution of problem (4.25). Inserting this factorization in the homogenized two-scale problem (4.21), after gluing the first and the third integral, it can be written in the simplified form

$$\int_{\Omega_T} \left(\int_{E_{\mathrm{s}}} a \, \mathrm{d}y + \lambda \right) (u_t + \overline{\chi}_s) w \, \mathrm{d}x \, \mathrm{d}t + \int_{\Omega_T} \int_{E_{\mathrm{s}}} \kappa (\nabla u + \nabla_y \widetilde{u}^1) \cdot (\nabla w + \nabla_y w^1) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t$$
$$= |E_{\mathrm{s}}| \int_{\Omega_T} f w \, \mathrm{d}x \, \mathrm{d}t \,, \qquad \text{for a.e. } s \in \Sigma, \tag{4.43}$$

for every $(w, w^1) \in W(\Omega_T; Q)$. □

Taking into account (4.41)–(4.42), the two-scale system (4.13)–(4.14) can be decoupled, leading to the main result of this paper.

Theorem 4.10. *The function $u \in L^2\big(0,T;H^1_0(\Omega)\big)\cap H^1(\Omega_T)$, given in Lemma 4.1, is the unique solution of the following single-scale problem*

$$\begin{aligned}&\frac{1}{\mathcal{M}_\Sigma(\mu)}u_t - \operatorname{div}(A_{\mathrm{hom}}\nabla u) = |E_{\mathrm{s}}|f\,, && \text{in } \Omega_T;\\ &u(x,0)=\overline{u}_0\,, && \text{in } \Omega,\end{aligned} \tag{4.44}$$

where the symmetric homogenized matrix A_{hom} is given by

$$A^{ij}_{\mathrm{hom}}(x) = \int_{E_{\mathrm{s}}} \kappa\nabla_y(y_i-\chi^i)\cdot\nabla_y(y_j-\chi^j)\,\mathrm{d}y = \int_Y \kappa\nabla_y(y_i-\chi^i)\cdot\nabla_y(y_j-\chi^j)\,\mathrm{d}y \tag{4.45}$$

and μ is defined by

$$\mu(x,t,s) = \left(\int_{E_{\mathrm{s}}} a(x,t,y,s)\,\mathrm{d}y + \lambda(x,t,s)\right)^{-1}. \tag{4.46}$$

Proof. We consider the second term in the left-hand side of (4.14), i.e.

$$\int_{\Omega_T}\int_{E_{\mathrm{s}}} \kappa(\nabla u+\nabla_y u^1)\cdot[\nabla w - \nabla_y\big((y^c\cdot\nabla w + (s-1/2)w_t)\psi\big)]\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\,, \qquad \text{a.e. in } \Sigma,$$

where $w\in L^2(0,T;H^1_0(\Omega))\cap H^1(\Omega_T)$ and $\psi\in H^1_0(Y)$, with $\psi\equiv 1$ in E_{v}. Then, we insert in it the factorization given in (4.41)–(4.42), thus obtaining

$$\begin{aligned}&\int_{\Omega_T}\int_{E_{\mathrm{s}}} \kappa\nabla_y(y-\chi)\nabla u\cdot[\nabla w - \nabla_y\big((y^c\cdot\nabla w+(s-1/2)w_t)\psi\big)]\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{\Omega_T}\left(\int_{E_{\mathrm{s}}} \kappa\nabla_y(y-\chi)\nabla_y(y-y^c\psi)\,\mathrm{d}y\right)\nabla u\cdot\nabla w\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_{\Omega_T}(s-1/2)w_t\left(\int_{E_{\mathrm{s}}}\kappa\nabla_y(y-\chi)\nabla_y\psi\,\mathrm{d}y\right)\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{\Omega_T}\left(\int_{E_{\mathrm{s}}} \kappa\nabla_y(y-\chi)\nabla_y(y-y^c\psi)\,\mathrm{d}y\right)\nabla u\cdot\nabla w\,\mathrm{d}x\,\mathrm{d}t\,,\end{aligned}$$

where, in the last equality, we took into account that ψ can be taken as a test function in (4.28) of Lemma 4.7.
We notice that $\chi^j - (\mathbf{e}_j\cdot y^c)\psi \in L^\infty(\Omega;H^\Gamma_\#(Y))$ (recall that $\psi\in\mathcal{C}^\infty_c(Y)$ and $\psi\equiv 1$ in E_{v}), and so it is an admissible test function in (4.28) of Lemma 4.7. Thus, we get

$$\int_Y \kappa\nabla_y(\chi^j-y_j)\cdot\nabla_y\chi^j\,\mathrm{d}y = \int_Y \kappa\nabla_y(\chi^j-y_j)\cdot\nabla_y\big((\mathbf{e}_j\cdot y^c)\psi\big)\,\mathrm{d}y\,,$$

and, hence, we obtain

$$\int\limits_{\Omega_T}\int\limits_{E_{\rm s}} \kappa(\nabla u + \nabla_y \widetilde{u}^1)\cdot [\nabla w - \nabla_y\big((y^c\cdot\nabla w + (s-1/2)w_t)\psi\big)]\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$

$$= \int\limits_{\Omega_T}\left(\int\limits_{E_{\rm s}} \kappa\nabla_y(y-\chi)\nabla_y(y-\chi)\,\mathrm{d}y\right)\nabla u\cdot\nabla w\,\mathrm{d}x\,\mathrm{d}t\,. \quad (4.47)$$

Using (4.41) and (4.47), taking into account (4.45) and gluing the first and the third term in the left-hand side of (4.14), we are led to

$$\int\limits_{\Omega_T}\left(\int\limits_{E_{\rm s}} a\,\mathrm{d}y + \lambda\right)(u_t + \overline{\chi}_s)w\,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega_T} A_{\rm hom}\nabla u\cdot\nabla w\,\mathrm{d}x\,\mathrm{d}t$$

$$= |E_{\rm s}|\int\limits_{\Omega_T} fw\,\mathrm{d}x\,\mathrm{d}t\,, \qquad \text{for a.e. } s\in\Sigma.$$

Localizing with respect to (x,t) and dividing by $\left(\int\limits_{E_{\rm s}} a\,\mathrm{d}y + \lambda\right)$, we arrive at

$$(u_t + \overline{\chi}_s(x,t,s)) - \mu(x,t,s)\,\mathrm{div}(A_{\rm hom}(x)\nabla u) = \mu(x,t,s)|E_{\rm s}|f\,, \qquad (4.48)$$

where μ is defined in (4.46). Finally, integrating over Σ and taking into account the Σ-periodicity of $\overline{\chi}(x,t,\cdot)$, it follows (4.44), after dividing again by $\mathcal{M}_\Sigma(\mu)$. We note that the homogenized matrix $A_{\rm hom}$ is symmetric. Moreover, thanks to Lemma 4.11 below, which gives the positive definiteness of the matrix $A_{\rm hom}$, it follows that problem (4.44) has a unique solution $u\in L^2\big(0,T;H_0^1(\Omega)\big)\cap H^1(\Omega_T)$. Therefore, the whole sequence $\{u_\varepsilon\}$, and not only a subsequence, converges to the homogenized limit function u. □

Lemma 4.11. *The matrix* $A_{\rm hom}$ *in* (4.45) *is positive definite.*

Proof. The proof is quite standard. Using Jensen's inequality, we obtain

$$\sum_{i,j=1}^{N} A_{\hom}^{ij}\xi_i\xi_j = \int_{E_s} \sum_{i,j=1}^{N} \kappa\nabla_y(y_i\xi_i - \chi^i\xi_i)\cdot\nabla_y(y_j\xi_j - \chi^j\xi_j)\,\mathrm{d}y$$

$$\geq \gamma_0 \int_{E_s} \Big|\sum_{j=1}^{N}\nabla_y(y_j\xi_j - \chi^j\xi_j)\Big|^2\,\mathrm{d}y = \gamma_0\int_{Y}\Big|\sum_{j=1}^{N}\nabla_y(y_j\xi_j - \chi^j\xi_j)\Big|^2\,\mathrm{d}y$$

$$\geq \gamma_0\left|\int_{Y}\sum_{j=1}^{N}\nabla_y(y_j\xi_j - \chi^j\xi_j)\,\mathrm{d}y\right|^2 = \gamma_0\sum_{h=1}^{N}\left(\sum_{j=1}^{N}\int_{Y}\delta_{hj}\xi_j - \sum_{j=1}^{N}\xi_j\int_{Y}\frac{\partial\chi_j}{\partial y_h}\,\mathrm{d}y\right)^2$$

$$\geq \gamma_0\sum_{h=1}^{N}\left(\xi_h - \sum_{j=1}^{N}\xi_j\int_{\partial Y}\chi^j\,n_h\,\mathrm{d}\sigma\right)^2 = \gamma_0|\xi|^2$$

where we have denoted by $n = (n_1, \dots, n_N)$ the outward unit normal to ∂Y and we have taken into account that, because of the Y-periodicity of $\chi^j(x,\cdot)$, the last integral in the previous formula is equal to zero. □

Remark 4.12. Notice that it is not necessary to further characterize $\overline{\chi}$, since in the homogenization process it disappears from (4.44). However, from (4.48), we get that $\overline{\chi}(x,t,s)$ satisfies

$$\overline{\chi}_s = \frac{|E_s|f + \operatorname{div}(A_{\hom}\nabla u)}{\displaystyle\int_{E_s} a\,\mathrm{d}y + \lambda} - u_t\,, \tag{4.49}$$

and, hence, it is uniquely determined. We point out that, since $\overline{\chi}$ is Σ-periodic, the mean value of $\overline{\chi}_s$ over Σ is equal to 0, so that we recover (4.44). □

4.2. **The case $\alpha > 1$.** We state the following compactness result.

Lemma 4.13. *Assume that $\|\overline{u}_{0\varepsilon}\|_{H_0^1(\Omega)} \leq \gamma$, with γ independent of ε, and that, for every $\varepsilon > 0$, u_ε is the unique solution of problem* (2.11)*. Then, up to a subsequence, still denoted by ε, there exist $u \in L^2(0,T;H_0^1(\Omega)) \cap H^1(\Omega_T)$, $u_1 \in L^2(\Omega_T; H^1_{per}(Q))$, with $\mathcal{M}_Q(u_1) = 0$ and $\frac{\partial u_1}{\partial s} = 0$ a.e. in $\Omega_T \times Q$, and $\mathring{u} \in L^2(\Omega_T \times Y; H^1(\Sigma))$, with $\mathcal{M}_\Sigma(\mathring{u}) = 0$, such that*

$$u_\varepsilon \to u \qquad \textit{strongly in } L^2(\Omega_T)\,, \tag{4.50}$$

$$u_\varepsilon \rightharpoonup u \qquad \textit{weakly in } H^1(\Omega_T), \tag{4.51}$$

$$\mathcal{T}_\varepsilon(u_{\varepsilon t}) \rightharpoonup u_t + \mathring{u}_s \qquad \textit{weakly in } L^2(\Omega_T\times Q), \tag{4.52}$$

$$\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla u + \nabla_y u_1 \qquad \textit{weakly in } L^2(\Omega_T\times Q_s), \tag{4.53}$$

$$\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup 0 \qquad \textit{weakly in } L^2(\Omega_T\times Q_v), \tag{4.54}$$

$$\frac{1}{\varepsilon}\mathcal{Z}_\varepsilon(u_\varepsilon) \rightharpoonup y^c\cdot\nabla u + u_1 \qquad \textit{weakly in } L^2(\Omega_T\times Q), \tag{4.55}$$

$$y^c\cdot\nabla u + u_1\,, \qquad \textit{is independent of } y \textit{ on } \Omega_T\times Q_v. \tag{4.56}$$

Remark 4.14. Notice that, due to the special time scaling ε^α ($\alpha > 1$), u_1 belongs in fact to $L^2(\Omega_T; H^1_\#(Y))$ and $\mathcal{M}_Y(u_1) = 0$. □

Proof. The proof follows the same lines as the one of Lemma 4.1, the only difference being the fact that we use here Theorem 3.15 instead of Theorem 3.13. □

For later use, we set

$$\mathcal{H}^\Gamma(\Omega_T; Y) := L^2((0,T)\times Y; H^1_0(\Omega)) \cap H^1(\Omega_T; L^2(Y)) \cap L^2(\Omega_T; H^\Gamma_\#(Y)) . \quad (4.57)$$

Moreover, we introduce the space

$$W(\Omega_T; Y) := \{(w, w^1) \; : \; w \in L^2(0,T; H^1_0(\Omega)) \cap H^1(\Omega_T), \; w^1 \in L^2(\Omega_T; H^1_\#(Y)) , \\ \mathcal{M}_Y(w^1) = 0 \, , \; y^c \cdot \nabla w + w^1 \text{ is independent of } y \text{ on } \Omega_T \times E_{\mathrm{v}}.\} \quad (4.58)$$

Notice that, by Remark 4.14, the pair (u, u^1) given in Lemma 4.13 belongs to the space $W(\Omega_T; Y)$.

Theorem 4.15. *Assume that* (4.1) *holds. Assume also that* $\|\overline{u}_{0\varepsilon}\|_{H^1_0(\Omega)} \le \gamma$, *with* γ *independent of* ε, *and that there exists a function* $\overline{u}_0 \in H^1_0(\Omega)$ *such that* $\overline{u}_{0\varepsilon} \to \overline{u}_0$ *strongly in* $L^2(\Omega)$. *Then,* $(u, u^1) \in W(\Omega_T; Y)$ *and* $\mathring{u} \in L^2(\Omega_T \times Y; H^1(\Sigma))$, *appearing in the statement of Lemma 4.13, form a weak solution of the two-scale problem*

$$\iint\limits_{\Omega_T Q_{\mathrm{s}}} a(u_t + \mathring{u}_s) w \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t + \iint\limits_{\Omega_T E_{\mathrm{s}}} \kappa(\nabla u + \nabla_y u^1)\cdot(\nabla w + \nabla_y w^1)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ + \frac{1}{|E_{\mathrm{v}}|} \iint\limits_{\Omega_T Q_{\mathrm{v}}} \lambda(u_t + \mathring{u}_s) w \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t = |E_{\mathrm{s}}| \int\limits_{\Omega_T} f w \,\mathrm{d}x\,\mathrm{d}t \, , \quad (4.59)$$

for every $(w, w^1) \in W(\Omega_T; Y)$, *with the initial condition* $u(x, 0) = \overline{u}_0(x)$, *a.e. in* Ω.

Proof. Similarly to [13, 17] and taking into account that u_1 does not depend on s, we can take as test function in (2.11) $\phi_\varepsilon(x,t) = \varepsilon\phi(x,t,\varepsilon^{-1}x)$, where

$$\phi(x,t,y) = \mathcal{M}^\varepsilon(w)(x,t)\psi(y) + w(x,t)\varphi(y), \quad (4.60)$$

with $w \in \mathcal{C}^\infty([0,T]; \mathcal{C}^\infty_c(\Omega))$, $\psi \in \mathcal{C}^\infty_c(\overline{Y}) \cap H^\Gamma_\#(Y)$ and $\varphi \in \mathcal{C}^\infty_\#(\overline{Y})$, satisfying $\varphi \,|_{E_{\mathrm{v}}} = 0$. Reasoning as in the proof of Theorem 4.4, we get

$$\iint\limits_{\Omega_T E_{\mathrm{s}}} \kappa(\nabla u + \nabla_y u^1)\cdot \nabla_y \Psi \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = 0 \, , \quad \text{for every } \Psi \in \mathcal{H}^\Gamma(\Omega_T; Y). \quad (4.61)$$

Now, let ϕ be as in (4.18); we take $\phi_\varepsilon(x,t) = \phi(x,t,\varepsilon^{-1}x,\varepsilon^{-\alpha}t)$ as test function in (2.11). Unfolding and then passing to the limit for $\varepsilon \to 0$, as in the proof of Theorem

4.4, we are led to

$$\iint_{\Omega_T Q_{\mathrm{s}}} a(u_t + \mathring{u}_s) wz \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega_T Q_{\mathrm{s}}} \kappa(\nabla u + \nabla_y u^1)\cdot[\nabla w - \nabla_y\big((y^c\cdot\nabla w)\psi\big)] z\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{|E_{\mathrm{v}}|}\iint_{\Omega_T Q_{\mathrm{v}}} \lambda(u_t + \mathring{u}_s) zw\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t = \iint_{\Omega_T Q_{\mathrm{s}}} fzw\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\,, \quad (4.62)$$

where we have used Lemma 4.13 and we have taken into account (3.29). Notice that, taking $z \equiv 1$ and summing (4.61) and (4.62), it follows

$$\iint_{\Omega_T Q_{\mathrm{s}}} a(u_t + \mathring{u}_s) w \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega_T E_{\mathrm{s}}} \kappa(\nabla u + \nabla_y u^1)\cdot[\nabla w + \nabla_y\big(\Psi - (y^c\cdot\nabla w)\psi\big)]\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{|E_{\mathrm{v}}|}\iint_{\Omega_T Q_{\mathrm{v}}} \lambda(u_t + \mathring{u}_s) w\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t = |E_{\mathrm{s}}|\int_0^T\int_{\Omega} fw\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\,, \quad (4.63)$$

which is equivalent to (4.59), by setting

$$w^1(x,y,t) = \Psi(x,y,t) - (y^c\cdot\nabla w(x,t))\psi(y) - \int_Y \big(\Psi(x,y,t) - (y^c\cdot\nabla w(x,t))\psi(y)\big)\,\mathrm{d}y.$$

The initial condition can be easily recovered since $u_\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega_T)$. □

Remark 4.16. Unfortunately, due to the presence of the function $\mathring{u}$ in equation (4.59), we cannot go further into our analysis. Indeed, we notice that this is not a classical two-scale problem, because it contains two different correctors u^1 and $\mathring{u}$. Moreover, it is not a variational problem and it does not lead to a strong formulation neither to a factorization, as usual. Therefore, we are forced to restrict ourselves to a special factorized case described below. More precisely, we assume that the capacity A can be split in the form

$$A(x,t,y,s) = A^1(x,t,y)A^2(t,s)\,,$$

with $A^1 \in L^2(\Omega\times Y; W^{1,\infty}(0,T))$ and $A^2 \in L^2((0,T)\times\Sigma)$. Moreover, we stipulate that the coefficient λ^ξ takes the same value on each inclusion, i.e. $\lambda^\xi(t,s) = \Lambda(t;s)$, and that

$$A^2(t,s) = \Lambda(t;s)\,.$$

□

Thus, setting $a^1_\varepsilon(x) = A^1(x,t,\varepsilon^{-1}x)$ and $\lambda_\varepsilon(t) = \Lambda(t,\varepsilon^{-\alpha}t)$, the problem (2.4)–(2.7) can be rewritten as

$$a^1_\varepsilon \lambda_\varepsilon u_{\varepsilon t} - \operatorname{div}(\kappa_\varepsilon \nabla u_\varepsilon) = f\,, \qquad \text{in } \Omega_\varepsilon \times (0,T); \tag{4.64}$$

$$\lambda_\varepsilon u_{\varepsilon t} = \frac{1}{\varepsilon^N} \int_{\Gamma^\varepsilon_\xi} \kappa_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \,\mathrm{d}\sigma\,, \qquad \text{on } \Gamma^\varepsilon_\xi \times (0,T),\ \xi \in \Xi_\varepsilon; \tag{4.65}$$

$$u_\varepsilon = 0\,, \qquad \text{on } \partial\Omega \times (0,T); \tag{4.66}$$

$$u_\varepsilon(x,0) = \overline{u}_{0\varepsilon}(x)\,, \qquad \text{on } \Omega, \tag{4.67}$$

whose weak formulation is given by

$$\int_0^T\!\!\int_{\Omega_\varepsilon} a^1_\varepsilon \lambda_\varepsilon u_{\varepsilon t} \phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_{\Omega_\varepsilon} \kappa_\varepsilon \nabla u_\varepsilon \cdot \nabla \phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{|E_{\mathrm{v}}|} \int_0^T\!\!\int_{T^\varepsilon} \lambda_\varepsilon\, u_{\varepsilon t} \phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\int_{\Omega_\varepsilon} f \phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t\,, \tag{4.68}$$

for every $\phi \in \mathcal{X}^\varepsilon$, with $\phi(x,T) = 0$ in $\overline{\Omega}$.

Theorem 4.17. *Assume that*

$$\mathcal{T}_\varepsilon(\kappa_\varepsilon) \to \kappa, \qquad \textit{strongly in } L^2(\Omega \times Y);$$
$$\mathcal{T}_\varepsilon(a^1_\varepsilon) \to a^1, \qquad \textit{strongly in } L^2(\Omega_T \times Y);$$
$$\mathcal{T}_\varepsilon(a^1_{\varepsilon t}) \to a^1_t, \qquad \textit{strongly in } L^2(\Omega_T \times Y);$$
$$\mathcal{T}_\varepsilon(\lambda_\varepsilon) \to \lambda, \qquad \textit{strongly in } L^2((0,T) \times Q).$$

Assume also that $\|\overline{u}_{0\varepsilon}\|_{H^1_0(\Omega)} \le \gamma$, *with* γ *independent of* ε, *and that there exists a function* $\overline{u}_0 \in H^1_0(\Omega)$ *such that* $\overline{u}_{0\varepsilon} \to \overline{u}_0$ *strongly in* $L^2(\Omega)$. *Then, the pair* $(u,u^1) \in W(\Omega_T;Y)$, *appearing in the statement of Lemma 4.13, is the unique weak solution of the two-scale problem*

$$\int_0^T\!\!\int_{\Omega} |E_{\mathrm{s}}| \mathcal{M}_{E_{\mathrm{s}}}(a) u_t w \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \mathcal{M}_\Sigma(\lambda^{-1}) \left(\int_\Omega\!\int_Y \kappa(\nabla u + \nabla_y u^1)\cdot(\nabla w + \nabla_y w^1) \,\mathrm{d}y\,\mathrm{d}x \right) \mathrm{d}t + \int_0^T\!\!\int_\Omega u_t w \,\mathrm{d}x\,\mathrm{d}t = |E_{\mathrm{s}}| \int_0^T \mathcal{M}_\Sigma(\lambda^{-1}) \left(\int_\Omega f w \,\mathrm{d}x \right) \mathrm{d}t\,, \tag{4.69}$$

with the initial condition $u(x,0) = \overline{u}_0(x)$, *a.e. in* Ω, *for every* $(w,w^1) \in W(\Omega_T;Y)$, *such that* $w(x,T) = 0$ *a.e. in* $\overline{\Omega}$.

Proof. Let us take as test function in (4.68) $\phi_\varepsilon(x,t) = \varepsilon\phi(x,t,\varepsilon^{-1}x)\lambda_\varepsilon(t)$, where

$$\phi(x,t,y) = \mathcal{M}^\varepsilon(w)(x,t)\psi(y) + w(x,t)\varphi(y)\,, \tag{4.70}$$

with $w \in \mathcal{C}^\infty([0,T];\mathcal{C}_c^\infty(\Omega))$, $w(x,T)=0$ in $\overline{\Omega}$, $\psi \in \mathcal{C}_c^\infty(Y)\cap H^\Gamma_\#(Y)$ and $\varphi \in \mathcal{C}^\infty_\#(\overline{Y})$, with $\varphi\,|_{E_\mathrm{v}}=0$. This implies, after integrating by parts with respect to t,

$$
\begin{aligned}
&-\varepsilon\int_0^T\!\!\int_{\Omega_\varepsilon} a^1_\varepsilon u_\varepsilon[\mathcal{M}^\varepsilon(w_t)\psi + w_t\varphi]\,\mathrm{d}x\,\mathrm{d}t - \varepsilon\int_0^T\!\!\int_{\Omega_\varepsilon} a^1_{\varepsilon t} u_\varepsilon[\mathcal{M}^\varepsilon(w)\psi + w\varphi]\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad + \int_0^T\!\!\int_{\Omega_\varepsilon} \lambda_\varepsilon^{-1}\kappa_\varepsilon\nabla u_\varepsilon\cdot[\mathcal{M}^\varepsilon(w)\nabla_y\psi + \varepsilon\nabla_x w\varphi + w\nabla_y\varphi]\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\qquad - \frac{\varepsilon}{|E_\mathrm{v}|}\int_0^T\!\!\int_{T^\varepsilon} u_\varepsilon[\mathcal{M}^\varepsilon(w_t)\psi + w_t\varphi]\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad = \varepsilon\int_{\Omega_\varepsilon} a^1_\varepsilon(x,0,\varepsilon^{-1}x)\overline{u}_{0\varepsilon}[\mathcal{M}^\varepsilon(w)(x,0)\psi + w(x,0)\varphi]\,\mathrm{d}x\\
&+\frac{\varepsilon}{|E_\mathrm{v}|}\int_{T^\varepsilon}\overline{u}_{0\varepsilon}[\mathcal{M}^\varepsilon(w)(x,0)\psi + w(x,0)\varphi]\,\mathrm{d}x + \varepsilon\int_0^T\!\!\int_{\Omega_\varepsilon}\lambda_\varepsilon^{-1} f[\mathcal{M}^\varepsilon(w)\psi + w(x)\varphi]\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}
\tag{4.71}
$$

Taking into account Lemma 4.13, unfolding and then passing to the limit for $\varepsilon\to 0$, we get

$$
\begin{aligned}
&\int_{\Omega_T}\!\int_{E_\mathrm{s}} \lambda^{-1}\kappa(\nabla u + \nabla_y u^1)\cdot\nabla_y(\psi+\varphi)w\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \longleftarrow\\
&\int_{\Omega_T}\!\int_{E_\mathrm{s}} \mathcal{T}_\varepsilon(\lambda_\varepsilon^{-1})\mathcal{T}_\varepsilon(\kappa_\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\cdot[\mathcal{T}_\varepsilon(\mathcal{M}^\varepsilon(w)\nabla_y\psi) + \mathcal{T}_\varepsilon(w\nabla_y\varphi)]\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = O(\varepsilon)\to 0\,,
\end{aligned}
\tag{4.72}
$$

where we used (3.18). Taking into account that a general function in $\mathcal{C}^\infty_\#(\overline{Y})\cap H^\Gamma_\#(Y)$ can always be split in the form $\psi+\varphi$, with ψ,φ as before, and recalling the density of product functions in $\mathcal{H}^\Gamma(\Omega_T;Y)$, we obtain

$$
\int_0^T \mathcal{M}_\Sigma(\lambda^{-1})\left(\int_\Omega\!\int_{E_\mathrm{s}}\kappa(\nabla u + \nabla_y u^1)\cdot\nabla_y\Psi\,\mathrm{d}y\,\mathrm{d}x\right)\mathrm{d}t = 0\,, \tag{4.73}
$$

for every $\Psi\in\mathcal{H}^\Gamma(\Omega_T;Y)$. Here, we also use the fact that $\frac{\partial u^1}{\partial s}=0$.
Now, we take as test function in (2.11) $\phi_\varepsilon(x,t)=\phi(x,t,\varepsilon^{-1}x)\lambda_\varepsilon^{-1}(t)$, where

$$
\phi(x,t,y) = \mathcal{M}^\varepsilon(w)(x,t)\psi(y) + w(x,t)(1-\psi(y))\,, \tag{4.74}
$$

with w, ψ as in (4.70) and $\psi \equiv 1$ on E_{v}. Clearly, $\phi_\varepsilon \to w\lambda^{-1}$ strongly in $L^2(\Omega_T \times Q)$. Inserting it in the weak formulation (4.68) and integrating by parts in time, it follows

$$
\begin{aligned}
&-\int_0^T\!\!\int_{\Omega_\varepsilon} a^1_\varepsilon u_\varepsilon [\mathcal{M}^\varepsilon(w_t)\psi + w_t(1-\psi)]\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_{\Omega_\varepsilon} a^1_{\varepsilon t} u_\varepsilon [\mathcal{M}^\varepsilon(w)\psi + w(1-\psi)]\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_0^T\!\!\int_{\Omega_\varepsilon} \lambda_\varepsilon^{-1}\kappa_\varepsilon \nabla u_\varepsilon \cdot \left[\frac{1}{\varepsilon}(\mathcal{M}^\varepsilon(w) - w)\nabla_y\psi + \nabla_x w(1-\psi)\right]\mathrm{d}x\,\mathrm{d}t \\
&- \frac{1}{|E_{\mathrm{v}}|}\int_0^T\!\!\int_{T^\varepsilon} u_\varepsilon \mathcal{M}^\varepsilon(w_t)\psi\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_\varepsilon} a^1_\varepsilon(x,0,\varepsilon^{-1}x)\overline{u}_{0\varepsilon}[\mathcal{M}^\varepsilon(w)(x,0)\psi + w(x,0)(1-\psi)]\,\mathrm{d}x \\
&\qquad + \frac{1}{|E_{\mathrm{v}}|}\int_{T^\varepsilon} \overline{u}_{0\varepsilon}\mathcal{M}^\varepsilon(w)(x,0)\,\mathrm{d}x + \int_0^T\!\!\int_{\Omega_\varepsilon} \lambda_\varepsilon^{-1} f[\mathcal{M}^\varepsilon(w)\psi + w(1-\psi)]\,\mathrm{d}x\,\mathrm{d}t\,. \qquad (4.75)
\end{aligned}
$$

Unfolding and then passing to the limit for $\varepsilon \to 0$, we get

$$
\begin{aligned}
&-\int_{\Omega_T}\!\int_{E_{\mathrm{s}}} a^1 u w_t\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T}\!\int_{E_{\mathrm{s}}} a^1_t u w\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_0^T \mathcal{M}_\Sigma(\lambda^{-1})\left(\int_\Omega\!\int_{E_{\mathrm{s}}} \kappa(\nabla u + \nabla_y u^1)\cdot[\nabla w - \nabla_y\big((y^c\cdot\nabla w)\psi\big)]\,\mathrm{d}y\,\mathrm{d}x\right)\mathrm{d}t \\
&\qquad\qquad - \int_{\Omega_T} u w_t\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega\!\int_{E_{\mathrm{s}}} a^1(x,0,y)\overline{u}_0 w(x,0)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad\qquad\qquad + \int_\Omega \overline{u}_0 w(x,0)\,\mathrm{d}x + |E_{\mathrm{s}}|\int_{\Omega_T} \mathcal{M}_\Sigma(\lambda^{-1}) f w\,\mathrm{d}x\,\mathrm{d}t\,, \qquad (4.76)
\end{aligned}
$$

where we have taken into account (3.29). Since, as before, the initial condition can be easily recovered, as a consequence of the convergence $u_\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega_T)$, we integrate again by parts with respect to t and sum the resulting equation with

(4.73), thus obtaining

$$\int\limits_{\Omega_T}\int\limits_{E_\mathrm{s}} a u_t w \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$

$$+ \int\limits_0^T \mathcal{M}_\Sigma(\lambda^{-1}) \left(\int\limits_{E_\mathrm{s}} \kappa(\nabla u + \nabla_y u^1)\cdot [\nabla w + \nabla_y \big(\Psi - (y^c\cdot\nabla w)\psi\big)] \,\mathrm{d}y\,\mathrm{d}x \right) \mathrm{d}t$$

$$+ \int\limits_{\Omega_T} u_t w \,\mathrm{d}x\,\mathrm{d}t = |E_\mathrm{s}| \int\limits_0^T \mathcal{M}_\Sigma(\lambda^{-1}) \left(\int\limits_{\Omega} f w \,\mathrm{d}x \right) \mathrm{d}t\,, \quad (4.77)$$

which is equivalent to (4.69), by setting

$$w^1(x,y,t) = \Psi(x,y,t) - (y^c\cdot\nabla w(x,t))\psi(y) - \int\limits_Y \big(\Psi(x,y,t) - (y^c\cdot\nabla w(x,t))\psi(y)\big)\,\mathrm{d}y$$

and recalling (4.56).
Finally, the variational character of equation (4.69) together with Poincaré or Gronwall inequality gives immediately the uniqueness of the solution. □

Despite the fact that the connection between the macroscopic and the microscopic test functions w and w^1 prevents the possibility to state a strong formulation for (4.69), we still can factorize in a standard way the corrector u^1, thus obtaining the single-scale homogenized equation satisfied by u.

Theorem 4.18. *Let $(u,u^1)\in W(\Omega_T;Y)$ be the unique solution of equation* (4.69)*, satisfying the initial condition $u(x,0)=\overline{u}_0(x)$ a.e. in Ω. Then, the two-scale problem* (4.69) *can be decoupled by setting*

$$u^1(x,t,y) = -\chi^j(x,y)\partial_j u(x,t)\,, \quad (4.78)$$

where χ^j, for $j=1,\dots,N$, satisfies problem (4.28)–(4.30) *and $u\in L^2(0,T;H^1_0(\Omega))\cap H^1(\Omega_T)$ is the unique solution of the single-scale equation*

$$\int\limits_0^T\int\limits_{\Omega} |E_\mathrm{s}|\mathcal{M}_{E_\mathrm{s}}(a^1) u_t w \,\mathrm{d}x\,\mathrm{d}t + \int\limits_0^T\int\limits_{\Omega} \mathcal{M}_\Sigma(\lambda^{-1}) A_\mathrm{hom}\nabla u\cdot\nabla w \,\mathrm{d}x\,\mathrm{d}t$$

$$+ \int\limits_0^T\int\limits_{\Omega} u_t w \,\mathrm{d}x\,\mathrm{d}t = |E_\mathrm{s}| \int\limits_0^T\int\limits_{\Omega} f\mathcal{M}_\Sigma(\lambda^{-1}) w \,\mathrm{d}x\,\mathrm{d}t\,, \quad (4.79)$$

for every test functions $w\in L^2(0,T;H^1_0(\Omega))\cap H^1(\Omega_T)$, with $w(x,T)=0$ a.e. in Ω, where the symmetric and positive definite homogenized matrix A_hom is given by (4.45).

Proof. Equation (4.79) follows from (4.69), after rearranging the second term in the left-hand side as done in the first part of the proof of Theorem 4.10. As a consequence of the positive definiteness of the homogenized matrix, the solution u of equation (4.79) is uniquely determined. □

Remark 4.19. One can see that the strong formulation of (4.79) reads like

$$\begin{aligned} &\left(\frac{|E_{\rm s}|\mathcal{M}_{E_{\rm s}}(a^1)+1}{\mathcal{M}_{\Sigma}(\lambda^{-1})}\right) u_t - \operatorname{div}(A_{\rm hom}\nabla u) = |E_{\rm s}| f\,, && \text{in } \Omega_T;\\ &u = 0\,, && \text{on } \partial\Omega\times(0,T);\\ &u(x,0)=\overline{u}_0\,, && \text{in } \Omega. \end{aligned} \tag{4.80}$$

□

4.3. **Final remarks.** After concluding our analysis, we are in a position to make some final considerations.
First, we emphasize that we treat in this paper only the case $\alpha \geq 1$, corresponding to the so-called *fast oscillations* (see [3]), while the case $\alpha \in (0,1)$, corresponding to the *slow oscillations*, deserves a completely different approach and is left to further investigations. Moreover, we collect some common features of the two classes of scalings treated above.

Remark 4.20. Assume that in problem (2.4)–(2.7) we take $a_\varepsilon(x,t) \equiv 1$, $\lambda^\xi_\varepsilon(t)$ a strictly positive constant λ and $\kappa_\varepsilon(x)$ independent of x (or, equivalently, in (4.64)–(4.67) we take $\Lambda(t,s)$ equal to the strictly positive constant λ, $A^1(x,t,y)=\lambda^{-1}$ and again $\kappa_\varepsilon(x)$ independent of x). Then, one can obtain that the limit function $u \in L^2(0,T;H^1_0(\Omega))\cap H^1(\Omega_T)$ is the unique solution of the homogenized problem

$$\begin{aligned} &(|E_{\rm s}|+\lambda)\, u_t - \operatorname{div}(A_{\rm hom}\nabla u) = |E_{\rm s}| f\,, && \text{in } \Omega_T;\\ &u(x,0)=\overline{u}_0\,, && \text{in } \Omega, \end{aligned} \tag{4.81}$$

where the matrix $A_{\rm hom}$ is obtained as in (4.45), but it is now constant.
We point out that this result can also be derived independently, by means of simpler techniques (see [5], where a more general initial condition can be considered). □

Remark 4.21. Our results can be generalized to the case in which the microscopic diffusion matrix depends also on the macro-time t, the only difference being that, in this situation, the corresponding homogenized matrix $A_{\rm hom}$ depends also, parametrically, on t. □

Remark 4.22. We notice that for the whole family $\alpha \geq 1$, the limit problems obtained above present the same elliptic part. Indeed, the homogenized matrix $A_{\rm hom}$ depends only on the solution χ^j, $j=1,\dots,N$, of the cell problem appearing in Lemma 4.7, which is independent of the micro-time s, since the microscopic diffusion matrix is not oscillating in time. □

Remark 4.23. Finally, we remark that, if we assume that the capacity A is chosen as in Remark 4.16 also for the case $\alpha = 1$, then the corresponding effective capacity appearing in front of the time-derivative in (4.44) coincides with the one arising in (4.80). Thus, the single-scale formulation is the same for all the scalings. □

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