



Oscillating solutions for nonlinear Helmholtz equations

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Abstract. Existence results for radially symmetric oscillating solutions for a class of nonlinear autonomous Helmholtz equations are given and their exact asymptotic behaviour at infinity is established. Some generalizations to nonautonomous radial equations as well as existence results for nonradial solutions are found. Our theorems prove the existence of standing waves solutions of nonlinear Klein–Gordon or Schrödinger equations with large frequencies.

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1. Introduction

The main aim of this paper is to give existence results for the following class of nonlinear equations

$$-\Delta u = g(u) \quad \text{in } \mathbb{R}^N \quad (1.1)$$

with $N \geq 1$ and assuming that the nonlinearity g is such that

$$g \in C_{\text{loc}}^{1,\sigma}(\mathbb{R}) \quad \text{for some } \sigma \in (0, 1), \quad (1.2)$$

$$g \text{ is odd}, \quad (1.3)$$

$$g'(0) > 0, \quad (1.4)$$

$$\exists \alpha_0 \in (0, +\infty] : g \text{ is positive on } (0, \alpha_0) \text{ and negative on } (\alpha_0, \infty). \quad (1.5)$$

There is a huge literature concerning (1.1) and nonautonomous variants of it under the assumption $g'(0) < 0$. Two seminal papers in this context are the contributions by Berestycki–Lions and Strauss [6, 21] who proved the existence of smooth radially symmetric and exponentially decaying solutions for a large class of nonlinearities with this property. We refer to the monographs [2, 3, 27] for more results in this context. One of the main interests in finding solutions of (1.1) is motivated by the relation with the existence of solutions to nonlinear time-dependent Klein–Gordon equations. Indeed, assuming g to have the special form $g(z) = (\lambda^2 - V_0)z + h(|z|)z$ and extending it to \mathbb{C} in the natural way, then any solution u of (1.1) gives rise to a standing wave, i.e. a solution of the form $\psi(x, t) = e^{i\lambda t}u(x)$ to

$$\frac{\partial^2 \psi}{\partial t^2} - \Delta \psi + V_0 \psi = h(|\psi|)\psi \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

Therefore, assuming $g'(0) < 0$ amounts to look for standing waves having low frequencies $\lambda^2 < V_0$ and numerous existence results for $H^1(\mathbb{R}^N)$ -solutions under this assumption can be found in the references

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mentioned above. In this paper, we deal with nonlinearities satisfying $g'(0) > 0$, which gives rise to standing waves with large frequencies $\lambda^2 > V_0$. Looking at the form of the linearized operator $-\Delta - g'(0)$, one realizes that $u_0 = 0$ lies in its essential spectrum and we are actually dealing with a class of nonlinear Helmholtz equations. Furthermore, as explained in subsection 2.2 in [6], the hypothesis (1.4) has the striking consequence that radially symmetric $H^1(\mathbb{R}^N)$ solutions of (1.1) cannot exist, and usual variational methods fail. On the other hand, (1.4) is naturally linked to (1.5); in particular, if $g(z)/z$ decreases in $(0, +\infty)$, then (1.4) turns out to be necessary in order to have $H^1(\mathbb{R}^N)$ solutions. Actually, the relevant solutions naturally lie outside this functional space. This fact can also be illustrated by an examination of the behaviour of the minimal energy solutions on a sequence of large bounded domains. Namely, in Theorem 4.4 we will show that if one takes a sequence of bounded domains Ω_n invading \mathbb{R}^N , then (1.4) guarantees the existence of a sequence (u_n) of global minimizers of the associated action functional over $H_0^1(\Omega_n)$ for sufficiently large n . But, it results that (u_n) converges in $C_{\text{loc}}^2(\mathbb{R}^N)$ to the constant solution $u \equiv \alpha_0$.

Therefore, under the assumption (1.4), one has to look for solutions in a broader class of functions. Our focus will be on oscillating and localized ones which we define as follows.

Definition 1.1. A distributional solution $u \in C^{1,\alpha}(\mathbb{R}^N)$ of (1.1) is called oscillating if it has an unbounded sequence of zeros. It is called localized when it converges to zero at infinity.

Let us notice that the strong maximum principle implies that oscillating solutions of (1.1) change sign at each of their zeros, so that we are going to find solutions that change sign infinitely many times. In our study, we will pay particular attention to the following model cases

$$g_1(z) = -\lambda z + \frac{z}{s + z^2} \quad \text{where } s > 0, \lambda < \frac{1}{s}. \tag{1.6}$$

$$g_2(z) = k^2 z - |z|^{p-2} z, \quad g_3(z) = k^2 z + |z|^{p-2} z \quad \text{for } k \neq 0, p > 2. \tag{1.7}$$

Our interest in these examples has various motivations. The nonlinearity g_1 is related to the study of the propagation of lights beams in a photorefractive crystals (see [9,28]) when a saturation effect is taken into account. Differently from the more frequently studied model

$$\tilde{g}(z) := -\lambda z + \frac{z^3}{1 + sz^2},$$

see e.g. [8], g_1 describes a transition from the linear propagation and the saturated one. This difference has important consequences, for instance, for $g = \tilde{g}$ there are $H^1(\mathbb{R}^N)$ solutions of (1.1) for $0 < \lambda < 1/s$ (e.g. see Theorem 3.6 in [25]), whereas this is not the case if $g = \tilde{g}, \lambda < 0$ or $g = g_1$ because of $g'(0) > 0$. Notice that, as $\lambda < 1/s$, Eq. (1.1) for $g = g_1$ can be rewritten in the following form

$$-\Delta u - k^2 u = -\frac{u^3}{s(s + u^2)} \quad \text{in } \mathbb{R}^N \quad \text{with } k^2 = \frac{1}{s} - \lambda$$

which allows to settle the problem in $H^1(\mathbb{R}^N)$ in every dimension N and that shows that also this saturable model is included in the class of the nonlinear Helmholtz equations. The principal difference between (1.7) and (1.6) is that the formers are superlinear and homogeneous nonlinearities, while the latter is not homogeneous and it is asymptotically linear. However, all of them satisfy our general assumptions, with $\alpha_0 \in (0, +\infty)$ for g_1 and g_2 and $\alpha_0 = +\infty$ for g_3 .

Up to now, nonlinear Helmholtz equations (1.1) have been mainly investigated for the model nonlinearity g_3 or more general superlinear nonlinearities, even not autonomous. In a series of papers, [10–13] Evéquoz and Weth proved the existence of radial and nonradial real, localized solutions of this equation under various different assumptions on the nonlinearity. Let us mention that some of the tools used in [13] had already appeared in a paper by Gutiérrez [17] where the existence of complex-valued solutions was proved for space dimensions $N = 3, 4$. Let us first focus our attention on radially symmetric solutions

and state our first result, which provides a complete description of the radially symmetric solutions of (1.1). In the statement, G will denote the primitive of g satisfying $G(0) = 0$.

Theorem 1.2. *Assume (1.2),(1.3),(1.4),(1.5). Then there is a maximal continuum $\mathcal{C} = \{u_\alpha \in C^2(\mathbb{R}^N) : |\alpha| < \alpha_0\}$ in $C^2_{loc}(\mathbb{R}^N)$ consisting of radially symmetric oscillating solutions of (1.1) having the following properties for all $|\alpha| < \alpha_0$:*

- (i) $u_\alpha(0) = \alpha$,
- (ii) $\|u_\alpha\|_{L^\infty(\mathbb{R}^N)} = |\alpha|, \|u'_\alpha\|_{L^\infty(\mathbb{R}^N)} \leq \sqrt{2G(\alpha)}$.

Moreover, for $N = 1$ all these solutions are periodic, whereas for $N \geq 2$ they are localized and satisfy the following asymptotic behaviour:

- (iii) *There are positive numbers $c_\alpha, C_\alpha > 0$ such that*

$$c_\alpha r^{(1-N)/2} \leq |u_\alpha(r)| + |u'_\alpha(r)| + |u''_\alpha(r)| \leq C_\alpha r^{(1-N)/2} \quad \text{for all } r \geq 1.$$

Here, a continuum in $C^2_{loc}(\mathbb{R}^N)$ is a connected subset of $C^2_{loc}(\mathbb{R}^N)$ with respect to the uniform convergence of the zeroth, first and second derivatives on compact subsets of \mathbb{R}^N . The maximality of the continuum refers to the fact that there are no further radially symmetric solutions and in particular no larger continuum in $C^2_{loc}(\mathbb{R}^N)$ as we will see in Sect. 2. Moreover, conclusion (iii) states that the property $u_\alpha \in L^s(\mathbb{R}^N)$ is equivalent to $u_\alpha \in W^{2,s}(\mathbb{R}^N)$, and this happens if and only if $s > \frac{2N}{N-1}$. Notice that this implies that $u_\alpha \notin L^2(\mathbb{R}^N)$, showing again that the solutions, as expected, live outside the commonly used energy space.

Furthermore, let us stress that the behaviour of the nonlinearity beyond α_0 is completely irrelevant; in particular, the negativity of g on (α_0, ∞) is actually not needed, because all solutions satisfy $|u_\alpha(x)| \leq |\alpha| < \alpha_0$ by item (ii) of the theorem. This is the reason why we do not need to assume any subcritical growth condition on the exponent p in the model nonlinearities g_2, g_3 . Moreover, this shows that $g'(0) > 0$ is the crucial hypothesis to show the existence of a continuum of solutions. Let us recall that in the autonomous setting Theorem 4 in [12] yields nontrivial radially symmetric solutions of (1.1) for superlinear nonlinearities such as g_3 , but not for g_1, g_2 .

Theorem 1.2 admits generalizations to some nonautonomous radially symmetric nonlinearities. In particular, we can prove a nonautonomous version of this result that applies to the nonlinearities

$$g_1(r, z) = -\lambda(r)z + \frac{z}{s(r) + z^2}, \tag{1.8}$$

$$g_2(r, z) = k(r)^2z \pm Q(r)|z|^{p-2}z, \tag{1.9}$$

under suitable assumptions on the coefficients $\lambda(r), s(r), k(r), Q(r)$, see Theorem 2.10 and Corollaries 2.12 and 2.14. Our results in this context extend Theorem 4 in [12] in several directions (see Remark 2.13).

The existence of nonradially symmetric solutions is clearly a more difficult topic, and here, we can give a partial positive answer in this direction, by exploiting the argument developed in [10–13], where the authors study nonlinear Helmholtz equations with general superlinear nonlinearities such as

$$-\Delta u - k^2u = Q(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.10}$$

for $Q(x) > 0$, which represents a nonautonomous version of our model nonlinearity g_3 . Among other results, in [13] (Theorem 1.1 and Theorem 1.2) it is shown that if Q is \mathbb{Z}^N -periodic or vanishing at infinity then there exist nontrivial solutions of (1.10) for p satisfying $\frac{2(N+1)}{N-1} < p < \frac{2N}{N-2}$ when $N \geq 3$.

Our contribution to this issue is that the positivity assumption on Q may be replaced by a negativity assumption in order to make the dual variational approach work, so that, using Fourier transform, we show that the main ideas from [13] may be modified in such a way that their main results remain true for negative Q . Our results read as follows.

Theorem 1.3. *Let $N \geq 3$, $\frac{2(N+1)}{N-1} < p < \frac{2N}{N-2}$ and let $Q \in L^\infty(\mathbb{R}^N)$ be periodic and negative almost everywhere. Then the Eq. (1.10) has a nontrivial localized oscillating strong solution in $W^{2,q}(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N)$ for all $q \in [p, \infty), \alpha \in (0, 1)$.*

Theorem 1.4. *Let $N \geq 3$, $\frac{2(N+1)}{N-1} \leq p < \frac{2N}{N-2}$ and let $Q \in L^\infty(\mathbb{R}^N)$ be negative almost everywhere with $Q(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Then Eq. (1.10) has a sequence of pairs $\pm u_m$ of nontrivial localized oscillating strong solutions in $W^{2,q}(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N)$ for all $q \in [p, \infty), \alpha \in (0, 1)$ such that*

$$\|u_m\|_{L^p(\mathbb{R}^N)} \rightarrow \infty \quad \text{as } m \rightarrow \infty.$$

Since the above results together with those from [13] provide some existence results for the nonlinear Helmholtz equation associated with the nonlinearity g_2 from (1.9), one leads to wonder whether similar results hold true for asymptotically linear nonlinearities like g_1 in (1.8). Here, the dual variational framework does not seem to be convenient since even the choice of the appropriate function spaces is not clear. A thorough discussion of such nonlinear Helmholtz equations leading to existence results for nonradial solutions still remains to be done.

Let us observe that there is a gap in the admissible range for p between Theorems 1.2, 1.3, and 1.4. Reading Theorem 1.2 one naturally leads to the conjecture that nontrivial nonradial solutions in $L^p(\mathbb{R}^N)$ may be found regardless of any sign condition on Q and for all exponents $p > \frac{2N}{N-1}$. On the contrary, Theorem 1.3 only holds for exponents $p > \frac{2(N+1)}{N-1}$, so that it is still an open question whether or not nonradial L^p -solutions exist for $p \in \left(\frac{2N}{N-1}, \frac{2(N+1)}{N-1}\right]$.

The paper is organized as follows: In Sect. 2, we present the proof of Theorem 1.2 as well as a generalization to the radial nonautonomous case (see Theorem 2.10 and Corollaries 2.12, 2.14). In Sect. 3, we present the proofs of Theorems 1.3 and 1.4. In Sect. 4, we will discuss in detail the attempt to obtain a solution by approximating \mathbb{R}^N by bounded domains.

2. Radial solutions

2.1. The autonomous case

Throughout this section, we will suppose that (1.2), (1.3), (1.4), (1.5) hold true. We will prove Theorem 1.2 by providing a complete understanding of the initial value problem

$$-u'' - \frac{N-1}{r}u' = g(u) \quad \text{in } (0, \infty), \quad u(0) = \alpha, \quad u'(0) = 0 \tag{2.1}$$

for $\alpha \in \mathbb{R}$ and $N \in \mathbb{N}$. Notice that assumptions (1.3) and (1.4) imply that there exists a $\delta > 0$ such that

$$g(z)z > 0 \quad \forall z \in (-\delta, \delta).$$

Such a positivity region is in fact almost necessary as the following result shows.

Proposition 2.1. *Assume that $g \in C(\mathbb{R})$ satisfies $g(z)z < 0$ for all $z \in \mathbb{R}$. Then there is no nontrivial localized solution and there is no nontrivial oscillating solution $u \in C^2(\mathbb{R}^N)$ of (1.1).*

Proof. Assume that $u \in C^2(\mathbb{R}^N)$ is a nontrivial localized or oscillating solution. Then it attains a positive local maximum or a negative local minimum in some point $x_0 \in \mathbb{R}^N$. Hence, we obtain

$$-\Delta u(x_0)u(x_0) = u(x_0)g(u(x_0)) < 0,$$

a contradiction. □

Remark 2.2. In view of elliptic regularity theory, the above result is also true for weak solutions $u \in H^1(\mathbb{R}^N)$ since these solutions coincide almost everywhere with classical solutions and decay to zero at infinity by Theorem C.3.1 in [19]. Notice that in case $N \geq 3$ we can deduce the nonexistence of $H^1(\mathbb{R}^N)$ solutions from the fact that $g(z)z < 0$ in \mathbb{R} violates the necessary condition (1.3) in [6], see section 2.2 in that paper. In the case $N = 2$, the same follows from Remarque 1 in [5].

First, we briefly address the one-dimensional initial value problem

$$-u'' = g(u) \quad \text{in } (0, \infty), \quad u(0) = \alpha, \quad u'(0) = 0. \tag{2.2}$$

In view of the oddness of g , it suffices to discuss the initial value problem for $\alpha \geq 0$. The uniquely determined solution of the initial value problem will be denoted by u_α with maximal existence interval $(-T_\alpha, T_\alpha)$ for $T_\alpha \in (0, \infty]$.

Proposition 2.3. *Let $N = 1$. Then the following holds:*

- (i) *If $\alpha = \alpha_0 \in \mathbb{R}$ then $u_\alpha \equiv \alpha_0$ and if $\alpha = 0$ then $u_\alpha \equiv 0$.*
- (ii) *If $\alpha > \alpha_0$ then u_α strictly increases to $+\infty$ on $(0, T_\alpha)$.*
- (iii) *If $0 < \alpha < \alpha_0$ then u_α is periodic and oscillating with $\|u_\alpha\|_\infty = \alpha$.*

Proof. Conclusion (i) immediately follows from (1.5). Then we only have to prove (ii) and (iii). For notational convenience, we write u, T instead of u_α, T_α . In the situation of (ii), we set $\xi := \sup\{s \in [0, T) : u''(s) > 0\}$. From $u(0) = \alpha > \alpha_0$ and (2.2), we get $u''(0) = -g(u(0)) = -g(\alpha) > 0$ and thus $\xi \in (0, T)$. We even have $\xi = T$, because otherwise

$$u(\xi) = \alpha + \int_0^\xi \int_0^t u''(s) \, ds \, dt > \alpha > \alpha_0$$

and thus $u''(\xi) > 0$ in view of assumption (1.5) and (2.2). This, however, would contradict that ξ is the supremum, hence $\xi = T$. As a consequence, u is strictly convex on $(0, T)$ which implies (ii).

In order to show (iii), we notice that (1.3) implies that solutions are symmetric about critical points and antisymmetric about zeros. Therefore, it suffices to show that u decreases until it attains a zero. By the choice of $\alpha \in (0, \alpha_0)$, we have $u''(0) < 0$ so that u decreases on a right neighbourhood of 0. Exploiting (1.5) and (2.2), we deduce that $u''(s)$ is negative whenever $0 < u(s) < u(0) < \alpha_0$. As a consequence, we obtain that u decreases as long as it remains positive. Moreover, it cannot be positive on $[0, \infty)$ since this would imply, thanks $0 \leq u(r) \leq \alpha < \alpha_0$ and the assumptions (1.4),(1.5),

$$u''(r) + c(r)u(r) = 0, \quad \text{with } c(r) := \frac{g(u(r))}{u(r)} \geq c_0 > 0.$$

Hence, Sturm’s comparison theorem (see p.2 in [23]) ensures that u vanishes somewhere, so that it cannot be positive in $[0, +\infty)$, a contradiction. Hence, u attains a zero and the proof is finished. □

Remark 2.4. There are many contributions concerning (1.1) in dimension $N = 1$, mainly related to some resonance phenomena. In this context, some “Landesman–Lazer” type conditions, joint with suitable hypotheses on the nonlinearity g , are assumed in order to obtain existence of bounded, periodic or oscillating solution, eventually with arbitrarily large L^∞ norm, by taking advantage of the presence on a forcing term in the equation (see [20,26] and the references therein). Here the situation is different, as we do not need any monotonicity assumption on g , nor the knowledge of the asymptotic behaviour at infinity of g is important, as it is in [20,26]. Moreover, our solutions satisfy a uniform L^∞ bound, so that the phenomenon we are dealing with is actually different from the resonant one.

Next, we consider the initial value problem (2.1) in the higher-dimensional case $N \geq 2$. Again, we may restrict our attention to the case $\alpha \geq 0$. As before we will denote by G the primitive of g with $G(0) = 0$. The following result furnishes the study of the solution set which are needed in the proof of Theorem 1.2.

Again, the uniquely determined solution of the initial value problem (2.1) will be denoted by u_α with maximal existence interval $(-T_\alpha, T_\alpha)$.

Lemma 2.5. *Let $N \geq 2$. Then the following holds:*

- (i) *If $\alpha = \alpha_0 \in \mathbb{R}$ then $u_\alpha \equiv \alpha_0$ and if $\alpha = 0$ then $u_\alpha \equiv 0$.*
- (ii) *If $\alpha > \alpha_0$ then u_α strictly increases to $+\infty$ on $[0, T_\alpha)$.*
- (iii) *If $0 < \alpha < \alpha_0$ then u_α is oscillating, localized and satisfies*

$$\|u_\alpha\|_{L^\infty(\mathbb{R})} = |\alpha| \quad \text{and} \quad \|u'_\alpha\|_{L^\infty(\mathbb{R})} \leq \sqrt{2G(\alpha)} \tag{2.3}$$

as well as

$$c_\alpha r^{(1-N)/2} \leq |u_\alpha(r)| + |u'_\alpha(r)| + |u''_\alpha(r)| \leq C_\alpha r^{(1-N)/2} \quad \text{for } r \geq 1 \tag{2.4}$$

for some $c_\alpha, C_\alpha > 0$ depending on the solution but not on r .

Proof. The existence and uniqueness of a twice continuously differentiable solution $u_\alpha : (-T_\alpha, T_\alpha) \rightarrow \mathbb{R}$ can be deduced from Theorem 1 and Theorem 2 in [18]. We write again u, T in place of u_α, T_α . The proof of (i) is direct and assertion (ii) follows similar to the one-dimensional case. Indeed, note that $u''(0) > 0$ because of

$$Nu''(0) = \lim_{r \rightarrow 0^+} u''(r) + \frac{N-1}{r}u'(r) = -g(u(0)) = -g(\alpha) > 0.$$

Then, letting $\xi := \sup\{s \in (0, T) : u'(s) > 0\}$, it results $\xi \in (0, T]$. Assuming by contradiction that $\xi < T$ and using that $\alpha > \alpha_0$, from (1.5) we obtain

$$\xi^{N-1}u'(\xi) = - \int_0^\xi t^{N-1}g(u(t)) dt > 0$$

which is impossible, i.e. $\xi = T$. Then, (2.1),(1.5) and the maximality of T yield (ii). The proof of (iii) is lengthy so that it will be subdivided into four steps.

Step 1: u decreases to a first zero For all $r > 0$ such that $0 < u < \alpha_0$ on $[0, r]$ we have

$$r^{N-1}u'(r) = - \int_0^r t^{N-1}g(u(t)) dt < 0,$$

showing that u decreases as long as it remains positive, as in the one-dimensional case. Moreover, the function u cannot remain positive on $[0, \infty)$ because otherwise $v(r) := r^{(N-1)/2}u(r)$ would be a positive solution of

$$v'' + c(r)v = 0 \quad \text{where } c(r) = \frac{g(u(r))}{u(r)} - \frac{(N-1)(N-3)}{4r^2}. \tag{2.5}$$

As in the proof of Proposition 2.3, we observe $c(r) \geq c_0 > 0$ for sufficiently large r so that Sturm's comparison theorem tells us that v vanishes somewhere. This is a contradiction to the positivity of u and thus u attains a first zero.

Step 2: u oscillates and satisfies (2.3) Let us first show that there are $0 = r_0 < r_1 < r_2 < r_3 < \dots$ such that all r_{4j} are local maximizers, all r_{4j+2} are local minimizers and all r_{2j+1} are zeros of u . Moreover, we will find that all zeros or critical points of u are elements of this sequence and

$$2G(u(r_0)) > u'(r_1)^2 > 2G(u(r_2)) > u'(r_3)^2 > 2G(u(r_4)) > \dots \tag{2.6}$$

In order to prove this, we consider the function

$$Z(r) := u'(r)^2 + 2G(u(r)), \tag{2.7}$$

and we observe that Z decreases as

$$Z'(r) = 2u'(r)(u''(r) + g(u(r))) = -\frac{2(N-1)}{r}u'(r)^2 < 0. \quad (2.8)$$

The existence of a first zero $r_1 > 0 = r_0$ of u has been shown in *Step 1*, and the strict monotonicity of Z implies $Z(r_1) < Z(r_0)$. Concerning the behaviour of u on $[r_1, \infty)$, there are now three alternatives:

- (a) u decreases until it attains $-u(r_0)$
- (b) u decreases on $[r_1, \infty)$ to some value $u_\infty \in [-u(r_0), 0)$
- (c) u decreases until it attains a critical point at some $r_2 > r_1$ with $-u(r_0) < u(r_2) < 0$.

Let us show that the cases (a) and (b) do not occur. Indeed, if there exists $r > r_0$ such that $u(r) = -u(r_0)$, then, by (2.7) we deduce that

$$Z(r) \geq 2G(u(r)) = 2G(u(r_0)) = Z(r_0)$$

which is forbidden by (2.8). Hence, the case (a) is impossible. Let us now suppose that (b) holds. Then u_∞ has to be a stationary solution of (2.1) and thus $u_\infty = -\alpha_0 = -u(r_0)$. But then

$$Z(r) \geq 2G(u(r)) \rightarrow 2G(u_\infty) = 2G(-u(r_0)) = Z(r_0) \quad \text{as } r \rightarrow \infty$$

which again contradicts (2.8). So the case (c) occurs and there must be a critical point r_2 with

$$2G(u(r_2)) = Z(r_2) < Z(r_1) = u'(r_1)^2 < Z(r_0) = 2G(u(r_0)),$$

so that (2.1), (1.5) and (1.3) yield

$$0 > u(r_2) > -u(r_0) \text{ and } u'(r_2) = 0, u''(r_2) > 0.$$

Hence, r_2 is a local minimizer. Using that Z is decreasing we can now repeat the argument to get a zero $r_3 > r_2$, a local maximizer $r_4 > r_3$, a zero $r_5 > r_4$ and so on. By the strict monotonicity of Z , one obtains (2.6) and thus (2.3). Notice that this reasoning also shows that there are no further zeros or critical points.

Step 3: u is localized First we show $u(r) \rightarrow 0$ as $r \rightarrow \infty$. Our proof is similar to the one of Lemma 4.1 in [16] and it will be presented for the convenience of the reader. Take the sequence of maximizers $\{r_{4j}\}$ and assume by contradiction that $u(r_{4j}) \rightarrow z \in (0, \alpha_0)$. Then (2.1) and the Ascoli–Arzelà theorem imply that $u(\cdot + r_{4j})$ converges locally uniformly to the unique solution w of (2.2) with $w(0) = z, w'(0) = 0$. Proposition 2.3, (iii) implies that this solution w is T -periodic with two zeroes at $T/4, 3T/4$. As a consequence, there exists $\delta > 0$ such that $|w'|^2 \geq 2\delta$ on $[T/4 - 2\delta, T/4 + 2\delta]$. Hence, for sufficiently large $j_0 \in \mathbb{N}$ we have for $j \geq j_0$

$$u'(r_{4j} + r)^2 \geq \delta \quad \text{for } r \in [T/4 - \delta, T/4 + \delta] \text{ and } r_{4(j+1)} - r_{4j} \leq T + \delta.$$

From this, we deduce for $j \geq j_0$

$$u'(r)^2 \geq \delta, \quad \text{for } r \in [r_{4j} + T/4 - \delta, r_{4j} + T/4 + \delta], \quad (2.9)$$

$$r_{4j} \leq r_{4j_0} + (j - j_0)(T + \delta) \quad \text{for } j \geq j_0. \quad (2.10)$$

Then, for $k \geq j_0$ and $r > r_{4k} + T/4 + \delta$ we may exploit (2.8) and (2.9) to obtain

$$\begin{aligned} Z(r) &= Z(0) - 2(N - 1) \int_0^r \frac{u'(t)^2}{t} dt \\ &\leq Z(0) - 2(N - 1) \sum_{j=j_0}^k \int_{r_{4j} + T/4 - \delta}^{r_{4j} + T/4 + \delta} \frac{u'(t)^2}{t} dt \\ &\leq Z(0) - 2(N - 1)\delta \sum_{j=j_0}^k \int_{r_{4j} + T/4 - \delta}^{r_{4j} + T/4 + \delta} \frac{1}{t} dt \\ &= Z(0) - 2(N - 1)\delta \sum_{j=j_0}^k \ln \left(\frac{r_{4j} + \frac{T}{4} + \delta}{r_{4j} + \frac{T}{4} - \delta} \right). \end{aligned}$$

Let us fix $c(\delta) > 0$ such that $\ln(1 + x) \geq c(\delta)x$ for $0 \leq x \leq 2\delta/(r_{4j_0} + \frac{T}{4} - \delta)$. Then (2.10) implies

$$\begin{aligned} Z(r) &\leq Z(0) - 2(N - 1)\delta c(\delta) \sum_{j=j_0}^k \frac{2\delta}{r_{4j} + \frac{T}{4} - \delta} \\ &\leq Z(0) - 2(N - 1)\delta c(\delta) \sum_{j=j_0}^k \frac{2\delta}{r_{4j_0} + (j - j_0)(T + \delta) + \frac{T}{4} - \delta}. \end{aligned}$$

Choosing now k, r sufficiently large, we obtain that $Z(r) \rightarrow -\infty$ because the harmonic series diverges, but (2.3) implies that $Z(r) \geq 2G(u(r)) \geq 0$, yielding a contradiction. As a consequence, $u(r_{4j})$ converges to zero as $j \rightarrow \infty$ and analogously we deduce that also $u(r_{4j+2}) \rightarrow 0$. In the end, we obtain $u(r) \rightarrow 0$ as $r \rightarrow +\infty$.

Since Z is decreasing and nonnegative, it follows that $Z(r) \rightarrow Z_\infty \in [0, Z(0))$ as $r \rightarrow \infty$. Hence, by (2.7), also $|u'|$ has a limit at infinity which must be zero because u converges to 0. Finally, from the differential equation we deduce that $u''(r) \rightarrow 0$ as $r \rightarrow \infty$, i.e.

$$u(r), u'(r), u''(r) \rightarrow 0 \quad (r \rightarrow \infty). \tag{2.11}$$

As in Lemma 4.2 in [16], we get that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|u(r)|, |u'(r)|, |u''(r)| \leq C_\varepsilon r^{\frac{1-N}{2} + \varepsilon} \quad (r \geq 1). \tag{2.12}$$

Step 4: Proof of (2.4) Slightly generalizing the approach from the proof of Theorem 4 in [12], we study the function

$$\psi(r) := v'(r)^2 + 2r^{N-1}G(u(r)), \quad \text{where } v(r) = r^{(N-1)/2}u. \tag{2.13}$$

Using the function c from (2.5) and taking into account (2.1), we obtain that ψ satisfies the following differential equation

$$\begin{aligned} \psi'(r) &= 2v'(r)[-c(r)v(r)] + 2r^{(N-1)/2}g(u(r)) \left[v'(r) - \frac{N-1}{2}r^{(N-3)/2}u(r) \right] \\ &\quad + 2(N-1)r^{N-2}G(u(r)) \\ &= (N-1)r^{N-2}(2G(u(r)) - u(r)g(u(r))) + \frac{(N-1)(N-3)}{2r^2}v(r)v'(r). \end{aligned}$$

Using (2.11) as well as (1.4), (1.5), we obtain that there exist $C, r_0 > 0$ such that

$$\frac{u(r)^2}{G(u(r))} \leq C \quad \forall r \geq r_0. \tag{2.14}$$

Then, exploiting (1.2) and (2.12), we find $C', C'', r^* > 0$ such that, for all $r \geq r^*$, it results

$$\begin{aligned} & |(N - 1)r^{N-2} (2G(u(r)) - u(r)g(u(r)))| \\ & \leq \frac{(N - 1)C}{2r} \frac{|2G(u(r)) - u(r)g(u(r))|}{u(r)^2} \cdot 2r^{N-1}G(u(r)) \\ & \leq \frac{C'}{r} |u(r)|^\sigma \psi(r) \\ & \leq C'' r^{-1+(\frac{1-N}{2}+\varepsilon)\sigma} \psi(r). \end{aligned}$$

Moreover, using (2.13) and (2.14), we get

$$\begin{aligned} \left| \frac{(N - 1)(N - 3)}{2r^2} v(r)v'(r) \right| & \leq \frac{|(N - 1)(N - 3)|}{r^2} \cdot (v(r)^2 + v'(r)^2) \\ & \leq \frac{|(N - 1)(N - 3)|}{r^2} \cdot (Cr^{N-1}G(u(r)) + v'(r)^2) \\ & \leq \frac{|(N - 1)(N - 3)|(C + 1)}{r^2} \cdot \psi(r). \end{aligned}$$

This yields $|\psi'(r)| \leq a(r)\psi(r)$ for $r \geq r^*$ and some positive integrable function a . Dividing this inequality by the positive function $\psi(r)$ and integrating the resulting inequality over $[r^*, \infty)$ shows that ψ is bounded from below and from above by a positive number. From this, we obtain the lower and upper bounds (2.4) and the proof is finished. \square

We are now ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let us define the set

$$\mathcal{C} = \{u_\alpha(|\cdot|) \in C^2(\mathbb{R}^N) : |\alpha| < \alpha_0\}$$

where u_α denotes the unique solution of the initial value problem (2.1). The set \mathcal{C} is a subset of $C^2(\mathbb{R}^N)$, and it is a continuum in $C^2_{loc}(\mathbb{R}^N)$ thanks to the Ascoli–Arzelà theorem. The claims for the one-dimensional situation $N = 1$ follow from Proposition 2.3. In the case $N \geq 2$, we get from Lemma 2.5 that all elements of \mathcal{C} are oscillating localized solutions satisfying (2.3) and (2.4). \square

Remark 2.6. Let us mention that an analogous result to Theorem 1.2 is Theorem 1 [15], which applies to a more restrictive class of nonlinearities. Moreover, the above theorem is related to Theorem 4 in [12] but we do not need their assumption (g2). Actually, this hypothesis is not satisfied in our model cases $g = g_1$ or $g = g_2$.

Remark 2.7. The arguments from the proof of Theorem 1.2 also show the existence of oscillating localized solutions to initial value problems which are not of nonlinear Helmholtz type. For instance, one can treat concave–convex problems such as

$$-\Delta u = \lambda|u|^{q-2}u + \mu|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{2.15}$$

for $1 < q < 2 < p < \infty$ with $\lambda > 0, \mu \in \mathbb{R}$, see, for instance, [1] or [4] for corresponding results on a bounded domain with homogeneous Dirichlet boundary conditions. The existence of solutions is provided by Theorem 1 in [18] so that the steps 1,2,3 are proven in the same way as above and we obtain infinitely many radially symmetric, oscillating, localized, solutions of (2.15).

Remark 2.8. Using nonlinear oscillation theorems instead of Sturm’s comparison theorem, we can even extend the above observation towards superlinear nonlinearities g satisfying $\lambda|z|^q \leq g(z)z \leq \Lambda|z|^q$ for

$$2 < q \leq \frac{2(N + 1)}{N - 1}, N \in \{1, 2, 3\} \quad \text{or} \quad 2 < q \leq \frac{2(N - 1)}{N - 2}, N \geq 4.$$

Indeed, in the first case the function c from (2.5) satisfies the estimate $c(r) \geq \lambda r^{(1-N)(q-2)/2} |v(r)|^{q-2}$ so that Atkinson’s oscillation criterion applies, see the first line and third column of the table on p.153 in [24]. In the second case, Noussair’s oscillation criterion result can be used in order finish step 1, see the third line and third column of the table on p.153 in [24].

Remark 2.9. If $z \mapsto g(z)/z$ is decreasing, then one can show that the first zero of u_α is greater than the first zero of $u_{\tilde{\alpha}}$ whenever $0 < \tilde{\alpha} < \alpha < \alpha_0$. Indeed, we set $u := u_\alpha, v := u_{\tilde{\alpha}}$. Then the interval

$$I := \{t > 0 : u(s) > v(s) > 0 \text{ for all } s \in (0, t)\}$$

is open, connected and nonempty and thus $I = (0, r^*)$ for some $r^* > 0$. On its right boundary, we have either $u(r^*) = v(r^*) \geq 0$ or $u(r^*) > v(r^*) = 0$; so it remains to exclude the first possibility. Using $u > v > 0$ on I and (2.1), we have

$$(r^{N-1}(u'v - v'u))' = r^{N-1}uv \left(\frac{g(v)}{v} - \frac{g(u)}{u} \right) > 0 \quad \text{on } I. \tag{2.16}$$

Integrating (2.16) from 0 to r^* , the assumption $u(r^*) = v(r^*)$ leads to

$$0 < (u'v - v'u)(r^*) = u(r^*)(u - v)'(r^*),$$

then $u(r^*) = v(r^*) > 0$ and $(u - v)'(r^*) > 0$. On the other hand, $u - v > 0$ on $I = (0, r^*)$ and $(u - v)(r^*) = 0$ implies $(u - v)'(r^*) \leq 0$, a contradiction. Thus $u(r^*) > v(r^*) = 0$ so that the first zero of v comes before the first zero of u .

2.2. The nonautonomous case

In this section, we generalize Theorem 1.2 to a nonautonomous setting. Our aim is to identify mild assumptions on a nonautonomous nonlinearity g that ensure the existence of a continuum of oscillating localized solutions of the initial value problems

$$-u'' - \frac{N - 1}{r}u' = g(r, u), \quad u(0) = \alpha, \quad u'(0) = 0 \tag{2.17}$$

that behave like $r^{(1-N)/2}$ at infinity in the sense of (2.4). Before formulating such assumptions and stating the corresponding existence result, let us mention that our result applies to the nonlinearities (1.8), (1.9) under suitable conditions on the coefficient functions. This will be seen in Corollary 2.12 and Corollary 2.14 at the end of this section. Our existence results for (2.17) will be proven assuming that

$$g \in C([0, +\infty) \times \mathbb{R}, \mathbb{R}) \text{ is continuously differentiable w.r.t. } r. \tag{2.18}$$

Moreover, we suppose that there exist positive numbers $\alpha_*, \alpha^*, \lambda, \Lambda$ and a locally Lipschitz continuous function $g_\infty : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\lim_{r \rightarrow \infty} g(r, \cdot) = g_\infty(\cdot) \quad \text{uniformly on } [-\alpha_*, \alpha^*] \tag{2.19}$$

$$g_r(r, z)z \leq 0 \quad \text{on } [0, +\infty) \times [-\alpha_*, \alpha^*], \tag{2.20}$$

$$\lambda z^2 \leq g_\infty(z)z \leq g(r, z)z \leq \Lambda z^2 \quad \text{on } [0, +\infty) \times [-\alpha_*, \alpha^*]. \tag{2.21}$$

These assumptions will allow us to prove the mere existence of an oscillating localized solution. In order to show the desired asymptotic behaviour, we need some extra condition “at infinity” where r is large

and the solution itself is small: We will assume that there exist $\varepsilon, \sigma, C > 0$ and some integrable function k such that

$$|2G(r, z) - zg(r, z)| \leq Cz^2 |\ln(z)|^{-1-\sigma}, \quad |z| \leq \varepsilon, r \geq \varepsilon^{-1} \tag{2.22}$$

$$g_r(r, z)z \geq -k(r)z^2, \quad |z| \leq \varepsilon, r \geq \varepsilon^{-1}. \tag{2.23}$$

These assumptions are rather technical but can be verified easily in concrete situations as we show in the proof of Corollary 2.12. Let us remark that our assumptions (1.2),(1.3),(1.4) and (1.5) in the autonomous case (for any choice $\alpha^* = \alpha_* \in (0, \alpha_0)$) are more restrictive than the assumptions used above. For instance, oddness of g is no longer required and (2.22) replaces the $C_{loc}^{1,\sigma}$ -assumption. In particular, the following theorem generalizes our result for the autonomous case.

Theorem 2.10. *Let $N \geq 2$ and assume (2.19),(2.20),(2.21) as well as*

$$G(0, \alpha) \leq \min\{G_\infty(-\alpha_*), G_\infty(\alpha^*)\} \quad \text{for } \alpha \in [-\alpha_*, \alpha^*]. \tag{2.24}$$

Then there is an oscillating, localized solution u of (2.17) that satisfies $u(0) = \alpha$ as well as

$$\|u\|_{L^\infty(\mathbb{R})} \leq \max\{\alpha_*, \alpha^*\} \quad \text{and} \quad \|u'\|_{L^\infty(\mathbb{R})} \leq \sqrt{2G(0, \alpha)}. \tag{2.25}$$

Moreover, if (2.22) and (2.23) hold, then we can find $c, C > 0$ such that

$$cr^{(1-N)/2} \leq |u(r)| + |u'(r)| + |u''(r)| \leq Cr^{(1-N)/2} \quad \text{for } r \geq 1. \tag{2.26}$$

Proof. The proof of our result follows the same argument of the proof of Theorem 1.2, so we only mention the main differences. For simplicity, we only treat the case $\alpha > 0$. The existence of a maximally extended solution of (2.17) follows from a Peano type existence theorem for singular initial value problems, see Theorem 1 in [18].

Step 1 is proven as in the autonomous case where the function c from (2.5) has to be replaced by $c(r) = g(r, u(r))/u(r) - (N-1)(N-3)/4r^2$. Assumption (2.21) ensures that c is bounded from below by a positive constant as long as $0 \leq u(r) < \alpha$ so that u has to attain a first zero. In step 2, one shows that $Z(r) := u'(r)^2 + 2G(r, u(r))$ is nondecreasing due to $G_r(r, u(r)) \leq 0$ for $-\alpha_* \leq u(r) \leq \alpha^*$, see (2.20). Arguing as in the autonomous case we find that u decreases until it attains a local minimum at some $r_2 > r_1$ with $-\alpha_* < u(r_2) < 0$. More precisely, one finds a sequence (r_j) such that all r_{2j} are critical points and all r_{2j+1} are zeros of u with the additional property (the counterpart to (2.6))

$$2G(r_0, u(r_0)) > u'(r_1)^2 > 2G(r_2, u(r_2)) > u'(r_3)^2 > 2G(r_4, u(r_4)) > \dots$$

This and $G(r_0, u(r_0)) = G(0, \alpha)$ yield the L^∞ -bounds for u' , whereas the L^∞ -bounds follow from $-\alpha_* < u(r_{2j}) < \alpha^*$ for all $j \in \mathbb{N}$. Hence, (2.25) is proved so that step 2 is finished. Step 3 is the same as in the proof of Theorem 1.2. Since the reasoning of Lemma 4.2 in [16] may be adapted to our nonautonomous (but asymptotically autonomous) problem, we also find (2.12), i.e.

$$|u(r)|, |u'(r)|, |u''(r)| \leq C_\varepsilon r^{\frac{1-N}{2} + \varepsilon} \quad (r \geq 1). \tag{2.27}$$

In step 4, we use (2.22),(2.23) in order to study the asymptotics of the function

$$\psi(r) := v'(r)^2 + 2r^{N-1}G(r, u(r))$$

where $v(r) := r^{(N-1)/2}u(r)$. One shows

$$\begin{aligned} \psi'(r) &= 2r^{N-1}G(r, u(r)) \left(\frac{N-1}{r} \frac{2G(r, u(r)) - u(r)g(r, u(r))}{2G(r, u(r))} + \frac{G_r(r, u(r))}{G(r, u(r))} \right) \\ &\quad + \frac{(N-1)(N-3)}{2r^2}v(r)v'(r). \end{aligned}$$

From (2.23), we get for sufficiently large $r \geq r^*$ the estimate $|G_r(r, u(r))| \leq \frac{k(r)}{\lambda} G(r, u(r))$. Moreover, with an analogous inequality as in (2.14) as well as (2.22), (2.27) we obtain

$$\begin{aligned} \left| \frac{2G(r, u(r)) - u(r)g(r, u(r))}{2G(r, u(r))} \right| &\leq \left| \frac{2G(r, u(r)) - u(r)g(r, u(r))}{\lambda u(r)^2} \right| \\ &\leq C |\ln(u(r))|^{-1-\sigma} \\ &\leq C' \ln(r)^{-1-\sigma} \end{aligned}$$

so that we may find as in the autonomous case a positive integrable function a such that $|\psi'(r)| \leq a(r)\psi(r)$. This shows that ψ is bounded from below and from above by a positive number. From this and

$$\lambda z^2 \leq 2G_\infty(z) \leq 2G(r, z) \leq \Lambda z^2 \quad \text{on } [0, +\infty) \times [-\alpha_*, \alpha^*],$$

which is a consequence of (2.21), we obtain the lower and upper bounds (2.26) and the proof is finished. \square

Remark 2.11. Let us stress that Theorem 2.10 is stated for $N \geq 2$ since we are focused on localized solution, but we could also prove the nonautonomous counterpart of Theorem 1.2 for $N = 1$.

Finally, let us apply Theorem 2.10 to the special nonlinearities g_1, g_2 given in (1.8), (1.9). We obtain the following results.

Corollary 2.12. *Let $N \geq 2, p > 2$ and suppose that $k, Q \in C^1([0, +\infty), \mathbb{R})$ are nonincreasing functions with limits $k_\infty > 0$ and $Q_\infty \in \mathbb{R}$, respectively. Then there is a nonempty open interval I containing 0 and a continuum $\mathcal{C} = \{u_\alpha \in C^2(\mathbb{R}^N) : \alpha \in I\}$ in $C_{loc}^2(\mathbb{R}^N)$ consisting of radially symmetric oscillating classical solutions of the equation*

$$-\Delta u - k(|x|)^2 u = Q(|x|)|u|^{p-2}u \quad \text{in } \mathbb{R}^N$$

having the properties (2.25), (2.26) stated in Theorem 2.10. In case $Q_\infty \geq 0$, we have $I = \mathbb{R}$.

Proof. We set

$$g(r, z) = k(r)^2 z + Q(r)|z|^{p-2}z, \quad g_\infty(z) = k_\infty^2 z + Q_\infty|z|^{p-2}z$$

and

$$G(r, z) = \frac{k(r)^2}{2} z^2 + \frac{Q(r)}{p} |z|^p, \quad G_\infty(z) = \frac{k_\infty^2}{2} z^2 + \frac{Q_\infty}{p} |z|^p.$$

By the regularity assumptions on k, Q we have (2.18). Moreover, $k', Q' \leq 0$ implies $g_r(r, z)z \leq 0$ for all $z \in \mathbb{R}$ and thus (2.20). We set

$$\begin{aligned} I &:= \left\{ \alpha \in \mathbb{R} : G(0, \alpha) < \sup_{\mathbb{R}} G_\infty \right\} \\ &= \left\{ \alpha \in \mathbb{R} : G(0, \alpha) < \left(\frac{1}{2} - \frac{1}{p} \right) k_\infty^2 \left(\frac{k_\infty^2}{(Q_\infty)^-} \right)^{\frac{2}{p-2}} \right\}. \end{aligned} \tag{2.28}$$

Here, $(Q_\infty)^- = \max\{-Q_\infty, 0\}$. For any given $\alpha \in I$, we can choose

$$0 < \alpha_* = \alpha^* < \left(\frac{k_\infty^2}{(Q_\infty)^-} \right)^{\frac{1}{p-2}} \text{ s.t. } G_\infty(-\alpha_*) = G_\infty(\alpha^*) = G(0, \alpha).$$

For this choice of α_*, α^* assumption (2.21) holds. Finally, (2.22) follows from $2G(r, z) - zg(r, z) = O(|z|^p)$ as $z \rightarrow 0$ uniformly with respect to r and (2.23) holds because $g_r(r, z)z \geq (2k(r)k'(r) + Q'(r))z^2$ for $|z| \leq 1$. Hence, all assumptions of Theorem 2.10 are satisfied and the existence of solutions of (2.17) follows. Due to the unique solvability of these initial value problems and the theorem of Ascoli–Arzelà, they form a continuum in $C_{loc}^2(\mathbb{R})$. Finally we remark that (2.28) implies $I = \mathbb{R}$ whenever $Q_\infty \geq 0$. \square

Remark 2.13. Theorem 2.10 extends Theorem 4 in [12] in various directions. First of all, it provides more qualitative information of the solutions such as the $W^{1,\infty}$ -bounds, the oscillating behaviour of the solutions and the lower bounds for their decay at infinity. Additionally, we do not assume any global positivity assumption on f . Furthermore, our assumption (2.22) is not covered by the hypotheses in [12].

The following result can be proved similarly and we state it for completeness.

Corollary 2.14. *Let $N \geq 2$ and suppose that $\lambda, s \in C^1([0, +\infty), \mathbb{R})$ are nondecreasing functions with limits λ_∞, s_∞ , respectively, such that s is positive and $\lambda_\infty < 1/s_\infty$. Then there is a nonempty open interval I containing 0 and a continuum $\mathcal{C} = \{u_\alpha \in C^2(\mathbb{R}^N) : \alpha \in I\}$ in $C^2_{loc}(\mathbb{R}^N)$ consisting of radially symmetric oscillating classical solutions of the equation*

$$-\Delta u + \lambda(|x|)u = \frac{u}{s(|x|) + u^2} \quad \text{in } \mathbb{R}^N$$

having the properties (2.25), (2.26) from Theorem 2.10. In the case $\lambda_\infty \leq 0$, we have $I = \mathbb{R}$.

3. Nonradial solutions

In this section, we study equation (1.10) proving Theorems 1.3 and 1.4. We will follow the argument introduced in [13] adapting their methods to our context. First note that, up to rescaling, we may assume $k = 1$ in the following. Let us introduce some notations in order to facilitate the reading. Let Ψ be the real part of the fundamental solution of the Helmholtz equation $-\Delta - 1$ on \mathbb{R}^N (see e.g. (11) in [13]). Performing the transformation $v = |Q|^{1/p'}|u|^{p-2}u$ for $\frac{1}{p} + \frac{1}{p'} = 1$, our problem amounts to solving

$$|v|^{p'-2}v = -|Q|^{1/p}[\Psi * (|Q|^{1/p}v)] \quad \text{in } \mathbb{R}^N. \tag{3.1}$$

Notice that the right-hand side comes with a negative sign in contrast to [13]. This is because we assume Q to be negative so that $Q = -|Q|$. Let us introduce the linear operators $\mathbf{R}, \mathbf{K}_p : L^{p'}(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)$ defined by

$$\mathbf{R}(v) = \Psi * v, \quad \mathbf{K}_p(v) = |Q|^{1/p}\mathbf{R}(|Q|^{1/p}v).$$

Both \mathbf{R} and \mathbf{K}_p are continuous and we have for all $f, g \in L^{p'}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} f\mathbf{R}(g) = \int_{\mathbb{R}^N} f(\Psi * g) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{(|\xi|^2 - 1)\hat{f}(\xi)\hat{g}(\xi)}{(|\xi|^2 - 1)^2 + \varepsilon^2} d\xi, \tag{3.2}$$

where \hat{f}, \hat{g} are the Fourier transforms of f and g , respectively. In view of the variational structure of (3.1), we define the functionals $J, \bar{J} : L^{p'}(\mathbb{R}^N) \rightarrow \mathbb{R}$ via the formulas

$$J(v) := \frac{1}{p'} \int_{\mathbb{R}^N} |v|^{p'} - \frac{1}{2} \int_{\mathbb{R}^N} v\mathbf{K}_p(v),$$

$$\bar{J}(v) := \frac{1}{p'} \int_{\mathbb{R}^N} |v|^{p'} + \frac{1}{2} \int_{\mathbb{R}^N} v\mathbf{K}_p(v),$$

so that the solutions of (3.1) are precisely the critical points of \bar{J} , see (49) in [13]. Notice that the functional J is used when Q is positive. Our main observation is that not only J but also \bar{J} has the mountain pass geometry. This follows from the following Lemmas which are the counterparts of Lemma 4.2 and Lemma 5.1 in [13]. In the following, we will denote with $\|\cdot\|_q$ the standard norm in the Lebesgue space $L^q(\mathbb{R}^N)$.

Lemma 3.1. *Under the assumptions of Theorem 1.3, there is a function $v_0 \in L^{p'}(\mathbb{R}^N)$ such that $\|v_0\|_{p'} > 1, \bar{J}(v_0) < 0$.*

Proof. As in Lemma 4.2 (ii) [13], it suffices to prove

$$\int_{\mathbb{R}^N} z \mathbf{K}_p z < 0$$

for some $z \in L^{p'}(\mathbb{R}^N)$ because then one may take $v_0 := tz$ for sufficiently large $|t|$. To this end, let $y \in \mathcal{S}(\mathbb{R}^N)$ be a nontrivial Schwartz function satisfying $\text{supp}(\hat{y}) \subset B_1(0)$. For $\delta > 0$ we set

$$z_\delta := y|Q|^{-1/p} 1_{\{|Q|>\delta\}}, \quad \mu := \int_{\text{supp}(\hat{y})} \frac{|\hat{y}(\xi)|^2}{|\xi|^2 - 1} d\xi < 0,$$

where $1_{\{|Q|>\delta\}}$ is the indicator function of the set $\{x \in \mathbb{R}^N : |Q(x)| > \delta\}$. Then we have $z_\delta \in L^{p'}(\mathbb{R}^N)$ and thus $\mathbf{K}_p z_\delta \in L^p(\mathbb{R}^N)$. Hence, by definition of \mathbf{K}_p , the function $y_\delta := |Q|^{1/p} z_\delta = y \cdot 1_{\{|Q|>\delta\}}$ satisfies

$$\int_{\mathbb{R}^N} z_\delta (\mathbf{K}_p z_\delta) = \int_{\mathbb{R}^N} |Q|^{1/p} z_\delta \mathbf{R}(|Q|^{1/p} z_\delta) = \int_{\mathbb{R}^N} y_\delta \mathbf{R}(y_\delta).$$

Since we have $|Q| > 0$ almost everywhere, we get $y_\delta \rightarrow y$ in $L^{p'}(\mathbb{R}^N)$ as $\delta \rightarrow 0^+$. Thus the continuity of \mathbf{R} implies that we can choose $\delta > 0$ so small that the following holds:

$$\int_{\mathbb{R}^N} z_\delta (\mathbf{K}_p z_\delta) < \int_{\mathbb{R}^N} y \mathbf{R}(y) + \frac{|\mu|}{2}.$$

From this and (3.2), we infer

$$\int_{\mathbb{R}^N} z_\delta (\mathbf{K}_p z_\delta) < \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{(|\xi|^2 - 1)|\hat{y}(\xi)|^2}{(|\xi|^2 - 1)^2 + \varepsilon^2} d\xi + \frac{|\mu|}{2} = \mu + \frac{|\mu|}{2} < 0$$

which is all we had to show. □

Lemma 3.2. *Let the assumptions of Theorem 1.4 hold. Then for every $m \in \mathbb{N}$ there is an m -dimensional subspace $\mathcal{W} \subset L^{p'}(\mathbb{R}^N)$ with the following properties:*

- (i) $\int_{\mathbb{R}^N} v \mathbf{K}_p v < 0$ for all $v \in \mathcal{W} \setminus \{0\}$.
- (ii) There exists $R = R(\mathcal{W}) > 0$ such that $\bar{J}(v) \leq 0$ for every $v \in \mathcal{W}$ with $\|v\|_{p'} \geq R$.

Proof. Let $y^1, \dots, y^m \in \mathcal{S}(\mathbb{R}^N)$ be nontrivial Schwartz functions such that

$$\bigcup_{j=1}^m \text{supp}(\hat{y}^j) \subset B_1(0), \quad \text{supp}(\hat{y}^j) \cap \text{supp}(\hat{y}^i) = \emptyset \quad (i \neq j). \tag{3.3}$$

For sufficiently small $\delta > 0$, we then define

$$\mathcal{W} := \text{span}\{z_\delta^1, \dots, z_\delta^m\} \quad \text{where } z_\delta^j := y^j |Q|^{-1/p} 1_{\{|Q|>\delta\}}.$$

Then (3.3) implies that \mathcal{W} is m -dimensional and similar calculations as above show (i) and (ii). □

With the aid of the above lemmas, the proofs of our theorems are essentially the same as in [13]. We indicate the main steps for the convenience of the reader.

Proof of Theorem 1.3: Under the given assumptions, \bar{J} has the mountain pass geometry. Indeed, as in the parts (i), (iii) of Lemma 4.2 in [13] one proves that 0 is a strict local minimum and the boundedness of Palais–Smale sequences of \bar{J} . In Lemma 3.1, we proved that there is a $v_0 \in L^{p'}(\mathbb{R}^N)$ such that $\|v_0\|_{p'} > 1, \bar{J}(v_0) < 0$. Hence, as in Lemma 6.1 [13] the Deformation Lemma implies the existence of a bounded Palais–Smale sequence (v_m) for \bar{J} at its mountain pass level $\bar{c} > 0$. Similar to the proof of Theorem 6.2 in [13], one has

$$\begin{aligned} \lim_{m \rightarrow \infty} \int_{\mathbb{R}^N} |Q|^{1/p} v_m \mathbf{R}(|Q|^{1/p} v_m) &= \frac{2p'}{2-p'} \lim_{m \rightarrow \infty} \left[-\bar{J}(v_m) + \frac{1}{p'} \bar{J}'(v_m)[v_m] \right] \\ &= -\frac{2p'}{2-p'} \bar{c} < 0. \end{aligned}$$

Then, Theorem 3.1 in [13] implies that there are $R, \zeta > 0$ and points $x_m \in \mathbb{R}^N$ and a subsequence, still denoted with (v_m) , such that

$$\int_{B_R(x_m)} |v_m|^{p'} \geq \zeta > 0.$$

From this point on the reasoning is the same as in [13] and we obtain that (v_m) converges weakly to a nontrivial solution v of (1.10) and the solution u of the original equation may be found via $u = \mathbf{R}(|Q|^{1/p} v) \in L^p(\mathbb{R}^N)$. In particular, u satisfies (1.10) and

$$\lim_{|x| \rightarrow \infty} \int_{|x-y| \leq 1} |u(y)|^p dy = 0$$

so that replacing 2 by p in the proof of Theorem C.3.1 in [19] one proves $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$, namely that u is localized. This implies

$$\Delta u + (k^2 + o(1))u = 0, \quad \text{as } |x| \rightarrow \infty$$

so that the PDE version of Sturm’s comparison principle (see, for instance, Theorem 5.1 in [23]) and the strong maximum principle shows that u is oscillating. □

Proof of Theorem 1.4: Lemma 3.2 yields all the required geometrical features of the symmetric mountain pass theorem (see Theorem 6.5 in [22]). Moreover, Lemma 5.2 in [13] implies that the Palais–Smale condition holds for \bar{J} , giving the existence of pairs of nontrivial localized solutions. The oscillation property follows again from Theorem 5.1 in [23]. □

4. On the approximation by bounded domains

In this section, we briefly address the question whether localized solutions of (1.1) can be approximated by solutions of the corresponding homogeneous Dirichlet problem on a large bounded domain. We are going to show that this method does not work in general. More precisely, we will prove that the positive minimizers of the Euler functionals associated with the problem on bounded domains diverge in $H^1(\mathbb{R}^N)$ as the domains approach \mathbb{R}^N . Even though the divergence will only be proved for the sequence of minimizers, we believe that the analogous phenomenon occurs for broader classes of finite energy solutions, e.g. constrained minimizers, or solutions with a given upper bound on their nodal domains or on their Morse index. Throughout this section, we will assume that the nonlinearity g satisfies the hypotheses (1.2), (1.3), (1.4) as well as (1.5) with $\alpha_0 \in (0, +\infty)$, in order to avoid some sub-critical growth conditions (see Remark 4.3).

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and consider the variational problem

$$c_\Omega := \inf_{H_0^1(\Omega)} I_\Omega \quad \text{where } I_\Omega(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega G(u)$$

where $G(z)$ denotes the primitive of g such that $G(0) = 0$. Notice that (1.2),(1.4),(1.5) imply $G(z) \leq C|z|^2$ for some $C > 0$ and for all $z \in \mathbb{R}$, so that $I_\Omega : H_0^1(\Omega) \mapsto \mathbb{R} \cup \{+\infty\}$ is well defined. Bounded critical points of I_Ω are classical solutions of the boundary value problem

$$\begin{cases} -\Delta w = g(w) & \text{in } \Omega, \\ w \in H_0^1(\Omega). \end{cases} \tag{4.1}$$

In the following proposition, we show that I_Ω admits a positive minimizer provided $g'(0) > \lambda_1(\Omega)$ holds. More precisely, we have the following result.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and in addition to (1.2),(1.3),(1.4),(1.5) assume $g'(0) > \lambda_1(\Omega)$. Then, there exists a global minimizer u_Ω of I_Ω in $H_0^1(\Omega)$ which is a solution of (4.1) satisfying $0 < u_\Omega < \alpha_0$ in Ω .*

Proof. Hypotheses (1.3) and (1.5) imply that $G(z) \leq G(\alpha_0)$ holds for every $z \in \mathbb{R}$. Hence, for all $u \in H_0^1(\Omega)$ we have

$$I_\Omega(u) \geq \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega G(\alpha_0) = \frac{1}{2} \int_\Omega |\nabla u|^2 - |\Omega|G(\alpha_0),$$

which shows that I_Ω is coercive and bounded from below. Moreover, if ϕ_1 denotes the eigenfunction associated to $\lambda_1(\Omega)$, then

$$\lim_{t \rightarrow 0} \frac{I_\Omega(t\phi_1)}{t^2} = \frac{1}{2} \int_\Omega |\nabla \phi_1|^2 - G''(0)\phi_1^2 = \frac{1}{2} (\lambda_1(\Omega) - g'(0)) \int_\Omega \phi_1^2 < 0,$$

so that $c_\Omega < 0 = I_\Omega(0)$. Additionally, I_Ω is weakly sequentially lower semicontinuous so that there exists a minimizer u_Ω , which must be nontrivial because of $c_\Omega < 0$. We may assume $0 \leq u_\Omega \leq \alpha_0$ because $\min\{|u_\Omega|, \alpha_0\} \in H_0^1(\Omega)$ is another minimizer of I_Ω . From the strong maximum principle, we deduce that u_Ω satisfies $0 < u_\Omega < \alpha_0$ in Ω as it is nontrivial. \square

Remarks 4.2. (a) If $z \mapsto g(z)/z$ is decreasing, then the condition $g'(0) > \lambda_1(\Omega)$ is even necessary for the existence of a positive solution $u \in H_0^1(\Omega)$. Indeed, testing (4.1) with u gives

$$\lambda_1(\Omega) \int_\Omega u^2 \leq \int_\Omega |\nabla u|^2 = \int_\Omega g(u)u < g'(0) \int_\Omega u^2.$$

In particular, note that our model nonlinearities g_1, g_2 given in (1.6) and (1.7) satisfy this monotonicity property.

(b) If Ω is smooth then Theorem 1 in [7] shows that the positive solution of (4.1) is unique provided $z \mapsto g(z)/z$ is decreasing.

Remark 4.3. In this section, we do not consider the case $\alpha_0 = +\infty$ in (1.5) because, without imposing additional growth conditions, the functional I_Ω may not be well defined in this case and, even if it were, it need not be bounded from below.

Next we study the convergence of the minimizers obtained in Proposition 4.1. To this end, we consider a sequence (Ω_n) of bounded domains satisfying $\Omega_n \subset \Omega_{n+1} \subset \mathbb{R}^N$ and $\bigcup_{n \in \mathbb{N}} \Omega_n = \mathbb{R}^N$. Since every compact subset of \mathbb{R}^N is covered by finitely many of those bounded domains, we observe that $\lambda_1(\Omega_n) \rightarrow 0$ as $n \rightarrow \infty$ so that, by the above proposition, the existence of positive minimizers is guaranteed for large n provided

that (1.4) holds. We show that the minimizers converge to the constant solution α_0 and therefore do not give any new finite energy solution.

Theorem 4.4. *Assume (1.2),(1.3),(1.4),(1.5) and let (Ω_n) be a sequence of bounded domains such that $\Omega_n \subseteq \Omega_{n+1}$ and $\cup_n \Omega_n = \mathbb{R}^N$. Then, for all sufficiently large n , there exists a nontrivial minimizer u_n of I_{Ω_n} on $H_0^1(\Omega_n)$ having the following properties:*

- (a) $0 < u_n < \alpha_0$ in Ω_n ,
- (b) $I_{\Omega_n}(u_n) \rightarrow -\infty$,
- (c) $u_n \rightarrow \alpha_0$ in $C_{loc}^\infty(\mathbb{R}^N)$ and $\|u_n\|_{L^q(\Omega_n)} \rightarrow \infty$ for all $q \in [1, \infty)$.

Proof. Since $\lambda_1(\Omega_n) \rightarrow 0$, taking into account (1.4) we find n_0 such that, for every $n \geq n_0$, $\lambda_1(\Omega_n) < g'(0)$. As a consequence, we can apply Proposition 4.1 to deduce that there exists a sequence $(u_n)_{n \geq n_0}$ of positive minimizers of I_{Ω_n} satisfying conclusion (a). In order to prove conclusion (b), let $\phi \in C_0^\infty(\mathbb{R}^n)$ be given with $\|\phi\|_2 = \|\phi\|_\infty = 1$. For every $k \in \mathbb{N}$ we set

$$\phi_k(x) = \frac{1}{k^{N/2}} \phi\left(\frac{x}{k} + ke_1\right),$$

so that $\|\phi_k\|_2 = 1, \|\phi_k\|_\infty \leq 1, \|\nabla \phi_k\|_2 \rightarrow 0$. Without loss of generality, we may assume $\sigma \in (0, 1)$ from hypothesis (1.2) to be so small that $2 + \sigma \in [2, \frac{2N}{N-2}]$ holds provided $N > 2$. Exploiting (1.2) and (1.5), we obtain positive constants A, C such that

$$g'(0)z^2 - 2G(z) \leq A|z|^{2+\sigma} \quad \text{for } |z| \leq 1, \quad \|\phi_k\|_{2+\sigma} \leq C.$$

Then, for a fixed positive $t \leq \min\{(g'(0)/(4AC))^{1/\sigma}, 1\}$ and sufficiently large $k \geq k_0$ we have $\|t\phi_k\|_\infty \leq 1$ so that the following estimate holds

$$\begin{aligned} 2I(t\phi_k) &= t^2 \int_{\mathbb{R}^N} |\nabla \phi_k|^2 - g'(0)\phi_k^2 + \int_{\mathbb{R}^N} (g'(0)(t\phi_k)^2 - 2G(t\phi_k)) \\ &\leq -\frac{g'(0)}{2}t^2 + A \int_{\mathbb{R}^N} |t\phi_k|^{2+\sigma} \leq \frac{t^2}{2} (-g'(0) + 2t^\sigma AC) =: -E, \end{aligned}$$

where $E > 0$ by the choice of t . Since the supports of (ϕ_k) go off to infinity we find some $k_1 \in \mathbb{N}$ such that for all $k \geq k_1$ it results

$$2I(t\phi_{k_0} + t\phi_k) \leq \frac{E}{2} + 2I(t\phi_{k_0}) + 2I(t\phi_k) \leq -\frac{3}{2}E.$$

Inductively, we find $k_2 < k_3 < \dots$ such that for all $k \geq k_m$ we have

$$\begin{aligned} 2I(t\phi_{k_0} + t\phi_{k_1} + \dots + t\phi_k) &\leq \frac{E}{2} + 2I(t\phi_{k_0} + t\phi_{k_1} + \dots + t\phi_{k_{m-1}}) + 2I(t\phi_k) \\ &\leq -(1 + m/2)E. \end{aligned}$$

Since for any given $m \in \mathbb{N}$ $\text{supp}(t\phi_{k_0} + t\phi_{k_1} + \dots + t\phi_k) \subset \Omega_n$ for sufficiently large n , the same estimate holds true for I_{Ω_n} , yielding conclusion (b).

In order to show (c), note that the sequence (u_n) is made of minimizers of I_{Ω_n} so that

$$\int_{\Omega_n} g'(u_n)\phi^2 \leq \int_{\Omega_n} |\nabla \phi|^2 \quad \text{for all } \phi \in C_c^1(\Omega_n). \tag{4.2}$$

By the Ascoli–Arzelà theorem and interior Schauder estimates, we find that (u_n) converges in $C_{loc}^2(\mathbb{R}^N)$ to some limit function $u \in C^2(\mathbb{R}^N)$ satisfying $0 \leq u(x) \leq \alpha_0$ for all $x \in \mathbb{R}^N$ as well as (1.1). The Dominated Convergence theorem allows to pass to the limit in (4.2) and we obtain

$$\int_{\mathbb{R}^N} g'(u)\phi^2 \leq \int_{\mathbb{R}^N} |\nabla \phi|^2 \quad \text{for all } \phi \in C_c^1(\mathbb{R}^N),$$

There are now three possibilities: either $u \equiv 0$ or u is nonconstant or $u \equiv \alpha_0$. As a consequence, it is left to show that the first two possibilities do not occur.

First, assume by contraction that $u \equiv 0$, so that for any given compact K we have $u_n \rightarrow u$ uniformly on K . Thanks to (1.4) we can find a sufficiently large n such that $g(u_n)/u_n \geq \delta^2 := g'(0)/2$. We may choose K so large such that the fundamental solution ψ of

$$\Delta\psi + \delta^2\psi = 0$$

changes sign within K . Then the PDE version of Sturm's comparison theorem (Theorem 5.1 in [23]) shows that u_n has a zero within K contradicting the positivity of u_n .

Assume now that u is nonconstant. Arguing as in the proof of Theorem 1.3 in [14] and applying Proposition 1.4 in [14], one shows that $u > 0$ and $\|u\|_\infty < \alpha_0$. As a consequence, the constant

$$c_0 := \min_{0 \leq s \leq \|u\|_\infty} \frac{g(s)}{s}$$

turns out to be positive and, for any compact set K we can find n such that $g(u_n)/u_n \geq c_0/2$. Choosing again K sufficiently large, we get a contradiction as above.

Hence, it turns out that $u \equiv \alpha_0$, and in particular, we get $\|u_n\|_{L^q(\Omega_n)} \rightarrow \infty$ for all $q \in [1, \infty)$. \square

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