

CHARACTERIZATION OF STADIUM-LIKE DOMAINS VIA BOUNDARY VALUE PROBLEMS FOR THE INFINITY LAPLACIAN

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ABSTRACT. We give a complete characterization, as “stadium-like domains”, of convex subsets Ω of \mathbb{R}^n where a solution exists to Serrin-type overdetermined boundary value problems in which the operator is either the infinity Laplacian or its normalized version. In case of the not-normalized operator, our results extend those obtained in a previous work, where the problem was solved under some geometrical restrictions on Ω . In case of the normalized operator, we also show that stadium-like domains are precisely the unique convex sets in \mathbb{R}^n where the solution to a Dirichlet problem is of class $C^{1,1}(\Omega)$.

1. INTRODUCTION

Consider the following Serrin-type problems for the infinity Laplace operator Δ_∞ or its normalized version Δ_∞^N :

$$(1) \quad \begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| = c & \text{on } \partial\Omega, \end{cases}$$

and

$$(2) \quad \begin{cases} -\Delta_\infty^N u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| = c & \text{on } \partial\Omega. \end{cases}$$

Aim of this paper is to provide a complete characterization of convex domains $\Omega \subset \mathbb{R}^n$ where such problems admit a solution.

Following the seminal paper by Serrin [34] and the huge amount of literature after it (see for instance [5, 6, 18, 19, 20, 21, 27, 35]), overdetermined boundary value problems involving the infinity Laplace operator were firstly considered only few years ago by Buttazzo and Kawohl (see [7]). In fact, due to the high degeneracy of the operator, all the different methods exploited in the literature to obtain symmetry results for overdetermined boundary value problems fail when applied to problems (1)-(2).

In [7], Buttazzo and Kawohl dealt with a simplified version of problems (1)-(2), which consists in looking for solutions having the same level lines as the distance function to the boundary of Ω , which are called *web-functions* (see Section 2 below). This simplification essentially reduces the problem to a one-dimensional setting, allowing to prove that the existence of a web-solution implies a precise geometric condition on Ω , which is the coincidence of its *cut locus* and *high ridge* (see again Section 2 for the definitions).

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In particular, such condition does not imply symmetry, at least if taken alone without any additional boundary regularity requirement. In our previous paper [12] we studied the geometry of domains whose cut locus and high ridge agree, by providing a complete characterization of them in dimension $n = 2$, and in higher dimensions under convexity constraint; in particular, these results reveal that planar convex sets with the same cut locus and high ridge are tubular neighborhoods of a line segment (possibly degenerated into a point). Moreover, in [14] we were able to carry over the study of problem (1) in the class of web-functions, by dropping all the regularity hypotheses on both the domain and the solution previously asked in [7].

The study of problems (1)–(2) in their full generality, namely without imposing the solution to be a web-function, turns out to be much more challenging. As for problem (2), to our knowledge it has never been undergone. As for problem (1), in a recent work we proved that, among convex sets, those having the same cut locus and high ridge - that we call “stadium-like domains” - are the only ones for which *whatever* solution (not necessarily of web type) exists, see [13, Thm. 5]. As a drawback, we needed to ask the following *a priori* geometrical hypothesis on the convex domain Ω : there exists an inner ball, of radius equal to the maximum of the distance from the boundary, touching $\partial\Omega$ at two diametral points. Moreover, we also needed the technical assumption that Ω satisfies an interior sphere condition at every point of the boundary.

The approach we adopted for the proof relies on the study of a suitable P -function along the gradient flow of the unique solution to the Dirichlet problem. In particular, the diametral ball condition was used as a fundamental picklock to get the result. Indeed, it allowed us to overcome the possible lack of regularity of the solution, which is an intrinsic phenomenon; we refer to [13, Sections 5 and 6] for more details, including regularity thresholds.

However, there was no reason to think that the geometric assumptions made on Ω should be really necessary, so that the conclusion reached in [13] was not completely satisfactory. We can now introduce the contents of this paper, by describing its main results:

- Theorem 2 improves the achievement of [13, Thm. 5], by showing that it continues to hold without any geometric assumptions on Ω , i.e. when both the diametral ball assumption and the interior sphere condition are removed. Contrarily to our previous belief, it is possible to arrive at this conclusion by completely circumventing regularity matters, but rather exploiting the observation that a suitable web-function is always a super-solution to our problem (see Proposition 12).
- Theorem 3 states that the same result (in its fully general version when no assumption is made on the convex set Ω), holds true in the case of the normalized infinity Laplace operator Δ_∞^N . Recently, such operator has attracted an increasing interest for its applications and connections with different areas, in particular “tug of war” differential games [2, 28, 32]. As we pointed out in [13, Remark 3], in order to deal with problem (3) a missing key ingredient was the C^1 -regularity of the solution to the corresponding Dirichlet problem, which has been established quite recently in [11]. More generally, the definition of Δ_∞^N via a dichotomy demands some care to adapt the different parts of the proof.
- Theorem 4 gives yet another characterization of stadium-like domains, as the only convex sets Ω where the unique solution to the homogeneous Dirichlet problem with constant source term for the normalized operator achieves its maximal regularity, namely is in $C^{1,1}(\Omega)$.

We address as an interesting and challenging task the problem of extending our results to non-convex domains.

The paper is organized as follows. In Section 2 we collect the required preliminary definitions and results. In Section 3 we state our main results (Theorems 2, 3, and 4), along with an outline of the proofs of Theorems 2 and 3, including the statement of the auxiliary results which serve as intermediate steps. In case of the operator Δ_∞ , the proofs of these intermediate steps can be found in [13], except for Proposition 12, which is precisely the key new ingredient allowing us to remove the diametral ball condition. In case of the operator Δ_∞^N , the proofs of all the intermediate steps must be adapted, and therefore we have chosen to present them separately in Section 4. Finally in Section 5 we prove Theorem 4.

2. PRELIMINARIES

Let us recall the basic notions and known results about the unique viscosity solution to the Dirichlet problems given by the first two equations in (1) or in (2).

For a C^2 function φ , we introduce the (*not normalized*) *infinity Laplacian* by

$$\Delta_\infty \varphi := \langle \nabla^2 \varphi \nabla \varphi, \nabla \varphi \rangle$$

and the operators

$$\Delta_\infty^+ \varphi(x) := \begin{cases} \Delta_\infty^N \varphi(x), & \text{if } \nabla \varphi(x) \neq 0, \\ \lambda_{\max}(\nabla^2 \varphi(x)), & \text{if } \nabla \varphi(x) = 0, \end{cases}$$

$$\Delta_\infty^- \varphi(x) := \begin{cases} \Delta_\infty^N \varphi(x), & \text{if } \nabla \varphi(x) \neq 0, \\ \lambda_{\min}(\nabla^2 \varphi(x)), & \text{if } \nabla \varphi(x) = 0. \end{cases}$$

Here $\Delta_\infty^N \varphi$ is the *normalized infinity Laplacian*:

$$\Delta_\infty^N \varphi := \frac{1}{|\nabla \varphi|^2} \langle \nabla^2 \varphi \nabla \varphi, \nabla \varphi \rangle$$

and, for a symmetric matrix $A \in \mathbb{R}_{\text{sym}}^{n \times n}$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote respectively the minimum and the maximum eigenvalue of A .

Let Ω be an open bounded subset of \mathbb{R}^n , and consider the infinity Laplace equations

$$(3) \quad -\Delta_\infty u = 1 \quad \text{in } \Omega$$

and

$$(4) \quad -\Delta_\infty^N u = 1 \quad \text{in } \Omega.$$

In order to recall the notion of viscosity solutions for these equations, according to [10], it is convenient to fix some notation. If $u, v: \Omega \rightarrow \mathbb{R}$ are two functions and $x \in \Omega$, by

$$u \prec_x v$$

we mean that $u(x) = v(x)$ and $u(y) \leq v(y)$ for every $y \in \Omega$.

Moreover we denote by $J_\Omega^{2,-} u(x)$ (resp. $J_\Omega^{2,+} u(x)$) the *second order sub-jet* (resp. *super-jet*), of a function $u \in C(\bar{\Omega})$ at a point $x \in \bar{\Omega}$, which is by definition the set of pairs $(p, A) \in \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n}$ such that, as $y \rightarrow x$, $y \in \bar{\Omega}$, it holds

$$u(y) \geq (\leq) u(x) + \langle p, y - x \rangle + \frac{1}{2} \langle A(y - x), y - x \rangle + o(|y - x|^2).$$

A *viscosity solution* to (3), or to (4), is a function $u \in C(\Omega)$ which is both a viscosity sub-solution and a viscosity super-solution to the same equation.

A *viscosity subsolution* to (3), or to (4), is an upper semicontinuous function u such that, for every $x \in \Omega$,

$$\forall \varphi \in C^2(\Omega) \text{ s.t. } u \prec_x \varphi, \quad -\Delta_\infty \varphi(x) \leq 1,$$

or

$$(5) \quad \forall \varphi \in C^2(\Omega) \text{ s.t. } u \prec_x \varphi, \quad -\Delta_\infty^+ \varphi(x) \leq 1, \text{ i.e. } \begin{cases} -\Delta_\infty \varphi(x) \leq |\nabla \varphi(x)|^2, \\ -\lambda_{\max}(\nabla^2 \varphi(x)) \leq 1, \text{ if } \nabla \varphi(x) = 0; \end{cases}$$

equivalently, in terms of superjets, this amounts to ask respectively that

$$\forall (p, X) \in J_\Omega^{2,+} u(x), \quad -\langle Xp, p \rangle \leq 1,$$

or

$$\forall (p, X) \in J_\Omega^{2,+} u(x), \quad \begin{cases} -\langle Xp, p \rangle \leq |p|^2, \\ -\lambda_{\max}(X) \leq 1, \text{ if } p = 0. \end{cases}$$

A *viscosity super-solution* to (3), or to (4), is a lower semicontinuous function u such that, for every $x \in \Omega$,

$$\forall \varphi \in C^2(\Omega) \text{ s.t. } \varphi \prec_x u, \quad -\Delta_\infty \varphi(x) \geq 1$$

or

$$(6) \quad \forall \varphi \in C^2(\Omega) \text{ s.t. } \varphi \prec_x u, \quad -\Delta_\infty^- \varphi(x) \geq 1, \text{ i.e. } \begin{cases} -\Delta_\infty \varphi(x) \geq |\nabla \varphi(x)|^2, \\ -\lambda_{\min}(\nabla^2 \varphi(x)) \geq 1, \text{ if } \nabla \varphi(x) = 0; \end{cases}$$

equivalently, in terms of subjets, this amounts to ask that

$$\forall (p, X) \in J_\Omega^{2,-} u(x), \quad -\langle Xp, p \rangle \geq 1,$$

or

$$\forall (p, X) \in J_\Omega^{2,-} u(x), \quad \begin{cases} -\langle Xp, p \rangle \geq |p|^2 \\ -\lambda_{\min}(X) \geq 1, \text{ if } p = 0. \end{cases}$$

Next consider the Dirichlet boundary value problems

$$(7) \quad \begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$$(8) \quad \begin{cases} -\Delta_\infty^N u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

A *viscosity solution* to (7) or to (8) is a function $u \in C(\bar{\Omega})$ such that $u = 0$ on $\partial\Omega$ and u is a viscosity solution to the pde $-\Delta_\infty u = 1$ or $-\Delta_\infty^N u = 1$, according to the above recalled definitions.

The existence and uniqueness of such a viscosity solution has been proved in [29, 4] for the Dirichlet problem (7) and in [32, 30, 31, 2] for the Dirichlet problem (8).

Concerning regularity, we proved in our previous papers [13] and [11] that, under the assumption that Ω is convex, the unique solution to the above Dirichlet problems is power-concave (precisely, $(\frac{3}{4})$ -concave in case of problem (7) and $(\frac{1}{2})$ -concave in case of problem (8)), locally semiconcave, and of class $C^1(\Omega)$. In case of the Dirichlet problem for the not-normalized operator, such regularity result was established in [13] under the additional assumption that Ω satisfies an interior sphere condition; we are going to remove this restriction in Lemma 13 below, using the fact that an appropriate web function is a supersolution (see Proposition 12).

Finally, we need to recall some definitions related to the distance function to the boundary of Ω , which will be denoted by $d_{\partial\Omega}$. We let $\Sigma(\Omega)$ be the set of points in Ω where $d_{\partial\Omega}$ is not differentiable, and we call *cut locus* and *high ridge* the sets given respectively by

$$(9) \quad \overline{\Sigma}(\Omega) := \text{the closure of } \Sigma(\Omega) \text{ in } \overline{\Omega}$$

$$(10) \quad M(\Omega) := \text{the set where } d_{\partial\Omega}(x) = \rho_\Omega := \max_{\overline{\Omega}} d_{\partial\Omega}.$$

Following [22, 15], we say that $u : \Omega \rightarrow \mathbb{R}$ is a *web-function* if u depends only on $d_{\partial\Omega}$, i.e. $u = g \circ d_{\partial\Omega}$ for some function $g : [0, \rho_\Omega] \rightarrow \mathbb{R}$.

Two web-functions will play a special role in the paper, in connection with problems (1)–(2). We denote them by ϕ^Ω and ϕ_N^Ω respectively:

$$(11) \quad \phi^\Omega(x) := c_0 \left[\rho_\Omega^{4/3} - (\rho_\Omega - d_{\partial\Omega}(x))^{4/3} \right], \quad \text{where } c_0 := 3^{4/3}/4,$$

$$(12) \quad \phi_N^\Omega(x) := \frac{1}{2} \left[\rho_\Omega^2 - (\rho_\Omega - d_{\partial\Omega}(x))^2 \right].$$

3. RESULTS

Throughout the paper, Ω is assumed to be an open bounded connected subset of \mathbb{R}^n . When the additional assumption that Ω is convex is needed, this is explicitly specified in the statements.

In our paper [12], we obtained some geometric information on the shape of domains $\Omega \subset \mathbb{R}^n$ whose cut locus $\overline{\Sigma}(\Omega)$ and high ridge $M(\Omega)$, defined respectively in (9) and (10), agree. In particular we proved that, in dimension $n = 2$, a domain Ω such that $\overline{\Sigma}(\Omega) = M(\Omega)$ is necessarily the tubular neighborhood of a line segment, possibly degenerated into a point. Inspired by this characterization, we set the following

Definition 1. We say that an open bounded convex subset of \mathbb{R}^n is a *stadium-like domain* if there holds $\overline{\Sigma}(\Omega) = M(\Omega)$.

Our main results state that being a stadium-like domain is a necessary and sufficient condition on a convex set Ω for the existence of a solution to any of the overdetermined problems (1) and (2).

Theorem 2. *Assume that Ω is convex. Then the overdetermined boundary value problem (1) admits a solution $u \in C^1(\overline{\Omega})$ if and only if Ω is a stadium-like domain (and in this case it holds $u = \phi^\Omega$, with $\rho_\Omega = c$).*

Theorem 3. *Assume that Ω is convex. Then the overdetermined boundary value problem (2) admits a solution $u \in C^1(\overline{\Omega})$ if and only if Ω is a stadium-like domain (and in this case it holds $u_N = \phi_N^\Omega$, with $\rho_\Omega = c$).*

As a companion result, which will be obtained as a consequence of Theorem 3, we establish that being a stadium-like domain is also a necessary and sufficient condition on a convex set Ω for the $C^{1,1}$ regularity of the unique solution to the Dirichlet problem (8):

Theorem 4. *Assume that Ω is convex. Then the unique solution to the Dirichlet boundary value problem (8) is of class $C^{1,1}(\Omega)$ if and only if Ω is a stadium-like domain (and in this case it holds $u_N = \phi_N^\Omega$, with $\rho_\Omega = c$).*

Remark 5. By combining Theorems 2, 3 and 4 with Theorem 6 in [12], we infer that, in dimension $n = 2$, domains Ω where any of the overdetermined problems (1) or (2) admits a solution (or where the unique solution to problem (8) is of class $C^{1,1}(\Omega)$) are geometrically characterized as

$$\Omega = \{x \in \mathbb{R}^2 : \text{dist}(x, S) < \rho_\Omega\},$$

being the set $S := \overline{\Sigma}(\Omega) = M(\Omega)$ a line segment (possibly degenerated into a point). If in addition $\partial\Omega$ is assumed to be of class C^2 , then Ω is a ball (see [12, Theorem 12]).

Remark 6. The same statement as Theorem 4 for the not normalized operator is clearly false. In fact, notice carefully that the function ϕ^Ω is merely of class $C^{1,1/3}(\Omega)$. We recall that, in the case of infinity harmonic functions, the works by Savin [33], Evans-Savin [16] and Evans-Smart [17] establish they are differentiable in any space dimension and $C^{1,\alpha}$ in dimension two.

Remark 7. We stress that asking that the solution is of class $C^1(\overline{\Omega})$ in Theorems 2 and 3 amounts to require merely that the $C^1(\Omega)$ -regularity known for the unique solution to problems (7)–(8) (cf. Section 2) is preserved at $\partial\Omega$. Notice that this is somehow necessary in order to give a pointwise meaning to the Neumann boundary condition in (1)–(2). We address as an open problem the question of establishing whether the C^1 regularity of the solution to problems (7)–(8) extends up to $\partial\Omega$ in dependence of the regularity of the boundary itself. For related boundary regularity results, see [36, 25, 26].

We now outline the proof of Theorems 2 and 3, by stating the results which serve as main intermediate steps and explaining how they allow to conclude. For convenience, the proof of such intermediate statements is postponed to Section 4, whereas the proof of Theorem 4 is given in the final Section 5.

The main idea to prove Theorems 2 and 3 is to make use of suitable P -functions, introduced hereafter.

Definition 8. For $x \in \Omega$, we set

$$(13) \quad P(x) := \frac{|\nabla u(x)|^4}{4} + u(x), \quad P_N(x) := \frac{|\nabla u_N(x)|^2}{2} + u_N(x),$$

where u and u_N denote respectively the unique solution to problems (7) and (8).

The choice of the above P -functions is due to the fact that their constancy on the whole Ω , if satisfied, gives the crucial information that u and u_N are web-functions, and more precisely that they agree with the functions ϕ_Ω and ϕ_Ω^N introduced in (11)–(12). We have indeed:

Proposition 9. *Assume that the unique viscosity solution to problem (7) or (8) is of class $C^1(\Omega)$, and that*

$$(14) \quad P(x) = \lambda \text{ a.e. on } \Omega \quad \text{or} \quad P_N(x) = \lambda_N \text{ a.e. on } \Omega,$$

where λ and λ_N are positive constants satisfying $\lambda \leq c_0 \rho_\Omega^{4/3}$ and $\lambda_N \leq \frac{1}{2} \rho_\Omega^2$.

Then we have respectively: $\lambda = c_0 \rho_\Omega^{4/3}$ and $u = \phi^\Omega$, or $\lambda_N = \frac{1}{2} \rho_\Omega^2$ and $u_N = \phi_\Omega^N$.

In turn, if the unique solution to problem (7) or (8) happens to be a web-function, we can prove that necessarily the cut locus and high ridge of Ω agree. Actually this geometric condition turns out to be necessary and sufficient for the solution being a web-function, according to the result below:

Proposition 10. *The unique viscosity solution to problem (7) or (8) is a web-function if and only if there holds $\bar{\Sigma}(\Omega) = M(\Omega)$.*

In view of Propositions 9 and 10, in order to prove Theorems 2 and 3, one is reduced to answer the following question: is it true that, if a solution to the overdetermined problems (1)–(2) exists, the corresponding P -function is constant?

In this respect, the pde interpreted pointwise at points of two-differentiability of u yields an elementary but important observation. Let u and u_N be the solutions to problems (7)–(8), and let γ and γ_N be local solutions on some interval $[0, \delta)$ to the gradient flow problems

$$(15) \quad \begin{cases} \dot{\gamma}(t) = \nabla u(\gamma(t)) \\ \gamma(0) = x \in \bar{\Omega}, \end{cases} \quad \begin{cases} \dot{\gamma}_N(t) = \nabla u_N(\gamma_N(t)) \\ \gamma_N(0) = x \in \bar{\Omega}. \end{cases}$$

We claim that, if u (resp. u_N) is twice differentiable at $\gamma(t)$ (resp. $\gamma_N(t)$) for \mathcal{L}^1 -a.e. $t \in [0, \delta)$, then it holds

$$(16) \quad \frac{d}{dt}(P(\gamma(t))) = 0 \quad \left(\text{resp. } \frac{d}{dt}(P_N(\gamma_N(t))) = 0 \right) \quad \mathcal{L}^1\text{-a.e. in } [0, \delta).$$

The proof of this claim is very simple and we limit ourselves to check it for the normalized operator, the other case being completely analogous. At every point x where u_N is twice differentiable, it holds $\nabla P_N(x) = D^2 u_N(x) \nabla u_N(x) + \nabla u_N(x)$; we infer that

$$\langle \nabla P_N(x), \nabla u_N(x) \rangle = \Delta_\infty u_N(x) + |\nabla u_N(x)|^2 = 0.$$

Thus, since by assumption u_N is twice differentiable at $\gamma_N(t)$ for \mathcal{L}^1 -a.e. $t \in [0, \delta)$, it holds

$$\frac{d}{dt}(P_N(\gamma_N(t))) = \langle \nabla P_N(\gamma(t)), \nabla u_N(\gamma_N(t)) \rangle = 0 \quad \mathcal{L}^1\text{-a.e. on } [0, \delta).$$

Unfortunately, we have not enough regularity at our disposal to infer from (16) that the P -functions are constant along the gradient flows. In fact, since u and u_N need not be of class $C^{1,1}(\Omega)$, the maps $t \mapsto P \circ \gamma$ and $t \mapsto P_N \circ \gamma_N$ may fail to be in $AC([0, \delta))$. To circumvent this lack of regularity, we argue as follows. In a first step, we proceed by finding some upper and lower bounds for the P -functions. They are obtained by approximating u and u_N by more regular functions (their supremal convolutions, see Section 4).

Proposition 11. *If Ω is convex, there holds*

$$(17) \quad \min_{\partial\Omega} \frac{|\nabla u|^4}{4} \leq P(x) \leq \max_{\Omega} u \quad \forall x \in \bar{\Omega},$$

$$(18) \quad \min_{\partial\Omega} \frac{|\nabla u_N|^2}{2} \leq P_N(x) \leq \max_{\Omega} u_N \quad \forall x \in \bar{\Omega}.$$

The above bounds enable us to arrive at the constancy of the P -functions when combined with a last key ingredient, which is stated below.

Proposition 12. *The web functions ϕ^Ω and ϕ_N^Ω are viscosity supersolutions respectively to the equation $-\Delta_\infty u = 1$ and $-\Delta_\infty^N u = 1$ in Ω .*

Based on the strategy outlined above and on the preliminary results stated so far, let us give more in detail the proof of Theorem 3. The proof of Theorem 2 is omitted since it is analogous, relying on the corresponding intermediate steps. The only difference is related with the removal of the interior sphere condition appearing in [13, Thm. 5], as mentioned in the Introduction. This is discussed in detail after the proof of Theorem 3.

Proof of Theorem 3. Let $B = B_{\rho_\Omega}(x_0)$ be an inner ball of radius ρ_Ω , let y_0 be a fixed point in $\partial B \cap \partial\Omega$ and let γ be the line segment $[x_0, y_0]$. Let ϕ_N^B and ϕ_N^Ω be the web-functions defined according to (12). By Proposition 12, applying the comparison principle proved in [2, Thm. 2.18], we infer that

$$(19) \quad \phi_N^B(x) \leq u_N(x) \leq \phi_N^\Omega(x) \quad \forall x \in B.$$

We can deduce several consequences from these inequalities. Firstly we observe that, since both the functions ϕ_N^B and ϕ_N^Ω have a relative maximum at x_0 , by (19) the same property holds true for u_N . Hence x_0 is a critical point of u_N . In turn, we observe that

$$(20) \quad x_0 \in \operatorname{argmax}_{\overline{\Omega}}(u_N).$$

Indeed, the set of critical points of u_N agrees with the set $\operatorname{argmax}_{\overline{\Omega}}(u_N)$ where u_N attains its maximum over $\overline{\Omega}$. This is because, by [11, Theorem 6], the function $u_N^{1/2}$ is concave in Ω ; hence its gradient vanishes only at maximum points of u_N .

Moreover we notice that, since the distance functions $d_{\partial B}$ and $d_{\partial\Omega}$ agree on the line segment γ , there holds

$$(21) \quad \phi_N^B(x) = \phi_N^\Omega(x) \quad \forall x \in \gamma.$$

As a consequence of (19) and (21), we deduce that $u_N(x) = \phi_N^\Omega(x) = \phi_N^B(x)$ for all $x \in \gamma$. Namely, there holds

$$(22) \quad u_N(x) = \frac{\rho_\Omega^2 - (\rho_\Omega - d_{\partial\Omega}(x))^2}{2} \quad \forall x \in \gamma.$$

It follows from (22) that $|\nabla u_N(y_0)| = \rho_\Omega$. Recalling that by assumption u_N satisfies the Neumann condition $|\nabla u_N(y)| = c$ for all $y \in \partial\Omega$, we deduce that the value of the parameter c is related to the inradius by the equality $c = \rho_\Omega$. Using such equality and (20), we get

$$\max_{\overline{\Omega}}(u_N) = u_N(x_0) = \frac{\rho_\Omega^2}{2} = \frac{c^2}{2}.$$

By Proposition 11, this implies that the P -function associated with u_N according to (13) satisfies

$$P_N(x) = \frac{c^2}{2} \quad \forall x \in \overline{\Omega}.$$

Since Ω is assumed to be convex, it follows from [11, Thm. 16], that u is of class $C^1(\Omega)$. Therefore, we are in a position to apply Proposition 9, to obtain that $u_N = \phi_N^\Omega$ (with $\rho_\Omega = c$), and finally Proposition 10 to conclude that $\overline{\Sigma}(\Omega) = M(\Omega)$. \square

Going through the above proof, we see that we have used all our intermediate results, stated in Propositions 9, 10, 11, and 12. Since such results have been established also in case of the not normalized operator Δ_∞ , this allows to obtain the proof of Theorem 2. Nevertheless, some attention must be paid, precisely when applying Proposition 9, because it requires that the unique solution to problem (7) is of class $C^1(\Omega)$. Whereas in case of problem (8) the C^1 -regularity of the solution was proved in [11, Thm. 16] for arbitrary convex domains, in case of problem (7), it was proved in [13, Cor. 10] under the additional assumption that Ω satisfies an *interior sphere condition*. Moreover, an inspection of the proof of [11, Thm. 16] reveals that it is not straightforward to adapt it to the case of the not normalized operator. However, relying on the new Proposition 12, we are now able to remove the interior sphere condition. This is done in Lemma 13 below. It ensures that,

also in case of problem (7), the C^1 -regularity condition asked in Proposition 9 is fulfilled for any convex domain, thus enabling us to conclude the proof of Theorem 2.

Lemma 13. *If Ω is convex, then the unique solution to problem (7) is of class $C^1(\Omega)$.*

Proof. By [13, Thm. 9], it is enough to show that the unique solution to problem (7) is power-concave. Let u be such a solution. For $\varepsilon \in (0, 1]$ let Ω_ε denote the outer parallel body of Ω defined by

$$\Omega_\varepsilon := \{x \in \mathbb{R}^n : \text{dist}(x, \Omega) < \varepsilon\},$$

and let u_ε denote the solution to

$$\begin{cases} -\Delta_\infty^N u_\varepsilon = 1 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Since Ω_ε satisfies an interior sphere condition (of radius ε), by [13, Cor. 10] the function $u_\varepsilon^{1/2}$ is concave in Ω_ε . Therefore, to show that $u^{1/2}$ is concave in $\overline{\Omega}$, it is enough to show that, as $\varepsilon \rightarrow 0$, $u_\varepsilon \rightarrow u$ uniformly in $\overline{\Omega}$. In turn, by [30, Thm. 5.3], this convergence holds true provided $u_\varepsilon|_{\partial\Omega}$ tends uniformly to 0.

To that aim we observe that, thanks to Proposition 12 and the comparison principle proved in [29, Thm. 3], there holds

$$0 < u_\varepsilon(x) \leq \phi^{\Omega_\varepsilon}(x) = c_0 \left[(\rho_\Omega + \varepsilon)^{4/3} - \rho_\Omega^{4/3} \right] \quad \forall x \in \partial\Omega,$$

which implies that $u_\varepsilon|_{\partial\Omega}$ converges uniformly to 0 on $\partial\Omega$. \square

4. PROOFS OF INTERMEDIATE RESULTS

4.1. Proof of Proposition 9. In case of the not normalized operator, the result has been proved in [13, Proposition 2]. Let us prove it for the normalized operator. It is clear that the constant λ_N is equal to $\max_{\overline{\Omega}} u_N$. On the other hand, $\max_{\overline{\Omega}} u_N \geq \max_{\overline{\Omega}} v = \rho_\Omega^2/2$, where v is the radial solution of the Dirichlet problem in a ball $B_{\rho_\Omega} \subseteq \Omega$. Hence $\lambda_N = \rho_\Omega^2/2$. Let $H: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ be the Hamiltonian defined by

$$H(u, p) := \frac{1}{2}|p|^2 + u - \lambda_N.$$

Then the second equality in (14) can be rewritten as

$$(23) \quad H(u_N(x), \nabla u_N(x)) = 0, \quad \mathcal{L}^n\text{-a.e. on } \Omega.$$

Since u_N is of class $C^1(\Omega)$, then it follows that it is a classical (hence also a viscosity) solution of the Dirichlet problem

$$(24) \quad \begin{cases} H(u_N, \nabla u_N) = 0, & \text{in } \Omega, \\ u_N = 0, & \text{on } \partial\Omega. \end{cases}$$

Since the solution to this Dirichlet problem is unique (see *e.g.* [3, Theorem III.1]), to prove that $u_N = \phi_N^\Omega$ it is enough to show that also ϕ_N^Ω is a viscosity solution to (24). Since ϕ_N^Ω is differentiable at every point $x \in \Omega \setminus S$, where $S := \Sigma(\Omega) \setminus M(\Omega)$, with

$$\nabla \phi_N^\Omega(x) = \begin{cases} (\rho_\Omega - d_{\partial\Omega}(x)) \nabla d_{\partial\Omega}(x), & \text{if } x \in \Omega \setminus \Sigma(\Omega), \\ 0, & \text{if } x \in M(\Omega), \end{cases}$$

we have $H(\phi_N^\Omega(x), \nabla \phi_N^\Omega(x)) = 0$ for every $x \in \Omega \setminus S$.

We remark that ϕ_N^Ω is a concave function, since it is the composition of the concave function

$$(25) \quad g(t) := \frac{1}{2}[\rho_\Omega^2 - (\rho_\Omega - t)^2], \quad t \in [0, \rho_\Omega],$$

with the distance function $d_{\partial\Omega}$, which in turn is concave because Ω is a convex set. Since $S \subseteq \Sigma(\Omega)$ has vanishing Lebesgue measure and H is convex with respect to the gradient variable, from Proposition 5.3.1 in [8] we conclude that ϕ_N^Ω is a viscosity solution to (24).

4.2. Proof of Proposition 10. For the case of problem (7), the result has been proved in [14], so we need to consider only the case of the normalized infinity Laplacian.

Assume that Ω is a stadium-like domain, and let us prove that ϕ_N^Ω is a viscosity solution to (8). Let $x \in \Omega$ and let us prove that both conditions (5) and (6) are satisfied. Let $p \in \bar{\Sigma}(\Omega) = M(\Omega)$ and $q \in \partial\Omega$ be such that $x \in [q, p]$ (we remark that we have $x = p$ if x belongs itself to the cut locus). Let us define $\nu := (p - q)/|p - q|$.

Let us first prove (5). By the comparison principle proved in [2, Thm. 2.18] we have

$$\phi_N^\Omega(y) \geq \frac{\rho_\Omega^2 - |y - p|^2}{2} =: v(y), \quad \forall y \in \Omega,$$

since the function v at the right-hand side is the solution of the Dirichlet problem in the ball $B_{\rho_\Omega}(p) \subseteq \Omega$. Moreover, the functions ϕ_N^Ω and v coincide on the segment $[q, p]$ and, in particular, at the point x . If $\phi_N^\Omega \prec_x \varphi$ we thus have

$$\varphi(x) = \phi_N^\Omega(x) = v(x), \quad v(y) \leq \phi_N^\Omega(y) \leq \varphi(y) \quad \forall y \in \Omega,$$

so that $v \prec_x \varphi$. Since v is a solution to $-\Delta_\infty^N v = 1$, this implies that $-\Delta_\infty^+ \varphi(x) \leq 1$.

Let us now prove (6). Let $\varphi \prec_x \phi_N^\Omega$.

If $x \in \bar{\Sigma}(\Omega) = M(\Omega)$, we must have $\nabla\varphi(x) = 0$ and

$$\langle \nabla^2\varphi(x)(y - x), y - x \rangle \leq -(\rho_\Omega - d_{\partial\Omega}(y))^2, \quad \forall y \in \bar{\Omega}.$$

Since, in this case, $x = q + \rho_\Omega \nu$ we get

$$\lambda_{\min}(\nabla^2\varphi(x)) \leq \langle \nabla^2\varphi(x)\nu, \nu \rangle = \frac{1}{\rho_\Omega^2} \langle \nabla^2\varphi(x)(q - x), q - x \rangle \leq -1.$$

If $x \notin \bar{\Sigma}(\Omega)$, then $\tau := d_{\partial\Omega}(x) = |x - q| < \rho_\Omega$, and

$$\nabla\varphi(x) = \nabla\phi_N^\Omega(x) = g'(\tau)\nabla d_{\partial\Omega}(x) = g'(\tau)\nu \neq 0.$$

Moreover, we have

$$h(t) := \varphi(q + t\nu) \leq \phi_N^\Omega(q + t\nu) = g(t), \quad \forall t \in [0, \rho_\Omega],$$

and

$$h(\tau) = g(\tau), \quad h'(\tau) = g'(\tau) > 0, \quad h''(\tau) \leq g''(\tau).$$

In particular we get

$$-\Delta_\infty^N \varphi(x) = -h''(\tau) \geq -g''(\tau) = 1,$$

so that we have proved that ϕ_N^Ω is a super-solution to (8).

It remains to prove the converse implication of the proposition. Let us assume that the unique viscosity solution to (8) is a web-function, and let us prove that Ω is a stadium-like domain and that the solution is given by ϕ_N^Ω .

Assume that the unique viscosity solution to (8) is of the form $u_N(x) := f(d_{\partial\Omega}(x))$.

We claim that the map $t \mapsto f(t)$ is monotone increasing on $[0, \rho_\Omega]$, and that the function $v(z) := f(\rho_\Omega - |z|)$ is a viscosity solution of

$$(26) \quad -\Delta_\infty^N v = 1 \quad \text{in } B_{\rho_\Omega}(0) \setminus \{0\}.$$

Namely, assume by contradiction that $t \mapsto f(t)$ is not monotone increasing on $[0, \rho_\Omega]$: let $t_1, t_2 \in [0, \rho_\Omega]$ be such that $t_1 < t_2$ but $f(t_1) > f(t_2)$. Then the absolute minimum of the continuous function f on the interval $[t_1, \rho_\Omega]$ is attained at some point $t_0 > t_1$; in particular, there exists a point $t_0 \in (0, \rho_\Omega]$ which is of local minimum for the map f . Let us show that this fact is not compatible with the assumption that $u_N(x) = f(d_{\partial\Omega}(x))$ is a web viscosity solution to $-\Delta_\infty^N u_N = 1$ in Ω . Since $t_0 > 0$, there exists a point x_0 lying in Ω such that $d_{\partial\Omega}(x_0) = t_0$. Since t_0 is a local minimum for the map f , the point x_0 is a local minimum for the function u_N . Then, we can construct a C^2 function φ with $\varphi \prec_{x_0} u_N$, which is locally constant in a neighbourhood of x_0 . Clearly it holds $-\Delta_\infty^- \varphi = -\lambda_{\min}(\nabla^2 \varphi(x_0)) = 0 < 1$, against the fact that u_N is a viscosity super-solution. To complete the proof of the claim, let us show that $v(z) := f(\rho_\Omega - |z|)$ is a viscosity solution to (26) at a fixed point $z_0 \in B_{\rho_\Omega}(0) \setminus \{0\}$. If ψ is a C^2 function with $v \prec_{z_0} \psi$, we have to show that

$$(27) \quad -\Delta_\infty^+ \psi(z_0) \leq 1.$$

We choose a maximal ray $[p_0, q_0]$, with $p_0 \in M(\Omega)$ and $q_0 \in \partial\Omega$, that is, p_0 is the center of a ball of radius $\rho_\Omega = |p_0 - q_0|$ contained into Ω . We pick a point $x_0 \in \Omega$ such that

$$x_0 \in]p_0, q_0[\quad \text{and} \quad d_{\partial\Omega}(x_0) = \rho_\Omega - |z_0|$$

and, for x belonging to a neighborhood of x_0 , we set

$$z(x) := [\rho_\Omega - |x - q_0|] \zeta_0, \quad \text{with } \zeta_0 := \frac{z_0}{|z_0|}.$$

In particular, notice that by construction there holds $z(x_0) = z_0$.

We now consider the composite map

$$\varphi(x) := \psi(z(x)).$$

Clearly it is of class C^2 in a neighborhood of x_0 , and it is easy to check that it satisfies the condition $u_N \prec_{x_0} \varphi$. Indeed, by the definitions of u_N, v , and z , and since $v \prec_{z_0} \psi$, there holds

$$u_N(x_0) = f(d_{\partial\Omega}(x_0)) = f(\rho_\Omega - |z_0|) = v(z_0) = \psi(z_0) = \varphi(x_0).$$

Moreover there exists $r > 0$ such that

$$u_N(x) = f(d_{\partial\Omega}(x)) \leq f(\rho_\Omega - |z(x)|) = v(z(x)) \leq \psi(z(x)) = \varphi(x) \quad \forall x \in B_r(x_0).$$

Notice that the first inequality in the line above follows from the fact already proved that f is monotone increasing, while the second one holds for r sufficiently small by the assumption that $v \prec_{z_0} \psi$ and the continuity the map z at x_0 .

Then, since $u_N \prec_{x_0} \varphi$ and by assumption u_N is a viscosity solution to $-\Delta_\infty^N u_N = 1$ in Ω , we deduce that

$$(28) \quad -\Delta_\infty^+ \varphi(x_0) \leq 1.$$

We now distinguish the two cases $\nabla \varphi(x_0) = 0$ and $\nabla \varphi(x_0) \neq 0$.

Case $\nabla \varphi(x_0) \neq 0$. Setting $\delta(x) := |x - q_0|$, a direct computation yields

$$\begin{aligned} \nabla \varphi(x) &= -\langle \nabla \psi(z(x)), \zeta_0 \rangle \nabla \delta(x), \\ D^2 \varphi(x) &= \langle D^2 \psi(z(x)) \zeta_0, \zeta_0 \rangle \nabla \delta(x) \otimes \nabla \delta(x) - \langle \nabla \psi(z(x)), \zeta_0 \rangle D^2 \delta(x). \end{aligned}$$

Taking into account the identities

$$\begin{aligned} |\nabla\delta(x)| &= 1, \\ [\nabla\delta(x) \otimes \nabla\delta(x)]\nabla\delta(x) &= \nabla\delta(x), \\ D^2\delta(x)\nabla\delta(x) &= 0, \end{aligned}$$

we obtain

$$(29) \quad \Delta_\infty^+\varphi(x_0) = \left\langle D^2\varphi(x_0) \frac{\nabla\varphi(x_0)}{|\nabla\varphi(x_0)|}, \frac{\nabla\varphi(x_0)}{|\nabla\varphi(x_0)|} \right\rangle = \langle D^2\psi(z_0)\zeta_0, \zeta_0 \rangle.$$

Now, from [14, Lemma 17(a)] we have

$$\nabla\psi(z_0) = \alpha\zeta_0, \quad \text{with } \alpha \in -D^+f(\rho_\Omega - |z_0|),$$

and our current assumption $\nabla\varphi(x_0) \neq 0$ implies $\alpha \neq 0$. Therefore,

$$(30) \quad \Delta_\infty^+\psi(z_0) = \left\langle D^2\psi(z_0) \frac{\nabla\psi(z_0)}{|\nabla\psi(z_0)|}, \frac{\nabla\psi(z_0)}{|\nabla\psi(z_0)|} \right\rangle = \langle D^2\psi(z_0)\zeta_0, \zeta_0 \rangle.$$

In view of (29) and (30), we conclude that, in case $\nabla\varphi(x_0) \neq 0$, (27) follows from (28).

Case $\nabla\varphi(x_0) = 0$. By (28), we know that $-\lambda_{\max}(D^2\varphi(x_0)) \leq 1$, and we have to prove that $-\lambda_{\max}(D^2\psi(z_0)) \leq 1$. From the relation $\nabla\varphi(x_0) = -\langle \nabla\psi(z_0), \zeta_0 \rangle \nabla\delta(x_0)$, we see that $\nabla\psi(z_0) = \alpha\zeta_0 = 0$, so that the Hessian matrices $D^2\varphi(x_0)$ and $D^2\psi(z_0)$ are related by

$$(31) \quad D^2\varphi(x_0) = \langle D^2\psi(z_0)\zeta_0, \zeta_0 \rangle \nabla\delta(x_0) \otimes \nabla\delta(x_0).$$

Since $\lambda_{\max}(D^2\varphi(x_0)) \geq -1$, there exists an eigenvector η such that $\langle D^2\varphi(x_0)\eta, \eta \rangle \geq -1$. Then (31) yields

$$-1 \leq \langle D^2\varphi(x_0)\eta, \eta \rangle = \langle D^2\psi(z_0)\zeta_0, \zeta_0 \rangle (\langle \nabla\delta(x_0), \eta \rangle)^2 \leq \langle D^2\psi(z_0)\zeta_0, \zeta_0 \rangle,$$

which shows that $\lambda_{\max}(D^2\psi(z_0)) \geq -1$.

In order to prove that v is a viscosity super-solution to (26) at z_0 , one can argue in a completely analogous way. More precisely, keeping the same definitions of ζ_0 , p_0 , q_0 , and x_0 as above, one has just to modify the auxiliary function $z(x)$ into $\tilde{z}(x) := |x - p_0|\zeta_0$, then replace the distance function $\delta(x)$ by $\tilde{\delta}(x) := |x - p_0|$, and finally apply part (b) in place of part (a) of [14, Lemma 17].

We are now ready to prove that u_N coincides with the function ϕ_Ω^N defined in (12). Let $f: [0, \rho_\Omega] \rightarrow \mathbb{R}$ be a continuous function such that $u_N(x) = f(d_{\partial\Omega}(x))$. We have to show that f agrees with the function $g: [0, \rho_\Omega] \rightarrow \mathbb{R}$ defined by (25). Since u_N is assumed to be a viscosity solution to the Dirichlet problem (8), according to what proved above we know that $v(z) := f(\rho_\Omega - |z|)$ is a viscosity solution to

$$(32) \quad \begin{cases} -\Delta_\infty^N v = 1 & \text{in } B_{\rho_\Omega}(0) \setminus \{0\}, \\ v = 0 & \text{on } \partial B_{\rho_\Omega}(0), \\ v(0) = f(\rho_\Omega). \end{cases}$$

Let us define, for every $r > 0$, the function

$$(33) \quad g_r(t) := \frac{1}{2} [r^2 - (r - t)^2], \quad t \in [0, r].$$

We claim that there exists $r \in [\rho_\Omega, +\infty)$ such that

$$(34) \quad g_r(\rho_\Omega) = f(\rho_\Omega).$$

To prove this claim, we observe that the function

$$r \mapsto g_r(\rho_\Omega) = \frac{1}{2} [r^2 - (r - \rho_\Omega)^2]$$

maps the interval $[\rho_\Omega, +\infty)$ onto $[\frac{1}{2}\rho_\Omega^2, +\infty)$. Thus in order to show the existence of some r such that (34) holds, it is enough to prove the inequality

$$(35) \quad f(\rho_\Omega) \geq \frac{1}{2}\rho_\Omega^2.$$

In turn, this inequality readily follows by the comparison principle holding for the Dirichlet problem (8) (see [2, Thm. 2.18]). Namely, let $x_0 \in M(\Omega)$. We observe that the function $w(x) := g(\rho_\Omega - |x - x_0|)$ solves $-\Delta_\infty^N w = 1$ in $B_{\rho_\Omega}(x_0)$ and $w = 0$ on $\partial B_{\rho_\Omega}(x_0)$. This is readily checked since, being w of class C^2 , it holds

$$-\Delta_\infty^N w(x) = -g''(\rho_\Omega - |x - x_0|) = 1 \quad \text{for } x \neq x_0$$

and

$$\begin{cases} -\Delta_\infty^+ w(x_0) = -\lambda_{\max}(D^2 w(x_0)) = -\lambda_{\max}(-\text{Id}) = 1, \\ -\Delta_\infty^- w(x_0) = -\lambda_{\min}(D^2 w(x_0)) = -\lambda_{\min}(-\text{Id}) = 1. \end{cases}$$

On the other hand, the function u_N solves $-\Delta_\infty^N u_N = 1$ in $B_{\rho_\Omega}(x_0)$ and $u_N \geq 0$ on $\partial B_{\rho_\Omega}(x_0)$. The latter inequality can be deduced by applying the comparison principle proved in [2, Thm. 2.18]. Again by applying the same result, we deduce that $u_N(x) \geq g(\rho_\Omega - |x - x_0|)$ in $B_{\rho_\Omega}(x_0)$. This implies in particular inequality (35), as

$$f(\rho_\Omega) = u_N(x_0) \geq g(\rho_\Omega) = \frac{1}{2}\rho_\Omega^2.$$

Now, we have that the function

$$g_r(\rho_\Omega - |z|), \quad z \in B_{\rho_\Omega}(0),$$

is a classical solution (and hence a viscosity solution) to problem (32). (Notice that in particular the third equation in (32) is satisfied thanks to (34)).

From [2, Thm. 2.18], [30, Thm. 1.8], [32, Cor. 1.9] we know that there exists a unique viscosity solution to (32). We conclude that, for some $r \geq \rho_\Omega$, it holds $v(z) = g_r(\rho_\Omega - |z|)$, that is

$$(36) \quad f(\rho_\Omega - |z|) = g_r(\rho_\Omega - |z|),$$

or equivalently $u_N(x) = g_r(d_{\partial\Omega}(x))$.

In order to show that $u_N = \phi_N^\Omega$, we are reduced to prove that $r = \rho_\Omega$. We recall that, since $r \geq \rho_\Omega$, then $g'_r(\rho_\Omega) \geq 0$, and that $g'_r(\rho_\Omega) = 0$ if and only if $r = \rho_\Omega$. Assume by contradiction that $g'_r(\rho_\Omega) > 0$. Let $x_0 \in M(\Omega)$. Without loss of generality, assume that $x_0 = 0$. Thanks to the concavity of g_r , we have

$$(37) \quad u_N(x) = g_r(d_{\partial\Omega}(x)) \leq u_N(0) + g'_r(\rho_\Omega)(d(x) - \rho_\Omega).$$

From Theorem 2 in [13], there exist vectors $p, \zeta \in \mathbb{R}^n$, with $\langle \zeta, p \rangle \neq 0$, and positive constants c, C, δ , such that

$$(38) \quad d_{\partial\Omega}(x) \leq d_{\partial\Omega}(0) + \langle p, x \rangle - c \langle \zeta, x \rangle^2 + \frac{C}{2}|x|^2 \quad \forall x \in \overline{B}_\delta(0).$$

By (37) and (38), it holds

$$u_N(x) \leq \varphi(x) := u_N(0) + g'_r(\rho_\Omega) \left[\langle p, x \rangle - c \langle \zeta, x \rangle^2 + \frac{1}{2\rho_\Omega} |x|^2 \right],$$

so that $u_N \prec_0 \varphi$. Since $\nabla\varphi(0) = g'_r(\rho_\Omega)p \neq 0$, via some straightforward computations we obtain

$$\Delta_\infty^+ \varphi(0) = \frac{g'_r(\rho_\Omega)}{|p|^2} \Delta_\infty \psi(0) = \frac{g'_r(\rho_\Omega)}{|p|^2} \left(-2c \langle \zeta, p \rangle^2 + \frac{1}{\rho_\Omega} |p|^2 \right).$$

Since $g'_r(\rho_\Omega) > 0$ and $\langle \zeta, p \rangle \neq 0$, it is enough to choose $c > 0$ large enough in order to have $\Delta_\infty^+ \varphi(0) < -1$, contradiction.

Since we have just proved that $u_N = \phi_N^\Omega$, we know that $u_N(x) = g(d_{\partial\Omega}(x))$, with g as in (25). Assume by contradiction that there exists $x_0 \in \Sigma(\Omega) \setminus M(\Omega)$. Without loss of generality, assume that $x_0 = 0$, and set $d_0 = d_{\partial\Omega}(0)$. Since we are assuming $x_0 \notin M(\Omega)$, it holds $d_0 < \rho_\Omega$, which implies $g'(d_0) > 0$. Then, we can reach a contradiction by arguing similarly as above. Namely, thanks to the concavity of g , we have

$$(39) \quad u_N(x) \leq u_N(0) + g'(d_0)(d_{\partial\Omega}(x) - d_0).$$

By (39) and (38), we have

$$u_N(x) \leq \varphi(x) := u_N(0) + g'(d_0) \left[\langle p, x \rangle - c \langle \zeta, x \rangle^2 + \frac{1}{2d_0} |x|^2 \right],$$

so that $u_N \prec_0 \varphi$. Moreover, since $\nabla\varphi(0) = g'(d_0)p \neq 0$, we have

$$\Delta_\infty^+ \varphi(0) = \frac{g'(d_0)}{|p|^2} \left(-2c \langle \zeta, p \rangle^2 + \frac{1}{d_0} |p|^2 \right).$$

Since $g'(d_0) > 0$ and $\langle \zeta, p \rangle \neq 0$, it is enough to choose $c > 0$ large enough in order to have $\Delta_\infty^+ \varphi(0) < -1$, contradiction. We have thus shown that $\Sigma(\Omega) \subseteq M(\Omega)$. Since the converse inclusion holds true for all Ω , and since $M(\Omega)$ is a closed set, we conclude that the required equality $\bar{\Sigma}(\Omega) = M(\Omega)$ holds. \square

4.3. Proof of Proposition 11. The estimates (17) for the not normalized infinity Laplacian have been proved in [13, Thm. 4], so we will prove only the estimates (18) for the normalized infinity Laplacian.

To that aim we need a number of preliminary results. We set for brevity

$$(40) \quad K := \operatorname{argmax}_{\bar{\Omega}} u_N, \quad \mu := \max_{\bar{\Omega}} u_N.$$

A first key step is the construction of the *gradient flow* \mathbf{X} associated with u_N , and the location of its terminal points, according to lemma below. The proof is omitted since it is completely analogous to that of Lemma 3 in [13]: we limit ourselves to mention that it is based on the local semiconcavity of u_N (see [9, Theorem 3.2 and Example 3.6]), which in case of the solution to problem (8) has been recently proved in [11, Prop. 13].

Lemma 14. *Assume that Ω is convex and that the unique solution u_N to (8) is of class $C^1(\bar{\Omega})$. Then, for every point $x \in \bar{\Omega} \setminus K$, there exists a unique solution $\mathbf{X}(\cdot, x)$ to (15) globally defined in $[0, +\infty)$. Moreover, if we set*

$$(41) \quad T(x) := \sup\{t \geq 0 : \nabla u_N(\mathbf{X}(t, x)) \neq 0\} \in (0, +\infty],$$

then

$$(42) \quad \lim_{t \rightarrow T(x)^-} \mathbf{X}(t, x) \in K, \quad \lim_{t \rightarrow T(x)^-} \nabla u_N(\mathbf{X}(t, x)) = 0.$$

Finally, there exist $x_0 \in \partial\Omega$ and $t_0 \in [0, T(x_0))$ such that $x = \mathbf{X}(t_0, x_0)$.

As we have already mentioned in Section 3, the above result cannot be directly exploited to infer the constancy of P_N along the flow \mathbf{X} , because of the possible lack of absolute continuity of P_N . In order to overcome this difficulty, we approximate u_N via its supremal convolutions, defined for $\varepsilon > 0$ by

$$(43) \quad u^\varepsilon(x) := \sup_{y \in \mathbb{R}^n} \left\{ \tilde{u}(y) - \frac{|x-y|^2}{2\varepsilon} \right\} \quad \forall x \in \mathbb{R}^n,$$

where \tilde{u} is a Lipschitz extension of u_N to \mathbb{R}^n with $\text{Lip}_{\mathbb{R}^n}(\tilde{u}) = \text{Lip}_{\overline{\Omega}}(u_N)$.

In the next lemma we state the basic properties of the functions u^ε which we are going to use in the sequel. Let us recall that, according to [8, Lemma 3.5.7], there exists $R > 0$, depending only on $\text{Lip}_{\mathbb{R}^n}(\tilde{u})$, such that any point y at which the supremum in (43) is attained satisfies $|y-x| < \varepsilon R$. Thus, setting

$$(44) \quad U_\varepsilon := \{x \in \Omega : u_N(x) > \varepsilon\}, \quad A_\varepsilon := \{x \in U_\varepsilon : d_{\partial U_\varepsilon}(x) > \varepsilon R\},$$

there holds

$$(45) \quad u^\varepsilon(x) = \sup_{y \in U_\varepsilon} \left\{ u_N(y) - \frac{|x-y|^2}{2\varepsilon} \right\} \quad \forall x \in A_\varepsilon.$$

Moreover, let us define

$$(46) \quad m_\varepsilon := \max_{\partial A_\varepsilon} u^\varepsilon, \quad \Omega_\varepsilon := \{x \in A_\varepsilon : u^\varepsilon(x) > m_\varepsilon\}.$$

Lemma 15. *Under the same assumptions of Lemma 14, let u^ε and Ω_ε be defined respectively as in (43) and (46). Then:*

- (i) u^ε is of class $C^{1,1}$ on Ω_ε ;
- (ii) u^ε is a sub-solution to $-\Delta_\infty^N u - 1 = 0$ in Ω_ε ;
- (iii) as $\varepsilon \rightarrow 0^+$, it holds

$$\begin{aligned} u^\varepsilon &\rightarrow u_N && \text{uniformly in } \overline{\Omega}, \\ \nabla u^\varepsilon &\rightarrow \nabla u_N && \text{uniformly in } \overline{\Omega} \end{aligned}$$

(so that $m_\varepsilon \rightarrow 0$ and Ω_ε converges to Ω in Hausdorff distance).

Proof. The proofs of (i) and (iii) are the same as those of the corresponding statements in [13, Lemma 4]. Let us check that also statement (ii) remains true for the normalized operator.

Let $x \in \Omega_\varepsilon$, and let $(p, X) \in J_{\Omega_\varepsilon}^{2,+} u^\varepsilon(x)$. It follows from magical properties of supremal convolution (cf. [10, Lemma A.5]) that $(p, X) \in J_{\Omega_\varepsilon}^{2,+} u_N(y)$, where y is a point at which the supremum which defines $u^\varepsilon(x)$ is attained. Since $y \in U_\varepsilon \subset \Omega_\varepsilon$, it holds $J_{\Omega}^{2,+} u_N(y) = J_{\Omega_\varepsilon}^{2,+} u^\varepsilon(x)$; therefore, we have $(p, X) \in J_{\Omega}^{2,+} u_N(y)$, which implies $-\langle Xp, p \rangle \leq 1$ in case $p \neq 0$ and $-\lambda_{\max}(X) \leq 1$ in case $p = 0$. \square

Next we observe that, for every $\varepsilon > 0$, one can consider the gradient flow \mathbf{X}_ε associated with u^ε . Namely, for every $x_\varepsilon \in \overline{\Omega}_\varepsilon$, the Cauchy problem

$$(47) \quad \begin{cases} \dot{\gamma}_\varepsilon(t) = \nabla u^\varepsilon(\gamma_\varepsilon(t)), \\ \gamma_\varepsilon(0) = x_\varepsilon \in \overline{\Omega}_\varepsilon, \end{cases}$$

admits a unique solution $\mathbf{X}_\varepsilon(\cdot, x_\varepsilon): [0, +\infty) \rightarrow \overline{\Omega}_\varepsilon$. Indeed, the fact that $\mathbf{X}_\varepsilon(\cdot, x_\varepsilon)$ is defined in $[0, +\infty)$ follows from the estimate

$$\frac{d}{dt}u^\varepsilon(\gamma_\varepsilon(t)) = |\nabla u^\varepsilon(\gamma_\varepsilon(t))|^2 \geq 0,$$

so that $\gamma_\varepsilon(t) \in \overline{\Omega}_\varepsilon$ for every $t \geq 0$, while uniqueness follows from the $C^{1,1}$ regularity of u^ε stated in Lemma 15(i).

The following lemma establishes the behavior, along the flow \mathbf{X}_ε , of the approximate P -function defined by

$$(48) \quad P_\varepsilon(x) := \frac{|\nabla u^\varepsilon(x)|^2}{2} + u^\varepsilon(x), \quad x \in \overline{\Omega}_\varepsilon,$$

showing that P_ε increases along \mathbf{X}_ε . For the proof, we refer to [13, Lemma 5].

Lemma 16. *Under the same assumptions of Lemma 14, let u^ε , Ω_ε , and P_ε be defined respectively as in (43), (46), and (48). Then, for \mathcal{H}^{n-1} -a.e. $x_\varepsilon \in \partial\Omega_\varepsilon$, it holds*

$$P_\varepsilon(\mathbf{X}_\varepsilon(t_1, x_\varepsilon)) \leq P_\varepsilon(\mathbf{X}_\varepsilon(t_2, x_\varepsilon)) \quad \forall t_1, t_2 \text{ with } 0 \leq t_1 \leq t_2.$$

We are finally in a position to give the

Proof of Proposition 11. By continuity, it is enough to show that the inequalities (18) hold for all $x \in \Omega \setminus K$. By Lemma 14, given $x \in \Omega \setminus K$, there exist $x_0 \in \partial\Omega$ and $t_0 \in [0, T(x_0))$ such that $x = \mathbf{X}(t_0, x_0)$. By Lemma 16, we may find a sequence of points $x_\varepsilon \in \partial\Omega_\varepsilon$ converging to x_0 such that, for every $t \geq t_0$, we have

$$P_\varepsilon(x_\varepsilon) \leq P_\varepsilon(\mathbf{X}_\varepsilon(t_0, x_\varepsilon)) \leq P_\varepsilon(\mathbf{X}_\varepsilon(t, x_\varepsilon)).$$

We now pass to the limit as $\varepsilon \rightarrow 0^+$ in the above inequalities: by using the continuous dependence for ordinary differential equations (see e.g. [23, Lemma 3.1]), and the uniform convergences stated in Lemma 15(iii), we get

$$(49) \quad P_N(x_0) \leq P_N(x) \leq P_N(\mathbf{X}(t, x_0)).$$

We have

$$P_N(x_0) = \frac{|\nabla u_N(x_0)|^2}{2} \geq \min_{\partial\Omega} \frac{|\nabla u_N|^2}{2};$$

on the other hand, from (42), it holds

$$\lim_{t \rightarrow T(x_0)^-} P_N(\mathbf{X}(t, x_0)) = \lim_{t \rightarrow T(x_0)^-} u_N(\mathbf{X}(t, x_0)) \leq \mu.$$

Then (18) follows from (49). \square

4.4. Proof of Proposition 12. The proof of this result is new for both the operators Δ_∞ and Δ_∞^N . Since it is analogous in the two cases, we present it just for the normalized operator. Let $\varphi \prec_x \phi_N^\Omega$. Let $p \in M(\Omega)$ and $q \in \partial\Omega$ be such that $x \in [q, p]$, and let $\nu := (p - q)/|p - q|$. We distinguish three cases.

Case 1: $x \in M(\Omega)$. In this case there holds necessarily $\nabla\varphi(x) = 0$ and

$$\langle \nabla^2\varphi(x)(y - x), y - x \rangle \leq -(\rho_\Omega - d_{\partial\Omega}(y))^2, \quad \forall y \in \overline{\Omega}.$$

Since, in this case, $x = q + \rho_\Omega \nu$ we get

$$\lambda_{\min}(\nabla^2\varphi(x)) \leq \langle \nabla^2\varphi(x)\nu, \nu \rangle = \frac{1}{\rho_\Omega^2} \langle \nabla^2\varphi(x)(q - x), q - x \rangle \leq -1.$$

Case 2: $x \notin \Sigma(\Omega)$. In this case we have $\tau := d_{\partial\Omega}(x) = |x - q| < \rho_\Omega$, and

$$\nabla\varphi(x) = \nabla\phi_N^\Omega(x) = g'(\tau)\nabla d_{\partial\Omega}(x) = g'(\tau)\nu \neq 0,$$

with g as in (25). Moreover, we have

$$h(t) := \varphi(q + t\nu) \leq \phi_N^\Omega(q + t\nu) = g(t), \quad \forall t \in [0, \rho_\Omega],$$

and

$$h(\tau) = g(\tau), \quad h'(\tau) = g'(\tau) > 0, \quad h''(\tau) \leq g''(\tau).$$

In particular we get

$$-\Delta_\infty^N \varphi(x) = -h''(\tau) \geq -g''(\tau) = 1.$$

Case 3: $x \in \Sigma(\Omega) \setminus M(\Omega)$. In this case, since the sub-differential of $d_{\partial\Omega}$ at x is empty, the same holds true for the sub-differential of ϕ_N^Ω at x . In particular, the second order sub-jet $J_\Omega^{2,-} \phi_N^\Omega(x)$ is empty, so that ϕ_N^Ω trivially satisfies the definition of viscosity super-solution at x . \square

5. PROOF OF THEOREM 4

The *sufficiency* part in the statement of Theorem 4 readily follows from Theorem 3 and formula (12). The *necessary* part is proved in Proposition 18 below, after the following preliminary lemma.

Lemma 17. *Assume that Ω is convex and that the unique solution u to problem (8) belongs to $C^{1,1}(\Omega \setminus K)$, with K as in (40). Then, for a.e. $x \in \Omega \setminus K$, there exists a unique solution $\mathbf{X}(\cdot, x)$ to (15), globally defined in $[0, +\infty)$, which satisfies*

$$(50) \quad \mathbf{X}(t, x) \notin K \quad \forall t \in [0, +\infty)$$

and

$$(51) \quad \lim_{t \rightarrow +\infty} \text{dist}(\mathbf{X}(t, x), K) = 0.$$

Proof. For every $x \in \Omega \setminus K$, any local solution γ to the second Cauchy problem in (15) cannot exit from $\{u \geq u(x)\}$ because we have

$$(52) \quad \frac{d}{dt} u(\gamma(t)) = \nabla u(\gamma(t)) \cdot \dot{\gamma}(t) = |\nabla u(\gamma(t))|^2,$$

so that u increases along the flow. Hence local solutions are actually global solutions, i.e. they are defined on $[0, +\infty)$. The uniqueness of the gradient flow associated with u in $\Omega \setminus K$ follows from the local Lipschitz regularity of ∇u assumed therein.

Let us now prove that, for a.e. $x \in \Omega \setminus K$, condition (50) is fulfilled. To that aim we are going to exploit the following claim, where the constant μ is defined according to (40):

Claim: *There exists a set $L \subseteq (0, \mu)$ with $|L| = \mu$ such that, for all $m \in L$, condition (50) is satisfied for \mathcal{H}^{n-1} -a.e. $x \in \{u = m\}$.*

Let us first show how the lemma follows from the claim. We point out that the set F given by points $x \in \Omega \setminus K$ such that (50) is *false* is \mathcal{L}^n -measurable. Indeed, F is open because its complement is given by $\bigcap_n G_n$, with

$$G_n := \left\{ x \in \Omega \setminus K : \mathbf{X}(t, x) \notin K \quad \forall t \in [0, n] \right\},$$

and every G_n is closed by continuous dependence on initial data. Then, we can integrate $|\nabla u|$ over F and we obtain

$$(53) \quad \int_F |\nabla u| dx = \int_0^\mu dm \int_{\{u=m\} \cap F} d\mathcal{H}^{n-1}(y) = 0,$$

where the first equality holds by the coarea formula, and the second one is consequence of our claim.

We now observe that $|\nabla u| > 0$ on $\Omega \setminus K$: this is due to the fact that $u \in C^1(\Omega)$ with $u^{1/2}$ concave, so that ∇u vanishes only at maximum points of u .

In view of this observation, (53) implies that F is \mathcal{L}^n -negligible, and the lemma is proved.

Finally, let us give the

Proof of the Claim: Let us define L as the set of values $m \in (0, \mu)$ such that u is twice differentiable \mathcal{H}^{n-1} -a.e. on $\{u = m\}$.

Firstly let us check that $|L| = \mu$. Namely, by the coarea formula, if Z is the set of points in $\Omega \setminus K$ where u is not twice differentiable, we have

$$0 = \int_Z |\nabla u| dx = \int_0^\mu dm \int_{\{u=m\} \cap Z} d\mathcal{H}^{n-1}(y).$$

We infer that, for \mathcal{L}^1 -a.e. $m \in (0, \mu)$, the set $\{u = m\} \cap Z$ is \mathcal{H}^{n-1} -negligible, so that L is of full measure in $(0, \mu)$.

From now on, let m denote a fixed value in L . For $x \in \{u = m\}$, let us define

$$(54) \quad N(x) := \left\{ t \in [0, T(x)] : u \text{ is not twice differentiable at } \mathbf{X}(t, x) \right\}$$

and let us show that

$$(55) \quad \mathcal{L}^1(N(x)) = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \{u = m\}.$$

By construction the set

$$E := \left\{ \mathbf{X}(t, x) : x \in \{u = m\}, t \in N(x) \right\}$$

is contained into the set of points where u is not twice differentiable. Then, since by assumption $u \in C^{1,1}(\Omega \setminus K)$, the set E is Lebesgue negligible. By the area formula, we have

$$0 = \mathcal{L}^n(E) = \int_{\{u=m\}} d\mathcal{H}^{n-1}(x) \int_{N(x)} J\mathbf{X}(t, x) dt,$$

where $J\mathbf{X}$ is the Jacobian of the function \mathbf{X} with respect to the second variable. Since this Jacobian is strictly positive (cf. [1, eq. (5)]), we infer that (55) holds true.

Let us prove that (50) holds for every $x_0 \in \{u = m\}$ such that both the conditions $\mathcal{L}^1(N(x_0)) = 0$ and u twice differentiable at x_0 hold.

Let x_0 be such a point, and set

$$\varphi(t) := u(\mathbf{X}(t, x_0)), \quad t \in [0, +\infty).$$

Since $\mathcal{L}^1(N(x_0)) = 0$, and since u is assumed to be in $C^{1,1}(\Omega \setminus K)$, the P -function is constant along γ . Therefore, the function $\varphi(t)$ (which is in $AC([0, +\infty))$, because $u \in C^1(\Omega)$ and $\gamma \in AC([0, +\infty))$), solves the Cauchy problem

$$\begin{cases} \frac{d\varphi}{dt}(t) = 2\lambda - 2\varphi(t) & \mathcal{L}^1\text{-a.e. on } [0, +\infty) \\ \varphi(0) = m, \end{cases}$$

where $m := u(x_0)$. Since this Cauchy problem admits a unique global solution, given by

$$\bar{\varphi}(t) := me^{-2t} + \lambda(1 - e^{-2t}),$$

we conclude that $u(\gamma(\cdot))$ agrees with $\bar{\varphi}(\cdot)$ on $[0, +\infty)$.

We now observe that

$$\frac{d\bar{\varphi}(t)}{dt} \neq 0 \quad \forall t \in [0, +\infty).$$

Since $\nabla u = 0$ on K , we infer that $\mathbf{X}(t, x_0) \notin K$ for $t \in [0, +\infty)$.

Eventually, we observe that (50) implies (51). Namely, assume that (51) is false. Since u is increasing along the flow, there exists some level set $\{u \leq m\}$, with $m < \mu$, which contains the whole trajectory $\mathbf{X}(t, x)$ for $t \in [0, +\infty)$. On the compact set $\{u \leq m\}$, the continuous function $|\nabla u|$ is bounded below by some strictly positive constant. Then, in view of (52), we deduce that (50) cannot hold. \square

Proposition 18. *Assume that Ω is convex. If the unique solution to problem (8) is in $C^{1,1}(\Omega \setminus K)$, then Ω is a stadium-like domain.*

Proof. Let u denote the unique solution to problem (8). As a first step we observe that, since by assumption $u \in C^{1,1}(\Omega \setminus K)$, there holds

$$(56) \quad P_N(x) = \mu \quad \forall x \in \Omega.$$

This can be obtained as a consequence of Lemma 17, by arguing as follows. Since P_N is continuous in Ω , it is enough to prove that the equality $P_N(x) = \mu$ holds a.e. on $\Omega \setminus K$. Namely, let us show that it holds for every $x \in \Omega \setminus K$ such that (50)-(51) hold and $\mathcal{L}^1(N(x)) = 0$, with $N(x)$ as in (54). (Actually, both these conditions are satisfied up to a \mathcal{L}^n -negligible set, by the same arguments used in the proof of Lemma 17). Let $x \in \Omega \setminus K$ be such that (50)-(51) hold and $\mathcal{L}^1(N(x)) = 0$. Since $\mathcal{L}^1(N(x)) = 0$, P is constant along γ and, since (50)-(51) hold, the constant is precisely equal to μ , yielding (56).

Now, for $m > 0$, consider the (convex) level sets

$$\Omega_m := \left\{ x \in \Omega : u(x) > m \right\}.$$

As a consequence of (56), and since $u \in C^1(\overline{\Omega_m})$, u satisfies on Ω_m the overdetermined boundary value problem

$$(57) \quad \begin{cases} -\Delta_\infty^N u = 1 & \text{in } \Omega_m, \\ u = m & \text{on } \partial\Omega_m, \\ |\nabla u| = \sqrt{2(\mu - m)} & \text{on } \partial\Omega_m. \end{cases}$$

By applying Theorem 3 (to the function $u - m$), we infer that Ω_m is a stadium-like domain for every $m > 0$.

To conclude, we notice that $\{\Omega_m\}$ is an increasing sequence of open sets contained into a fixed ball; therefore, as $m \rightarrow 0^+$, it converges in Hausdorff distance to their union (see for instance [24, Section 2.2.3]). Taking into account that $\Omega = \{u > 0\} = \bigcup_m \Omega_m$, we infer that $d_H(\Omega_m, \Omega) \rightarrow 0$, so that also the limit set Ω is a stadium-like domain. \square

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