ASYMPTOTIC DECAY UNDER NONLINEAR AND NONCOERCIVE DISSIPATIVE EFFECTS FOR ELECTRICAL CONDUCTION IN BIOLOGICAL TISSUES

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Abstract. We consider a nonlinear model for electrical conduction in biological tissues. The nonlinearity appears in the interface condition prescribed on the cell membrane.
The purpose of this paper is proving asymptotic convergence for large times to a periodic solution when time-periodic boundary data are assigned. The novelty here is that we allow the nonlinearity to be noncoercive. We consider both the homogenized and the non-homogenized version of the problem.

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1. Introduction

We study here a problem arising in electrical conduction in biological tissues with the purpose of obtaining some useful results for applications in electrical tomography, see [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]. Our interest in this framework is motivated by the fact that composite materials have widespread applications in science and technology and, for this reason, they have been extensively studied especially using homogenization techniques.
From a physical point of view our problem consists in the study of the electrical currents crossing a living tissue when an electrical potential is applied at the boundary (see [17], [19], [22], [25], [29]). Here the living tissue is regarded as a composite periodic domain made of extracellular and intracellular materials (both assumed to be conductive, possibly with different conductivities) separated by a lipidic membrane which experiments prove to exhibit both conductive (due to ionic channels in the membrane) and capacitive behavior. The periodic microstructure calls for the use of an homogenization technique. Among the wide literature on this topic, we recall for instance [1], [2], [3], [15], [16], [18], [20], [21], [28], [30], [31], [32], [33], [34]. As a result of the homogenization procedure we obtain a system of partial differential equations satisfied by the macroscopic electrical potential u, which is the limit of the electrical potential u_ε in the tissue as ε (the characteristic length of the cell) tends to zero.
Different scalings may appear in this homogenization procedure and they are studied in [9] and [13]. We study here further developments of the model proposed in [4],

[5], [7], [9], [12], [11], [13], where the magnetic field is neglected (as suggested by experimental evidence) and the potential u_ε is assumed to satisfy an elliptic equation both in the intracellular and in the extracellular domain (see, (2.1) below) while, on the membranes it satisfies the equation

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u_\varepsilon] + f\left(\frac{[u_\varepsilon]}{\varepsilon}\right) = \sigma^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon$$

where $[u_\varepsilon]$ denotes the jump of the potential across the membranes and $\sigma^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon$ is the current crossing the membranes. From a mathematical point of view a big difference does exist between the case of linear f and the nonlinear case, as already pointed out in [13] and [14].

At least in the linear case, the asymptotic behavior of the potentials u_ε and u is crucial in order to validate the phenomenological model employed in bioimpedance tomography devices, which currently relies on the use of complex elliptic equations, see [10]–[12].

Motivated by the previous considerations, in [14] and in this paper we investigate the behavior as $t \to +\infty$ of the nonlinear problem introduced in [13].

In [14], we proved that, if periodic boundary data are assigned and f is coercive in the following sense

$$f \in \mathcal{C}^1(\boldsymbol{R}), \qquad f'(s) \geq \kappa > 0, \qquad \forall s \in \boldsymbol{R}, \tag{1.1}$$

for a suitable $\kappa > 0$, then the solution of the ε-problem converges as $t \to +\infty$ to a periodic function solving a suitable system of equations. In that case such a convergence was proved to be exponential. A similar asymptotic exponential behavior was proved for the solution of the homogenized problem. Similar results in different frameworks can be found in [23], [24], [26], [27].

It is important to note that in [10]–[12], where f is linear, our approach was based on eigenvalue estimates which made it possible to keep into account (as far as the asymptotic rate of convergence is concerned) both the dissipative properties of the intra/extra cellular phases and the dissipative properties of the membranes. Namely, we looked at the eigenvalue problem associated to the static version of our problem (3.1)–(3.5) where $f(s) = \kappa s$. The relevant eigenvalue is positive and bounded from below uniformly in ε if σ^ε or κ are positive. This allows us to obtain a differential inequality for the L^2-norm of the solution u_ε in space, which implies its exponential asymptotic decay, again uniformly in ε.

Instead, in the nonlinear but coercive case (i.e., in [14]), we proceed by exploiting the coercivity of f, hence the electrical properties of the intra/extra cellular phases do not appear in the rate of convergence. Indeed the eigenvalue approach, which is peculiar to the linear case, cannot be applied here due to the lack of good comparison results. Thus the differential inequality mentioned above can rely only on the coercive dissipative effect of f (i.e., the exponential decay rate vanishes as $\kappa \to 0$).

If f is not coercive we must drop the approach via differential inequalities applied in [10]–[12] and in [14]. Instead we assume f to be monotone increasing and we proceed via a Liapunov-style technique so that the rate of convergence is not quantified.

The paper is organized as follows: in Section 2 we present the geometrical setting and the nonlinear differential model governing our problem at the microscale ε. In Section 3 we prove the decay in time of the solution of the microscopic problem. Finally, in Section 4 we prove the decay in time of the solution of the macroscopic (or homogenized) problem, providing also the differential system satisfied by such asymptotic limit.

2. Preliminaries

Let Ω be an open bounded subset of $\boldsymbol{R}^N$. In the sequel γ or $\widetilde{\gamma}$ will denote constants which may vary from line to line and which depend on the characteristic parameters of the problem, but which are independent of the quantities tending to zero, such as ε, δ and so on, unless explicitly specified.

2.1. **The geometrical setting.** The typical geometry we have in mind is depicted in Figure 1. In order to be more specific, assume $N \geq 2$ and let us introduce a

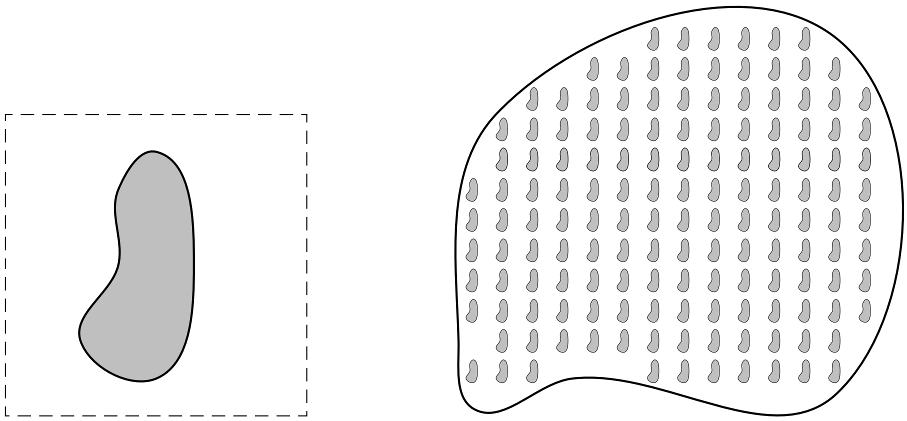

Figure 1. On the left: an example of admissible periodic unit cell $Y = E_1 \cup E_2 \cup \Gamma$ in $\boldsymbol{R}^2$. Here E_1 is the shaded region and Γ is its boundary. The remaining part of Y (the white region) is E_2. On the right: the corresponding domain $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$. Here Ω_1^ε is the shaded region and Γ^ε is its boundary. The remaining part of Ω (the white region) is Ω_2^ε.

periodic open subset E of $\boldsymbol{R}^N$, so that $E + z = E$ for all $z \in \boldsymbol{Z}^N$. For all $\varepsilon > 0$ define $\Omega_1^\varepsilon = \Omega \cap \varepsilon E$, $\Omega_2^\varepsilon = \Omega \setminus \overline{\varepsilon E}$. We assume that Ω, E have regular boundary, say of class $\mathcal{C}^\infty$ for the sake of simplicity, and $dist(\Gamma^\varepsilon, \partial\Omega) \geq \gamma\varepsilon$, where $\Gamma^\varepsilon = \partial\Omega_1^\varepsilon$. We also employ the notation $Y = (0,1)^N$, and $E_1 = E \cap Y$, $E_2 = Y \setminus \overline{E}$, $\Gamma = \partial E \cap \overline{Y}$. As a simplifying assumption, we stipulate that E_1 is a connected smooth subset of Y such that $dist(\Gamma, \partial Y) > 0$. We denote by ν the normal unit vector to Γ pointing into E_2, so that $\nu_\varepsilon(x) = \nu(\varepsilon^{-1}x)$.

For later use, we introduce also the conductivity

$$\sigma(y)=\begin{cases}\sigma_1 & \text{if } y\in E_1,\\ \sigma_2 & \text{if } y\in E_2,\end{cases}\qquad \text{and}\qquad \sigma_0=|E_1|\sigma_1+|E_2|\sigma_2\,,$$

where σ_1,σ_2 are positive constants, and we also set $\sigma^\varepsilon(x)=\sigma(\varepsilon^{-1}x)$. Moreover, let us set

$$\mathfrak{C}^k_\#(Y):=\{u:Y\setminus\Gamma\to\boldsymbol{R}\mid u_{|E_1}\in\mathcal{C}^k(\overline{E_1})\,,\ u_{|E_2}\in\mathcal{C}^k(\overline{E_2})\,,\ \text{and } u \text{ is } Y-\text{periodic}\}\,,$$

for every $0\leq k\leq+\infty$, and

$$\mathcal{X}^1_\#(Y):=\{u\in L^2(Y)\mid u_{|E_1}\in H^1(E_1)\,,\ u_{|E_2}\in H^1(E_2)\,,\ \text{and } u \text{ is } Y-\text{periodic}\}\,.$$

More generally, the subscript $\#$ in the definition of a function space will denote periodicity with respect to the first domain, in such a way that the extended function remains (locally) in the same space.
We set also

$$\mathcal{X}^1(\Omega_\varepsilon):=\{u\in L^2(\Omega)\mid u_{|\Omega^\varepsilon_1}\in H^1(\Omega^\varepsilon_1),\ u_{|\Omega^\varepsilon_2}\in H^1(\Omega^\varepsilon_2)\}\,.$$

We note that, if $u\in\mathcal{X}^1_\#(Y)$ then the traces of $u_{|E_i}$ on Γ, for $i=1,2$, belong to $H^{1/2}(\Gamma)$, as well as $u\in\mathcal{X}^1(\Omega_\varepsilon)$ implies that the traces of $u_{|\Omega^\varepsilon_i}$ on Γ^ε, for $i=1,2$, belong to $H^{1/2}(\Gamma^\varepsilon)$.

2.2. **Statement of the problem.** We write down the model problem:

$$-\operatorname{div}(\sigma^\varepsilon\nabla u_\varepsilon)=0\,,\qquad \text{in } (\Omega^\varepsilon_1\cup\Omega^\varepsilon_2)\times(0,T); \tag{2.1}$$

$$[\sigma^\varepsilon\nabla u_\varepsilon\cdot\nu_\varepsilon]=0\,,\qquad \text{on } \Gamma^\varepsilon\times(0,T); \tag{2.2}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u_\varepsilon]+f\left(\frac{[u_\varepsilon]}{\varepsilon}\right)=\sigma^\varepsilon\nabla u_\varepsilon\cdot\nu_\varepsilon\,,\qquad \text{on } \Gamma^\varepsilon\times(0,T); \tag{2.3}$$

$$[u_\varepsilon](x,0)=S_\varepsilon(x)\,,\qquad \text{on } \Gamma^\varepsilon; \tag{2.4}$$

$$u_\varepsilon(x)=\Psi(x,t)\,,\qquad \text{on } \partial\Omega\times(0,T), \tag{2.5}$$

where σ^ε is defined in the previous subsection and $\alpha>0$ is a constant. We note that, by the definition already given in the previous section, ν_ε is the normal unit vector to Γ^ε pointing into Ω^ε_2. Since u_ε is not in general continuous across Γ^ε we set

$$u^{(1)}_\varepsilon:=\ \text{trace of } u_{\varepsilon|\Omega^\varepsilon_1} \text{ on } \Gamma^\varepsilon\times(0,T);\quad u^{(2)}_\varepsilon:=\ \text{trace of } u_{\varepsilon|\Omega^\varepsilon_2} \text{ on } \Gamma^\varepsilon\times(0,T).$$

Indeed we refer conventionally to Ω^ε_1 as to the *interior domain*, and to Ω^ε_2 as to the *outer domain.* We also denote

$$[u_\varepsilon]:=u^{(2)}_\varepsilon-u^{(1)}_\varepsilon\,.$$

Similar conventions are employed for other quantities, for example in (2.2). In this framework we will assume that

$$i)\quad S_\varepsilon\in H^{1/2}(\Gamma^\varepsilon)\,,\qquad ii)\quad \int_{\Gamma^\varepsilon}S^2_\varepsilon(x)\,\mathrm{d}\sigma\leq\gamma\varepsilon\,, \tag{2.6}$$

where the second assumption in (2.6) is needed in order that the solution of system (2.1)–(2.5) satisfies the classical energy inequality. (see (3.1) in [14]).

Moreover, $f:\boldsymbol{R}\to\boldsymbol{R}$ satisfies

$$f\in\mathcal{C}^1(\boldsymbol{R})\,, \tag{2.7}$$

$$f \text{ is a strictly monotone increasing function,} \tag{2.8}$$

$$f(0)=0\,, \tag{2.9}$$

$$f'(s)\geq\delta_0\,,\qquad \text{for a suitable } \delta_0>0 \text{ and } \forall|s| \text{ sufficiently large.} \tag{2.10}$$

The previous assumptions imply also

$$f(s)s\geq\lambda_1 s^2-\lambda_2|s|\,,\qquad \text{for some constants } \lambda_1>0 \text{ and } \lambda_2\geq 0. \tag{2.11}$$

Notice that the results presented in this paper hold also in a more general case, namely if we replace condition (2.10) with the assumption that f^{-1} is uniformly continuous in $\boldsymbol{R}$, for example when $f(s)=s+\sin s$.
Finally, $\Psi:\Omega\times\boldsymbol{R}\to\boldsymbol{R}$ is a function satisfying the following assumptions

$$\begin{aligned} &i) && \Psi\in L^2_{loc}\big(\boldsymbol{R};H^2(\Omega)\big)\,;\\ &ii) && \Psi_t\in L^2_{loc}\big(\boldsymbol{R};H^1(\Omega)\big)\,;\\ &iii) && \Psi(x,\cdot) \text{ is 1-periodic} \qquad \text{for a.e. } x\in\Omega. \end{aligned} \tag{2.12}$$

Existence and uniqueness for problem (2.1)–(2.5) has been proved in [8]. Moreover, by [14, Lemma 4.1 and Remark 4.2] it follows that the solution $u_\varepsilon\in\mathcal{C}^0\big((0,T];\mathcal{X}^1(\Omega_\varepsilon)\big)$ and $[u_\varepsilon]\in\mathcal{C}^0\big((0,T];L^2(\Gamma^\varepsilon)\big)$, uniformly with respect to ε, and $[u_\varepsilon]\in\mathcal{C}^0\big([0,T];L^2(\Gamma^\varepsilon)\big)$, but with non uniform estimates.

3. Asymptotic convergence to a periodic solution of the ε-problem

The purpose of this section is to prove the asymptotic convergence of the solution of problem (2.1)–(2.5) to a periodic function $u^\#_\varepsilon$ when $t\to+\infty$. The function $u^\#_\varepsilon$ is, in turn, a solution of the system

$$-\operatorname{div}(\sigma^\varepsilon\nabla u^\#_\varepsilon)=0\,,\qquad \text{in } (\Omega^\varepsilon_1\cup\Omega^\varepsilon_2)\times\boldsymbol{R}; \tag{3.1}$$

$$[\sigma^\varepsilon\nabla u^\#_\varepsilon\cdot\nu_\varepsilon]=0\,,\qquad \text{on } \Gamma^\varepsilon\times\boldsymbol{R}; \tag{3.2}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u^\#_\varepsilon]+f\left(\frac{[u^\#_\varepsilon]}{\varepsilon}\right)=(\sigma^\varepsilon\nabla u^\#_\varepsilon\cdot\nu_\varepsilon)\,,\quad \text{on } \Gamma^\varepsilon\times\boldsymbol{R}; \tag{3.3}$$

$$u^\#_\varepsilon(x,t)=\Psi(x,t)\,,\qquad \text{on } \partial\Omega\times\boldsymbol{R}; \tag{3.4}$$

$$u^\#_\varepsilon(x,\cdot) \text{ is 1-periodic,}\qquad \text{in } \Omega. \tag{3.5}$$

Indeed, this problem is derived from (2.1)–(2.5) replacing equation (2.4) with (3.5). The rigorous definition of weak solution of (3.1)–(3.5) is standard (see for instance [14, Definition 4.13]
As a first step we will prove the following result.

Proposition 3.1. *Under the assumptions* (2.7)–(2.10) *and* (2.12)*, problem* (3.1)–(3.5) *admits a solution* $u^\#_\varepsilon\in\mathcal{C}^0_\#([0,1];\mathcal{X}^1(\Omega_\varepsilon))$.

Proof. For $\delta > 0$, let us denote by $f_\delta(s) := f(s) + \delta s$, for every $s \in \boldsymbol{R}$, and consider the problem

$$
\begin{aligned}
-\operatorname{div}(\sigma^\varepsilon \nabla u^{\#}_{\varepsilon,\delta}) &= 0\,, && \text{in } (\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \times \boldsymbol{R}; && (3.6)\\
[\sigma^\varepsilon \nabla u^{\#}_{\varepsilon,\delta} \cdot \nu_\varepsilon] &= 0\,, && \text{on } \Gamma^\varepsilon \times \boldsymbol{R}; && (3.7)\\
\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u^{\#}_{\varepsilon,\delta}] + f_\delta\left(\frac{[u^{\#}_{\varepsilon,\delta}]}{\varepsilon}\right) &= \sigma^\varepsilon \nabla u^{\#}_{\varepsilon,\delta} \cdot \nu_\varepsilon\,, && \text{on } \Gamma^\varepsilon \times \boldsymbol{R}; && (3.8)\\
u^{\#}_{\varepsilon,\delta}(x,t) &= \Psi(x,t)\,, && \text{on } \partial\Omega \times \boldsymbol{R}; && (3.9)\\
u^{\#}_{\varepsilon,\delta}(x,\cdot) &\text{ is 1-periodic,} && \text{in } \Omega. && (3.10)
\end{aligned}
$$

For any positive ε and δ, the previous problem admits a unique time-periodic solution because of the results already proved in [14].
On the other hand, multiplying equation (3.6) by $u^{\#}_{\varepsilon,\delta} - \Psi$, integrating by parts on $\Omega \times [0,1]$, using the periodicity and taking into account equations (3.7)–(3.9), we get

$$
\int_0^1 \int_\Omega \frac{\sigma^\varepsilon}{2} |\nabla u^{\#}_{\varepsilon,\delta}|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^1 \int_{\Gamma^\varepsilon} f_\delta\left(\frac{[u^{\#}_{\varepsilon,\delta}]}{\varepsilon}\right)[u^{\#}_{\varepsilon,\delta}] \,\mathrm{d}\sigma\,\mathrm{d}t \le \int_0^1 \int_\Omega \frac{\sigma^\varepsilon}{2} |\nabla \Psi|^2 \,\mathrm{d}x\,\mathrm{d}t\,. \quad (3.11)
$$

Finally, using (2.11) we obtain

$$
\int_0^1 \int_\Omega \frac{\sigma^\varepsilon}{2} |\nabla u^{\#}_{\varepsilon,\delta}|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^1 \int_{\Gamma^\varepsilon} \frac{\lambda_1}{2\varepsilon}[u^{\#}_{\varepsilon,\delta}]^2 \,\mathrm{d}\sigma\,\mathrm{d}t \le \int_0^1 \int_\Omega \frac{\sigma^\varepsilon}{2} |\nabla \Psi|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon}{2\lambda_1}\lambda_2^2 |\Gamma^\varepsilon|\,. \quad (3.12)
$$

Multiplying now equation (3.6) by $u^{\#}_{\varepsilon,\delta,t} - \Psi_t$, integrating by parts on $\Omega \times [0,1]$, using the periodicity and taking into account equations (3.7)–(3.9), we get

$$
\begin{aligned}
&\frac{\alpha}{\varepsilon} \int_0^1 \int_{\Gamma^\varepsilon} [u^{\#}_{\varepsilon,\delta,t}]^2 \,\mathrm{d}\sigma\,\mathrm{d}t + \int_0^1 \int_{\Gamma^\varepsilon} f_\delta\left(\frac{[u^{\#}_{\varepsilon,\delta}]}{\varepsilon}\right)[u^{\#}_{\varepsilon,\delta,t}] \,\mathrm{d}\sigma\,\mathrm{d}t \\
&\quad \le \int_0^1 \int_\Omega \sigma^\varepsilon \nabla u^{\#}_{\varepsilon,\delta} \nabla \Psi_t \,\mathrm{d}x\,\mathrm{d}t \le \int_0^1 \int_\Omega \frac{\sigma^\varepsilon}{2} |\nabla u^{\#}_{\varepsilon,\delta}|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^1 \int_\Omega \frac{\sigma^\varepsilon}{2} |\nabla \Psi_t|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le \int_0^1 \int_\Omega \frac{\sigma^\varepsilon}{2} |\nabla \Psi|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon}{2\lambda_1}\lambda_2^2 |\Gamma^\varepsilon| + \int_0^1 \int_\Omega \frac{\sigma^\varepsilon}{2} |\nabla \Psi_t|^2 \,\mathrm{d}x\,\mathrm{d}t\,, \quad (3.13)
\end{aligned}
$$

where we used (3.12). Notice that the second integral on the left-hand side is equal to zero by periodicity and trivial integration. Hence

$$
\frac{\alpha}{2\varepsilon} \int_0^1 \int_{\Gamma^\varepsilon} [u^{\#}_{\varepsilon,\delta,t}]^2 \,\mathrm{d}\sigma\,\mathrm{d}t \le \int_0^1 \int_\Omega \frac{\sigma^\varepsilon}{2} |\nabla \Psi_t|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^1 \int_\Omega \frac{\sigma^\varepsilon}{2} |\nabla \Psi|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon}{2\lambda_1}\lambda_2^2 |\Gamma^\varepsilon|\,.
$$
(3.14)

Inequalities (3.12) and (3.14), for $\varepsilon > 0$ fixed, yield the weak convergence of $u^{\#}_{\varepsilon,\delta}$ and $\nabla u^{\#}_{\varepsilon,\delta}$ in $L^2(\Omega^\varepsilon_i \times (0,1))$, $i = 1,2$, and respectively the strong convergence of $[u^{\#}_{\varepsilon,\delta}]$ in $L^2(\Gamma^\varepsilon \times (0,1))$, for $\delta \to 0$. Since all the functions $u^{\#}_{\varepsilon,\delta}$ are 1-periodic, denoting as usual with $u^{\#}_\varepsilon$ the limit of $u^{\#}_{\varepsilon,\delta}$ we have that the same periodicity holds true for $u^{\#}_\varepsilon$. Moreover we can pass to the limit, as $\delta \to 0$, in the weak formulation of problem (3.6)–(3.10), thus obtaining that $u^{\#}_\varepsilon$ is a 1-periodic solution of problem (3.1)–(3.5), under the assumptions (2.7)–(2.10) and (2.12).
Differentiating formally with respect to t (3.6)–(3.9), multiplying the first equation thus obtained by $(u^{\#}_{\varepsilon,\delta,t} - \Psi_t)$ and finally integrating by parts, we obtain

$$\int_0^1 \int_\Omega \frac{\sigma^\varepsilon}{2} |\nabla u^{\#}_{\varepsilon,\delta,t}|^2 \,\mathrm{d}x\,\mathrm{d}t \le \int_0^1 \int_\Omega \frac{\sigma^\varepsilon}{2} |\nabla \Psi_t|^2 \,\mathrm{d}x\,\mathrm{d}t \,.$$

Since the estimates above are uniform in δ, we have that $u^{\#}_\varepsilon$ belongs to the class claimed in the statement. □

Given $\varepsilon > 0$, it remains to prove the asymptotic convergence of the solution u_ε of (2.1)–(2.5) to $u^{\#}_\varepsilon$, for $t \to +\infty$.

Theorem 3.2. *Let $\varepsilon > 0$ be fixed and let u_ε be the solution of problem* (2.1)–(2.5). *Then, for $t \to +\infty$, $u_\varepsilon \to u^{\#}_\varepsilon$ in the following sense:*

$$\lim_{t\to+\infty} \|u_\varepsilon(\cdot,t) - u^{\#}_\varepsilon(\cdot,t)\|_{L^2(\Omega)} = 0\,; \tag{3.15}$$

$$\lim_{t\to+\infty} \|\nabla u_\varepsilon(\cdot,t) - \nabla u^{\#}_\varepsilon(\cdot,t)\|_{L^2(\Omega)} = 0\,; \tag{3.16}$$

$$\lim_{t\to+\infty} \|[u_\varepsilon](\cdot,t) - [u^{\#}_\varepsilon](\cdot,t)\|_{L^2(\Gamma^\varepsilon)} = 0\,. \tag{3.17}$$

Proof. Setting $r_\varepsilon := u^{\#}_\varepsilon - u_\varepsilon$, we obtain that r_ε satisfies

$$-\operatorname{div}(\sigma^\varepsilon \nabla r_\varepsilon) = 0\,, \qquad \text{in } (\Omega^\varepsilon_1 \cup \Omega^\varepsilon_2) \times (0,+\infty); \tag{3.18}$$

$$[\sigma^\varepsilon \nabla r_\varepsilon \cdot \nu_\varepsilon] = 0\,, \qquad \text{on } \Gamma^\varepsilon \times (0,+\infty); \tag{3.19}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[r_\varepsilon] + g_\varepsilon(x,t)\frac{[r_\varepsilon]}{\varepsilon} = \sigma^\varepsilon \nabla r_\varepsilon \cdot \nu_\varepsilon\,, \qquad \text{on } \Gamma^\varepsilon \times (0,+\infty); \tag{3.20}$$

$$[r_\varepsilon](x,0) = [u^{\#}_\varepsilon(x,0)] - S_\varepsilon(x) =: \widehat{S}_\varepsilon(x)\,, \qquad \text{on } \Gamma^\varepsilon; \tag{3.21}$$

$$r_\varepsilon(x) = 0\,, \qquad \text{on } \partial\Omega \times (0,+\infty); \tag{3.22}$$

where

$$g_\varepsilon(x,t) := \begin{cases} f'\left(\dfrac{[u_\varepsilon]}{\varepsilon}(x,t)\right) & \text{if } [u_\varepsilon](x,t) = [u^{\#}_\varepsilon](x,t), \\[2ex] \dfrac{f\left(\frac{[u^{\#}_\varepsilon]}{\varepsilon}(x,t)\right) - f\left(\frac{[u_\varepsilon]}{\varepsilon}(x,t)\right)}{\frac{[u^{\#}_\varepsilon]}{\varepsilon}(x,t) - \frac{[u_\varepsilon]}{\varepsilon}(x,t)} & \text{if } [u_\varepsilon](x,t) \neq [u^{\#}_\varepsilon](x,t), \end{cases}$$

so that $g_\varepsilon(x,t) \geq 0$, and $\widehat{S}_\varepsilon(x)$ still satisfies assumption *ii)* in (2.6) because of (3.12) and (3.14). Multiplying equation (3.18) by r_ε and integrating by parts we have

$$\int_\Omega \sigma^\varepsilon |\nabla r_\varepsilon|^2 \,\mathrm{d}x + \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} [r_{\varepsilon,t}][r_\varepsilon] \,\mathrm{d}\sigma + \int_{\Gamma^\varepsilon} \frac{g_\varepsilon(x,t)}{\varepsilon} [r_\varepsilon]^2 \,\mathrm{d}\sigma = 0 \,. \tag{3.23}$$

Equation (3.23) implies that the function $t \mapsto \frac{\alpha}{\epsilon} \int_{\Gamma^\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma$ is a positive, decreasing function of t; hence, it tends to a limit value $\overline{r}_\varepsilon \geq 0$ as $t \to +\infty$. We claim that the value $\overline{r}_\varepsilon$ must be zero. Otherwise, for every $t > 0$, $\frac{\alpha}{\epsilon} \int_{\Gamma^\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma \geq \overline{r}_\varepsilon > 0$. On the other hand, setting $\Gamma^\varepsilon_{\overline{r}_\varepsilon}(t) := \{x \in \Gamma^\varepsilon : [r_\varepsilon(x,t)]^2 \leq \frac{\overline{r}_\varepsilon \varepsilon}{2\alpha |\Gamma^\varepsilon|}\}$, we have that

$$\frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon \setminus \Gamma^\varepsilon_{\overline{r}_\varepsilon}(t)} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma \geq \frac{\overline{r}_\varepsilon}{2} \,, \qquad \forall t > 0 \,. \tag{3.24}$$

Indeed, by definition,

$$\begin{aligned}
\overline{r}_\varepsilon &\leq \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma = \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon \setminus \Gamma^\varepsilon_{\overline{r}_\varepsilon}(t)} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma + \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon_{\overline{r}_\varepsilon}(t)} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma \\
&\leq \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon \setminus \Gamma^\varepsilon_{\overline{r}_\varepsilon}(t)} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma + \frac{\alpha}{\varepsilon} \frac{\overline{r}_\varepsilon \varepsilon}{2\alpha |\Gamma^\varepsilon|} |\Gamma^\varepsilon_{\overline{r}_\varepsilon}| \leq \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon \setminus \Gamma^\varepsilon_{\overline{r}_\varepsilon}(t)} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma + \frac{\overline{r}_\varepsilon}{2} \,,
\end{aligned}$$

which implies (3.24). Moreover, we have that, on $\Gamma^\varepsilon \setminus \Gamma^\varepsilon_{\overline{r}_\varepsilon}(t)$, $g_\varepsilon(x,t) \geq \chi > 0$, where χ is a suitable positive constant depending only on $(\overline{r}_\varepsilon, \varepsilon, \alpha, |\Gamma^\varepsilon|)$ (this last result follows from assumption (2.8)–(2.10)). Hence, using (3.23), it follows

$$\begin{aligned}
\frac{d}{dt} \left(\frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma \right) &\leq - \int_{\Gamma^\varepsilon \setminus \Gamma^\varepsilon_{\overline{r}_\varepsilon}(t)} \frac{g_\varepsilon(x,t)}{\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma \\
&\leq -\chi \int_{\Gamma^\varepsilon \setminus \Gamma^\varepsilon_{\overline{r}_\varepsilon}(t)} \frac{1}{\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma \leq -\frac{\overline{r}_\varepsilon}{2\alpha} \chi < 0 \,.
\end{aligned} \tag{3.25}$$

Inequality (3.25) clearly contradicts the asymptotic convergence in t of the function $t \mapsto \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma$, hence

$$\lim_{t \to +\infty} \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma = 0 \,. \tag{3.26}$$

In particular, this gives (3.17). Integrating (3.23) in $[\hat{t}, \infty)$ and taking into account (3.26), we get

$$\int_{\hat{t}}^{+\infty} \int_\Omega \sigma^\varepsilon |\nabla r_\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}t \leq \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [r_\varepsilon(x, \hat{t})]^2 \,\mathrm{d}\sigma \,, \tag{3.27}$$

which implies

$$\lim_{t\to+\infty} \int_{\hat t}^{+\infty} \int_{\Omega} \sigma^\varepsilon |\nabla r_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t = 0\,. \tag{3.28}$$

Condition (3.28) guarantees that for every positive η there exists a $\hat t(\eta) > 0$, such that

$$\int_{\hat t(\eta)}^{+\infty} \int_{\Omega} \sigma^\varepsilon |\nabla r_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \le \eta\,,$$

which, in turn implies that, for every natural number n, there exists a $t_n \in (\hat t(\eta) + n, \hat t(\eta) + (n+1))$, such that

$$\int_{\Omega} \sigma^\varepsilon |\nabla r_\varepsilon(x,t_n)|^2 \,\mathrm{d}x \le \eta\,. \tag{3.29}$$

Now, we multiply (3.18) by $r_{\varepsilon,t}$ and integrate in Ω, so that

$$\int_{\Omega} \sigma^\varepsilon \nabla r_\varepsilon \nabla r_{\varepsilon,t}(x,t)\,\mathrm{d}x + \frac{\alpha}{\varepsilon}\int_{\Gamma^\varepsilon} [r_{\varepsilon,t}(x,t)]^2\,\mathrm{d}\sigma + \int_{\Gamma^\varepsilon} \frac{g_\varepsilon(x,t)}{\varepsilon} [r_\varepsilon(x,t)]\,[r_{\varepsilon,t}(x,t)]\,\mathrm{d}\sigma = 0\,, \tag{3.30}$$

which implies

$$\int_{\Omega} \sigma^\varepsilon \nabla r_\varepsilon \nabla r_{\varepsilon,t}\,\mathrm{d}x \le \int_{\Gamma^\varepsilon} \frac{g_\varepsilon^2(x,t)}{2\alpha\varepsilon}[r_\varepsilon]^2\,\mathrm{d}\sigma\,. \tag{3.31}$$

Moreover, integrating (3.31) in $[t_n, t^*]$ with $t^* \in [t_n, t_n+2]$ and using (3.29), we have

$$\sup_{t\in[t_n,t_n+2]} \left(\int_{\Omega} \frac{\sigma^\varepsilon}{2} |\nabla r_\varepsilon(x,t)|^2\,\mathrm{d}x \right) \le \frac{\eta}{2} + \frac{L^2}{\alpha^2} \sup_{t\in[t_n,+\infty)} \left(\frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} [r_\varepsilon(x,t)]^2\,\mathrm{d}\sigma \right)\,, \quad \forall n \in \boldsymbol{N}\,.$$

Since $t_{n+1} - t_n < 2$, the intervals of the form $[t_n, t_n+2]$, when n varies in $\boldsymbol{N}$, are overlapping; hence, we obtain

$$\sup_{t\in[\hat t+1,+\infty)} \left(\int_{\Omega} \frac{\sigma^\varepsilon}{2} |\nabla r_\varepsilon(x,t)|^2\,\mathrm{d}x \right) \le \frac{\eta}{2} + \frac{L^2}{\alpha^2} \sup_{t\in[\hat t,+\infty)} \left(\frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} [r_\varepsilon(x,t)]^2\,\mathrm{d}\sigma \right)\,. \tag{3.32}$$

Because of (3.26) the integral in the right-hand side of (3.32) can be made smaller than $\frac{\eta}{2}\left(\frac{L^2}{\alpha^2}\right)^{-1}$, provided $\hat t$ is chosen sufficiently large in dependence of η. This means that

$$\sup_{t\in[\hat t+1,+\infty)} \left(\int_{\Omega} \frac{\sigma^\varepsilon}{2} |\nabla r_\varepsilon(x,t)|^2\,\mathrm{d}x \right) \le \eta\,, \tag{3.33}$$

so that

$$\lim_{t\to+\infty} \int_{\Omega} \sigma^\varepsilon |\nabla r_\varepsilon(x,t)|^2\,\mathrm{d}x = 0\,. \tag{3.34}$$

In particular, this gives (3.16). Finally, Poincare's inequality together with (3.26) and (3.34) yield

$$\lim_{t\to+\infty}\int_{\Omega}|r_\varepsilon(x,t)|^2\,\mathrm{d}x=0\,, \tag{3.35}$$

which gives (3.15). □

Remark 3.3. More in general, the previous procedure allows us to prove that solutions of (2.1)–(2.5) having different initial data satisfying (2.6) but the same boundary condition tend asymptotically one to the other (such convergence being exponential if f is coercive in the sense of (1.1)). □

Remark 3.4. Observe that, thanks to previous remark, Theorem 3.2 implies uniqueness of the periodic solution of problem (3.1)–(3.5) in $\mathcal{C}^0_\#([0,1];\mathcal{X}^1(\Omega_\varepsilon))$. □

4. Asymptotic decay of the solution of the homogenized problem

The aim of this section is to prove asymptotic decay of the solution of the homogenized problem. To this purpose, let $(u,u^1)\in L^2\big((0,T);\,H^1(\Omega)\big)\times L^2\big(\Omega\times(0,T);\mathcal{X}^1_\#(Y)\big)$ be the two-scale limit of the solution u_ε of problem (2.1)–(2.5), where the initial data S_ε satisfies the additional condition that $S_\varepsilon/\varepsilon$ two-scale converges in $L^2\big(\Omega;L^2(\Gamma)\big)$ to a function S_1 such that $S_1(x,\cdot)=S_{|\Gamma}(x,\cdot)$ for some $S\in\mathcal{C}\big(\overline{\Omega};\mathcal{C}^1_\#(Y)\big)$, and

$$\lim_{\varepsilon\to0}\varepsilon\int_{\Gamma^\varepsilon}\left(\frac{S_\varepsilon}{\varepsilon}\right)^2(x)\,\mathrm{d}\sigma=\int_{\Omega}\int_{\Gamma}S_1^2(x,y)\,\mathrm{d}\sigma\,\mathrm{d}x\,. \tag{4.1}$$

We recall that, under these assumptions, by [13, Theorem 2.1], the pair (u,u^1) is the weak solution of the two-scale problem

$$-\operatorname{div}\left(\sigma_0\nabla u+\int_Y\sigma\nabla_y u^1\,\mathrm{d}y\right)=0\,,\qquad \text{in } \Omega\times(0,T); \tag{4.2}$$

$$-\operatorname{div}_y(\sigma\nabla u+\sigma\nabla_y u^1)=0\,,\qquad \text{in } \Omega\times(E_1\cup E_2)\times(0,T); \tag{4.3}$$

$$[\sigma(\nabla u+\nabla_y u^1)\cdot\nu]=0\,,\qquad \text{on } \Omega\times\Gamma\times(0,T); \tag{4.4}$$

$$\alpha\frac{\partial}{\partial t}[u^1]+f\left([u^1]\right)=\sigma(\nabla u+\nabla_y u^1)\cdot\nu\,,\qquad \text{on } \Omega\times\Gamma\times(0,T); \tag{4.5}$$

$$[u^1](x,y,0)=S_1(x,y)\,,\qquad \text{on } \Omega\times\Gamma; \tag{4.6}$$

$$u(x,t)=\Psi(x,t)\,,\qquad \text{on } \partial\Omega\times(0,T); \tag{4.7}$$

in the sense of the following definition.

Definition 4.1. A pair $(u,u^1)\in L^2\big((0,T);\,H^1(\Omega)\big)\times L^2\big(\Omega\times(0,T);\mathcal{X}^1_\#(Y)\big)$ is a weak solution of (4.2)–(4.7) if

$$\int_0^T\int_\Omega\int_Y \sigma\left(\nabla u+\nabla_y u^1\right)(\nabla\phi+\nabla_y\Phi)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega\int_\Gamma f([u^1])[\Phi]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t$$
$$-\alpha\int_0^T\int_\Omega\int_\Gamma [u^1]\frac{\partial}{\partial t}[\Phi]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t-\alpha\int_\Omega\int_\Gamma[\Phi]S_1\,\mathrm{d}\sigma\,\mathrm{d}x=0\,,\quad(4.8)$$

for any function $\phi\in\mathcal{C}^0\big(0,T;H^1_0(\Omega)\big)$ and any function $\Phi\in\mathcal{C}^0\big([0,T];L^2(\Omega;\mathcal{X}^1_\#(Y))\big)$ with $[\Phi_t]\in\mathcal{C}^0\big([0,T];L^2(\Omega\times\Gamma)\big)$ which vanishes at $t=T$.
Moreover, u satisfies the boundary condition on $\partial\Omega\times[0,T]$ in the trace sense (i.e. $u(x,t)=\Psi(x,t)$ a.e. on $\partial\Omega\times(0,T)$) and u^1 is periodic in Y and has zero mean value in Y for a.e. $(x,t)\in\Omega\times(0,T)$ (see [14, Definition 5.1]).

For later use, let us define

$$|||\big(h(\cdot,t),h^1(\cdot,t)\big)|||:=\|h\|_{\mathcal{C}^0([0,1];H^1(\Omega))}+\|h^1\|_{\mathcal{C}^0([0,1];L^2(\Omega\times Y))}$$
$$+\|\nabla_y h^1\|_{\mathcal{C}^0([0,1];L^2(\Omega\times Y))}+\|[h^1]\|_{\mathcal{C}^0([0,1];L^2(\Omega\times\Gamma))}\,,\quad(4.9)$$

where $(h,h^1)\in\mathcal{C}^0([0,T];H^1(\Omega))\times\mathcal{C}^0([0,T];L^2(\Omega;\mathcal{X}^1_\#(Y)))$, and

$$||||(\widetilde h,\widetilde h^1)||||:=\|\widetilde h\|_{H^1(\Omega)}+\|\widetilde h^1\|_{L^2(\Omega\times Y)}+\|\nabla_y\widetilde h^1\|_{L^2(\Omega\times Y)}+\|[\widetilde h^1]\|_{L^2(\Omega\times\Gamma)}\,,\quad(4.10)$$

where $(\widetilde h,\widetilde h^1)\in H^1(\Omega)\times L^2(\Omega;\mathcal{X}^1_\#(Y))$.
As in the previous section, first we prove that there exists a time-periodic weak solution of the two-scale problem

$$-\operatorname{div}\left(\sigma_0\nabla u^\#+\int_Y\sigma\nabla_y u^{1,\#}\,\mathrm{d}y\right)=0\,,\qquad\text{in }\Omega\times\boldsymbol{R};\quad(4.11)$$
$$-\operatorname{div}_y(\sigma\nabla u^\#+\sigma\nabla_y u^{1,\#})=0\,,\qquad\text{in }\Omega\times(E_1\cup E_2)\times\boldsymbol{R};\quad(4.12)$$
$$[\sigma(\nabla u^\#+\nabla_y u^{1,\#})\cdot\nu]=0\,,\qquad\text{on }\Omega\times\Gamma\times\boldsymbol{R};\quad(4.13)$$
$$\alpha\frac{\partial}{\partial t}[u^{1,\#}]+f\left([u^{1,\#}]\right)=\sigma(\nabla u^\#+\nabla_y u^{1,\#})\cdot\nu\,,\text{on }\Omega\times\Gamma\times\boldsymbol{R};\quad(4.14)$$
$$[u^{1,\#}](x,y,\cdot)\text{ is 1-periodic},\qquad\text{on }\Omega\times\Gamma;\quad(4.15)$$
$$u^\#(x,t)=\Psi(x,t)\,,\qquad\text{on }\partial\Omega\times\boldsymbol{R};\quad(4.16)$$

in the sense of the following definition.

Definition 4.2. A pair $(\mathrm{v}^\#,\mathrm{v}^{1,\#})\in\mathcal{C}^0_\#([0,1];H^1(\Omega))\times\mathcal{C}^0_\#([0,1];L^2(\Omega;\mathcal{X}^1_\#(Y))$ with $[\mathrm{v}^{1,\#}_t]\in L^2_\#\big(0,1;L^2(\Omega\times\Gamma)\big)$ is a time-periodic weak solution (with period 1) of

(4.11)–(4.16) if

$$\begin{aligned}\int_{\boldsymbol{R}}\int_{\Omega}\int_{Y} \sigma\left(\nabla \mathrm{v}^{\#}(x,t)+\nabla_y \mathrm{v}^{1,\#}(x,y,t)\right)\left(\nabla\phi(x,t)+\nabla_y\Phi(x,y,t)\right)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ +\int_{\boldsymbol{R}}\int_{\Omega}\int_{\Gamma} f([\mathrm{v}^{1,\#}(x,y,t)])[\Phi(x,y,t)]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t \\ -\alpha\int_{\boldsymbol{R}}\int_{\Omega}\int_{\Gamma}[\mathrm{v}^{1,\#}(x,y,t)]\frac{\partial}{\partial t}[\Phi(x,y,t)]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t=0 \qquad (4.17)\end{aligned}$$

for every $(\phi,\Phi)\in\mathcal{C}^0_c(\boldsymbol{R};H^1_0(\Omega))\times\mathcal{C}^0_c(\boldsymbol{R};L^2(\Omega;\mathcal{X}^1_{\#}(Y))$, $[\Phi_t]\in L^2\big(\boldsymbol{R};L^2(\Omega\times\Gamma)\big)$ and $\mathrm{v}^{1,\#}$ has zero mean value in Y for a.e. $(x,t)\in\Omega\times\boldsymbol{R}$ and $\mathrm{v}^{\#}$ satisfies (4.16) in the trace sense (see [14, Definition 5.7]).

Remark 4.3. We note that by a standard approximation of periodic testing functions with functions compactly supported in a period, the weak formulation (4.17) can be equivalently rewritten as

$$\begin{aligned}\int_0^1\int_{\Omega}\int_{Y} \sigma\left(\nabla \mathrm{v}^{\#}(x,t)+\nabla_y \mathrm{v}^{1,\#}(x,y,t)\right)\left(\nabla\phi(x,t)+\nabla_y\Phi(x,y,t)\right)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ +\int_0^1\int_{\Omega}\int_{\Gamma} f([\mathrm{v}^{1,\#}(x,y,t)])[\Phi(x,y,t)]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t \\ -\alpha\int_0^1\int_{\Omega}\int_{\Gamma}[\mathrm{v}^{1,\#}(x,y,t)]\frac{\partial}{\partial t}[\Phi(x,y,t)]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t=0\end{aligned}$$

for every $(\phi,\Phi)\in\mathcal{C}^0_{\#}([0,1];H^1_0(\Omega))\times\mathcal{C}^0_{\#}([0,1];L^2(\Omega;\mathcal{X}^1_{\#}(Y))$, $[\Phi_t]\in L^2_{\#}\big(0,1;L^2(\Omega\times\Gamma)\big)$. Hence, when it is more convenient, we replace compactly supported testing functions with 1-periodic testing functions. □

Proposition 4.4. *Under the assumptions* (2.7)–(2.10) *and* (2.12)*, problem* (4.11)–(4.16) *admits a 1-periodic in time solution.*

Proof. For $\delta > 0$, let us denote by $f_\delta(s) := f(s) + \delta s$, for every $s \in \boldsymbol{R}$, and consider the problem

$$-\operatorname{div}\left(\sigma_0 \nabla u_\delta^{\#} + \int_Y \sigma \nabla_y u_\delta^{1,\#}\,\mathrm{d}y\right) = 0\,, \qquad \text{in } \Omega \times \boldsymbol{R}\,; \tag{4.18}$$

$$-\operatorname{div}_y(\sigma \nabla u_\delta^{\#} + \sigma \nabla_y u_\delta^{1,\#}) = 0\,, \qquad \text{in } \Omega \times (E_1 \cup E_2) \times \boldsymbol{R}\,; \tag{4.19}$$

$$[\sigma(\nabla u_\delta^{\#} + \nabla_y u_\delta^{1,\#}) \cdot \nu] = 0, \qquad \text{on } \Omega \times \Gamma \times \boldsymbol{R}\,; \tag{4.20}$$

$$\alpha \frac{\partial}{\partial t}[u_\delta^{1,\#}] + f_\delta\left([u_\delta^{1,\#}]\right) = \sigma(\nabla u_\delta^{\#} + \nabla_y u_\delta^{1,\#}) \cdot \nu\,, \quad \text{on } \Omega \times \Gamma \times \boldsymbol{R}\,; \tag{4.21}$$

$$[u_\delta^{1,\#}](x, y, \cdot) \text{ is 1-periodic}, \qquad \text{on } \Omega \times \Gamma\,; \tag{4.22}$$

$$u_\delta^{\#}(x, t) = \Psi(x, t)\,, \qquad \text{on } \partial\Omega \times \boldsymbol{R}\,; \tag{4.23}$$

where $u_\delta^{1,\#}$ has zero mean value on Y for a.e. $(x, t) \in \Omega \times \boldsymbol{R}$.
Since f_δ has a strictly positive derivative on $\boldsymbol{R}$, by the results proved in [14, Section 5], a unique periodic solution $(u_\delta^{\#}, u_\delta^{1,\#})$ of problem (4.18)– (4.23) does exist, i.e. $(u_\delta^{\#}, u_\delta^{1,\#})$ satisfies

$$\begin{aligned}
&\int_0^1 \int_\Omega \int_Y \sigma \left(\nabla u_\delta^{\#}(x,t) + \nabla_y u_\delta^{1,\#}(x,y,t)\right) (\nabla \phi(x,t) + \nabla_y \Phi(x,y,t))\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_0^1 \int_\Omega \int_\Gamma f_\delta([u_\delta^{1,\#}(x,y,t)])[\Phi(x,y,t)]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad - \alpha \int_0^1 \int_\Omega \int_\Gamma [u_\delta^{1,\#}(x,y,t)] \frac{\partial}{\partial t}[\Phi(x,y,t)]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t = 0\,,
\end{aligned} \tag{4.24}$$

for every $(\phi, \Phi) \in \mathcal{C}^0_{\#}([0,1]; H^1_0(\Omega)) \times \mathcal{C}^0_{\#}([0,1]; L^2(\Omega; \mathcal{X}^1_{\#}(Y))$, $[\Phi_t] \in L^2_{\#}\big(0,1; L^2(\Omega \times \Gamma)\big)$ (recall Remark 4.3). Moreover $u_\delta^{1,\#}$ has zero mean value in Y for a.e. $(x,t) \in \Omega \times \boldsymbol{R}$ and $u_\delta^{\#}$ satisfies (4.23) in the trace sense. By (4.24) we get that $(u_\delta^{\#}, u_\delta^{1,\#})$ satisfies an energy estimate, easily obtained replacing (ϕ, Φ) with $(u_\delta^{\#} - \Psi, u_\delta^{1,\#})$, which implies

$$\int_0^1\int_\Omega\int_Y \frac{\sigma}{2}|\nabla u_\delta^\# + \nabla_y u_\delta^{1,\#}|^2 \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t + \int_0^1\int_\Omega\int_\Gamma f_\delta([u_\delta^{1,\#}])[u_\delta^{1,\#}]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^1\int_\Omega\int_Y \frac{\sigma}{2}|\nabla\Psi|^2\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\,, \quad (4.25)$$

where we take into account

$$\int_0^1 [u_{\delta,t}^{1,\#}][u_\delta^{1,\#}]\,\mathrm{d}t = \frac{1}{2}\int_0^1 \frac{\partial}{\partial t}[u_\delta^{1,\#}]^2\,\mathrm{d}t = 0\,, \quad (4.26)$$

which is a consequence of the periodicity of $u_\delta^{1,\#}$.
From (4.25), working as done in (3.11)–(3.12) of Section 3 and taking into account (2.11) we get

$$\int_0^1\int_\Omega\int_Y \sigma|\nabla u_\delta^\# + \nabla_y u_\delta^{1,\#}|^2\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t + \int_0^1\int_\Omega\int_\Gamma \lambda_1[u_\delta^{1,\#}]^2\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t \le \gamma\,, \quad (4.27)$$

where γ is a constant depending on $\lambda_1, \lambda_2, |\Gamma|$ and the H^1-norm of Ψ.
Replacing (ϕ,Φ) in (4.24) with $(u_{\delta,t}^\# - \Psi_t, u_{\delta,t}^{1,\#})$, by (4.27), (2.11) and taking into account the fact that

$$\int_0^1 (\nabla u_\delta^\# + \nabla_y u_\delta^{1,\#})(\nabla u_{\delta,t}^\# + \nabla_y u_{\delta,t}^{1,\#})\,\mathrm{d}t = \frac{1}{2}\int_0^1 \frac{\partial}{\partial t}|\nabla u_\delta^\# + \nabla_y u_\delta^{1,\#}|^2\,\mathrm{d}t = 0$$

and, denoting by F_δ a primitive of f_δ,

$$\int_0^1\int_\Omega\int_\Gamma f_\delta([u_\delta^{1,\#}])[u_{\delta,t}^{1,\#}]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t = \int_0^1\int_\Omega\int_\Gamma \frac{\partial F_\delta([u_\delta^{1,\#}])}{\partial t}\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t = 0\,,$$

because of the periodicity, we get

$$\alpha\int_0^1\int_\Omega\int_\Gamma [u_{\delta,t}^{1,\#}]^2\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t \le \gamma\,, \quad (4.28)$$

where, again γ depends on $\lambda_1, \lambda_2, |\Gamma|$ and the H^1-norms of Ψ and Ψ_t. From (4.27), we obtain

$$\int_0^1 \int_\Omega |\nabla u_\delta^{\#}|^2 \, \mathrm{d}x \, \mathrm{d}t \le \gamma \,, \tag{4.29}$$

$$\int_0^1 \int_\Omega \int_Y |\nabla_y u_\delta^{1,\#}|^2 \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t \le \gamma \,. \tag{4.30}$$

Indeed,

$$\begin{aligned}
&\int_0^1 \int_\Omega \int_Y |\nabla_y u_\delta^{1,\#}(x,y,t)|^2 \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t + \int_0^1 \int_\Omega |\nabla u_\delta^{\#}|^2 \, \mathrm{d}x \, \mathrm{d}t \\
\le &\gamma - 2 \int_0^1 \int_\Omega \int_Y \nabla_y u_\delta^{1,\#}(x,y,t) \nabla u_\delta^{\#}(x,t) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t \\
= &\gamma - 2 \int_0^1 \int_\Omega \nabla u_\delta^{\#} \left(\int_Y \nabla_y u_\delta^{1,\#}(x,y,t) \, \mathrm{d}y \right) \mathrm{d}x \, \mathrm{d}t \\
\le &\gamma + 2 \int_0^1 \int_\Omega |\nabla u_\delta^{\#}| \left(\int_\Gamma |[u_\delta^{1,\#}(x,y,t)]| \, \mathrm{d}\sigma \right) \mathrm{d}x \, \mathrm{d}t \\
\le &\gamma + \frac{1}{2} \int_0^1 \int_\Omega |\nabla u_\delta^{\#}|^2 \, \mathrm{d}x \, \mathrm{d}t + 8|\Gamma| \int_0^1 \int_\Omega \int_\Gamma [u_\delta^{1,\#}(x,y,t)]^2 \, \mathrm{d}\sigma \, \mathrm{d}x \, \mathrm{d}t \\
\le &\gamma + \frac{1}{2} \int_0^1 \int_\Omega |\nabla u_\delta^{\#}(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t + \gamma \,.
\end{aligned} \tag{4.31}$$

In order to be able to pass to the limit $\delta \to 0$ we need a formulation with vanishing boundary data. To this purpose we set $v_\delta^\# = u_\delta^\# - \Psi$; clearly $v_\delta^\#$ satisfies

$$-\operatorname{div}\left(\sigma_0 \nabla v_\delta^\# + \int_Y \sigma \nabla_y u_\delta^{1,\#}\,\mathrm{d}y\right) = \operatorname{div}(\sigma_0 \nabla \Psi)\,, \qquad \text{in } \Omega \times \boldsymbol{R}; \tag{4.32}$$

$$-\operatorname{div}_y(\sigma \nabla v_\delta^\# + \sigma \nabla_y u_\delta^{1,\#}) = 0\,, \qquad \text{in } \Omega \times (E_1 \cup E_2) \times \boldsymbol{R}; \tag{4.33}$$

$$[\sigma(\nabla v_\delta^\# + \nabla_y u_\delta^{1,\#})\cdot\nu] = -[\sigma \nabla \Psi \cdot \nu]\,, \qquad \text{on } \Omega \times \Gamma \times \boldsymbol{R}; \tag{4.34}$$

$$\alpha \frac{\partial}{\partial t}[u_\delta^{1,\#}] + f_\delta\left([u_\delta^{1,\#}]\right) = \sigma_2(\nabla v_\delta^\# + \nabla_y u_\delta^{1,\#})\cdot\nu + \sigma_2 \nabla\Psi\cdot\nu\,, \quad \text{on } \Omega \times \Gamma \times \boldsymbol{R}; \tag{4.35}$$

$$[u_\delta^{1,\#}](x,y,\cdot) \text{ is 1-periodic}, \qquad \text{on } \Omega \times \Gamma; \tag{4.36}$$

$$v_\delta^\#(x,t) = 0\,, \qquad \text{on } \partial\Omega \times \boldsymbol{R}, \tag{4.37}$$

or, in the weak form,

$$\begin{aligned}
&\int_0^1 \int_\Omega \int_Y \sigma \left(\nabla v_\delta^\#(x,t) + \nabla_y u_\delta^{1,\#}(x,y,t)\right)(\nabla\phi(x,t) + \nabla_y \Phi(x,y,t))\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_0^1 \int_\Omega \int_\Gamma f_\delta([u_\delta^{1,\#}(x,y,t)])[\Phi(x,y,t)]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \alpha \int_0^1 \int_\Omega \int_\Gamma [u_\delta^{1,\#}(x,y,t)] \frac{\partial}{\partial t}[\Phi(x,y,t)]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^1 \int_\Omega \sigma_0 \nabla\Psi(x,t)\nabla\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^1 \int_\Omega \int_\Gamma [\sigma\nabla\Psi(x,t)\cdot\nu]\Phi^{(1)}(x,y,t)\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_0^1 \int_\Omega \int_\Gamma \sigma_2 \nabla\Psi(x,t)\cdot\nu[\Phi(x,y,t)]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned} \tag{4.38}$$

for (ϕ, Φ) as in Remark 4.3. At this point, (4.27)–(4.30) allow us to pass to the limit with respect to δ in the weak formulation (4.38), thus proving that there exists a periodic (in time) pair of functions $(v^\#, u^{1,\#}) \in L^2_\#\big(0,1; H^1_0(\Omega)\big) \times L^2_\#\big(0,1; L^2(\Omega; \mathcal{X}^1_\#(Y))\big)$

such that $u^{1,\#}$ has zero mean value on Y, for a.e. $(x,t) \in \Omega \times \boldsymbol{R}$, and $(v^{\#}, u^{1,\#})$ satisfies the homogenized problem

$$\int_0^1 \int_{\Omega\times Y} \sigma(\nabla v^{\#} + \nabla_y u^{1,\#}) \cdot \nabla\phi \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}t + \int_0^1 \int_{\Omega\times Y} \sigma(\nabla v^{\#} + \nabla_y u^{1,\#}) \cdot \nabla_y \Phi \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}t$$

$$+ \int_0^1 \int_\Omega \int_\Gamma \mu[\Phi] \,\mathrm{d}\sigma \,\mathrm{d}x \,\mathrm{d}t - \alpha \int_0^1 \int_\Omega \int_\Gamma [u^{1,\#}] \frac{\partial}{\partial t}[\Phi] \,\mathrm{d}\sigma \,\mathrm{d}x \,\mathrm{d}t$$

$$= - \int_0^1 \int_\Omega \sigma_0 \nabla\Psi \nabla\phi \,\mathrm{d}x \,\mathrm{d}t + \int_0^1 \int_\Omega \int_\Gamma [\sigma\nabla\Psi \cdot \nu]\Phi^{(1)} \,\mathrm{d}\sigma \,\mathrm{d}x \,\mathrm{d}t$$

$$+ \int_0^1 \int_\Omega \int_\Gamma \sigma_2 \nabla\Psi \cdot \nu[\Phi] \,\mathrm{d}\sigma \,\mathrm{d}x \,\mathrm{d}t \quad (4.39)$$

for every test function $(\phi, \Phi) \in \mathcal{C}^0_{\#}([0,1]; H^1_0(\Omega)) \times \mathcal{C}^0_{\#}([0,1]; L^2(\Omega; \mathcal{X}^1_{\#}(Y))$ with $[\Phi_t] \in L^2_{\#}\big(0,1; L^2(\Omega \times \Gamma)\big)$, where we have taken into account that (2.7) and (4.27) imply

$$f_\delta([u^{1,\#}_\delta]) = f([u^{1,\#}_\delta]) + \delta[u^{1,\#}_\delta] \rightharpoonup \mu \,, \qquad \text{weakly in } L^2\big((0,T) \times \Omega \times \Gamma^\varepsilon\big), \text{ when } \delta \to 0.$$

It remains to identify μ. To this purpose, we follow the Minty monotone operators method. Let us consider a sequence of 1-periodic in time test functions $\psi_k(x,y,t) = \phi^k_0(x,t) + \phi^k_1(x,y,t) + \lambda\phi_2(x,y,t)$, with $\phi^k_0 \in \mathcal{C}^\infty(\Omega \times \boldsymbol{R})$, $\phi^k_1 \in \mathcal{C}^\infty\big(\Omega \times \boldsymbol{R}; \mathfrak{C}^\infty_{\#}(Y)\big)$, $\phi_2 \in \mathcal{C}^1_c\big(\Omega \times (0,1); \mathfrak{C}^1_{\#}(Y)\big)$, with $\phi^k_0(\cdot,t)$ vanishing on $\partial\Omega$ for $t \in \boldsymbol{R}$, $\phi^k_0 \to v^{\#}$ strongly in $L^2_{loc}\big(\boldsymbol{R}; H^1_0(\Omega)\big)$, $\phi^k_1 \to u^{1,\#}$ strongly in $L^2_{loc}(\boldsymbol{R}; L^2(\Omega; \mathcal{X}^1_{\#}(Y)))$, $[\phi^k_1] \to [u^{1,\#}]$ and $[\phi^k_{1,t}] \to [u^{1,\#}_t]$ strongly in $L^2((0,1) \times \Omega \times \Gamma)$, i.e.

$$\int_0^1 \int_\Omega \|\phi^k_1(x,\cdot,t) - u^{1,\#}(x,\cdot,t)\|^2_{H^1(E_i)} \,\mathrm{d}x \,\mathrm{d}t$$

$$+ \int_0^1 \int_\Omega \|[\phi^k_1(x,\cdot,t)] - [u^{1,\#}(x,\cdot,t)]\|^2_{L^2(\Gamma)} \,\mathrm{d}x \,\mathrm{d}t$$

$$+ \int_0^1 \int_\Omega \|[\phi^k_{1,t}(x,\cdot,t)] - [u^{1,\#}_t(x,\cdot,t)]\|^2_{L^2(\Gamma)} \,\mathrm{d}x \,\mathrm{d}t \to 0 \,, \qquad \text{for } k \to +\infty,\ i = 1,2.$$

Clearly, ϕ^k_0 can be constructed by means of standard convolutions with regular kernels; instead, in order to construct ϕ^k_1 we proceed as follows. Taking into account that, passing to the limit for $\delta \to 0$ in (4.28), we have

$$[u^{1,\#}_t] \in L^2((0,1) \times \Omega \times \Gamma) \,, \qquad (4.40)$$

by standard arguments we can approximate the jump $[u^{1,\#}]$ with a sequence of 1-periodic in time functions $\widetilde{\phi}^k_1 \in \mathcal{C}^\infty(\Omega \times \Gamma \times \boldsymbol{R})$ such that $\widetilde{\phi}^k_1 \to [u^{1,\#}]$ strongly in

$L^2\big((0,1)\times\Omega;H^{1/2}(\Gamma)\big)$ and $\widetilde{\phi}^k_{1,t}\to[u^{1,\#}_t]$ strongly in $L^2\big((0,1)\times\Omega\times\Gamma\big)$. Now, define ϕ^k_1 as the (1-periodic in time) solution of the problem

$$-\operatorname{div}_y\big(\sigma(\nabla\phi^k_0+\nabla_y\phi^k_1)\big)=0\,,\qquad \text{in } (E_1\cup E_2)\times\Omega\times\boldsymbol{R}; \tag{4.41}$$

$$[\sigma(\nabla\phi^k_0+\nabla_y\phi^k_1)\cdot\nu]=-[\sigma\nabla\Psi\cdot\nu]\,,\qquad \text{on } \Gamma\times\Omega\times\boldsymbol{R}; \tag{4.42}$$

$$[\phi^k_1]=\widetilde{\phi}^k_1\,,\qquad \text{on } \Gamma\times\Omega\times\boldsymbol{R}; \tag{4.43}$$

and $\phi^k_1(x,\cdot,t)$ is Y-periodic with zero mean value on Y for $(x,t)\in\Omega\times\boldsymbol{R}$. By Lemma 7.3 in [7], it follows that $\phi^k_1\in\mathcal{C}^\infty\big(\Omega\times[0,1];\mathfrak{C}^\infty_\#(Y)\big)$. Here, for the sake of simplicity, we work as if Ψ has enough regularity, otherwise we proceed with a standard regularization procedure also on Ψ. Moreover, by [8, Lemma 5] applied to $\phi^k_1-u^{1,\#}$ with $P=\operatorname{div}_y\big(\sigma(\nabla\phi^k_0-\nabla v^\#)\big)=0$ in $E_1\cup E_2$, $Q=[\sigma(\nabla\phi^k_0-\nabla v^\#)]$, and $S=\widetilde{\phi}^k_1-[u^{1,\#}]$, we obtain

$$||\phi^k_1-u^{1,\#}||_{L^2((0,1)\times\Omega;\mathcal{X}^1_\#(Y))}\le\gamma(||\widetilde{\phi}^k_1-[u^{1,\#}]||_{L^2((0,1)\times\Omega;H^{1/2}(\Gamma))}+\|\nabla\phi^k_0-\nabla v^\#\|_{L^2((0,1)\times\Omega)})\,. \tag{4.44}$$

Since the right-hand side of (4.44) tends to zero for $k\to+\infty$ we obtain the desired approximation.
Taking only into account the monotonicity assumption on f, the periodicity in time of ϕ^k_0 and ϕ^k_1 and Remark 4.3, we calculate

$$\begin{aligned}
&\int_0^1\int_{\Omega\times Y}\sigma(\nabla v^\#_\delta+\nabla_y u^{1,\#}_\delta-\nabla\phi^k_0-\nabla_y\phi^k_1-\lambda\nabla_y\phi_2)\cdot(\nabla v^\#_\delta-\nabla\phi^k_0)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^1\int_{\Omega\times Y}\sigma(\nabla v^\#_\delta+\nabla_y u^{1,\#}_\delta-\nabla\phi^k_0-\nabla_y\phi^k_1-\lambda\nabla_y\phi_2)\cdot(\nabla_y u^{1,\#}_\delta-\nabla_y\phi^k_1-\lambda\nabla_y\phi_2)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\alpha\int_0^1\int_{\Omega\times\Gamma}\frac{\partial}{\partial t}\Big([u^{1,\#}_\delta]-[\phi^k_1+\lambda\phi_2]\Big)\Big([u^{1,\#}_\delta]-[\phi^k_1+\lambda\phi_2]\Big)\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_0^1\int_{\Omega\times\Gamma}\Big(f_\delta([u^{1,\#}_\delta])-f_\delta\big([\phi^k_1+\lambda\phi_2]\big)\Big)\Big([u^{1,\#}_\delta]-[\phi^k_1+\lambda\phi_2]\Big)\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad=\int_0^1\int_{\Omega\times Y}\sigma|\nabla v^\#_\delta+\nabla_y u^{1,\#}_\delta-\nabla\phi^k_0-\nabla_y\phi^k_1-\lambda\nabla_y\phi_2|^2\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^1\int_{\Omega\times\Gamma}\Big(f_\delta([u^{1,\#}_\delta])-f_\delta\big([\phi^k_1+\lambda\phi_2]\big)\Big)\Big([u^{1,\#}_\delta]-[\phi^k_1+\lambda\phi_2]\Big)\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t\ge0\,,
\end{aligned}\tag{4.45}$$

where we have taken into account that the time-periodicity of $u_\delta^{1,\#}$, ϕ_1^k and ϕ_2 implies

$$\begin{aligned}
\alpha \int_0^1 \int_{\Omega\times\Gamma} & \frac{\partial}{\partial t}\Big([u_\delta^{1,\#}] - [\phi_1^k + \lambda\phi_2]\Big)\Big([u_\delta^{1,\#}] - [\phi_1^k + \lambda\phi_2]\Big) \, \mathrm{d}\sigma \, \mathrm{d}x \, \mathrm{d}t \\
&= \frac{\alpha}{2} \int_0^1 \int_{\Omega\times\Gamma} \frac{\partial}{\partial t}\Big([u_\delta^{1,\#}] - [\phi_1^k + \lambda\phi_2]\Big)^2 \, \mathrm{d}\sigma \, \mathrm{d}x \, \mathrm{d}t \\
&= \frac{\alpha}{2} \int_{\Omega\times\Gamma} \Big([u_\delta^{1,\#}(x,y,1)] - [\phi_1^k(x,y,1)]\Big)^2 \, \mathrm{d}\sigma \, \mathrm{d}x \\
&\quad - \frac{\alpha}{2} \int_{\Omega\times\Gamma} \Big([u_\delta^{1,\#}(x,y,0)] - [\phi_1^k(x,y,0)]\Big)^2 \, \mathrm{d}\sigma \, \mathrm{d}x = 0 \, .
\end{aligned}$$

Taking the function $(v_\delta^\# - \phi_0^k, u_\delta^{1,\#} - \phi_1^k - \lambda\phi_2)$ as a test function (φ, Φ) in (4.38), inequality (4.45) can be rewritten as

$$\begin{aligned}
- \int_0^1 \int_{\Omega\times Y} & \sigma(\nabla\phi_0^k + \nabla_y\phi_1^k + \lambda\nabla_y\phi_2)\cdot(\nabla v_\delta^\# - \nabla\phi_0^k) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t \\
& - \int_0^1 \int_{\Omega\times Y} \sigma(\nabla\phi_0^k + \nabla_y\phi_1^k + \lambda\nabla_y\phi_2)\cdot(\nabla_y u_\delta^{1,\#} - \nabla_y\phi_1^k - \lambda\nabla_y\phi_2) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t \\
& - \alpha \int_0^1 \int_{\Omega\times\Gamma} \frac{\partial}{\partial t}[\phi_1^k + \lambda\phi_2]\Big([u_\delta^{1,\#}] - [\phi_1^k + \lambda\phi_2]\Big) \, \mathrm{d}\sigma \, \mathrm{d}x \, \mathrm{d}t \\
& - \int_0^1 \int_{\Omega\times\Gamma} f_\delta\left([\phi_1^k + \lambda\phi_2]\right)\Big([u_\delta^{1,\#}] - [\phi_1^k + \lambda\phi_2]\Big) \, \mathrm{d}\sigma \, \mathrm{d}x \, \mathrm{d}t \\
\geq \int_0^1 \int_\Omega & \sigma_0 \nabla\Psi\cdot(\nabla v_\delta^\# - \nabla\phi_0^k) \, \mathrm{d}x \, \mathrm{d}t - \int_0^1 \int_\Omega \int_\Gamma [\sigma\nabla\Psi\cdot\nu](u_\delta^{1,\#} - \phi_1^k - \lambda\phi_2)^{(1)} \, \mathrm{d}\sigma \, \mathrm{d}x \, \mathrm{d}t \\
& - \int_0^1 \int_\Omega \int_\Gamma \sigma_2 \nabla\Psi\cdot\nu([u_\delta^{1,\#}] - [\phi_1^k + \lambda\phi_2]) \, \mathrm{d}\sigma \, \mathrm{d}x \, \mathrm{d}t \, . \qquad (4.46)
\end{aligned}$$

Hence, passing to the limit as $\delta \to 0$ and using (4.28), it follows

$$
\begin{aligned}
&-\int_0^1 \int_{\Omega\times Y} \sigma(\nabla \phi_0^k + \nabla_y \phi_1^k + \lambda \nabla_y \phi_2) \cdot (\nabla v^\# - \nabla \phi_0^k) \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}t \\
&\quad -\int_0^1 \int_{\Omega\times Y} \sigma(\nabla \phi_0^k + \nabla_y \phi_1^k + \lambda \nabla_y \phi_2) \cdot (\nabla_y u^{1,\#} - \nabla_y \phi_1^k - \lambda \nabla_y \phi_2) \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad - \alpha \int_0^1 \int_{\Omega\times\Gamma} \frac{\partial}{\partial t}[\phi_1^k + \lambda \phi_2] \left([u^{1,\#}] - [\phi_1^k + \lambda\phi_2] \right) \,\mathrm{d}\sigma \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad - \int_0^1 \int_{\Omega\times\Gamma} f\left([\phi_1^k + \lambda\phi_2] \right) \left([u^{1,\#}] - [\phi_1^k + \lambda\phi_2] \right) \,\mathrm{d}\sigma \,\mathrm{d}x \,\mathrm{d}t \\
&\geq \int_0^1 \int_\Omega \sigma_0 \nabla\Psi \cdot (\nabla v^\# - \nabla\phi_0^k) \,\mathrm{d}x \,\mathrm{d}t - \int_0^1 \int_\Omega \int_\Gamma [\sigma \nabla \Psi \cdot \nu] (u^{1,\#} - \phi_1^k - \lambda \phi_2)^{(1)} \,\mathrm{d}\sigma \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad - \int_0^1 \int_\Omega \int_\Gamma \sigma_2 \nabla\Psi \cdot \nu ([u^{1,\#}] - [\phi_1^k + \lambda\phi_2]) \,\mathrm{d}\sigma \,\mathrm{d}x \,\mathrm{d}t \,. \qquad (4.47)
\end{aligned}
$$

Now, letting $k \to +\infty$, we obtain

$$
\begin{aligned}
&\int_0^1 \int_\Omega \int_Y \sigma(\nabla v^\# + \nabla_y u^{1,\#} + \lambda \nabla_y \phi_2) \cdot \lambda \nabla_y \phi_2 \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}t + \\
&\alpha \int_0^1 \int_\Omega \int_\Gamma \frac{\partial}{\partial t}[u^{1,\#} + \lambda\phi_2] \lambda [\phi_2] \,\mathrm{d}\sigma \,\mathrm{d}x \,\mathrm{d}t + \int_0^1 \int_\Omega \int_\Gamma f\left([u^{1,\#} + \lambda \phi_2] \right) \lambda [\phi_2] \,\mathrm{d}\sigma \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad \geq \int_0^1 \int_\Omega \int_\Gamma [\sigma \nabla\Psi \cdot \nu] \lambda \phi_2^{(1)} \,\mathrm{d}\sigma \,\mathrm{d}x \,\mathrm{d}t + \int_0^1 \int_\Omega \int_\Gamma \sigma_2 \nabla \Psi \cdot \nu [\lambda \phi_2] \,\mathrm{d}\sigma \,\mathrm{d}x \,\mathrm{d}t \,. \qquad (4.48)
\end{aligned}
$$

Taking into account (4.39) with $\phi \equiv 0$ and $\Phi = \phi_2$, (4.48) becomes

$$
\begin{aligned}
&\lambda^2 \int_0^1 \int_\Omega \int_Y \sigma \nabla_y \phi_2 \cdot \nabla_y \phi_2 \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}t + \alpha\lambda^2 \int_0^1 \int_\Omega \int_\Gamma \frac{\partial}{\partial t}[\phi_2][\phi_2] \,\mathrm{d}\sigma \,\mathrm{d}x \,\mathrm{d}t \\
&\quad - \lambda \int_0^1 \int_\Omega \int_\Gamma \mu [\phi_2] \,\mathrm{d}\sigma \,\mathrm{d}x \,\mathrm{d}t + \lambda \int_0^1 \int_\Omega \int_\Gamma f\left([u^{1,\#} + \lambda \phi_2] \right) [\phi_2] \,\mathrm{d}\sigma \,\mathrm{d}x \,\mathrm{d}t \geq 0 \,. \qquad (4.49)
\end{aligned}
$$

Assuming firstly that $\lambda > 0$ and then $\lambda < 0$, dividing by λ the previous equation and then letting $\lambda \to 0$, we obtain

$$\int_0^1 \int_\Omega \int_\Gamma \mu[\phi_2] \, \mathrm{d}\sigma \, \mathrm{d}x \, \mathrm{d}t = \int_0^1 \int_\Omega \int_\Gamma f\left([u^{1,\#}]\right)[\phi_2] \, \mathrm{d}\sigma \, \mathrm{d}x \, \mathrm{d}t \,,$$

which gives

$$\mu = f\left([u^{1,\#}]\right) \,. \tag{4.50}$$

By (4.39) and (4.50), setting $v^\# = u^\# + \Psi$ and taking into account Remark 4.3, we obtain exactly the weak formulation of problem (4.11)–(4.16). □

Remark 4.5. Note that (4.28) is uniform with respect to δ. Moreover, we can obtain also estimates for $\nabla u^\#_{\delta,t}$ and $\nabla_y u^{1,\#}_{\delta,t}$ uniformly in δ. Indeed, differentiating formally with respect to t problem (4.18)–(4.23), multiplying equation (4.18) (differentiated with respect to t) by $\left((u^\#_{\delta,t} - \Psi_t), u^{1,\#}_{\delta,t}\right)$, and finally integrating by parts, we obtain, exploiting also the periodicity in time,

$$\int_0^1 \int_\Omega \int_Y \sigma |\nabla u^\#_{\delta,t} + \nabla_y u^{1,\#}_{\delta,t}|^2 \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t \le \gamma \tag{4.51}$$

where we used assumptions (2.7), (2.12) and inequality (4.28). Now, proceeding as in the proof of (4.29) and (4.30), we obtain

$$\int_0^1 \int_\Omega |\nabla u^\#_{\delta,t}|^2 \, \mathrm{d}x \, \mathrm{d}t \le \gamma \,, \tag{4.52}$$

$$\int_0^1 \int_\Omega \int_Y |\nabla_y u^{1,\#}_{\delta,t}|^2 \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t \le \gamma \,. \tag{4.53}$$

Therefore, passing to the limit for $\delta \to 0^+$, in (4.28), (4.52) and (4.53), we obtain that the same estimates hold for $(u^\#, u^{1,\#})$.
This implies that $(u^\#, u^{1,\#})$ belongs to $\mathcal{C}^0_\#([0,1]; H^1(\Omega)) \times \mathcal{C}^0_\#([0,1]; L^2(\Omega; \mathcal{X}^1_\#(Y))$. □

It remains to prove that any solution (u, u^1) of the homogenized problem converges to $(u^\#, u^{1,\#})$ as $t \to \infty$. This is the purpose of the next theorem.

Theorem 4.6. *Let* $(u, u^1) \in L^2\big(0,T); H^1(\Omega)\big) \times L^2\big(\Omega \times (0,T); \mathcal{X}^1_\#(Y)\big)$ *be the solution of problem* (4.2)–(4.7). *Then, for* $t \to +\infty$, $(u, u^1) \to (u^\#, u^{1,\#})$ *in the following sense:*

$$\lim_{t \to +\infty} \|u(\cdot,t) - u^\#(\cdot,t)\|_{H^1(\Omega)} = 0\,; \tag{4.54}$$

$$\lim_{t \to +\infty} \left[\|u^1(\cdot,\cdot,t) - u^{1,\#}(\cdot,\cdot,t)\|_{L^2(\Omega\times Y)} + \|\nabla_y u^1(\cdot,\cdot,t) - \nabla_y u^{1,\#}(\cdot,\cdot,t)\|_{L^2(\Omega\times Y)}\right] = 0\,; \tag{4.55}$$

$$\lim_{t \to +\infty} \|[u^1](\cdot,\cdot,t) - [u^{1,\#}](\cdot,\cdot,t)\|_{L^2(\Omega\times\Gamma)} = 0\,. \tag{4.56}$$

Proof. Firstly we recall that, by [14, Lemma 5.2] which holds even in the present case, $(u,u^1)\in\mathcal{C}^0\big((0,T];H^1(\Omega)\big)\times\mathcal{C}^0\big((0,T];L^2(\Omega;\mathcal{X}^1_\#(Y))\big)$ and $[u^1]\in\mathcal{C}^0\big((0,T];L^2(\Omega\times\Gamma)\big)$.
As usual, let $(r,r^1):=(u^\#-u,u^{1,\#}-u^1)$, so that the pair (r,r^1) satisfies:

$$\int_0^t\int_\Omega\int_Y \sigma\left(\nabla r+\nabla_y r^1\right)\left(\nabla\phi+\nabla_y\Phi\right)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^t\int_\Omega\int_\Gamma \frac{f([u^{1,\#}])-f([u^1])}{[u^{1,\#}]-[u^1]}[r^1][\Phi]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t+\alpha\int_0^t\int_\Omega\int_\Gamma [r^1_t][\Phi]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t=0\,,\quad\forall t\in(0,T)\,,\tag{4.57}$$

where $r=0$ on $\partial\Omega\times[0,T]$ in the trace sense, r^1 is periodic in Y and has zero mean value in Y for almost every $(x,t)\in\Omega\times(0,T)$. Here ϕ is any regular function depending on (x,t), with compact support in Ω and Φ is a any function depending on (x,y,t) which jumps across Γ, is zero when $t=T$ and is regular elsewhere. Differentiating (4.57) with respect to t, we get

$$\int_\Omega\int_Y \sigma\left(\nabla r+\nabla_y r^1\right)\left(\nabla\phi+\nabla_y\Phi\right)\,\mathrm{d}y\,\mathrm{d}x+\int_\Omega\int_\Gamma \frac{f([u^{1,\#}])-f([u^1])}{[u^{1,\#}]-[u^1]}[r^1][\Phi]\,\mathrm{d}\sigma\,\mathrm{d}x$$
$$+\alpha\int_\Omega\int_\Gamma [r^1_t][\Phi]\,\mathrm{d}\sigma\,\mathrm{d}x=0\,.\tag{4.58}$$

Replacing (ϕ,Φ) with (r,r^1) in (4.58), we get

$$\int_\Omega\int_Y \sigma|\nabla r+\nabla_y r^1|^2\,\mathrm{d}y\,\mathrm{d}x+\int_\Omega\int_\Gamma \frac{f([u^{1,\#}])-f([u^1])}{[u^{1,\#}]-[u^1]}[r^1]^2\,\mathrm{d}\sigma\,\mathrm{d}x$$
$$+\alpha\int_\Omega\int_\Gamma [r^1_t][r^1]\,\mathrm{d}\sigma\,\mathrm{d}x=0\,.\tag{4.59}$$

As in Section 3, equation (4.59) implies that the function $t\mapsto\alpha\int_\Omega\int_\Gamma[r^1(x,t)]^2\,\mathrm{d}\sigma\,\mathrm{d}x$ is a positive, decreasing function of t, hence it tends to a limit value $\overline{r}^1\geq 0$ as $t\to+\infty$. The value $\overline{r}^1$ must be zero otherwise $\alpha\int_\Omega\int_\Gamma[r^1]^2\,\mathrm{d}\sigma\,\mathrm{d}x\geq\overline{r}^1>0$ for every $t>0$. On the other hand, for $t>0$ and setting $\Gamma_{\overline{r}^1}(t):=\Big\{(x,y)\in\Omega\times\Gamma:[r^1]^2(x,y,t)\leq\frac{\overline{r}^1}{2\alpha|\Gamma|\,|\Omega|}\Big\}$, reasoning as in the proof of Theorem 3.2, it follows that

$$\alpha\int_\Omega\int_{\Gamma\setminus\Gamma_{\overline{r}^1}(t)}[r^1(x,y,t)]^2\,\mathrm{d}\sigma\,\mathrm{d}x\geq\frac{\overline{r}^1}{2}\,,\quad\forall t>0\,.$$

However, on $\Gamma\setminus\Gamma_{\overline{r}^1}$, $g(x,y,t):=\frac{f([u^{1,\#}])-f([u^1])}{[u^{1,\#}]-[u^1]}\geq\chi>0$, where χ is a suitable positive constant depending only on $\overline{r}^1,\alpha,|\Gamma|,|\Omega|$ (this last result follows from the assumptions (2.8)–(2.10)). Hence, using (4.59), we get

$$\frac{d}{dt}\left(\frac{\alpha}{2}\int_\Omega\int_\Gamma [r^1(x,y,t)]^2 \,\mathrm{d}\sigma\,\mathrm{d}x\right) \le -\int_\Omega \int_{\Gamma\setminus\Gamma_{\overline{r}^1}(t)} g(x,y,t)[r^1(x,y,t)]^2 \,\mathrm{d}\sigma\,\mathrm{d}x$$

$$\le -\chi\int_\Omega \int_{\Gamma\setminus\Gamma_{\overline{r}^1}(t)} [r^1(x,y,t)]^2 \,\mathrm{d}\sigma\,\mathrm{d}x \le -\frac{\overline{r}^1}{2\alpha}\chi < 0\,. \quad (4.60)$$

Inequality (4.60) clearly contradicts the asymptotic convergence for $t \to +\infty$ of $\alpha\int_\Omega\int_\Gamma [r^1]^2(x,y,t)\,\mathrm{d}\sigma\,\mathrm{d}x$ to a positive number, hence

$$\lim_{t\to+\infty}\alpha\int_\Omega\int_\Gamma [r^1(x,y,t)]^2 \,\mathrm{d}\sigma\,\mathrm{d}x = 0\,, \quad (4.61)$$

which is exactly (4.56). Integrating (4.59) in $[t,\infty)$ and taking into account (4.61), we get

$$\int_t^{+\infty}\int_\Omega\int_Y \sigma|\nabla r + \nabla_y r^1|^2 \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \le \frac{\alpha}{2}\int_\Omega\int_\Gamma [r^1(x,y,t)]^2 \,\mathrm{d}\sigma\,\mathrm{d}x\,, \quad (4.62)$$

which implies

$$\lim_{t\to+\infty}\int_t^{+\infty}\int_\Omega\int_Y \sigma|\nabla r + \nabla_y r^1|^2 \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = 0\,. \quad (4.63)$$

This last condition guarantees that for every positive η there exists a $\widehat{t}(\eta) > 0$, such that

$$\int_{\widehat{t}}^{+\infty}\int_\Omega\int_Y \sigma|\nabla r + \nabla_y r^1|^2 \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \le \eta\,,$$

which in turn implies that, for every $n \in \boldsymbol{N}$, there exists a $t_n \in (\widehat{t}+n, \widehat{t}+(n+1))$, such that

$$\int_\Omega\int_Y \sigma|\nabla r(x,t_n) + \nabla_y r^1(x,y,t_n)|^2 \,\mathrm{d}y\,\mathrm{d}x \le \eta\,. \quad (4.64)$$

Hence, replacing (ϕ,Φ) with (r_t, r^1_t) in (4.58), we get

$$\int_\Omega\int_Y \sigma(\nabla r + \nabla_y r^1)(\nabla r_t + \nabla_y r^1_t)\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega\int_\Gamma g(x,y,t)[r^1][r^1_t]\,\mathrm{d}\sigma\,\mathrm{d}x$$

$$+\,\alpha\int_\Omega\int_\Gamma [r^1_t(x,y,t)]^2\,\mathrm{d}\sigma\,\mathrm{d}x = 0\,, \quad (4.65)$$

and

$$\int_\Omega\int_Y \sigma(\nabla r + \nabla_y r^1)(\nabla r_t + \nabla_y r^1_t)\,\mathrm{d}y\,\mathrm{d}x \le \int_\Omega\int_\Gamma \frac{g^2(x,y,t)}{2\alpha}[r^1(x,y,t)]^2\,\mathrm{d}\sigma\,\mathrm{d}x\,. \quad (4.66)$$

Moreover, integrating (4.66) in $[t_n, t^*]$, with $t^* \in [t_n, t_n + 2]$, we have

$$\sup_{t\in[t_n,t_n+2]} \left(\int_\Omega \int_Y \frac{\sigma}{2} |\nabla r(x,t) + \nabla_y r^1(x,y,t)|^2 \,\mathrm{d}y \,\mathrm{d}x \right) \leq \frac{\eta}{2} + \frac{2L^2}{2\alpha^2} \sup_{t\in[t_n,+\infty)} \left(\alpha \int_\Omega \int_\Gamma [r^1(x,y,t)]^2 \,\mathrm{d}\sigma \,\mathrm{d}x \right) , \qquad \forall n \in \boldsymbol{N} ; \tag{4.67}$$

i.e.,

$$\sup_{t\in[\widehat{t}+1,+\infty)} \left(\int_\Omega \int_Y \frac{\sigma}{2} |\nabla r(x,t) + \nabla_y r^1(x,y,t)|^2 \,\mathrm{d}y \,\mathrm{d}x \right) \leq \frac{\eta}{2} + \frac{L^2}{\alpha^2} \sup_{t\in[\widehat{t},+\infty)} \left(\alpha \int_\Omega \int_\Gamma [r^1(x,y,t)]^2 \,\mathrm{d}\sigma \,\mathrm{d}x \right) . \tag{4.68}$$

Because of (4.61) the integral in the right-hand side of (4.68) can be made smaller than $\frac{\eta}{2}\left(\frac{L^2}{\alpha^2}\right)^{-1}$, provided $\widehat{t}$ is chosen sufficiently large in dependence of η. This means that

$$\sup_{t\in[\widehat{t}+1,+\infty)} \left(\int_\Omega \int_Y \frac{\sigma}{2} |\nabla r(x,t) + \nabla_y r^1(x,y,t)|^2 \,\mathrm{d}y \,\mathrm{d}x \right) \leq \eta . \tag{4.69}$$

Inequality (4.69) implies

$$\lim_{t\to+\infty} \int_\Omega \int_Y \sigma |\nabla r(x,t) + \nabla_y r^1(x,y,t)|^2 \,\mathrm{d}y \,\mathrm{d}x = 0 . \tag{4.70}$$

Now, working as done in (4.31), we get

$$\lim_{t\to+\infty} \int_\Omega |\nabla r(x,t)|^2 \,\mathrm{d}x = 0 ; \qquad \text{and} \qquad \lim_{t\to+\infty} \int_\Omega \int_Y |\nabla_y r^1(x,y,t)|^2 \,\mathrm{d}y \,\mathrm{d}x = 0 .$$

Finally, the previous results together with (4.61) and Poincare's inequalities yield

$$\lim_{t\to+\infty} \int_\Omega |r(x,t)|^2 \,\mathrm{d}x = 0 ; \qquad \text{and} \qquad \lim_{t\to+\infty} \int_\Omega \int_Y |r^1(x,y,t)|^2 \,\mathrm{d}y \,\mathrm{d}x = 0 ,$$

which give (4.54) and (4.55) and conclude the proof. □

Remark 4.7. More in general, the previous procedure allows us to prove that solutions of (4.2)–(4.7) having different initial data satisfying the assumptions stated at the beginning of this section but with the same boundary condition tend asymptotically one to the other (such convergence being exponential if f is coercive in the sense of (1.1)). □

Remark 4.8. Observe that, thanks to previous remark, Theorem 3.2 implies uniqueness of the periodic solution $(u^{\#}, u^{1,\#})$ of problem (4.11)–(4.16) in $\mathcal{C}^0_{\#}\big([0,1]; H^1(\Omega)\big) \times \mathcal{C}^0_{\#}\big([0,1]; L^2(\Omega; \mathcal{X}^1_{\#}(Y))\big)$. □

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