LARGE TIME BEHAVIOR FOR THE POROUS MEDIUM EQUATION WITH CONVECTION

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Abstract. We consider the Cauchy problem for the porous medium equation with nonlinear convection, when the nonlinearities are the same in the convection and in the diffusion terms. We get a new sharp bound of the solution for large times.
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1. Introduction

We consider the Cauchy problem for the porous medium equation with convection in the form

$$\frac{\partial u}{\partial t} = \Delta\, u^m - (u^m)_{x_N}\,, \qquad (x,t) \in \boldsymbol{R}^N \times (0,+\infty)\,, \tag{1}$$

$$u(x,0) = u_0(x)\,, \qquad x \in \boldsymbol{R}^N\,. \tag{2}$$

Here we assume $m > 1$, $N \geq 2$, $u_0 \in L^1(\boldsymbol{R}^N)$, $u_0 \geq 0$.

Theorem 1.1. *There exists a non-negative weak solution u of* (1)–(2) *such that for all $t > 0$*

$$\|u(t)\|_{L^\infty(\boldsymbol{R}^N)} \leq \gamma \|u_0\|^{\frac{2}{H}}_{L^1(\boldsymbol{R}^N)} t^{-\frac{N+1}{H}}\,. \tag{3}$$

Here $H = (N-1)(m-1) + 2m$.

We confine ourselves to prove the a-priori sup bound for the (weak) solution, whose definition is standard. Then the existence follows via a routine regularization procedure (see [5], [11] and also [1], [3]).

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Note that the known estimate (see for example [5], [11] and comments in [2])

$$\|u(t)\|_{L^\infty(\boldsymbol{R}^N)} \le \gamma \|u_0\|_{L^1(\boldsymbol{R}^N)}^{\frac{2}{N(m-1)+2}} t^{-\frac{N}{N(m-1)+2}}\,, \tag{4}$$

certainly holds for all positive times. However, it is easy to check that for $t \to +\infty$ (3) gives a faster decay in time than (4).

Remark 1.2. In the case $N = 1$ (3) follows from [10]. Estimates like (3) were obtained also in [2] for a class of doubly degenerate parabolic equations. Though the cases studied in [2] do not include the case of (1), the sup estimate obtained there formally coincides with the one found here, and is proven to be sharp in [2, Subsection 1.3]. □

The proof of Theorem 1.1 is based on the approach of [2] and [5], relying on suitable iterative estimates of integral norms. Classically, this approach makes use of Sobolev embedding theorems. In the case of equation (1) the presence of the convective term causes a marked anisotropy in the behavior of solutions. Though this does not appear explicitly in our main estimate, we must deal with this feature and in some sense take advantage of it. This is the purpose of the entropy type inequality stated below in Lemma 2.2, where we obtain integral estimates on $(N-1)$-dimensional hyperplanes orthogonal to the convection direction.

Estimates for large times of the anisotropic behavior of solutions to diffusion-convection equations were obtained in [6], [7], [8], [9] (where diffusion is linear). The use of entropy type inequalities goes back to those papers, but we use a weaker form of entropy type inequality than the one used there.

2. Auxiliary results

In the following, u denotes a non-negative solution to (1).

Lemma 2.1. *For any $t > 0$ we have*

$$u_t \ge -\frac{u}{(m-1)t}\,. \tag{5}$$

in the integral sense.

For the proof of this well known result we refer to [4] or to [11, Lemma 8.1]. Estimate (5) is used in the proof of our next Lemma.

Denote for $a > 0$, $k > 0$,

$$G_k(u) = \int_k^u s^{m-1}(s-k)_+^a \,\mathrm{d}s\,,$$

and for $a = 0$, $k > 0$

$$G_k(u) = \frac{(u-k)_+^m}{m} \,.$$

Lemma 2.2. *For any $a > 0$, $k > 0$, for any hyperplane $\boldsymbol{R}^{N-1}_{x'} = \{x_N = y\}$, where we denote by x' the variable in $\boldsymbol{R}^{N-1}$, for any $t > 0$,*

$$\int\limits_{\boldsymbol{R}^{N-1}_{x'}} G_k(u(x', y, t))\,\mathrm{d}x' \le (m(m-1)t)^{-1} \int\limits_{\boldsymbol{R}^N} u(u-k)_+^a \,\mathrm{d}x \,. \tag{6}$$

When $a = 0$ we have, denoting by χ_A the characteristic function of the set A,

$$\int\limits_{\boldsymbol{R}^{N-1}_{x'}} G_k(u(x', y, t))\,\mathrm{d}x' \le (m(m-1)t)^{-1} \int\limits_{\boldsymbol{R}^N} u\chi_{\{u>k\}} \,\mathrm{d}x \,. \tag{7}$$

Proof. Assume first that $a > 0$. Multiply both sides of (1) by $(u-k)_+^a$ and integrate over $\boldsymbol{R}^{N-1} \times (-\infty, y)$. We get on using the bound for u_t proved in Lemma 2.1 and integrating by parts

$$m \int\limits_{-\infty}^{y} \mathrm{d}x_N \int\limits_{\boldsymbol{R}^{N-1}_{x'}} \left(\int\limits_{k}^{u} s^{m-1}(s-k)_+^a \,\mathrm{d}s \right)_{x_N} \mathrm{d}x' \le$$

$$m\frac{d}{dy} \int\limits_{\boldsymbol{R}^{N-1}_{x'}} G_k(u(x', y, t))\,\mathrm{d}x' + \frac{1}{(m-1)t} \int\limits_{\boldsymbol{R}^N} u(u-k)_+^a \,\mathrm{d}x \,,$$

whence integrating the left hand side

$$m \int\limits_{\boldsymbol{R}^{N-1}_{x'}} G_k(u(x', y, t))\,\mathrm{d}x' \le$$

$$m\frac{d}{dy} \int\limits_{\boldsymbol{R}^{N-1}_{x'}} G_k(u(x', y, t))\,\mathrm{d}x' + \frac{1}{(m-1)t} \int\limits_{\boldsymbol{R}^N} u(u-k)_+^a \,\mathrm{d}x \,.$$

Multiplying both sides of this inequality by $\exp\{-y\}$ and integrating the result between y_1 and y_2 we obtain

$$E_k(y_1) := \int\limits_{\boldsymbol{R}^{N-1}_{x'}} G_k(u(x', y_1, t))\,\mathrm{d}x' \le \exp\{-(y_2 - y_1)\}\, E_k(y_2) +$$

$$(m(m-1)t)^{-1} \int\limits_{\boldsymbol{R}^N} u(u-k)_+^a \,\mathrm{d}x \,.$$

Letting $y_2 \to \infty$, we arrive at (6), for $y = y_1$.

Finally the case $a = 0$ follows on letting $a \to 0$ above, which is possible since the quantities involved do not critically depend on a. □

3. Proof of Theorem 1.1

We denote by $\gamma > 0$, $b > 1$ constants depending only on N, m, which may vary from line to line.

Let us state first the following energy inequality which can be proved by means of routine calculations.

Lemma 3.1. *For all $h_1 > h_2 > 0$, $t > t_1 > t_2 > 0$ and $\omega > 0$ we have*

$$\sup_{t_1<\tau<t} \int_{\boldsymbol{R}^N} (u - h_1)_+^{1+\omega} \, \mathrm{d}x + \int_{t_1}^{t} \int_{\boldsymbol{R}^N} \left| \nabla (u - h_1)_+^{\frac{m+\omega}{2}} \right|^2 \mathrm{d}x \, \mathrm{d}\tau \leq \frac{\gamma}{t_1 - t_2} \int_{t_2}^{t} \int_{\boldsymbol{R}^N} (u - h_2)_+^{1+\omega} \, \mathrm{d}x \, \mathrm{d}\tau \, . \quad (8)$$

Fix $t > 0$. Introduce then the decreasing sequences

$$\vartheta_n = \tau_2 + (\tau_1 - \tau_2) 2^{-n} \, , \qquad k_n = a_2 + (a_1 - a_2) 2^{-n} \, , \qquad n \geq 0 \, ,$$

for arbitrarily fixed $t > \tau_1 > \tau_2 > t/4$ and $a_1 > a_2 > 0$. Next we apply Lemma 3.1 with $t_1 = \vartheta_n$, $t_2 = \vartheta_{n+1}$, $h_1 = k_n$, $h_2 = k_{n+1}$. We obtain

$$\sup_{\vartheta_n<\tau<t} \int_{\boldsymbol{R}^N} (u - k_n)_+^{1+\omega} \, \mathrm{d}x + \iint_{E_n} \left| \nabla (u - k_n)_+^{\frac{m+\omega}{2}} \right|^2 \mathrm{d}x \, \mathrm{d}\tau \leq \frac{\gamma b^n}{\tau_1 - \tau_2} \iint_{E_{n+1}} (u - k_{n+1})_+^{1+\omega} \, \mathrm{d}x \, \mathrm{d}\tau \, ,$$

where $E_n = \boldsymbol{R}^N \times (\vartheta_n, t)$. Here $\omega > 0$ is any fixed constant, whose value is in practice not relevant. Set $v_n = (u - k_n)_+^{\frac{m+\omega}{2}}$. Then the last inequality leads to

$$\sup_{\vartheta_n<\tau<t} \int_{\boldsymbol{R}^N} v_n^{\beta} \, \mathrm{d}x + \iint_{E_n} |\nabla v_n|^2 \, \mathrm{d}x \, \mathrm{d}\tau \leq \frac{\gamma b^n}{\tau_1 - \tau_2} \iint_{E_{n+1}} v_{n+1}^{\beta} \, \mathrm{d}x \, \mathrm{d}\tau \, ; \qquad (9)$$

we employ here the notation

$$\beta = 2 \frac{1 + \omega}{m + \omega} \, , \qquad \delta = \frac{2}{m + \omega} \, .$$

On applying next Lemma 2.2 with $k = k_n$, $a = 0$ we can easily get at every time level $\tau > t/4$

$$\int_{\boldsymbol{R}^{N-1}_{x'}} v_n^{m\delta}\,\mathrm{d}x' \leq \gamma b^n \frac{a_1}{a_1 - a_2} t^{-1} \int_{\boldsymbol{R}^N} v_{n+1}^{\delta}\,\mathrm{d}x\,. \tag{10}$$

Next we use the $N-1$ dimensional Nirenberg-Gagliardo type inequality

$$\int_{\boldsymbol{R}^{N-1}_{x'}} v_{n+1}^{\beta}\,\mathrm{d}x' \leq \gamma \left(\int_{\boldsymbol{R}^{N-1}_{x'}} \left|\nabla v_{n+1}\right|^2 \mathrm{d}x' \right)^{\frac{\alpha\beta}{2}} \left(\int_{\boldsymbol{R}^{N-1}_{x'}} v_{n+1}^{\mu'}\,\mathrm{d}x' \right)^{\frac{(1-\alpha)\beta}{\mu'}}. \tag{11}$$

where $\mu' = (1+\mu)\delta$, $0 < \mu < \min(m-1, \omega)$ will be chosen and α by dimensional analysis is calculated as

$$\frac{N-1}{\beta} = \alpha \frac{N-3}{2} + (1-\alpha)\frac{N-1}{\mu'}\,. \tag{12}$$

By the Hölder inequality with $\mu < m-1$ and by (10) we infer

$$\begin{aligned}\int_{\boldsymbol{R}^{N-1}_{x'}} v_{n+1}^{\mu'}\,\mathrm{d}x' &\leq \left(\int_{\boldsymbol{R}^{N-1}_{x'}} v_{n+1}^{m\delta}\,\mathrm{d}x' \right)^{\frac{\mu}{m-1}} \left(\int_{\boldsymbol{R}^{N-1}_{x'}} v_{n+1}^{\delta}\,\mathrm{d}x' \right)^{1-\frac{\mu}{m-1}} \\ &\leq \left(\gamma b^n \frac{a_1}{a_1-a_2} t^{-1} \int_{\boldsymbol{R}^N} v_{n+2}^{\delta}\,\mathrm{d}x \right)^{\frac{\mu}{m-1}} \left(\int_{\boldsymbol{R}^{N-1}_{x'}} v_{n+1}^{\delta}\,\mathrm{d}x' \right)^{1-\frac{\mu}{m-1}}.\end{aligned}$$

Therefore by (11) we get

$$\begin{aligned}\int_{\boldsymbol{R}^{N-1}_{x'}} v_{n+1}^{\beta}\,\mathrm{d}x' &\leq \gamma \left(\int_{\boldsymbol{R}^{N-1}_{x'}} \left|\nabla v_{n+1}\right|^2 \mathrm{d}x' \right)^{\frac{\alpha\beta}{2}} \\ \times &\left(b^n \frac{a_1}{a_1-a_2} t^{-1} \int_{\boldsymbol{R}^N} v_{n+2}^{\delta}\,\mathrm{d}x \right)^{\frac{\mu}{m-1}\frac{(1-\alpha)\beta}{\mu'}} \left(\int_{\boldsymbol{R}^{N-1}_{x'}} v_{n+1}^{\delta}\,\mathrm{d}x' \right)^{\left(1-\frac{\mu}{m-1}\right)\frac{(1-\alpha)\beta}{\mu'}}.\end{aligned} \tag{13}$$

Integrate (13) in x_N between $-\infty$ and ∞ and apply the Hölder inequality to get

$$\int_{\boldsymbol{R}^N} v_{n+1}^{\beta}\,\mathrm{d}x \le \gamma b^n \left(\int_{\boldsymbol{R}^N} |\nabla v_{n+1}|^2\,\mathrm{d}x \right)^{\frac{\alpha\beta}{2}}$$
$$\times \left(\frac{a_1}{a_1 - a_2} t^{-1} \int_{\boldsymbol{R}^N} v_{n+2}^{\delta}\,\mathrm{d}x \right)^{A(1-\frac{\alpha\beta}{2})} \left(\int_{-\infty}^{\infty} \left(\int_{\boldsymbol{R}^{N-1}_{x'}} v_{n+1}^{\delta}\,\mathrm{d}x' \right)^{B} \mathrm{d}x_N \right)^{1-\frac{\alpha\beta}{2}} . \tag{14}$$

We have used the fact $\alpha\beta < 2$ which can be computed to actually hold true according to the value of μ selected below. Here we denote

$$A = \frac{\mu}{m-1} \frac{(1-\alpha)\beta}{\mu'} \left(1 - \frac{\alpha\beta}{2}\right)^{-1} , \quad B = \left(1 - \frac{\mu}{m-1}\right) \frac{A(m-1)}{\mu} . \tag{15}$$

Choose now the free parameter μ by imposing the relation $B = 1$. Then from (15) and (12) one gets for $K = (N-1)(m-1) + 2$

$$\mu = \frac{2\omega(m-1)}{K + 2(\omega + m - 1)} < \min(m-1, \omega) , \quad A = \frac{2\omega}{K + 2(m-1)} .$$

Integrating now (14) in time over (ϑ_{n+1}, t) and applying the Young inequality we obtain

$$\frac{b^n}{\tau_1 - \tau_2} \iint_{E_{n+1}} v_{n+1}^{\beta}\,\mathrm{d}x\,\mathrm{d}\tau \le \epsilon \iint_{E_{n+1}} |\nabla v_{n+1}|^2\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+ \gamma(\epsilon) b^n (\tau_1 - \tau_2)^{-\frac{2}{2-\alpha\beta}} \left(\frac{a_1 t^{-1}}{a_1 - a_2} \right)^{A} \tag{16}$$
$$\times (t - \vartheta_{n+1}) \left(\sup_{\vartheta_{n+1} < \tau < t} \int_{\boldsymbol{R}^N} v_{n+2}^{\delta}\,\mathrm{d}x \right)^{1+A} .$$

Finally, from (9) and (16) we infer

$$L_n := \sup_{\vartheta_n<\tau<t} \int\limits_{\boldsymbol{R}^N} v_n^{\beta}\,\mathrm{d}x + \iint\limits_{E_n} |\nabla v_n|^2\,\mathrm{d}x\,\mathrm{d}\tau \le \epsilon \iint\limits_{E_{n+1}} |\nabla v_{n+1}|^2\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+ \gamma_1 b^n (\tau_1-\tau_2)^{-\frac{2}{2-\alpha\beta}} t^{-A} \left(\frac{a_1}{a_1-a_2}\right)^{A}$$
$$\times (t-\vartheta_{n+1}) \left(\sup_{\vartheta_{n+1}<\tau<t} \int\limits_{\boldsymbol{R}^N} v_{n+2}^{\delta}\,\mathrm{d}x \right)^{1+A} .$$

Here we denote by γ_1 a constant depending on suitable powers of $\epsilon > 0$. Iterating this inequality with respect to n, we obtain

$$L_0 \le \epsilon^n L_n + \left(\sum_{i=0}^{n-1} (\epsilon b)^i\right) \gamma_1 (\tau_1-\tau_2)^{-\frac{2}{2-\alpha\beta}} t^{-A} \left(\frac{a_1}{a_1-a_2}\right)^{A}$$
$$\times (t-\tau_2) \left(\sup_{\tau_2<\tau<t} \int\limits_{\boldsymbol{R}^N} (u-a_2)_+\,\mathrm{d}x \right)^{1+A} .$$

Therefore on selecting $\epsilon < b^{-1}$ we have as $n \to \infty$

$$\sup_{\tau_1<\tau<t} \int\limits_{\boldsymbol{R}^N} (u-a_1)_+^{1+\omega}\,\mathrm{d}x \le \gamma_1 (\tau_1-\tau_2)^{-\frac{2}{2-\alpha\beta}} t^{-A} \left(\frac{a_1}{a_1-a_2}\right)^{A}$$
$$\times (t-\tau_2) \left(\sup_{\tau_2<\tau<t} \int\limits_{\boldsymbol{R}^N} (u-a_2)_+\,\mathrm{d}x \right)^{1+A} . \quad (17)$$

For $k > 0$ to be chosen define for $n \ge 0$

$$h_n = k - 2^{-n-1}k\,, \quad t_n = 2^{-1}t - 2^{-n-2}t\,, \quad \widetilde{h}_n = 2^{-1}(h_n + h_{n+1})\,,$$

and note that

$$(u-h_{n+1})_+ \le 2^{\omega(n+3)} k^{-\omega} (u-\widetilde{h}_n)_+^{1+\omega} .$$

Therefore, on applying (17) with

$$a_1 = \widetilde{h}_n\,, \quad a_2 = h_n\,, \quad \tau_1 = t_{n+1}\,, \quad \tau_2 = t_n\,,$$

so that

$$\frac{a_1}{a_1-a_2} \le 2^{n+3}\,,$$

we obtain

$$Y_{n+1} := \sup_{t_{n+1}<\tau<t} \int\limits_{\boldsymbol{R}^N} (u-h_{n+1})_+\,\mathrm{d}x \le \gamma_1 b^n k^{-\omega} t^{-\frac{\alpha\beta}{2-\alpha\beta}-A} Y_n^{1+A} .$$

Finally, by means of a standard iterative lemma (see e.g., [5] p.12) we conclude that $Y_n \to 0$ as $n \to \infty$ provided

$$k^{-\omega} t^{-\frac{\alpha\beta}{2-\alpha\beta} - A} Y_0^A \le \sigma \,, \tag{18}$$

for a small enough $\sigma = \sigma(m, N)$. After a lenghty but elementary calculation (see the Appendix below) the last inequality leads to

$$k^{-H} t^{-N-1} Y_0^2 \le \sigma^{\frac{K+2(m-1)}{\omega}} \,. \tag{19}$$

Finally, noting that

$$Y_0 \le \|u_0\|_{L^1(\boldsymbol{R}^N)} \,,$$

and choosing

$$k = \sigma^{-\frac{K+2(m-1)}{\omega}} t^{-\frac{N+1}{H}} \|u_0\|_{L^1(\boldsymbol{R}^N)}^{\frac{2}{H}} \,,$$

we conclude the proof of Theorem 1.1.

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4. Appendix: Calculation of the exponent in the main estimate

We show here how (18) yields (19); this follows from (24), (25) which we prove below.
Recall our definitions

$$\delta = \frac{2}{m+\omega} \,, \qquad \beta = \frac{2(1+\omega)}{m+\omega} \,, \qquad \mu' = (1+\mu)\delta = \frac{2(1+\mu)}{m+\omega} \,.$$

From the definition of α in (12) we obtain

$$\alpha = \frac{\frac{N-1}{\mu'} - \frac{N-1}{\beta}}{\frac{N-1}{\mu'} - \frac{N-3}{2}} = \frac{2}{\beta} \frac{(N-1)(\beta - \mu')}{(N-1)(2-\mu') + 2\mu'} \,.$$

Hence

$$\begin{aligned}\frac{\alpha\beta}{2} &= \frac{(N-1)(\omega-\mu)}{(N-1)(m-1) + (N-1)(\omega-\mu) + 2(1+\mu)} \\ &= \frac{(N-1)(\omega-\mu)}{K + (N-1)(\omega-\mu) + 2\mu} \,.\end{aligned} \tag{20}$$

Next we compute from (20)

$$1 - \frac{\alpha\beta}{2} = \frac{K+2\mu}{K + (N-1)(\omega-\mu) + 2\mu} \,. \tag{21}$$

We have from the definition of α

$$1-\alpha=\frac{\frac{N-1}{\beta}-\frac{N-3}{2}}{\frac{N-1}{\mu'}-\frac{N-3}{2}}=\frac{\mu'}{\beta}\frac{(N-1)(2-\beta)+2\beta}{(N-1)(2-\mu')+2\mu'}$$
$$=\frac{\mu'}{\beta}\frac{K+2\omega}{K+(N-1)(\omega-\mu)+2\mu}\,,$$

that is

$$(1-\alpha)\frac{\beta}{\mu'}=\frac{K+2\omega}{K+(N-1)(\omega-\mu)+2\mu}\,. \tag{22}$$

Therefore from the definition of A in (15), from (21) and (22) we get

$$A\frac{m-1}{\mu}=(1-\alpha)\frac{\beta}{\mu'}\left(1-\frac{\alpha\beta}{2}\right)^{-1}=\frac{K+2\omega}{K+2\mu}\,.$$

From this equality we infer imposing that $B=1$

$$1=B:=\left(1-\frac{\mu}{m-1}\right)A\frac{m-1}{\mu}=\frac{m-1-\mu}{m-1}\frac{K+2\omega}{K+2\mu}\,.$$

Thus with the choice $B=1$ we solve the last equality for μ as

$$\mu=\frac{2\omega(m-1)}{K+2(\omega+m-1)}=\frac{2\omega(m-1)}{H+2\omega}\,, \tag{23}$$

and therefore (using $B=1$ again and (23))

$$A=\frac{\frac{\mu}{m-1}}{1-\frac{\mu}{m-1}}=\frac{2\omega}{K+2(m-1)}=\frac{2\omega}{H}\,. \tag{24}$$

Next we have from (20), (21) and (23)

$$\frac{\alpha\beta}{2-\alpha\beta}=\frac{\alpha\beta}{2(1-\alpha\beta/2)}=\frac{(N-1)(\omega-\mu)}{K+2\mu}=$$
$$\frac{(N-1)(\omega-\frac{2\omega(m-1)}{H+2\omega})}{K+2\frac{2\omega(m-1)}{H+2\omega}}=\frac{(N-1)(K+2\omega)\omega}{H(K+2\omega)}=\frac{(N-1)\omega}{H}\,.$$

Finally from the last equality and (24) we conclude that

$$\frac{\alpha\beta}{2-\alpha\beta}+A=\frac{(N+1)\omega}{H}\,, \tag{25}$$

so that by substituting (24), (25) in inequality (18) we get the desired result.

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