# DOUBLE POISSON VERTEX ALGEBRAS AND NON-COMMUTATIVE HAMILTONIAN EQUATIONS 

ALBERTO DE SOLE, VICTOR G. KAC, DANIELE VALERI


#### Abstract

We develop the formalism of double Poisson vertex algebras (local and non-local) aimed at the study of non-commutative Hamiltionan PDEs. This is a generalization of the theory of double Poisson algebras, developed by Van den Bergh, which is used in the study of Hamiltonian ODEs. We apply our theory of double Poisson vertex algebras to non-commutative KP and Gelfand-Dickey hierarchies. We also construct the related non-commutative de Rham and variational complexes.


## Contents

0 . Introduction ..... 2

1. Preliminaries and notation for associative algebras ..... 7
1.1. "Double" linear algebra ..... 7
1.2. Trace map and reduction ..... 8
1.3. $V$-module structures of $V^{\otimes n}$ ..... 8
1.4. Some further notation ..... 9
1.5. $n$-fold derivations ..... 10
1.6. $n$-fold brackets ..... 11
2. Double Poisson algebras and non-commutative Hamiltonian ODEs ..... 12
2.1. Definition of double Poisson algebras ..... 12
2.2. The trace map and connection to Lie algebras ..... 13
2.3. Double Poisson brackets on an algebra of (non-commutative) ordinary differential functions ..... 14
2.4. Evolution ODE and Hamiltonian ODE ..... 17
2.5. Lenard-Magri scheme for Hamiltonian ODE ..... 18
2.6. Examples ..... 19
2.7. Basic de Rham complex over an algebra of ordinary differential functions ..... 21
2.8. Reduced de Rham complex ..... 22
3. Double Poisson vertex algebras and non-commutative Hamiltonian PDEs ..... 24
3.1. Definition of double Poisson vertex algebra ..... 24
3.2. The trace map and connection to Lie conformal algebras ..... 26
3.3. Double Poisson vertex algebra structure on an algebra of (non- commutative) differential functions ..... 28
3.4. Evolution PDE and Hamiltonian PDE over an algebra of differential functions ..... 29
3.5. de Rham complex over an algebra of differential functions ..... 31
3.6. Variational complex over an algebra of differential functions ..... 31
3.7. Poisson vertex algebra structure on $\mathcal{V}_{m}$ ..... 35
3.8. Examples: affine and AGD double PVA structures ..... 38
4. Non-local double Poisson vertex algebras and their Dirac reduction ..... 43
4.1. Non-local double Poisson vertex algebras ..... 43
4.2. Dirac reduction for non-local double Poisson vertex algebras44
5. Adler-Gelfand-Dickey non-commutative integrable hierarchies ..... 52
References ..... 56

## 0 . Introduction

First, we recall the notion of a Poisson vertex algebra. Given a unital commutative associative algebra $\mathcal{V}$ with a derivation $\partial$, a $\lambda$-bracket on $\mathcal{V}$ is a linear map $\mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}[\lambda], a \otimes b \rightarrow\left\{a_{\lambda} b\right\}$, satisfying the following axioms:

$$
\begin{array}{cl}
\left\{\partial a_{\lambda} b\right\}=-\lambda\left\{a_{\lambda} b\right\}, \quad\left\{a_{\lambda} \partial b\right\}=(\lambda+\partial)\left\{a_{\lambda} b\right\} & \text { (sesquilinearity) } \\
\left\{a_{\lambda} b c\right\}=\left\{a_{\lambda} b\right\}_{c}+\left\{a_{\lambda} c\right\} b & \text { (left Leibniz rule) } \\
\left\{a b_{\lambda} c\right\}=\left\{a_{\lambda+\partial} c\right\}_{\rightarrow} b+\left\{b_{\lambda+\partial} c\right\}_{\rightarrow} a & \text { (right Leibniz rule) . } \tag{0.3}
\end{array}
$$

The algebra $\mathcal{V}$, endowed with a $\lambda$-bracket satisfying

$$
\begin{array}{cl}
\left\{a_{\lambda} b\right\}=-\left\{b_{-\lambda-\partial} a\right\} & \text { (skewsymmetry) } \\
\left\{a_{\lambda}\left\{b_{\mu} c\right\}\right\}-\left\{b_{\mu}\left\{a_{\lambda} c\right\}\right\}=\left\{\left\{a_{\lambda} b\right\}_{\lambda+\mu} c\right\} & \text { (Jacobi identity) } \tag{0.5}
\end{array}
$$

is called a Poisson vertex algebra (PVA). (Of course, if $\partial=0$, then (0.2)-(0.5) turn into axioms of a Poisson algebra if we let $\lambda=\mu=0$.)

In the case when $\mathcal{V}$ is the algebra of functions on the space of jets of a finitedimensional manifold, the notion of a PVA is equivalent to that of a local Poisson bracket [BDSK09]. In order to accommodate non-local Poisson brackets one needs to introduce the non-local PVA, for which the $\lambda$-bracket takes values in $\mathcal{V}\left(\left(\lambda^{-1}\right)\right)$, the formal Laurent series in $\lambda^{-1}$ (instead of polynomials in $\lambda$ ). The axioms (0.1)(0.4) can be easily interpreted, but for the Jacobi identity (0.5) to make sense in the non-local case, one needs, in addition, the admissibility property of the $\lambda$-bracket:

$$
\begin{equation*}
\left\{a_{\lambda}\left\{b_{\mu} c\right\}\right\} \in \mathcal{V}\left[\left[\lambda^{-1}, \mu^{-1},(\lambda+\mu)^{-1}\right]\right][[\lambda, \mu]], \text { for all } a, b, c \in \mathcal{V} \tag{0.6}
\end{equation*}
$$

(see [DSK13] for details).
Local and non-local Poisson brackets play a fundamental role in the theory of integrable systems (see e.g. [FT86]). In recent years there have been attempts to develop a theory of integrable systems on non-commutative associative algebras (see e.g. [DF92, OS98, Kup00, MS00, ORS13]). An important advance in this direction was made by Van den Bergh, who in [VdB08] introduced the notion of a double Poisson algebra structure in a non-commutative associative algebra $V$. His basic idea was to consider a 2 -fold bracket $\{[-,-\}$ on $V$, with values in $V \otimes V$. The Leibniz rules of a 2 -fold bracket are almost identical to the usual Leibniz rules:

$$
\begin{align*}
& \{a, b c\}=\{\{a, b\} c+b\{\{a, c\},  \tag{0.7}\\
& \{a b, c\}=\{a a, c\} \star_{1} b+a \star_{1}\{b, c\}, \tag{0.8}
\end{align*}
$$

where $\left(a_{1} \otimes a_{2}\right) \star_{1} a_{3}=a_{1} a_{3} \otimes a_{2}, a_{1} \star_{1}\left(a_{2} \otimes a_{3}\right)=a_{2} \otimes a_{1} a_{3}$. The skewsymmetry axiom is

$$
\begin{equation*}
\{a, b\}\}=-\{\{b, a\}\}^{\sigma}, \tag{0.9}
\end{equation*}
$$

where $\sigma$ is the permutation of factors in $V \otimes V$, and the Jacobi identity is

$$
\begin{equation*}
\left\{\{a,\{\{b, c\}\}\}_{L}-\{\{b,\{\{a, c\}\}\}\}_{R}=\left\{\{\{a a, b\}, c\}_{L},\right.\right. \tag{0.10}
\end{equation*}
$$

where we denote $\left\{\left\{a_{1}, a_{2} \otimes a_{3}\right\}_{L}=\left\{\left\{a_{1}, a_{2}\right\}\right\} \otimes a_{3},\left\{\left\{a_{1}, a_{2} \otimes a_{3}\right\}_{R}=a_{2} \otimes\left\{\left\{a_{1}, a_{3}\right\}\right\}\right.\right.$, $\left\{\left\{a_{1} \otimes a_{2}, a_{3}\right\}_{L}=\left\{\left\{a_{1}, a_{3}\right\}\right\} \otimes_{1} a_{2}\right.$, and $(a \otimes b) \otimes_{1} c=a \otimes c \otimes b$. (Formula (0.10) is slightly different from but equivalent to the Jacobi identity in [VdB08].)

An associative algebra $V$, endowed with a 2 -fold bracket $\{\{-,-\}$ satisfying axioms (0.9) and (0.10), is called a double Poisson algebra.

Given a double Poisson algebra $V$, Van den Bergh defines the following bracket on $V$ with values in $V$ :

$$
\begin{equation*}
\{a, b\}=\mathrm{m}\{\{a, b\}, \tag{0.11}
\end{equation*}
$$

where $\mathrm{m}: V \otimes V \rightarrow V$ is the multiplication map. The bracket (0.11) still satisfies the left Leibniz rule, but does not satisfy, in general, other axioms of a Poisson bracket. However, this bracket induces well-defined linear maps

$$
V /[V, V] \otimes V \rightarrow V \quad \text { and } \quad V /[V, V] \otimes V /[V, V] \rightarrow V /[V, V]
$$

given by

$$
\begin{equation*}
\{\operatorname{tr}(a), b\}=\{a, b\} \quad \text { and } \quad\{\operatorname{tr}(a), \operatorname{tr}(b)\}=\operatorname{tr}\{a, b\}, \tag{0.12}
\end{equation*}
$$

where $\operatorname{tr}: V \rightarrow V /[V, V]$ is the quotient map and $[V, V]$ is the linear span of commutators $a b-b a$. These maps have the following properties, important for the theory of non-commutative Hamiltonian ODEs: the vector space $V /[V, V]$ is a Lie algebra and one has its representation on $V$ by derivations, both defined by (0.12) (see Proposition 2.5).

Brackets (0.12) allow one to define the basic notions of a Hamiltonian theory. Given a Hamiltonian function $\operatorname{tr}(h), h \in V$, one defines the associated Hamiltonian equation

$$
\begin{equation*}
\frac{d x}{d t}=\{\operatorname{tr}(h), x\}, \quad x \in V \tag{0.13}
\end{equation*}
$$

Two Hamiltonian functions $\operatorname{tr}(f)$ and $\operatorname{tr}(g)$ are said to be in involution if

$$
\begin{equation*}
\{\operatorname{tr}(f), \operatorname{tr}(g)\}=0 \tag{0.14}
\end{equation*}
$$

In the case of $V=R_{\ell}$, the algebra of non-commutative polynomials in $\ell$ variables $x_{1}, \ldots, x_{\ell}$, any 2 -fold bracket can be written in a traditional form $\left(f, g \in R_{\ell}\right)$ :

$$
\begin{equation*}
\{f, g\}\}=\nabla g \bullet H \bullet(\nabla f)^{\sigma}, \tag{0.15}
\end{equation*}
$$

where $\nabla f$, the gradient of $f$, is the vector of 2-fold derivatives $\frac{\partial f}{\partial x_{i}}: R_{\ell} \rightarrow R_{\ell} \otimes R_{\ell}$, defined by $\frac{\partial f}{\partial x_{i}}\left(x_{j}\right)=\delta_{i j}(1 \otimes 1)$ (see [CBEG07]), $H=\left(H_{i j}\right)_{i, j=1}^{\ell}$, where $H_{i j}=$ $\left\{\left\{x_{j}, x_{i}\right\}\right\} \in V \otimes V$, and $\bullet$ denotes the multiplication in $V \otimes V^{o p}$. Moreover, the 2fold bracket (0.15) is skewsymmetric if and only if skewsymmetry holds on any pair of generators $x_{i}, x_{j}$ (equivalently, if the matrix $H$ is skewadjoint: $\left.\left(H^{t}\right)^{\sigma}=-H\right)$, and, provided that $H$ is skewadjoint, the Jacobi identity (0.10) holds if and only if it holds on any triple of generators $x_{i}, x_{j}, x_{k}$ (see Theorem 2.8).

In the case of $V=R_{\ell}$ the Hamiltonian equation (0.13) becomes the following evolution ODE, where $x=\left(x_{i}\right)_{i=1}^{\ell}$ :

$$
\begin{equation*}
\frac{d x}{d t}=\mathrm{m}\left(H(\nabla h)^{\sigma}\right), \tag{0.16}
\end{equation*}
$$

and the bracket $\{-,-\}$ on $R_{\ell} /\left[R_{\ell}, R_{\ell}\right]$ becomes:

$$
\begin{equation*}
\{\operatorname{tr}(f), \operatorname{tr}(g)\}=\operatorname{tr}\left(\mathrm{m}\left(\nabla g \bullet H \bullet(\nabla h)^{\sigma}\right)\right) \tag{0.17}
\end{equation*}
$$

Developing the ideas of [MS00], we study in detail the Euler hierarchy, obtained via the Lenard-Magri scheme from the following pair of compatible double Poisson brackets on $R_{2}=\mathbb{F}\langle x, y\rangle$, where $y$ is central for both 2-fold brackets:

$$
\begin{equation*}
\left\{\{x, x\}_{0}=1 \otimes y-y \otimes 1, \quad\{x, x\}_{1}=x \otimes y-y \otimes x .\right. \tag{0.18}
\end{equation*}
$$

Letting

$$
\begin{equation*}
h_{0}=1 \quad \text { and } \quad h_{n}=\frac{1}{n}(x+y)^{n} \quad \text { for } n \geq 1 \tag{0.19}
\end{equation*}
$$

it is easy to establish the following recursion relations:

$$
\left\{\operatorname{tr}\left(h_{0}\right), x\right\}_{0}=0 \quad \text { and } \quad\left\{\operatorname{tr}\left(h_{n}\right), x\right\}_{1}=\left\{\operatorname{tr}\left(h_{n+1}\right), x\right\}_{0} \quad \text { for } n \in \mathbb{Z}_{+} .
$$

Hence, by the Lenard-Magri scheme, all $\operatorname{tr}\left(h_{n}\right)$ are in involution with respect to both brackets, and we obtain, by (0.16), the following compatible set of Hamiltonian ODEs $\left(m \in \mathbb{Z}_{+}\right)$:

$$
\begin{equation*}
\frac{d x}{d t_{m}}=x(x+y)^{m} y-y(x+y)^{m} x, \quad \frac{d y}{d t_{m}}=0 \tag{0.20}
\end{equation*}
$$

for which all $h_{n}, n \in \mathbb{Z}_{+}$, are conserved densities. Thus, by definition in [MS00], the hierarchy of Hamiltonian ODEs (0.20) is integrable (see Section 2.6 for details).

We develop this theory in the natural generality of a (non-commutative) algebra of ordinary differential functions. This is a unital associative algebra $V$ endowed with $\ell$ strongly commuting 2 -fold derivations $\frac{\partial}{\partial x_{i}}$. We say that the 2-fold derivations $D$ and $E$ strongly commute if $D_{L} \circ E=E_{R} \circ D$, where $D_{L}\left(a_{1} \otimes a_{2}\right)=\left(D_{L} a_{1}\right) \otimes a_{2}$ and $D_{R}\left(a_{1} \otimes a_{2}\right)=a_{1} \otimes\left(D_{R} a_{2}\right)$ (then $D$ and $E$ commute).

Formula (0.15) defines a 2 -fold bracket on any algebra of ordinary differential functions $V$, and $V$ is a double Poisson algebra if and only if the matrix $H$ is skewadjoint and (0.10) holds on any "triple of generators" (see Theorem 2.8 for details).

The assumption of strong commutativity of the $\frac{\partial}{\partial x_{i}}$ is used in the proof of the Jacobi identity for the 2 -fold bracket (0.15) and the proof that $d^{2}=0$ for the de Rham complex $\widetilde{\Omega}(V)$, discussed below.

For an algebra of differential functions $V$ we construct in Section 2.7 the de Rham complex $\widetilde{\Omega}(V)$ as a free product of $V$ and the algebra of non-commutative polynomials in the indeterminates $d x_{1}, \ldots, d x_{\ell}$. This is a superalgebra with the $\mathbb{Z}_{+}$-grading given by $\operatorname{deg}(V)=0, \operatorname{deg}\left(d x_{i}\right)=1$, compatible with the parity, and with the de Rham differential $d$ being the odd derivation such that $d\left(d x_{i}\right)=0$ and $d f=\sum_{i=1}^{\ell} \mathrm{m}\left(\frac{\partial f}{\partial x_{i}} \otimes_{1} d x_{i}\right)$ for $f \in \widetilde{\Omega}^{0}(V)=V$.

The relevant to the theory of non-commutative evolution ODEs is the reduced complex

$$
\begin{equation*}
\Omega(V)=\widetilde{\Omega}(V) /[\widetilde{\Omega}(V), \widetilde{\Omega}(V)] \tag{0.21}
\end{equation*}
$$

In Section 2.8 we give an explicit description of the space of $k$-forms $\Omega^{k}(V)$. In particular: $\Omega^{0}(V)=V /[V, V], \Omega^{1}(V)=V^{\oplus \ell}, \Omega^{2}(V)=\left(\operatorname{Mat}_{\ell \times \ell}(V \otimes V)\right)_{-}$, where the subscript _ means that $\left(A_{i j}\right)_{i, j=1}^{\ell}=-\left(A_{j i}^{\sigma}\right)_{i, j=1}^{\ell}$. Under these identifications the formula for $d$ becomes:

$$
\begin{equation*}
d(\operatorname{tr} f)=\mathrm{m}(\nabla f)^{\sigma}, \quad d F=\frac{1}{2}\left(J_{F}^{t}-J_{F}^{\sigma}\right) \tag{0.22}
\end{equation*}
$$

where for $F=\left(F_{i}\right)_{i=1}^{\ell} \in V^{\oplus \ell}$ we define the 2 -fold Jacobian matrix as $J_{F}=$ $\left(\frac{\partial F_{i}}{\partial x_{j}}\right)_{i, j=1}^{\ell}$. We also prove that both complexes $\left(\widetilde{\Omega}\left(R_{\ell}\right), d\right)$ and $\left(\Omega\left(R_{\ell}\right), d\right)$ are acyclic.

The theory of double Poisson algebras and non-commutative Hamiltonian ODEs, over an algebra of ordinary differential functions $V$, developed in Section 2 sets the stage for the theory of double PVAs and non-commutative Hamiltonian PDEs over an algebra of differential functions $\mathcal{V}$, which we develop in Section 3.

In a nutshell a double PVA is a cross between a double Poisson algebra and a PVA. Let $\mathcal{V}$ be unital associative algebra with a derivation $\partial$. We define a 2 -fold $\lambda$-bracket on $\mathcal{V}$ as a map $\left\{\left\{-_{\lambda}-\right\}: \mathcal{V} \otimes \mathcal{V} \rightarrow(\mathcal{V} \otimes \mathcal{V})[\lambda]\right.$, satisfying sesquilinearity (3.5) and left and right Leibniz rules (3.6), similar to (0.1) and (0.2),(0.3) for PVA. A double PVA is the algebra $\mathcal{V}$, endowed with a 2 -fold $\lambda$-bracket satisfying skewsymmetry (3.8) and Jacobi identity (3.9) ( $=$ cross between (0.4) and (0.8), and between $(0.5)$ and ( 0.10 ), respectively). If $\partial=0$, the axioms of a double PVA turn into the axioms of a double Poisson algebra if we let $\lambda=\mu=0$.

A non-local double PVA is defined in the same way as a non-local PVA (but the one cannot put $\lambda=0$ ).

For a (local) double PVA $\mathcal{V}$, we denote by $\int$ the quotient map $\mathcal{V} \rightarrow \overline{\mathcal{V}}:=$ $\mathcal{V} /([\mathcal{V}, \mathcal{V}]+\partial \mathcal{V})$. Similarly to (0.11), define the map $\left\{-_{\lambda}-\right\}: \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}[\lambda]$ by

$$
\begin{equation*}
\left\{a_{\lambda} b\right\}=\mathrm{m}\left\{\left\{a_{\lambda} b\right\}\right. \tag{0.23}
\end{equation*}
$$

As for the (local) PVA (see [BDSK09]), (0.23) induces well-defined maps

$$
\overline{\mathcal{V}} \otimes \mathcal{V} \rightarrow \mathcal{V} \quad \text { and } \quad \overline{\mathcal{V}} \otimes \overline{\mathcal{V}} \rightarrow \overline{\mathcal{V}}
$$

given by formulas, similar to (0.12):

$$
\begin{equation*}
\left\{\int a, b\right\}=\left.\left\{a_{\lambda} b\right\}\right|_{\lambda=0} \quad \text { and } \quad\left\{\int a, \int b\right\}=\left.\int\left\{a_{\lambda} b\right\}\right|_{\lambda=0} \tag{0.24}
\end{equation*}
$$

As before, these maps have properties important for the theory of non-commutative Hamiltonian PDEs: $\overline{\mathcal{V}}$ is a Lie algebra and one has its representation on $\mathcal{V}$ by derivations, both defined by (0.24) (see Theorem 3.6). We have definitions of a Hamiltonian function $\int h \in \overline{\mathcal{V}}$, the corresponding Hamiltonian equation in $\mathcal{V}$, similar to $(0.13)$, and involutiveness, similar to $(0.14)$, where tr is replaced by $\int$.

We define an algebra of (non-commutative) differential functions as a unital associative algebra $\mathcal{V}$ with a derivation $\partial$ and strongly commuting 2 -fold derivations $\frac{\partial}{\partial u_{i}^{(n)}}, i=1, \ldots, \ell, n \in \mathbb{Z}_{+}$, such that the following two properties hold (cf. [BDSK09]):

$$
\begin{gather*}
\text { for each } f \in \mathcal{V}, \frac{\partial f}{\partial u_{i}^{(n)}}=0 \text { for all but finitely many }(i, n)  \tag{0.25}\\
{\left[\frac{\partial}{\partial u_{i}^{(n)}}, \partial\right]=\frac{\partial}{\partial u_{i}^{(n-1)}} .} \tag{0.26}
\end{gather*}
$$

The most important example of an algebra of non-commutative differential functions is the algebra of non-commutative differential polynomials $\mathcal{R}_{\ell}$ in the indeterminates $u_{i}^{(n)}, i=1, \ldots, \ell, n \in \mathbb{Z}_{+}$, with the derivation $\partial$ defined by $\partial u_{i}^{(n)}=u_{i}^{(n+1)}$.

First, we prove a PVA analogue of Theorem 2.8. Namely, any 2 -fold $\lambda$-bracket on $\mathcal{R}_{\ell}$ has the form (cf. (0.15)):

$$
\begin{equation*}
\left.\left\{f_{\lambda} g\right\}\right\}=\sum_{\substack{i, j \in I \\ m, n \in \mathbb{Z}_{+}}} \frac{\partial g}{\partial u_{j}^{(n)}} \bullet(\lambda+\partial)^{n} H_{i j}(\lambda+\partial)(-\lambda-\partial)^{m} \bullet\left(\frac{\partial f}{\partial u_{i}^{(m)}}\right)^{\sigma} \tag{0.27}
\end{equation*}
$$

where $H_{i j}(\lambda)=\left\{\left\{u_{j \lambda} u_{i}\right\}\right\} \in\left(\mathcal{R}_{\ell} \otimes \mathcal{R}_{\ell}\right)[\lambda]$. Formula (0.27) defines a 2-fold $\lambda$-bracket on any algebra of differential functions $\mathcal{V}$. This 2 -fold $\lambda$-bracket is skewsymmetric if and only if $H(\partial)=\left(H_{i j}(\partial)\right)_{i, j=1}^{\ell}$ is a skewadjoint differential operator over $\mathcal{V} \otimes \mathcal{V}$ and, provided that $H(\partial)$ is skewadjoint, the Jacobi identity holds if and only if it holds on any triple of generators (see Theorem 3.10).

Next, the non-commutative Hamiltonian PDE, associated to the matrix differential operator $H(\partial)$, defining the 2 -fold $\lambda$-bracket, and to the Hamiltionan functional $\int h \in \overline{\mathcal{V}}$ is the following evolution PDE, where $u=\left(u_{i}\right)_{i=1}^{\ell}($ cf. (0.16)):

$$
\begin{equation*}
\frac{d u}{d t}=\mathrm{m}\left(H(\partial) \bullet(\delta h)^{\sigma}\right) \tag{0.28}
\end{equation*}
$$

where $\delta h=\left(\frac{\delta h}{\delta u_{i}}\right)_{i=1}^{\ell} \in(\mathcal{V} \otimes \mathcal{V})^{\oplus \ell}$ is the vector of 2-fold variational derivatives

$$
\frac{\delta h}{\delta u_{i}}=\sum_{n \in \mathbb{Z}_{+}}(-\partial)^{n} \frac{\partial h}{\partial u_{i}^{(n)}}
$$

The bracket $\{-,-\}$ on $\overline{\mathcal{V}}$ is given by a formula, similar to (0.17):

$$
\begin{equation*}
\left\{\int f, \int g\right\}=\int \mathrm{m}\left(\delta g \bullet H(\partial) \bullet(\delta f)^{\sigma}\right) \tag{0.29}
\end{equation*}
$$

Furthermore the de Rham complex $\widetilde{\Omega}(\mathcal{V})$ over an algebra of differential functions $\mathcal{V}$ is defined in the same way as the de Rham complex $\widetilde{\Omega}(V)$ over an algebra of ordinary differential functions $V$. Moreover, the action of $\partial$ on $\mathcal{V}$ naturally extends to an action on $\widetilde{\Omega}(\mathcal{V})$, commuting with the de Rham differential $\delta$ (see Section 3.5). This allows us to define the variational complex

$$
\Omega(\mathcal{V})=\widetilde{\Omega}(\mathcal{V}) /(\partial \widetilde{\Omega}(\mathcal{V})+[\widetilde{\Omega}(\mathcal{V}), \widetilde{\Omega}(\mathcal{V})])
$$

In Section 3.6 we give an explicit description of the space of the variational $k$-forms $\Omega^{k}(\mathcal{V})$. In particular,

$$
\Omega^{0}(\mathcal{V})=\mathcal{V} /(\partial \mathcal{V}+[\mathcal{V}, \mathcal{V}]), \quad \Omega^{1}(\mathcal{V})=\mathcal{V}^{\oplus \ell}, \quad \Omega^{2}(\mathcal{V})=\left(\operatorname{Mat}_{\ell \times \ell}(\mathcal{V} \otimes \mathcal{V})[\lambda]\right)_{-}
$$

where the subscript _ means that $\left(A_{i j}(\lambda)\right)_{i, j=1}^{\ell}=-\left(A_{j i}(-\lambda-\partial)^{\sigma}\right)_{i, j=1}^{\ell}$. Under these identifications, the formula for $\delta$ is similar to (0.22):

$$
\begin{equation*}
\delta\left(\int f\right)=\mathrm{m}(\delta f)^{\sigma}, \quad(\delta F)(\partial)=\frac{1}{2}\left(D_{F}(\partial)^{t}-D_{F}(\partial)^{* \sigma}\right) \tag{0.30}
\end{equation*}
$$

where $\left(D_{F}(\partial)\right)_{i j}=\sum_{n \in \mathbb{Z}_{+}} \frac{\partial F_{i}}{\partial u_{j}^{(n)}} \partial^{n}$ is the 2-fold Frechet derivative.
We also prove that both complexes $\left(\widetilde{\Omega}\left(\mathcal{R}_{\ell}\right), \delta\right)$ and $\left(\Omega\left(\mathcal{R}_{\ell}\right), \delta\right)$ are acyclic. Hence, in particular, we obtain the description of the kernel and the image of the 2 -fold variational derivative $\delta$.

The main motivation for introducing double Poisson algebras was the observation that, given a unital finitely generated associative algebra $V$, each double Poisson algebra structure on $V$ associates for each positive integer $m$ a Poisson algebra structure on the algebra of polynomial functions $V_{m}$ on the affine algebraic variety of $m$-dimensional representations of the algebra $V[\mathrm{VdB} 08]$.

Note that $V_{m}$ is isomorphic to the commutative associative algebra with generators $\left\{a_{i j} \mid a \in V, i, j=1, \ldots, m\right\}$, subject to the relations (3.51) in Section 3.7.

If $\mathcal{V}$ is a (non-commutative) differential associative algebra with derivation $\partial$, then $\mathcal{V}_{m}$ is a commutative differential associative algebra with derivation defined on generators by $\partial\left(a_{i j}\right)=(\partial a)_{i j}$. Similarly to [VdB08], we show that, given a double PVA structure on $\mathcal{V}$, one can associate for each positive integer $m$ a PVA structure on $\mathcal{V}_{m}$ (see Theorem 3.22).

The simplest example of a double PVA is the following family of compatible double PVA structures on $\mathcal{R}_{1}=\mathbb{F}\left\langle u, u^{\prime}, u^{\prime \prime}, \ldots\right\rangle$ :

$$
\left\{\left\{u_{\lambda} u\right\}=1 \otimes u-u \otimes 1+c(1 \otimes 1) \lambda,\right.
$$

where $c \in \mathbb{F}$ is a parameter. In this case the associated PVA structure on $\left(\mathcal{R}_{1}\right)_{m}=$ $\mathbb{F}\left[u_{i j}^{(n)} \mid i, j=1, \ldots, m, n \in \mathbb{Z}_{+}\right]$is just the affine PVA for $\mathfrak{g l}_{m}$ (see Section 3.8.1). The extension of this double PVA structure on $\mathcal{R}_{1}$ to the double PVA structure on $\mathcal{R}_{2}=\mathbb{F}\left\langle u, v, u^{\prime}, v^{\prime}, \ldots\right\rangle$, where $v$ is central, produces the following hierarchy of non-commutative Hamiltonian PDE's $\left(n \in \mathbb{Z}_{+}\right)$:

$$
\frac{d u}{d t_{n}}=v(u+v)^{n} u-u(u+v)^{n} v+c \partial(u+v)^{n+1}, \quad \frac{d v}{d t_{n}}=0 .
$$

Our next example is a compatible pair of double PVA structures on the algebra $\mathcal{R}_{N}$ (resp. $\mathcal{R}_{\infty}$ ), associated to the "generic" differential (resp. pseudodifferential) operator with coefficients in $\mathcal{R}_{N}$ (resp. $\mathcal{R}_{\infty}$ ), similar to the PVA structures in the commutative case, considered in [DSKV14a] (see Section 3.8.2).

We apply for the latter example the theory of Dirac reduction for (non-local) double PVA developed in Section 4.2, similar to that for (non-local) PVA, developed in [DSKV14b]. As a result, we obtain compatible non-local double Poisson vertex algebra structures on $\mathcal{V}=\mathcal{R}_{N}$ and $\mathcal{R}_{\infty}$, such that the corresponding non-local PVA structures on $\mathcal{V}_{m}$ coincide with those in [DSKV14a] (see Example 4.11).

Finally, in Section 5 we apply the Lenard-Magri scheme to the double PVA constructed in Section 3.8.2 to prove integrability of the non-commutative KP hierarchy and of non-commutative Gelfand-Dickey hierarchies. In particular, we prove integrability of the non-commutative KdV and Boussinesq hierarchies.

Throughout the paper we let $\mathbb{F}$ be a field of characteristic 0 , and, unless otherwise specified, we consider all vector spaces, tensor products, etc., over the field $\mathbb{F}$.

Acknowledgments. We are grateful to Vladimir Sokolov for introducing us to the fascinating subject of non-commutative integrable equations.

Part of the work was completed during the visits of the second and third author to the University of Rome La Sapienza, during the visit of the first author to SISSA in Trieste, and during the visits of all three authors to IHES, France. We thank all these institutions for their kind hospitality. The first author is supported by the national FIRB grant RBFR12RA9W "Perspectives in Lie Theory", and the national PRIN grant number 2012KNL88Y_001. The second author is supported by a national NSF grant. The third author was supported by the ERC grant "FroM-PDE: Frobenius Manifolds and Hamiltonian Partial Differential Equations".

## 1. Preliminaries and notation for associative algebras

Throughout this and the next section, we let $V$ be an associative algebra over the field $\mathbb{F}$.
1.1. "Double" linear algebra. Consider the associative product • on $V \otimes V$ given, in Sweedler's notation, by the formula

$$
\begin{equation*}
A \bullet B=A^{\prime} B^{\prime} \otimes B^{\prime \prime} A^{\prime \prime} \tag{1.1}
\end{equation*}
$$

(It is the product in $V \otimes V^{o p}$.) The endomorphism $\sigma$ of $V \otimes V$ obtained by exchanging the two factors:

$$
\begin{equation*}
(a \otimes b)^{\sigma}=b \otimes a \tag{1.2}
\end{equation*}
$$

is an antiautomorphism of the $\bullet$-product:

$$
\begin{equation*}
(A \bullet B)^{\sigma}=B^{\sigma} \bullet A^{\sigma} \tag{1.3}
\end{equation*}
$$

We define an inner product $(\cdot \mid \cdot)$ on the vector space $(V \otimes V)^{\oplus \ell}$ with values in the algebra $V \otimes V$, letting for $F=\left(F_{i}\right)_{i=1}^{\ell}$ and $G=\left(G_{i}\right)_{i=1}^{\ell}\left(F_{i}, G_{i} \in V \otimes V\right)$ :

$$
\begin{equation*}
(F \mid G)=\sum_{i=1}^{\ell} F_{i} \bullet G_{i}^{\sigma} \tag{1.4}
\end{equation*}
$$

By (1.3), it satisfies the following symmetry property:

$$
\begin{equation*}
(G \mid F)^{\sigma}=(F \mid G) \tag{1.5}
\end{equation*}
$$

Let $H=\left(H_{i j}\right)_{i, j=1}^{\ell} \in \operatorname{Mat}_{\ell \times \ell}(V \otimes V)$. We define its action on $(V \otimes V)^{\oplus \ell}$ by the following formula:

$$
\begin{equation*}
(H F)_{i}=\sum_{j=1}^{\ell}\left(H_{i j} \bullet F_{j}^{\sigma}\right)^{\sigma} . \tag{1.6}
\end{equation*}
$$

In Sweedler's notation, it is $(H F)_{i}=\sum_{j=1}^{\ell} F_{j}^{\prime} H_{i j}^{\prime \prime} \otimes H_{i j}^{\prime} F_{j}^{\prime \prime}$. The adjoint of the matrix $H$ is defined by

$$
\begin{equation*}
\left(H^{\dagger}\right)_{i j}=\left(H_{j i}\right)^{\sigma} \tag{1.7}
\end{equation*}
$$

It is adjoint with respect to the inner product (1.4) in the usual sense:

$$
\begin{equation*}
(F \mid H G)=\left(H^{\dagger} F \mid G\right), \quad F, G \in(V \otimes V)^{\oplus \ell} \tag{1.8}
\end{equation*}
$$

1.2. Trace map and reduction. Denote by $\mathrm{m}: V \otimes V \rightarrow V$ (or, in general, $\left.\mathrm{m}: V^{\otimes n} \rightarrow V\right)$ the multiplication map:

$$
\mathrm{m}(a \otimes b)=a b
$$

We define the following "reduction" map $(V \otimes V)^{\oplus \ell} \rightarrow V^{\oplus \ell}$ by

$$
\begin{equation*}
F \mapsto \mathrm{~m}\left(F^{\sigma}\right)\left(=\left(\mathrm{m}\left(F_{i}^{\sigma}\right)\right)_{i=1}^{\ell}\right) \tag{1.9}
\end{equation*}
$$

Furthermore, let [ $V, V$ ] be the subspace of $V$ spanned by commutators $a b-b a$, for $a, b \in V$, and denote by

$$
\begin{equation*}
\operatorname{tr}: V \rightarrow V /[V, V] \tag{1.10}
\end{equation*}
$$

the canonical quotient map.
The inner product (1.4) induces a well defined inner product $(\cdot \mid \cdot): V^{\oplus \ell} \times V^{\oplus \ell} \rightarrow$ $V /[V, V]$ so that the following diagram is commutative:


It is given explicitly by

$$
\begin{equation*}
(F \mid G)=\sum_{i=1}^{\ell} \operatorname{tr}\left(F_{i} G_{i}\right), \quad F, G \in V^{\oplus \ell} \tag{1.11}
\end{equation*}
$$

This bilinear form is obviously symmetric.
Furthermore, the action (1.6) of the matrix $H \in \operatorname{Mat}_{\ell \times \ell}(V \otimes V)$ induces an action on $V^{\oplus \ell}$ via the commutative diagram


Explicitly, it is given, in Sweedler's notation, by

$$
\begin{equation*}
(H F)_{i}=\sum_{j=1}^{\ell} H_{i j}^{\prime} F_{j} H_{i j}^{\prime \prime}, \quad F \in V^{\oplus \ell} \tag{1.12}
\end{equation*}
$$

1.3. $V$-module structures of $V^{\otimes n}$. The space $V^{\otimes n}$ has the following outer $V$ bimodule structure:

$$
\begin{equation*}
a\left(b_{1} \otimes \cdots \otimes b_{n}\right) c=\left(a b_{1}\right) \otimes b_{2} \cdots \otimes b_{n-1} \otimes\left(b_{n} c\right) \tag{1.13}
\end{equation*}
$$

More generally, for every $i=0, \ldots, n-1$, we define the $i$-th left and right $V$-module structures of $V^{\otimes n}$ by

$$
\begin{align*}
& a \star_{i}\left(b_{1} \otimes \cdots \otimes b_{n}\right)=b_{1} \otimes \cdots \otimes b_{i} \otimes a b_{i+1} \otimes \cdots \otimes b_{n}, \\
& \left(a_{1} \otimes \cdots \otimes a_{n}\right) \star_{i} b=a_{1} \otimes \cdots \otimes a_{n-i} b \otimes \cdots \otimes a_{n} . \tag{1.14}
\end{align*}
$$

(The index denotes the number of "jumps".) In particular, the bimodule structure (1.13) is given by $a B c=a \star_{0} B \star_{0} c$.

We use a similar notation for $\otimes$-product of an element of $V$ and an element of $V^{\otimes n}$ :

$$
\begin{align*}
& a \otimes_{i}\left(b_{1} \otimes \cdots \otimes b_{n}\right)=b_{1} \otimes \cdots \otimes b_{i} \otimes a \otimes b_{i+1} \otimes \cdots \otimes b_{n},  \tag{1.15}\\
& \left(a_{1} \otimes \cdots \otimes a_{n}\right) \otimes_{i} b=a_{1} \otimes \cdots \otimes a_{n-i} \otimes b \otimes \cdots \otimes a_{n} .
\end{align*}
$$

For example, the action (1.12) of $H \in \operatorname{Mat}_{\ell \times \ell}(V \otimes V)$ on $V^{\oplus \ell}$ can be rewritten, using this notation, as $(H F)_{i}=\sum_{j=1}^{\ell} \mathrm{m}\left(H_{i j} \otimes_{1} F_{j}\right)\left(=\sum_{j=1}^{\ell} \mathrm{m}\left(F_{j} \otimes_{1} H_{i j}\right)=\right.$ $\left.\sum_{j=1}^{\ell} \mathrm{m}\left(H_{i j} \star_{1} F_{j}\right)=\sum_{j=1}^{\ell} \mathrm{m}\left(F_{j} \star_{1} H_{i j}\right)\right)$.

Note that, for $f \in V$ and $A, B \in V^{\otimes 2}$, we have

$$
\begin{align*}
& (f A) \bullet B=f(A \bullet B), \quad(A f) \bullet B=(A \bullet B) f,  \tag{1.16}\\
& A \bullet\left(f \star_{1} B\right)=f \star_{1}(A \bullet B), \quad A \bullet\left(B \star_{1} f\right)=(A \bullet B) \star_{1} f .
\end{align*}
$$

We also will be using the multiplication map $V^{\otimes m} \times V^{\otimes n} \rightarrow V^{\otimes(m+n-1)}$, given by

$$
\begin{equation*}
\left(a_{1} \otimes \cdots \otimes a_{m}\right)\left(b_{1} \otimes \cdots \otimes b_{n}\right)=a_{1} \otimes \cdots \otimes a_{m-1} \otimes\left(a_{m} b_{1}\right) \otimes b_{2} \cdots \otimes b_{n} \tag{1.17}
\end{equation*}
$$

1.4. Some further notation. We define three possible left and right actions of the algebra $V^{\otimes 2}$ on $V^{\otimes 3}$, denoted by $\bullet_{i}, i=1,2,3$, as follows (here the subscript indicates the place where no multiplication occurs)

$$
\begin{align*}
& (a \otimes b) \bullet_{1}(x \otimes y \otimes z)=x \otimes a y \otimes z b,(x \otimes y \otimes z) \bullet_{1}(a \otimes b)=x \otimes y a \otimes b z, \\
& (a \otimes b) \bullet_{2}(x \otimes y \otimes z)=a x \otimes y \otimes z b,(x \otimes y \otimes z) \bullet_{2}(a \otimes b)=x a \otimes y \otimes b z, \\
& (a \otimes b) \bullet_{3}(x \otimes y \otimes z)=a x \otimes y b \otimes z,(x \otimes y \otimes z) \bullet_{3}(a \otimes b)=x a \otimes b y \otimes z . \tag{1.18}
\end{align*}
$$

Lemma 1.1. (a) The $\bullet_{i}$ left (and right) actions of $V^{\otimes 2}$ on $V^{\otimes 3}$ are indeed actions, i.e. they are associative with respect to the $\bullet$-product of $V^{\otimes 2}$ :

$$
A \bullet_{i}\left(B \bullet_{i} X\right)=(A \bullet B) \bullet_{i} X \quad \text { and }\left(X \bullet_{i} A\right) \bullet_{i} B=X \bullet_{i}(A \bullet B),
$$ for every $A, B \in V^{\otimes 2}$ and $X \in V^{\otimes 3}$.

(b) The left $\bullet_{i}$ and the right $\bullet_{j}$ actions commute for every $i, j=1,2,3$ :

$$
A \bullet_{i}\left(X \bullet_{j} B\right)=\left(A \bullet_{i} X\right) \bullet_{j} B
$$

for every $A, B \in V^{\otimes 2}$ and $X \in V^{\otimes 3}$.
(c) The $\bullet_{1}$ and $\bullet_{3}$ left (resp. right) actions of $V^{\otimes 2}$ on $V^{\otimes 3}$ commute:

$$
A \bullet_{1}\left(B \bullet_{3} X\right)=B \bullet_{3}\left(A \bullet_{1} X\right) \text { and }\left(X \bullet_{1} A\right) \bullet_{3} B=\left(X \bullet_{3} B\right) \bullet_{1} A,
$$

for every $A, B \in V^{\otimes 2}$ and $X \in V^{\otimes 3}$. (In general, the $\bullet_{i}$ and $\bullet_{j}$ left (resp. right) actions do NOT commute if $|i-j|=1$.

Proof. Straightforward.
Lemma 1.2. For every $A \in V^{\otimes 2}$ and $X \in V^{\otimes 3}$ we have

$$
\begin{array}{lll}
\left(A \bullet_{1} X\right)^{\sigma}=X^{\sigma} \bullet_{2} A^{\sigma}, & \left(A \bullet_{2} X\right)^{\sigma}=X^{\sigma} \bullet_{3} A^{\sigma}, & \left(A \bullet_{3} X\right)^{\sigma}=A \bullet_{1} X^{\sigma}, \\
\left(A \bullet_{1} X\right)^{\sigma^{2}}=A \bullet_{3} X^{\sigma^{2}}, & \left(A \bullet_{2} X\right)^{\sigma^{2}}=X^{\sigma^{2}} \bullet_{1} A^{\sigma}, & \left(A \bullet_{3} X\right)^{\sigma^{2}}=X^{\sigma^{2}} \bullet_{2} A^{\sigma}, \\
\left(X \bullet_{1} A\right)^{\sigma}=A^{\sigma} \bullet_{2} X^{\sigma}, & \left(X \bullet_{2} A\right)^{\sigma}=A^{\sigma} \bullet_{3} X^{\sigma}, & \left(X \bullet_{3} A\right)^{\sigma}=X^{\sigma} \bullet_{1} A, \\
\left(X \bullet_{1} A\right)^{\sigma^{2}}=X^{\sigma^{2}} \bullet_{3} A, & \left(X \bullet_{2} A\right)^{\sigma^{2}}=A^{\sigma} \bullet_{1} X^{\sigma^{2}}, & \left(X \bullet_{3} A\right)^{\sigma^{2}}=A^{\sigma} \bullet_{2} X^{\sigma^{2}} . \tag{1.19}
\end{array}
$$

Proof. Straightforward.

## 1.5. $n$-fold derivations.

Definition 1.3. An $n$-fold derivation of $V$ is a linear map $D: V \rightarrow V^{\otimes n}$ such that

$$
\begin{equation*}
D(a b)=(D a) b+a(D b), \quad a, b \in V \tag{1.20}
\end{equation*}
$$

where, in the RHS, we use the bimodule structure (1.13).
We can extend the $n$-fold derivation $D: V \rightarrow V^{\otimes n}$ to a map $D: V^{\otimes m} \rightarrow$ $V^{\otimes(m+n-1)}$ for every $m \in \mathbb{Z}_{+}$, by

$$
\begin{equation*}
D\left(a_{1} \otimes \cdots \otimes a_{m}\right)=\sum_{i=1}^{m} a_{1} \otimes \cdots \otimes\left(D a_{i}\right) \otimes \cdots \otimes a_{m} \tag{1.21}
\end{equation*}
$$

where, in the RHS, we use the concatenation product in the tensor algebra over $V$. We can also extend the $n$-fold derivation $D$ to a map $V^{\otimes m} \rightarrow V^{\otimes m+n-1}$ by applying $D$ to only the $i$-th factor, and we denote the corresponding map by $D_{(i)}$ :

$$
\begin{equation*}
D_{(i)}\left(a_{1} \otimes \cdots \otimes a_{m}\right)=a_{1} \otimes \cdots \otimes D\left(a_{i}\right) \otimes \cdots \otimes a_{m} \tag{1.22}
\end{equation*}
$$

We will use the special notation for the derivation on the leftmost and the rightmost factors:

$$
\begin{equation*}
D_{L}:=D_{(1)} \text { and } D_{R}:=D_{(m)} \tag{1.23}
\end{equation*}
$$

The symmetric group $S_{n}$ acts on $V^{\otimes n}$ in the usual way: for $\tau \in S_{n}$, we have

$$
\begin{equation*}
\left(a_{1} \otimes \cdots \otimes a_{n}\right)^{\tau}=a_{\tau^{-1}(1)} \otimes \cdots \otimes a_{\tau^{-1}(n)} \tag{1.24}
\end{equation*}
$$

For every $n$, we shall denote by $\sigma$ the cyclic permutation $(1, \ldots, n)$, so that

$$
\begin{equation*}
\left(a_{1} \otimes \cdots \otimes a_{n}\right)^{\sigma}=a_{n} \otimes a_{1} \otimes \cdots \otimes a_{n-1} \tag{1.25}
\end{equation*}
$$

For convenience, we write here the formulas for the action of the whole cyclic group $C_{n}$ on $A^{\otimes n}(s=1, \ldots, n)$ :

$$
\begin{equation*}
\left(a_{1} \otimes \cdots \otimes a_{n}\right)^{\sigma^{s}}=a_{n+1-s} \otimes \cdots \otimes a_{n} \otimes a_{1} \otimes \cdots \otimes a_{n-s} \tag{1.26}
\end{equation*}
$$

and on an $n$-tuple of indices $\left(i_{1}, \ldots, i_{n}\right) \in\{1, \ldots, \ell\}^{n}$ :

$$
\begin{equation*}
\left(i_{\sigma^{-s}(1)}, \ldots, i_{\sigma^{-s}(n)}\right)=\left(i_{n+1-s}, \ldots, i_{n}, i_{1}, \ldots, i_{n-s}\right) \tag{1.27}
\end{equation*}
$$

Lemma 1.4. Let $D: V \rightarrow V^{\otimes n}$ be an $n$-fold derivation of $V$. For $A \in V^{\otimes m}$, we have

$$
\begin{align*}
& \left(D_{(i)} A\right)^{\sigma}=D_{(i+1)}\left(A^{\sigma}\right) \text { for } 1 \leq i \leq m-1, \\
& \left(D_{(m)} A\right)^{\sigma^{n}}=D_{(1)}\left(A^{\sigma}\right) \tag{1.28}
\end{align*}
$$

Proof. Straightforward.
Lemma 1.5. If $D: V \rightarrow V^{\otimes n}$ is an $n$-fold derivation of $V$, then, for every $A \in V^{\otimes h}$ and $B \in V^{\otimes k}\left(h, k \in \mathbb{Z}_{+}\right)$, we have

$$
\begin{equation*}
D(A B)=(D A) B+A(D B) \tag{1.29}
\end{equation*}
$$

where, in the RHS, we use the multiplication map (1.17).
Proof. Straightforward.
Given an $m$-fold derivation $D_{1}: V \rightarrow V^{\otimes m}$ and an $n$-fold derivation $D_{2}: V \rightarrow$ $V^{\otimes n}$, we can compose them, using their extension (1.21), to get a map

$$
\begin{equation*}
D_{1} \circ D_{2}: V \rightarrow V^{\otimes m+n-1}, \quad\left(D_{1} \circ D_{2}\right)(a)=D_{1}\left(D_{2} a\right) \tag{1.30}
\end{equation*}
$$

Proposition 1.6. Let $D_{1}$ and $D_{2}$ be respectively an $m$-fold and an $n$-fold derivation of $V$. Then:
(a) The commutator $\left[D_{1}, D_{2}\right]=D_{1} \circ D_{2}-D_{2} \circ D_{1}$ is an $m+n-1$-fold derivation of $V$.
(b) $\left(D_{1}\right)_{L} \circ D_{2}-\left(D_{2}\right)_{R} \circ D_{1}$ is an $m+n-1$-fold derivation of $V$.

Proof. Part (a) follows from Lemma 1.5. Using the fact that $D_{2}$ is an $n$-fold derivation, and by the definition (1.23) of $\left(D_{1}\right)_{L}$, we have $(a, b \in V)$

$$
\begin{align*}
& \left(\left(D_{1}\right)_{L} \circ D_{2}\right)(a b)=\left(D_{1}\right)_{L}\left(D_{2}(a) b\right)+\left(D_{1}\right)_{L}\left(a D_{2}(b)\right)  \tag{1.31}\\
& =\left(\left(D_{1}\right)_{L}\left(D_{2} a\right)\right) b+\left(D_{1} a\right)\left(D_{2} b\right)+a\left(\left(D_{1}\right)_{L}\left(D_{2} b\right)\right),
\end{align*}
$$

where in the second term we are using the multiplication in the tensor algebra given by (1.17). Similarly we get

$$
\begin{equation*}
\left(\left(D_{2}\right)_{R} \circ D_{1}\right)(a b)=\left(\left(D_{2}\right)_{R}\left(D_{1} a\right)\right) b+\left(D_{1} a\right)\left(D_{2} b\right)+a\left(\left(D_{2}\right)_{R}\left(D_{1} b\right)\right) . \tag{1.32}
\end{equation*}
$$

Combining equations (1.31) and (1.32) we get part (b).
Remark 1.7. Suppose that the algebra $V$ is counital (with counity $\epsilon \in V^{*}$ ), so that $\mathbb{F}$ is an $A$-bimodule (using the counit). Then, by Proposition 1.6, we have a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}=\bigoplus_{n=-1}^{\infty} \mathfrak{g}_{n}$, where $\mathfrak{g}_{n}$ is the space of $n+1$-fold derivations of $V$.

## 1.6. $n$-fold brackets.

Definition 1.8. An $n$-fold bracket on $V$ is a linear map $\left\{\{-, \cdots,-\}: A^{\otimes n} \rightarrow A^{\otimes n}\right.$ satisfying the following Leibniz rules (using the notation (1.14)):

$$
\begin{equation*}
\left.\left\{\left\{a_{1}, \ldots, b c, \ldots, a_{n}\right\}\right\}=b \star_{i}\left\{\left\{a_{1}, \ldots, c, \ldots, a_{n}\right\}\right\}+\left\{a_{1}, \ldots, b, \ldots, a_{n}\right\}\right\} \star_{n-i} c, \tag{1.33}
\end{equation*}
$$

for all $i=1 \ldots, n$ (in the first term of the RHS we let $\star_{n}=\star_{0}$ ).
Note that, if $\left\{\{-, \cdots,-\}\right.$ is an $n$-fold bracket, then, for every $a_{1}, \ldots, a_{n-1} \in$ $V$, the map $\left\{\left\{a_{1}, \ldots, a_{n-1},-\right\}: V \rightarrow V^{\otimes n}\right.$ is an $n$-fold derivation of $V$. Then, according to the notation (1.23), we let, for $a_{1}, \ldots, a_{n-1}, b_{1}, \ldots, b_{m} \in V$,

$$
\begin{align*}
& \left\{\left\{a_{1}, \ldots, a_{n-1}, b_{1} \otimes \cdots \otimes b_{m}\right\}_{L}=\left\{\left\{a_{1}, \ldots, a_{n-1}, b_{1}\right\}\right\} \otimes b_{2} \otimes \cdots \otimes b_{m},\right.  \tag{1.34}\\
& \left\{\left\{a_{1}, \ldots, a_{n-1}, b_{1} \otimes \cdots \otimes b_{m}\right\}_{R}=b_{1} \otimes \cdots \otimes b_{m-1} \otimes\left\{a_{1}, \ldots, a_{n-1}, b_{m}\right\} .\right.
\end{align*}
$$

As a special case of Lemma 1.4, we get the following result, which will be used later
Corollary 1.9. Given a 2 -fold bracket $\{\{-,-\}$ on $V$, we have

$$
\begin{equation*}
\{\{a,\{\{b, c\}\}\}\}_{R}=\left\{\left\{a,\{\{b, c\}\}^{\sigma}\right\}_{L}^{\sigma}\right. \text {. } \tag{1.35}
\end{equation*}
$$

Proof. Equation (1.35) is a special case of equation (1.28) for $D=\{a,-\}: V \rightarrow$ $V^{\otimes 2}$.

The following result will be used later as well.
Lemma 1.10. Let $\{[-,-\}$ be a 2-fold bracket on the associative algebra $V$. For every $a \in V$ and $B, C \in V^{\otimes 2}$, we have

$$
\{\{a, B \bullet C\}\}_{L}=B \bullet_{2}\left\{\{a, C\}_{L}+\left\{\{a, B\}_{L} \bullet_{1} C,\right.\right.
$$

where in the LHS we use the product (1.1) while in the RHS we use notation (1.18). Proof. Straightforward.

## 2. Double Poisson algebras and non-commutative Hamiltonian ODEs

### 2.1. Definition of double Poisson algebras.

Definition 2.1 ([VdB08]). A double Poisson algebra is an associative algebra $V$ endowed with a 2-fold bracket $\{-,-\}: V \otimes V \rightarrow V \otimes V$ satisfying the following axioms:
(i) skewsymmetry: $\left\{\{a, b\}=-\{\{b, a\}\}^{\sigma}\right.$,
(ii) Jacobi identity:

$$
\begin{equation*}
\{a,\{\{b, c\}\}\}_{L}+\left\{\left\{b,\{\{c, a\}\}_{L}^{\sigma}+\left\{\{c,\{\{a, b\}\}\}_{L}^{\sigma^{2}}=0\right. \text {. }\right.\right. \tag{2.1}
\end{equation*}
$$

(Here we are using notation (1.34), i.e. we let $\{a, b \otimes c\}\}_{L}=\{a, b\} \otimes c \in$ $V \otimes V \otimes V$.

We can write explicitly the Leibniz rules for a double Poisson algebra bracket: equation (1.33) with $i=2$ and 1 reads, respectively,

$$
\begin{equation*}
\{\{a, b c\}=\{\{a, b\} c+b\{\{a, c\}, \tag{2.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\{a b, c\}\}=\left\{\{a, c\} \star_{1} b+a \star_{1}\{\{b, c\}\} .\right. \tag{2.3}
\end{equation*}
$$

Note that equation (2.3) follows from (2.2) and the skewsymmetry axiom.
Remark 2.2. We can write the Jacobi identity (2.1) in an alternative way, analogous to the usual "representation theoretic form" of the Jacobi identity for Lie algebras. For this we extend the notation (1.34) of $\left\{\{-,-\}_{L}\right.$ and $\left\{\{-,-\}_{R}\right.$ to the case when we have an element of $V \otimes V$ in the first entry. We do this in analogy to the two terms in the RHS of (2.3): using notation (1.15), we let

$$
\begin{equation*}
\{a \otimes b, c\}_{L}:=\left\{\{a, c\} \otimes_{1} b, \quad\{a \otimes b, c\}_{R}:=a \otimes_{1}\{\{b, c\}\} .\right. \tag{2.4}
\end{equation*}
$$

With this notation, it is not hard to check, using skewsymmetry, that the Jacobi identity (2.1) can be equivalently written as follows:

$$
\begin{equation*}
\left\{\{a,\{\{b, c\}\}\}_{L}-\{\{b,\{\{a, c\}\}\}\}_{R}=\{\{\{a, b\}\}, c\}\right\}_{L} . \tag{2.5}
\end{equation*}
$$

Lemma 2.3 ([VdB08]). If $\left\{\{-,-\}: V \rightarrow V^{\otimes 2}\right.$ is a 2-fold bracket satisfying the skewsymmetry axiom, then the LHS of the Jacobi identity (2.1)

$$
\begin{equation*}
\{a, b, c\}\}:=\{a,\{\{b, c\}\}\}_{L}+\left\{\left\{b,\{\{c, a\}\}_{L}^{\sigma}+\left\{\{c,\{\{a, b\}\}\}_{L}^{\sigma^{2}}\right.\right.\right. \tag{2.6}
\end{equation*}
$$

is a 3-fold bracket on $V$.
Proof. Using the Leibniz rules for the 2-fold bracket \{\{-, -\}, we have

$$
\begin{align*}
& \left\{\{a,\{\{b, c d\}\}\}_{L}=\left\{\{a,\{\{b, c\}\} d\}_{L}+\left\{\{a, c\{\{b, d\}\}\}_{L}\right.\right.\right.  \tag{2.7}\\
& =\left\{\{a,\{\{b, c\}\}\}_{L} d+\{\{a, c\}\}\{\{b, d\}\}+c\left\{\{a,\{\{b, d\}\}\}_{L},\right.\right.
\end{align*}
$$

$$
\begin{align*}
& \left\{\{b,\{\{c d, a\}\}\}_{L}^{\sigma}=\left(\left\{\left\{b, c \star_{1}\{\{d, a\}\}\right\}_{L}+\left\{\left\{b,\left\{\{c, a\} \star_{1} d\right\}_{L}\right)^{\sigma}\right.\right.\right.\right. \\
& =\left(c \star _ { 2 } \left\{\{b,\{\{d, a\}\}\}_{L}+\left\{\left\{b,\{\{c, a\}\}_{L} \star_{1} d+\{\{c, a\}\}^{\prime}\{b, d\}\right\}\{c, a\}^{\prime \prime}\right)^{\sigma}\right.\right.  \tag{2.8}\\
& =c\left\{\{b,\{\{d, a\}\}\}_{L}^{\sigma}+\left\{\{b,\{\{c, a\}\}\}_{L}^{\sigma} d+\{\{c, a\}\}^{\sigma}\{\{b, d\}\} .\right.\right.
\end{align*}
$$

Hereafter we use Sweedler's notation: for $A \in V^{\otimes 2}$, we let $A=A^{\prime} \otimes A^{\prime \prime}$ (omitting the sign of summation). Similarly,

$$
\begin{align*}
& \left\{\{c d,\{\{a, b\}\}\}_{L}^{\sigma^{2}}=\left(c \star _ { 1 } \left\{\{d,\{\{a, b\}\}\}_{L}+\left\{\{c,\{\{a, b\}\}\}_{L} \star_{2} d\right)^{\sigma^{2}}\right.\right.\right. \\
& =c\left\{\{d,\{\{a, b\}\}\}_{L}^{\sigma^{2}}+\{\{c,\{\{a, b\}\}\}\}_{L}^{\sigma^{2}} d .\right. \tag{2.9}
\end{align*}
$$

Combining equations (2.7), (2.8) and (2.9), and using the skewsymmetry assumption on $\{\{-,-\}$, we get

$$
\begin{equation*}
\{\{a, b, c d\}\}=c\{\{a, b, d\}\}+\{\{a, b, c\}\} d, \tag{2.10}
\end{equation*}
$$

i.e. the Leibniz rule holds on the third entry. The Leibniz rules (1.33) on the first and second entry follow from (2.10) and the following obvious identities $(a, b, c \in V)$

$$
\begin{equation*}
\{a, b, c\}\}=\{\{b, c, a\}\}^{\sigma}=\{\{c, a, b\}\}^{\sigma^{2}} . \tag{2.11}
\end{equation*}
$$

2.2. The trace map and connection to Lie algebras. Let $\{\{-,-\}$ be a 2 -fold bracket on the associative algebra $V$. By composing it with the multiplication map $\mathrm{m}: V \otimes V \rightarrow V$, we obtain the corresponding map $\{-,-\}: V \otimes V \rightarrow V:$

$$
\begin{equation*}
\{a, b\}=\mathrm{m}\{\{a, b\}\} . \tag{2.12}
\end{equation*}
$$

It is immediate to check that this bracket satisfies the Leibniz rule on the second entry:

$$
\begin{equation*}
\{a, b c\}=b\{a, c\}+\{a, b\} c . \tag{2.13}
\end{equation*}
$$

(While, in general, it does not satisfies the Leibniz rule on the first entry.) Hence, for every $a \in V$, the map $\{a,-\}: V \rightarrow V$ is a (1-fold) derivation, and we extend it to a map $\{a,-\}: V^{\otimes n} \rightarrow V^{\otimes n}$, for every $n \in \mathbb{Z}_{+}$, via (1.21).

Lemma 2.4 ([VdB08]). If $\{[-,-\}$ is a skewsymmetric 2 -fold bracket on $V$, then the following identity holds in $V^{\otimes 2}(a, b, c \in V)$

$$
\begin{equation*}
\{a,\{\{b, c\}\}\}-\{\{a, b\}, c\}\}-\{\{b,\{a, c\}\}\}=(\mathrm{m} \otimes 1)\{a, b, c\}\}-(1 \otimes \mathrm{~m})\{\{b, a, c\}, \tag{2.14}
\end{equation*}
$$

where $\{a, b, c\}$ is as in (2.6).
Proof. Let us compute the three terms appearing in the LHS of equation (2.14). Using the Leibniz rules we get, after a straightforward computation,

$$
\begin{align*}
& \{a,\{\{b, c\}\}\}=(\mathrm{m} \otimes 1)\{a,\{\{b, c\}\}\}_{L}+(1 \otimes \mathrm{~m})\left\{\{a,\{\{b, c\}\}\}_{R}\right.  \tag{2.15}\\
& =(\mathrm{m} \otimes 1)\left\{\{a,\{\{b, c\}\}\}_{L}-(1 \otimes \mathrm{~m})\left\{\{a,\{\{c, b\}\}\}_{L}^{\sigma} .\right.\right.
\end{align*}
$$

For the last equality we used Corollary 1.9 and the skewsymmetry assumption. With similar computations, we get

$$
\begin{equation*}
\{\{b,\{a, c\}\}\}=(1 \otimes \mathrm{~m})\{b,\{\{a, c\}\}\}_{L}-(\mathrm{m} \otimes 1)\left\{\{b,\{\{c, a\}\}\}_{L}^{\sigma},\right. \tag{2.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\{\{a, b\}, c\}\}=-\{c,\{a, b\}\}\}^{\sigma}=-(\mathrm{m} \otimes 1)\left\{\{c,\{\{a, b\}\}\}_{L}^{\sigma^{2}}+(1 \otimes \mathrm{~m})\left\{\{c,\{\{b, a\}\}\}_{L}^{\sigma^{2}} .\right.\right. \tag{2.17}
\end{equation*}
$$

Combining equations (2.15), (2.17) and (2.16), and using the definition (2.6) of $\{a, b, c\}$, equation (2.14) follows, thus concluding the proof.
Proposition 2.5 ([VdB08]). (a) If $\{\{-,-\}$ is a 2 -fold bracket on $V$, then the associated bracket $\{-,-\}$ defined in (2.12) induces well defined maps (which we denote by the same symbol) $\{-,-\}: V /[V, V] \otimes V \rightarrow V$, given by (recalling (1.10))

$$
\begin{equation*}
\{\operatorname{tr}(a), b\}:=\{a, b\} \tag{2.18}
\end{equation*}
$$

and $\{-,-\}: V /[V, V] \otimes V /[V, V] \rightarrow V /[V, V]$, given by

$$
\begin{equation*}
\{\operatorname{tr}(a), \operatorname{tr}(b)\}:=\operatorname{tr}\{a, b\} \tag{2.19}
\end{equation*}
$$

(b) If $V$ is a double Poisson algebra, then $V /[V, V]$ is a Lie algebra with the bracket defined by (2.19), and (2.18) defines a representation of this Lie algebra by derivations of $V$.

Proof. First, we have, using Sweedler's notation,

$$
\begin{equation*}
\{a b, c\}=\left\{\{ a , c \} ^ { \prime } b \left\{\{a, c\}^{\prime \prime}+\{\{b, c\}\}^{\prime} a\{\{b, c\}\}^{\prime \prime}=\{b a, c\} .\right.\right. \tag{2.20}
\end{equation*}
$$

Moreover, by (2.13), we have

$$
\begin{equation*}
\{a, b c\}-\{a, c b\}=[\{a, b\}, c]+[b,\{a, c\}] \in[V, V] . \tag{2.21}
\end{equation*}
$$

Equations (2.20) and (2.21) imply that both brackets (2.18) and (2.19) are well defined, proving (a).

The skewsymmetry for the bracket $\{-,-\}$ on $V /[V, V]$ follows immediately by the skewsymmetry axiom for the 2 -fold bracket $\{[-,-\}$. By the Jacobi identity axiom (in Definition 2.1), the RHS of equation (2.14) is identically zero. Applying the multiplication map $m$ to its LHS, we obtain the Jacobi identity for the bracket $\{-,-\}$ of $V /[V, V]$, and, at the same time, the claim that (2.18) defines a representation of the Lie algebra $V /[V, V]$ on $V$. Finally, this Lie algebra action is by derivations of the associative algebra $V$, thanks to equation (2.13).
2.3. Double Poisson brackets on an algebra of (non-commutative) ordinary differential functions. Consider the algebra of non-commutative polynomials $R_{\ell}=\mathbb{F}\left\langle x_{1}, \ldots, x_{\ell}\right\rangle$, i.e. the free unital associative algebra in the indeterminates $x_{1}, \ldots, x_{\ell}$. Let $\frac{\partial}{\partial x_{i}}: R_{\ell} \rightarrow R_{\ell} \otimes R_{\ell}, i=1, \ldots, \ell$, be the linear maps defined on monomials by

$$
\begin{equation*}
\frac{\partial}{\partial x_{i}}\left(x_{i_{1}} \ldots x_{i_{s}}\right)=\sum_{k=1}^{s} \delta_{i_{k}, i} x_{i_{1}} \ldots x_{i_{k-1}} \otimes x_{i_{k+1}} \ldots x_{i_{s}} . \tag{2.22}
\end{equation*}
$$

The partial derivatives $\frac{\partial}{\partial x_{i}}$ are clearly 2 -fold derivations of $R_{\ell}$. Furthermore, by Proposition 1.6, $\left[\frac{\partial}{\partial x_{i}}, \frac{\partial}{\partial x_{j}}\right]$ is a 3 -fold derivation. Since it is zero on generators $x_{k}$, $k=1, \ldots, \ell$, we obtain

$$
\begin{equation*}
\frac{\partial}{\partial x_{i}} \circ \frac{\partial}{\partial x_{j}}=\frac{\partial}{\partial x_{j}} \circ \frac{\partial}{\partial x_{i}} \tag{2.23}
\end{equation*}
$$

Lemma 2.6. We have the following equality of maps $R_{\ell} \rightarrow R_{\ell}^{\otimes 3}\left(i, j \in \mathbb{Z}_{+}\right)$:

$$
\begin{equation*}
\left(\frac{\partial}{\partial x_{i}}\right)_{L} \circ \frac{\partial}{\partial x_{j}}=\left(\frac{\partial}{\partial x_{j}}\right)_{R} \circ \frac{\partial}{\partial x_{i}} . \tag{2.24}
\end{equation*}
$$

Proof. By Proposition 1.6(b) it follows that

$$
\left(\frac{\partial}{\partial x_{i}}\right)_{L} \circ \frac{\partial}{\partial x_{j}}-\left(\frac{\partial}{\partial x_{j}}\right)_{R} \circ \frac{\partial}{\partial x_{i}}
$$

is a 3 -fold derivation. The claim follows since this 3 -fold derivation is obviously zero on the generators $x_{k}, k=1, \ldots, \ell$.

Note that equation (2.24) implies (2.23), by the definition (1.30) of the composition of 2 -fold derivations. Hence, we will say that the 2 -fold derivations $\frac{\partial}{\partial x_{i}}$ and $\frac{\partial}{\partial x_{j}}$ strongly commute.
Definition 2.7. An algebra of ordinary differential functions is a unital associative algebra $V$, endowed with $\ell$ strongly commuting 2 -fold derivations $\frac{\partial}{\partial x_{i}}: V \rightarrow V \otimes V$, i.e. satisfying (2.24).

An example is the algebra $R_{\ell}=\mathbb{F}\left\langle x_{1}, \ldots, x_{\ell}\right\rangle$ of non-commutative polynomials in the variables $x_{1}, \ldots, x_{\ell}$, endowed with the 2 -fold derivations (2.22). Other examples can be obtained as extensions of $R_{\ell}$ by localization by one or more non-zero elements.

Theorem 2.8. (a) Any 2-fold bracket on $R_{\ell}$ has the following form $(f, g \in V)$

$$
\begin{equation*}
\{f, g\}=\sum_{i, j=1}^{\ell} \frac{\partial g}{\partial x_{j}} \bullet\left\{x_{i}, x_{j}\right\} \bullet\left(\frac{\partial f}{\partial x_{i}}\right)^{\sigma}, \tag{2.25}
\end{equation*}
$$

where • is as in (1.1).
(b) Let $V$ be an algebra of ordinary differential functions, and let $H$ be an $\ell \times \ell$ matrix with entries in $V \otimes V$. Denoting its entries by $H_{j i}=\left\{\left\{x_{i}, x_{j}\right\}, i, j=\right.$ $1, \ldots, \ell$, formula (2.25) defines a 2 -fold bracket on $V$.
(c) Equation (2.25) defines a structure of double Poisson algebra on $V$ if and only if the skewsymmetry axiom (i) and the Jacobi identity (ii) hold on the $x_{i}$ 's, namely, for all $i, j, k=1, \ldots, \ell$, we have

$$
\begin{equation*}
\left\{\left\{x_{i}, x_{j}\right\}\right\}=-\left\{\left\{x_{j}, x_{i}\right\}\right\}^{\sigma}\left(\text { equivalently, recalling }(1.7), H^{\dagger}=-H\right), \tag{2.26}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\{\left\{x_{i},\left\{\left\{x_{j}, x_{k}\right\}\right\}\right\}_{L}+\left\{\left\{x_{j},\left\{\left\{x_{k}, x_{i}\right\}\right\}\right\}_{L}^{\sigma}+\left\{\left\{x_{k},\left\{\left\{x_{i}, x_{j}\right\}\right\}\right\}_{L}^{\sigma^{2}}=0,\right.\right.\right. \tag{2.27}
\end{equation*}
$$

where each term is understood, in terms of the matrix $H$, via (2.25), for example

$$
\begin{equation*}
\left\{\left\{x_{i},\left\{\left\{x_{j}, x_{k}\right\}\right\}\right\}_{L}=\sum_{h=1}^{\ell}\left(\left(\frac{\partial}{\partial x_{h}}\right)_{L} H_{k j}\right) \bullet_{3} H_{h i}\right. \tag{2.28}
\end{equation*}
$$

Proof. We prove that the RHS of (2.25) satisfies the Leibniz rules (2.2) and (2.3). Using the fact that partial derivatives are 2-fold derivations, the LHS of (2.2) is

$$
\begin{equation*}
\sum_{i, j=1}^{\ell}\left(\frac{\partial g}{\partial x_{j}} h\right) \bullet\left\{\left\{x_{i}, x_{j}\right\} \bullet\left(\frac{\partial f}{\partial x_{i}}\right)^{\sigma}+\sum_{i, j=1}^{\ell}\left(g \frac{\partial h}{\partial x_{j}}\right) \bullet\left\{\left\{x_{i}, x_{j}\right\} \bullet\left(\frac{\partial f}{\partial x_{i}}\right)^{\sigma}\right.\right. \tag{2.29}
\end{equation*}
$$

By (1.16) we have that (2.29) is the same as RHS of (2.2). The Leibniz rule (2.3) is proved in the same way using again (1.16). Since any 2 -fold bracket is extended uniquely from the generators to $R_{\ell}$ via the Leibniz rules, this concludes the proof of (a) and (b).

The fact that the bracket given by equation (2.25) satisfies the skewsymmetry axiom is immediate from (1.3) and the assumption that $\left.\left\{\left\{x_{i}, x_{j}\right\}\right\}=-\left\{x_{j}, x_{i}\right\}\right\}^{\sigma}$ for every $i, j=1, \ldots, \ell$.

To conclude the proof of part (c) we are left to prove Jacobi identity (2.1). First, for $f \in V$ and $G \in V^{\otimes 2}$ we have, by (2.25) and the definition of the $\bullet_{i}$ actions,

$$
\begin{equation*}
\left\{\{f, G\}_{L}=\sum_{i, j=1}^{\ell}\left(\left(\frac{\partial}{\partial x_{j}}\right)_{L} G\right) \bullet 3\left(\left\{\left\{x_{i}, x_{j}\right\} \bullet \bullet\left(\frac{\partial f}{\partial x_{i}}\right)^{\sigma}\right) .\right.\right. \tag{2.30}
\end{equation*}
$$

Then, by equation (2.25), Lemma 1.10, and equation (2.30), we get

$$
\begin{align*}
& \left\{\{f,\{\{g, h\}\}\}_{L}\right. \\
& =\sum_{i, j, h, k=1}^{\ell}\left(\frac{\partial}{\partial x_{h}}\right)_{L}\left(\frac{\partial h}{\partial x_{k}}\right) \bullet \bullet_{3}\left(\left\{\left\{x_{i}, x_{h}\right\}\right\} \bullet\left(\frac{\partial f}{\partial x_{i}}\right)^{\sigma}\right) \bullet_{1}\left(\left\{\left\{x_{j}, x_{k}\right\} \bullet\left(\frac{\partial g}{\partial x_{j}}\right)^{\sigma}\right)\right.  \tag{2.31}\\
& +\sum_{i, j, k=1}^{\ell} \frac{\partial h}{\partial x_{k}} \bullet_{2}\left\{\left\{x_{i},\left\{\left\{x_{j}, x_{k}\right\}\right\}\right\}\right\}_{L} \bullet 3\left(\frac{\partial f}{\partial x_{i}}\right)^{\sigma} \bullet_{1}\left(\frac{\partial g}{\partial x_{j}}\right)^{\sigma}  \tag{2.32}\\
& \left.+\sum_{i, j, h, k=1}^{\ell}\left(\frac{\partial h}{\partial x_{k}} \bullet\left\{x_{j}, x_{k}\right\}\right\}\right) \bullet_{2}\left(\frac{\partial}{\partial x_{h}}\right)_{L}\left(\frac{\partial g}{\partial x_{j}}\right)^{\sigma} \bullet_{3}\left(\left\{\left\{x_{i}, x_{h}\right\} \bullet \bullet\left(\frac{\partial f}{\partial x_{i}}\right)^{\sigma}\right) .\right. \tag{2.33}
\end{align*}
$$

Note that in the RHS above we avoided using unnecessary parentheses, in view of Lemma 1.1. From the above equation, permuting $f, g$ and $h$, applying the permutation $\sigma$, and using Lemma 1.2 several times, we get

$$
\begin{align*}
& \{g,\{\{h, f\}\}\}_{L}^{\sigma} \\
& =\sum_{i, j, h, k=1}^{\ell}\left(\frac{\partial h}{\partial x_{k}} \bullet\left\{\left\{x_{k}, x_{i}\right\}\right\}^{\sigma}\right) \bullet_{2}\left(\left(\frac{\partial}{\partial x_{h}}\right)_{L}\left(\frac{\partial f}{\partial x_{i}}\right)\right)^{\sigma} \bullet_{1}\left(\left\{\left\{x_{j}, x_{h}\right\}\right\} \bullet\left(\frac{\partial g}{\partial x_{j}}\right)^{\sigma}\right)  \tag{2.34}\\
& +\sum_{i, j, k=1}^{\ell} \frac{\partial h}{\partial x_{k}} \bullet_{2}\left\{\left\{x_{j},\left\{\left\{x_{k}, x_{i}\right\}\right\}\right\}_{L}^{\sigma} \bullet_{1}\left(\frac{\partial g}{\partial x_{j}}\right)^{\sigma} \bullet_{3}\left(\frac{\partial f}{\partial x_{i}}\right)^{\sigma}\right.  \tag{2.35}\\
& +\sum_{i, j, h, k=1}^{\ell}\left(\left(\frac{\partial}{\partial x_{h}}\right)_{L}\left(\frac{\partial h}{\partial x_{k}}\right)^{\sigma}\right)^{\sigma} \bullet_{1}\left(\left\{\left\{x_{j}, x_{h}\right\} \bullet\left(\frac{\partial g}{\partial x_{j}}\right)^{\sigma}\right) \bullet_{3}\left(\left\{\left\{x_{k}, x_{i}\right\}\right\}^{\sigma} \bullet\left(\frac{\partial f}{\partial x_{i}}\right)^{\sigma}\right) .\right. \tag{2.36}
\end{align*}
$$

and, similarly,

$$
\begin{align*}
& \{\{h,\{\{f, g\}\}\}\}_{L}^{\sigma^{2}} \\
& =\sum_{i, j, h, k=1}^{\ell}\left(\frac{\partial h}{\partial x_{k}} \bullet\left\{\left\{x_{k}, x_{h}\right\}\right\}^{\sigma}\right) \bullet_{2}\left(\left(\frac{\partial}{\partial x_{h}}\right)_{L}\left(\frac{\partial g}{\partial x_{j}}\right)\right)^{\sigma^{2}} \bullet_{3}\left(\left\{\left\{x_{i}, x_{j}\right\} \bullet\left(\frac{\partial f}{\partial x_{i}}\right)^{\sigma}\right)\right.  \tag{2.37}\\
& +\sum_{i, j, k=1}^{\ell} \frac{\partial h}{\partial x_{k}} \bullet_{2}\left\{\left\{x_{k},\left\{\left\{x_{i}, x_{j}\right\}\right\}\right\}\right\}_{L}^{\sigma^{2}} \bullet 3\left(\frac{\partial f}{\partial x_{i}}\right)^{\sigma} \bullet 1\left(\frac{\partial g}{\partial x_{j}}\right)^{\sigma}  \tag{2.38}\\
& \left.+\sum_{i, j, h, k=1}^{\ell}\left(\frac{\partial h}{\partial x_{k}} \bullet\left\{\left\{x_{k}, x_{h}\right\}\right\}\right)^{\sigma}\right) \bullet_{2}\left(\left(\frac{\partial}{\partial x_{h}}\right)_{L}\left(\frac{\partial f}{\partial x_{i}}\right)^{\sigma}\right)^{\sigma^{2}} \bullet_{1}\left(\left\{\left\{x_{i}, x_{j}\right\}\right\}^{\sigma} \bullet\left(\frac{\partial g}{\partial x_{j}}\right)^{\sigma}\right) . \tag{2.39}
\end{align*}
$$

Since, by assumption, the Jacobi identity (2.1) holds on generators, we have that $(2.32),(2.35)$ and (2.38) sum up to zero. Note that here we are using the fact that the $\bullet_{1}$ and $\bullet_{3}$ actions commute, by Lemma 1.1(c). Furthermore, by Lemma 1.4 and assumption (2.24), we have

$$
\left(\left(\frac{\partial}{\partial x_{k}}\right)_{L}\left(\frac{\partial h}{\partial x_{h}}\right)^{\sigma}\right)^{\sigma}=\left(\frac{\partial}{\partial x_{k}}\right)_{R}\left(\frac{\partial h}{\partial x_{h}}\right)=\left(\frac{\partial}{\partial x_{h}}\right)_{L}\left(\frac{\partial h}{\partial x_{k}}\right) .
$$

It follows that (2.31) and (2.36) are opposite to each other by the skewsymmetry on generators. Similarly, (2.33) and (2.37) (respectively (2.34) and (2.39)) are opposite to each other. This concludes the proof.

Definition 2.9. The matrix $H=\left(H_{i j}\right)_{i, j=1}^{\ell} \in \operatorname{Mat}_{\ell \times \ell}(V \otimes V)$ satisfying (2.26) and (2.27) is called a Poisson structure on the algebra of ordinary differential functions $V$.

Remark 2.10. Formula (2.25) for the 2 -fold bracket of $f, g \in V$ can be written in a "traditional" form as

$$
\begin{equation*}
\{f, g\}=(\nabla g \mid H \nabla f) \tag{2.40}
\end{equation*}
$$

where $\nabla f=\left(\frac{\partial f}{\partial x_{i}}\right)_{i=1}^{\ell} \in(V \otimes V)^{\oplus \ell}, H=\left(H_{j i}\right)_{i, j=1}^{\ell} \in \operatorname{Mat}_{\ell \times \ell}(V \otimes V)$ is the matrix with entries $H_{j i}=\left\{\left\{x_{i}, x_{j}\right\}\right.$, its action on $(V \otimes V)^{\oplus \ell}$ is defined by (1.6), and $(\cdot \mid \cdot)$ is the inner product (1.4).

Remark 2.11. One can define a double Lie algebra as a vector space $\mathfrak{g}$ endowed with a double bracket $\{\{-,-\}: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ satisfying the skewsymmetry and Jacobi identity axioms from Definition 2.1. Then, by Theorem 2.8 the tensor algebra $\mathcal{T}(\mathfrak{g})$ is a double Poisson algebra, with the 2 -fold bracket given by (2.25).
2.4. Evolution ODE and Hamiltonian ODE. Let $V$ be an algebra of ordinary differential functions. An evolution $O D E$ over $V$ is an equation of the form

$$
\begin{equation*}
\frac{d x_{i}}{d t}=P_{i} \in V, \quad i=1, \ldots, \ell \tag{2.41}
\end{equation*}
$$

By the chain rule, a function $f \in V$ evolves, in virtue of (2.41), according to the equation

$$
\begin{equation*}
\frac{d f}{d t}=\sum_{i=1}^{\ell} \mathrm{m}\left(P_{i} \otimes_{1} \frac{\partial f}{\partial x_{i}}\right) . \tag{2.42}
\end{equation*}
$$

It is immediate to check that the time derivative $\frac{d}{d t}: V \rightarrow V$ as given by (2.42) is a derivation of $V$. In particular, it induces a well defined map on $V /[V, V]$. An integral of motion for equation (2.41) is an element $\operatorname{tr}(h) \in V /[V, V]$ constant in time, i.e. such that

$$
\begin{equation*}
\frac{d \operatorname{tr} h}{d t}=\operatorname{tr} \sum_{i=1}^{\ell} \mathrm{m}\left(P_{i} \otimes_{1} \frac{\partial h}{\partial x_{i}}\right)=0 . \tag{2.43}
\end{equation*}
$$

Given $P \in V^{\ell}$, we define the corresponding evolutionary vector field $X_{P}$ as the derivation of $V$ given by the RHS of (2.42):

$$
\begin{equation*}
X_{P}(f)=\sum_{i=1}^{\ell} \mathrm{m}\left(P_{i} \otimes_{1} \frac{\partial f}{\partial x_{i}}\right) \tag{2.44}
\end{equation*}
$$

Evolutionary vector fields form the Lie algebra Vect $(V)$, with commutator given by the usual formula:

$$
\begin{equation*}
\left[X_{P}, X_{Q}\right]=X_{[P, Q]}, \quad \text { where }[P, Q]_{i}=X_{P}\left(Q_{i}\right)-X_{Q}\left(P_{i}\right), \quad i=1, \ldots, \ell \tag{2.45}
\end{equation*}
$$

This can be easily checked using condition (2.24) on $V$.
Equation (2.41) is called compatible with another evolution ODE $\frac{d x_{i}}{d \tau}=Q_{i}$, $i=1, \ldots, \ell$, if the corresponding evolutionary vector fields commute: $\left[X_{P}, X_{Q}\right]=0$.

A Hamiltonian $O D E$ on an arbitrary double Poisson algebra $V$, associated to the Hamiltonian function $\operatorname{tr} h \in V /[V, V]$ is an equation of the form

$$
\begin{equation*}
\frac{d u}{d t}=\{\operatorname{tr} h, u\}, u \in V \tag{2.46}
\end{equation*}
$$

An integral of motion for the Hamiltonian ODE (2.46) is an element $\operatorname{tr} f \in V /[V, V]$ such that $\{\operatorname{tr} h, \operatorname{tr} f\}=0$. In this case, by Proposition 2.5, the derivations $X^{h}=$ $\{\operatorname{tr} h,-\}$ and $X^{f}=\{\operatorname{tr} f,-\}$ (called Hamiltonian vector fields) commute, and equations $\frac{d u}{d t}=\{\operatorname{tr} f, u\}$ and (2.46) are compatible.

In the special case when $V$ is an algebra of ordinary differential functions and the 2 -fold bracket $\{-,-\}$ is given by a Poisson structure $H$ on $V$ via (2.25), where $\left\{\left\{x_{i}, x_{j}\right\}\right\}=H_{j i}$, the Hamiltonian ODE (2.46) becomes the following evolution ODE

$$
\begin{equation*}
\frac{d x_{i}}{d t}=\mathrm{m} \sum_{j=1}^{\ell} H_{i j} \bullet\left(\frac{\partial h}{\partial x_{j}}\right)^{\sigma} \tag{2.47}
\end{equation*}
$$

Furthermore the Lie algebra bracket $\{-,-\}$ on $V /[V, V]$ associated to the double Poisson bracket (2.25), defined by Proposition 2.5, is:

$$
\begin{equation*}
\{\operatorname{tr}(f), \operatorname{tr}(g)\}=\operatorname{tr}\left(\mathrm{m} \sum_{i, j=1}^{\ell} \frac{\partial g}{\partial x_{j}} \bullet H_{j i} \bullet\left(\frac{\partial f}{\partial x_{i}}\right)^{\sigma}\right), \quad f, g \in V . \tag{2.48}
\end{equation*}
$$

In this case the notions of compatibility and of integrals of motion are consistent with those for general evolution ODEs, due to Theorem 2.8(c).
Remark 2.12. Recalling the definition (1.12) of the action of the matrix $H \in$ Mat $_{\ell \times \ell}(V \otimes V)$ on $V^{\oplus \ell}$ formula (2.47) can be written in the "traditional" form

$$
\begin{equation*}
\frac{d x}{d t}=H d h \tag{2.49}
\end{equation*}
$$

and similarly formula (2.48) can be written as (see (2.40))

$$
\begin{equation*}
\{\operatorname{tr}(f), \operatorname{tr}(g)\}=(d g \mid H d f), \tag{2.50}
\end{equation*}
$$

where $(\cdot \mid \cdot)$ is the inner product (1.11) on $V^{\oplus \ell}$, and $d f=\mathrm{m}(\nabla f)^{\sigma}$ (cf. (1.9)). The latter notation is compatible with the theory of the reduced de Rham complex, cf. (2.70).
2.5. Lenard-Magri scheme for Hamiltonian ODE. Let $H \in \operatorname{Mat}_{\ell \times \ell}(V \otimes V)$. We define a bilinear form associated to $H,\langle-,-\rangle_{H}: V^{\oplus \ell} \otimes V^{\oplus \ell} \rightarrow V /[V, V]$ as follows

$$
\begin{equation*}
\langle F, G\rangle_{H}=(F \mid H G), \quad F, G \in V^{\oplus \ell} . \tag{2.51}
\end{equation*}
$$

Here we are using notation (1.11) and (1.12). Recalling (1.8), we obtain that the form (2.51) is symmetric (resp. skewsymmetric) if $H$ is selfadjoint (resp. skewadjoint) (see (1.7)).
Lemma 2.13. Let $H_{0}, H_{1} \in \operatorname{Mat}_{\ell \times \ell}(V \otimes V)$ be skewadjoint, and let $\left\{F_{n}\right\}_{n=0}^{N} \subset$ $V^{\oplus \ell}$ be such that

$$
H_{0} F^{n+1}=H_{1} F^{n} \quad \text { for } n=0, \ldots, N-1
$$

Then, the following orthogonality relations hold:

$$
\left\langle F^{n}, F^{m}\right\rangle_{H_{0}}=\left\langle F^{n}, F^{m}\right\rangle_{H_{1}}=0 \text { for } n, m=0, \ldots, N
$$

Proof. Same proof as in [Mag78] or Lemma 2.6 in [BDSK09].
Corollary 2.14. Let $\left\{\{-,-\}_{0}\right.$ and $\left\{\{-,-\}_{1}\right.$ be two skewsymmetric 2 -fold brackets on an algebra of ordinary differential functions $V$ given by the equation (2.25) for $H_{0}, H_{1} \in \operatorname{Mat}_{\ell \times \ell}(V \otimes V)$ respectively. Let $\left\{h_{n}\right\}_{n=0}^{N} \subset V$ be such that

$$
\begin{equation*}
\left\{\operatorname{tr}\left(h_{n}\right), u\right\}_{0}=\left\{\operatorname{tr}\left(h_{n+1}\right), u\right\}_{1} \quad \text { for } u \in V, n=0, \ldots, N-1 . \tag{2.52}
\end{equation*}
$$

Then we have

$$
\left\{\operatorname{tr}\left(h_{n}\right), \operatorname{tr}\left(h_{m}\right)\right\}_{0}=\left\{\operatorname{tr}\left(h_{n}\right), \operatorname{tr}\left(h_{m}\right)\right\}_{1}=0 \quad \text { for } n, m=0, \ldots, N
$$

(See (2.12), (2.18) and (2.19) for notation).
Proof. It follows from Lemma 2.13 and equation (2.50).
Remark 2.15. The analogues of Propositions 2.9 and 2.10 in [BDSK09] still hold in the present setup with the same proof.

Theorem 2.16. Let $V=R_{\ell}$ be the algebra of non-commutative polyomials in $\ell$ variables. Let $H_{0}, H_{1}$ be compatible Poisson structures on $V$, and assume that $K$ is a non-degenerate matrix. Let $h_{0}, h_{1} \in V$ and $F \in V^{\ell}$ be such that

$$
\begin{equation*}
H_{1} d h_{0}=H_{0} d h_{1}, \quad H_{1} d h_{1}=H_{0} F \tag{2.53}
\end{equation*}
$$

where, as before, $d h=m(\nabla h)^{\sigma}$, and we are using the notation (1.12). Then, there exists $h_{2} \in V$ such that $F=d h_{2}$.
Proof. First, one checks that the Jacobian of $F_{2}$ (defined after equation (0.22)), is self-adjoint, i.e. $J_{F}=J_{F}^{\dagger}$. The proof of this claim is long but straightforward, and the argument is the same as in the commutative case, cf. e.g. [Olv93, Lemma 7.25]. The proof uses non-degeneracy of the bilinear form (1.11), which follows from the well-known property of the algebra $V=R_{\ell}$ that for $a \in V, \operatorname{tr} a V=0$ implies $a=0$. Then, one uses the exactness of the reduced de Rham complex, proved in Theorem 2.18 below, to conclude that the 1 -form $F$ is exact.

Using this theorem and Remark 2.15, one shows that the Lenard-Magri scheme works for $V=R_{\ell}$, i.e. the Lenard-Magri recursive relation in Corollary 2.14 for $N=1$ can be extended to $N=\infty$, provided that the orthogonality condition, similar to that in [BDSK09, Proposition 2.9], holds.
2.6. Examples. It is claimed in [VdB08] that the only non-zero double Poisson brackets on $R_{1}=\mathbb{F}[x]$ (up to automorphisms of $R_{1}$ ) are given by

$$
\{\{x, x\}\}=x \otimes 1-1 \otimes x \text { and }\{\{x, x\}\}=x^{2} \otimes x-x \otimes x^{2} .
$$

The corresponding Lie algebra brackets on $R_{1} /\left[R_{1}, R_{1}\right]$ given by (2.48) are trivial. Hence, we do not get non-trivial Hamiltonian equations.

The simplest double Poisson bracket on $R_{2}=\mathbb{F}\langle x, y\rangle$ has the following form:

$$
\begin{equation*}
\{x, x\}=p \otimes q-q \otimes p \text { where } p, q \in R_{2}, \quad\{\{x, y\}=\{\{y, y\}\}=0 \tag{2.54}
\end{equation*}
$$

The double Poisson bracket given by (2.25) and (2.54) is obviously skewsymmetric and it satisfies Jacobi identity if and only if equation (2.27) holds for $x_{i}=x_{j}=$ $x_{k}=x$ (see Theorem 2.8). Using equation (2.28), this condition can be rewritten as follows (using Sweedler's notation)

$$
\begin{align*}
& \left(\frac{\partial p}{\partial x}\right)^{\prime} p \otimes q\left(\frac{\partial p}{\partial x}\right)^{\prime \prime} \otimes q-\left(\frac{\partial p}{\partial x}\right)^{\prime} q \otimes p\left(\frac{\partial p}{\partial x}\right)^{\prime \prime} \otimes q-\left(\frac{\partial q}{\partial x}\right)^{\prime} p \otimes q\left(\frac{\partial q}{\partial x}\right)^{\prime \prime} \otimes p \\
& +\left(\frac{\partial q}{\partial x}\right)^{\prime} q \otimes p\left(\frac{\partial q}{\partial x}\right)^{\prime \prime} \otimes p+\text { cyclic permutations }=0 \tag{2.55}
\end{align*}
$$

We found the following three types of solutions:
(a) $p=p(y), q=q(y)$;
(b) $p=x, q=q(y)$;
(c) $p=x, q=x f(y) x$.

In fact, one can show that, up to automorphisms of $R_{2}$, these are all solutions.
Replacing $x$ by $x+1$ in (b) we get the following compatible double Poisson structures (where $y$ is central for both structures):

$$
\{\{x, x\}\}_{0}=1 \otimes y-y \otimes 1 \text { and }\left\{\{x, x\}_{1}=x \otimes y-y \otimes x .\right.
$$

Let

$$
\begin{equation*}
h_{0}=1 \text { and } h_{n}=\frac{1}{n}(x+y)^{n} \text { for every integer } n \geq 1 \tag{2.56}
\end{equation*}
$$

We have the following recursion relations

$$
\begin{equation*}
\left\{\operatorname{tr}\left(h_{0}\right), x\right\}_{0}=0 \text { and }\left\{\operatorname{tr}\left(h_{n}\right), x\right\}_{1}=\left\{\operatorname{tr}\left(h_{n+1}\right), x\right\}_{0}, n \in \mathbb{Z}_{+} \tag{2.57}
\end{equation*}
$$

which can be proved using equation (2.48) and the fact that for $n \geq 1$

$$
\frac{\partial h_{n}}{\partial x}=\frac{1}{n} \sum_{k=0}^{n-1}(x+y)^{k} \otimes(x+y)^{n-1-k}
$$

Hence, by Corollary 2.14, we get that all $\operatorname{tr}\left(h_{n}\right)$ are in involution with respect to both brackets $\{-,-\}_{0,1}$ on $R_{2} /\left[R_{2}, R_{2}\right]$, given by equation (2.48). Furthermore all the $\operatorname{tr}\left(h_{n}\right)$ are integrals of motion for the corresponding compatible Hamiltonian ODEs, which are computed using (2.47):

$$
\begin{equation*}
\frac{d x}{d t_{n}}=x(x+y)^{n} y-y(x+y)^{n} x, \quad \frac{d y}{d t_{n}}=0, \quad n \in \mathbb{Z}_{+} \tag{2.58}
\end{equation*}
$$

Note that the algebra $R_{2}$ is $\mathbb{Z}_{+}^{2}$-graded in the obvious way. The homogeneous components of the $h_{n}$ 's are still in involution with respect to both brackets and the corresponding Hamiltonian vector fields still commute. Thus, for $n \in \mathbb{Z}_{+}$, equation (2.58) produces $n+1$ compatible Hamiltonian ODEs, and they are compatible for all $n$. For example, for $n=0$, equation (2.58) is

$$
\frac{d x}{d t_{0}}=x y-y x, \quad \frac{d y}{d t_{0}}=0
$$

The homogeneous components of equation (2.58) for $n=1$ produce the following two Hamiltonian ODEs:

$$
\begin{array}{ll}
\frac{d x}{d t_{1}}=x^{2} y-y x^{2}, & \frac{d y}{d t_{1}}=0 \\
\frac{d x}{d t_{\tilde{1}}}=x y^{2}-y^{2} x, & \frac{d y}{d t_{\tilde{1}}}=0
\end{array}
$$

As explained in [MS00] the first of the above equations is a higher symmetry of the Euler equation and they find infinitely many its conserved densities. For $n=2$, the homogeneous (2,2)-component of equation (2.58) produces the following equation which also was found in [MS00]:

$$
\frac{d x}{d t_{0}}=x^{2} y^{2}+x y x y-y x y x-y^{2} x^{2}, \quad \frac{d y}{d t_{0}}=0
$$

Furthermore, let us consider on $R_{2}$ the following double Poisson structure (of type (c)):

$$
\{\{x, x\}\}=x \otimes x y x-x y x \otimes x, \quad\{\{x, y\}\}=\{\{y, y\}\}=0 .
$$

Let $h_{0}=1$ and

$$
\begin{equation*}
h_{n}=\frac{x^{n}}{n}, \tilde{h}_{n}=x^{n} y, \bar{h}_{n}=x(y x)^{n} \text { for every integer } n \geq 1 \tag{2.59}
\end{equation*}
$$

Using the fact that

$$
\begin{gathered}
\frac{\partial h_{n}}{\partial x}=\frac{1}{n} \sum_{k=0}^{n-1} x^{k} \otimes x^{n-1-k}, \quad \frac{\partial \tilde{h}_{n}}{\partial x}=\sum_{k=0}^{n-1} x^{k} \otimes x^{n-1-k} y \\
\frac{\partial \bar{h}_{n}}{\partial x}=1 \otimes(y x)^{n}+\sum_{k=0}^{n-1} x(y x)^{k} y \otimes(y x)^{n-1-k}
\end{gathered}
$$

and equation (2.48), it is straightforward to show that

$$
\left\{\operatorname{tr}\left(X_{n}\right), \operatorname{tr}\left(Y_{m}\right)\right\}=0 \text { for every integers } n, m \geq 1
$$

where $X$ and $Y$ can be $h, \tilde{h}$ or $\bar{h}$. Hence, we get the following integrable hierarchy of Hamiltonian ODEs:

$$
\begin{align*}
\frac{d x}{d t_{n}} & =x^{n+1} y x-x y x^{n+1}, \quad \frac{d y}{d t_{n}}=0, \quad n \geq 1 \\
\frac{d x}{d t_{\tilde{n}}} & =\sum_{k=0}^{\tilde{n}-2} x^{\tilde{n}-k} y x^{k+1} y x-x y x^{k+1} y x^{\tilde{n}-k}, \quad \frac{d y}{d t_{\tilde{n}}}=0, \quad \tilde{n} \geq 1  \tag{2.60}\\
\frac{d x}{d t_{\bar{n}}} & =x(x y)^{\bar{n}+1} x-x(y x)^{\bar{n}+1} x, \quad \frac{d y}{d t_{\bar{n}}}=0, \quad \bar{n} \geq 1
\end{align*}
$$

The first equation of the hierarchy, corresponding to $n=1$ (and the Hamiltonian density $h_{1}=x$ ), is

$$
\frac{d x}{d t_{1}}=x^{2} y x-x y x^{2}, \quad \frac{d y}{d t_{1}}=0
$$

This equation already appeared in [MS00], where its higher symmetries of degree $n+2$ in the variable $x$ and degree 1 in the variable $y$ were found (which correspond to the first line in equation (2.60)).
2.7. Basic de Rham complex over an algebra of ordinary differential functions. Let $V$ be an algebra of ordinary differential functions. We define the basic de Rham complex $\widetilde{\Omega}(V)$ as the free product of the algebra $V$ and the algebra $\mathbb{F}\left\langle d x_{1}, \ldots, d x_{\ell}\right\rangle$ of non-commutative polynomials in $\ell$ variables $d x_{1}, \ldots, d x_{\ell}$. It is a $\mathbb{Z}_{+}$-graded unital associative algebra, where $f \in V$ has degree 0 and the $d x_{i}$ 's have degree 1. We consider it as a superalgebra, with superstructure compatible with the $\mathbb{Z}_{+}$-grading. The subspace $\widetilde{\Omega}^{n}(V)$ of degree $n$ consists of linear combinations of elements of the form

$$
\begin{equation*}
\widetilde{\omega}=f_{1} d x_{i_{1}} f_{2} d x_{i_{2}} \ldots f_{n} d x_{i_{n}} f_{n+1}, \text { where } f_{1}, \ldots, f_{n+1} \in V \tag{2.61}
\end{equation*}
$$

In particular, $\widetilde{\Omega}^{0}(V)=V$ and $\widetilde{\Omega}^{1}(V) \simeq \widetilde{\Omega}_{i=1}^{\ell} V d x_{i} V$.
Define the de Rham differential $d$ on $\widetilde{\Omega}(V)$ as the odd derivation of degree 1 on the superalgebra $\widetilde{\Omega}(V)$ by letting

$$
\begin{equation*}
d f=\sum_{i=1}^{\ell}\left(\frac{\partial f}{\partial x_{i}}\right)^{\prime} d x_{i}\left(\frac{\partial f}{\partial x_{i}}\right)^{\prime \prime} \in \widetilde{\Omega}^{1}(V) \text { for } f \in \widetilde{\Omega}^{0}(V), \quad d\left(d x_{i}\right)=0 . \tag{2.62}
\end{equation*}
$$

In order to check that $d$ is a differential on $\widetilde{\Omega}(V)$, i.e. $d^{2}=0$, it suffices to check that $d^{2} f=0$ for $f \in V$. Using Sweedler's notation and equation (2.62) we have

$$
\begin{aligned}
& d^{2} f=\sum_{i, j=1}^{\ell}\left(\frac{\partial}{\partial x_{j}}\left(\frac{\partial f}{\partial x_{i}}\right)^{\prime}\right)^{\prime} d x_{j}\left(\frac{\partial}{\partial x_{j}}\left(\frac{\partial f}{\partial x_{i}}\right)^{\prime}\right)^{\prime \prime} d x_{i}\left(\frac{\partial f}{\partial x_{i}}\right)^{\prime \prime} \\
& -\sum_{i, j=1}^{\ell}\left(\frac{\partial f}{\partial x_{i}}\right)^{\prime} d x_{i}\left(\frac{\partial}{\partial x_{j}}\left(\frac{\partial f}{\partial x_{i}}\right)^{\prime \prime}\right)^{\prime} d x_{j}\left(\frac{\partial}{\partial x_{j}}\left(\frac{\partial f}{\partial x_{i}}\right)^{\prime \prime}\right)^{\prime \prime} \\
& =\sum_{i, j=1}^{\ell} \mathrm{m}\left(d x_{j} \star_{1}\left(\left(\frac{\partial}{\partial x_{j}}\right)_{L}\left(\frac{\partial f}{\partial x_{i}}\right)\right) \star_{1} d x_{i}\right) \\
& -\sum_{i, j=1}^{\ell} \mathrm{m}\left(d x_{i} \star_{1}\left(\left(\frac{\partial}{\partial x_{j}}\right)_{R}\left(\frac{\partial f}{\partial x_{i}}\right)\right) \star_{1} d x_{j}\right)=0
\end{aligned}
$$

The last equality follows by condition (2.24) on $V$. So we can consider the corresponding cohomology complex $(\widetilde{\Omega}(V), d)$.

Given an evolutionary vector field $X_{P} \in \operatorname{Vect}(V)$ (cf. (2.44)), we define the corresponding Lie derivative $L_{P}: \widetilde{\Omega}(V) \rightarrow \widetilde{\Omega}(V)$ as the even derivation of degree 0 extending $X_{P}$ from $V$, and such that $L_{P}\left(d x_{i}\right)=d P_{i}, i=1, \ldots, \ell$. In other words, if $\tilde{\omega}$ is as in (2.61), then, recalling (2.62), we have

$$
\begin{align*}
& L_{P}(\widetilde{\omega})=\sum_{s=1}^{n+1} f_{1} d x_{i_{1}} f_{2} \ldots d x_{i_{s-1}} X_{P}\left(f_{s}\right) d x_{i_{s}} \ldots d x_{i_{n}} f_{n+1}  \tag{2.63}\\
& +\sum_{s=1}^{n} \sum_{i=1}^{\ell} f_{1} d x_{i_{1}} f_{2} \ldots f_{s}\left(\frac{\partial P_{i_{s}}}{\partial x_{i}}\right)^{\prime} d x_{i}\left(\frac{\partial P_{i_{s}}}{\partial x_{i}}\right)^{\prime \prime} f_{s+1} \ldots d x_{i_{n}} f_{n+1} .
\end{align*}
$$

Next, we define the corresponding contraction operator $\iota_{P}: \widetilde{\Omega}(V) \rightarrow \widetilde{\Omega}(V)$ as the odd derivation of degree -1 defined on generators by $\iota_{P}(f)=0$, for $f \in V$, and

$$
\begin{align*}
\iota_{P}\left(d x_{i}\right) & =P_{i} . \text { In other words, if } \tilde{\omega} \text { is as in (2.61), then } \\
\iota_{X}(\widetilde{\omega}) & =\sum_{s=1}^{n}(-1)^{s+1} f_{1} d x_{i_{1}} f_{2} \ldots d x_{i_{s-1}} f_{s} P_{i_{s}} f_{s+1} d x_{i_{s+1}} f_{s+2} \ldots d x_{i_{n}} f_{n+1} . \tag{2.64}
\end{align*}
$$

Proposition 2.17. The Lie derivatives (2.63) and the contraction operators (2.64) define a structure of $a \operatorname{Vect}(V)$-complex on $\widetilde{\Omega}(V)$. In other words the following relations hold for $P, Q \in V^{\ell}$ (recalling (2.45)):
(a) $\left[\iota_{P}, \iota_{Q}\right]=0$;
(b) $\left[L_{P}, \iota_{Q}\right]=\iota_{[P, Q]}$;
(c) $\left[L_{P}, L_{Q}\right]=L_{[P, Q]}$;
(d) $L_{P}=\left[\iota_{P}, d\right]\left(=\iota_{P} d+d \iota_{P}\right)($ Cartan's formula).

Proof. Both sides in each formula are derivations of the superalgebra $\widetilde{\Omega}(V)$, and they are obviously equal on generators $d x_{i}$ and $f \in V$.
Theorem 2.18. The complex $\left(\widetilde{\Omega}\left(R_{\ell}\right), d\right)$ is acyclic, i.e. $H^{n}\left(\widetilde{\Omega}\left(R_{\ell}\right), d\right)=\delta_{n, 0} \mathbb{F}$.
Proof. Let $\Delta=\left(x_{i}\right)_{i=1}^{\ell} \in R_{\ell}^{\ell}$ (it corresponds to the degree derivation $X_{\Delta}$ of $R_{\ell}$ ). Then $L_{\Delta}$ is a diagonalizable operator on $\widetilde{\Omega}\left(R_{\ell}\right)$, with eigenvalues given by the total degree in the $x_{i}$ 's and the $d x_{i}$ 's. Moreover, $h=\iota_{\Delta}$ is a homotopy operator for the complex $\left(\widetilde{\Omega}\left(R_{\ell}\right), d\right)$ : by Cartan's formula we have $h(d \widetilde{\omega})+d(h \widetilde{\omega})=k \widetilde{\omega}$, for every $\widetilde{\omega} \in \widetilde{\Omega}\left(R_{\ell}\right)$ of total degree $k$. In particular, if $k \neq 0$ and $d \widetilde{\omega}=0$, then $\widetilde{\omega}=\frac{1}{k} d(h \widetilde{\omega}) \in d \widetilde{\Omega}\left(R_{\ell}\right)$. The claim follows.
Remark 2.19. Let $\Delta_{i}=\left(\delta_{i j} x_{i}\right)_{j=1}^{\ell} \in V^{\ell}$. Consider the filtration $V_{0} \subset V_{1} \subset \cdots \subset$ $V_{\ell}=V$, where

$$
V_{i}=\left\{f \in V \left\lvert\, \frac{\partial f}{\partial x_{j}}=0\right. \text { for all } j>i\right\},
$$

and extend it to a filtration of $\widetilde{\Omega}(V)$, letting $\widetilde{\Omega}_{i}^{n}(V)$ contain elements of the form (2.61), with $f_{1}, \ldots, f_{n+1} \in V_{i}$ and $i_{1}, \ldots, i_{n} \leq i$. Then, $L_{\Delta_{i}}$ preserves the space $\widetilde{\Omega}_{i}^{k}(V)$, and $\iota_{\Delta_{i}}\left(\widetilde{\Omega}_{i}^{k}(V)\right) \subset \widetilde{\Omega}_{i}^{k-1}(V)$. Assume that $L_{\Delta_{i}}: \widetilde{\Omega}_{i}^{k}(V) \rightarrow \widetilde{\Omega}_{i}^{k}(V)$ is surjective and its kernel is $\widetilde{\Omega}_{i-1}^{k}(V)$. This is true, for example, for the algebra $R_{\ell}$ of non-commutative polynomials in $\ell$ variables. Then, it is easy to check that the operator $h_{i}=L_{\Delta_{i}}^{-1} \circ \iota_{\Delta_{i}}: \widetilde{\Omega}_{i}^{k}(V) \rightarrow \widetilde{\Omega}_{i}^{k-1}(V)$, defined up to elements of $\widetilde{\Omega}_{i-1}^{k-1}(V)$, is a local homotopy operator, in the sense that

$$
\left(d \circ h_{i}+h_{i} \circ d\right)(\widetilde{\omega})-\widetilde{\omega} \in \widetilde{\Omega}_{i-1}^{k}(V) \text { for every } \widetilde{\omega} \in \widetilde{\Omega}_{i}^{k}(V) .
$$

As a consequence, under the above assumptions the basic de Rham complex is acyclic.
2.8. Reduced de Rham complex. Obviously the subspace $[\widetilde{\Omega}(V), \widetilde{\Omega}(V)] \subset \widetilde{\Omega}(V)$ is compatible with the $\mathbb{Z}_{+}$-grading and it is is preserved by $d$. Hence, we can consider the reduced de Rham complex

$$
\begin{equation*}
\Omega(V)=\widetilde{\Omega}(V) /[\widetilde{\Omega}(V), \widetilde{\Omega}(V)]=\oplus_{n \in \mathbb{Z}_{+}} \Omega^{n}(V), \tag{2.65}
\end{equation*}
$$

with the induced action of $d$. Since Lie derivatives $L_{P}$ and contraction operators $\iota_{P}$, for $P \in V^{\ell}$, are derivations of the superalgebra $\widetilde{\Omega}(V)$, they preserve the commutator subspace $[\widetilde{\Omega}(V), \widetilde{\Omega}(V)]$. Hence, they induce well defined maps on the reduced de Rham complex $\Omega(V)$ satisfying all relations of Proposition 2.17.

It follows by the same argument as in the proof of Theorem 2.18 that the complex $\left(\Omega\left(R_{\ell}\right), d\right)$ is acyclic as well:

$$
\begin{equation*}
H^{n}\left(\Omega\left(R_{\ell}\right), d\right)=\delta_{n, 0} \mathbb{F} . \tag{2.66}
\end{equation*}
$$

We obviously have $\Omega^{0}(V)=V /[V, V]$. Elements of $\Omega^{1}(V)$ can be understood as 1-forms $\omega=\sum_{i=1}^{\ell} f_{i} d x_{i}$, and elements of $\Omega^{n}(V)$ as sort of cyclically skewsymmetric "products" of 1-forms. We provide now an alternative explicit description of the spaces $\Omega^{n}(V)$ for $n \geq 1$ as follows. We identify the space $\Omega^{n}(V)$ with the space $\Sigma^{n}(V)$ of arrays $\left(A_{i_{1} \ldots i_{n}}\right)_{i_{1}, \ldots, i_{n}=1}^{\ell}$ with entries $A_{i_{1} \ldots i_{n}} \in V^{\otimes n}$, satisfying the following skewsymmetry condition:

$$
\begin{equation*}
A_{i_{1} \ldots i_{n}}=-(-1)^{n}\left(A_{i_{2} \ldots i_{n} i_{1}}\right)^{\sigma} \quad \text { for all } i_{1}, \ldots, i_{n} \in\{1, \ldots, \ell\} \tag{2.67}
\end{equation*}
$$

where $\sigma$ denotes the action of the cyclic permutation on $V^{\otimes n}$ as in (1.25). To prove the isomorphism $\Omega^{n}(V) \simeq \Sigma^{n}(V)$ we write explicitly the maps in both directions. Given the coset $\omega=[\widetilde{\omega}] \in \Omega^{n}(V)$, with $\widetilde{\omega}$ as in (2.61), we map it to the array (recall (1.26) and (1.27)) $\left(A_{i_{1} \ldots i_{n}}\right)_{i_{1}, \ldots, i_{n}=1}^{\ell} \in \Sigma^{n}(V)$, with entries $A_{j_{1} \ldots j_{n}}=0$, unless $\left(j_{1}, \ldots, j_{n}\right)$ is a cyclic permutation of $\left(i_{1}, \ldots, i_{n}\right)$, and

$$
\begin{equation*}
A_{j_{1} \ldots j_{n}}=\frac{1}{n}(-1)^{s(n-s)} a^{s+1} \otimes \cdots \otimes a^{n} \otimes a^{n+1} a^{1} \otimes a^{2} \otimes \cdots \otimes a^{s} \tag{2.68}
\end{equation*}
$$

for $\left(j_{1}, \ldots, j_{s}\right)=\left(i_{\sigma^{s}(1)}, \ldots, i_{\sigma^{s}(n)}\right)$, The inverse map $\Sigma^{n}(V) \rightarrow \Omega^{n}(V)$ is given by (in Sweedler's notation):

$$
\begin{equation*}
\left(A_{i_{1} \ldots i_{n}}\right)_{i_{1}, \ldots, i_{n}=1}^{\ell} \mapsto \sum_{i_{1}, \ldots, i_{n}=1}^{\ell}\left[A_{i_{1} \ldots i_{n}}^{\prime} d x_{i_{1}} A_{i_{1} \ldots i_{n}}^{\prime \prime} d x_{i_{2}} \ldots A_{i_{1} \ldots i_{n}}^{\prime \cdots \ldots \prime} d x_{i_{n}}\right] \tag{2.69}
\end{equation*}
$$

It is easy to check that the maps (2.68) and (2.69) are well defined, and that they are inverse to each other, thus proving that the space $\Omega^{n}(V)$ and the space of arrays $\Sigma^{n}(V)$ can be identified using these maps.

It is also not hard to find the formula for the differential $d: \Sigma^{n}(V) \rightarrow \Sigma^{n+1}(V)$ corresponding to the differential $d$ of the reduced complex $\Omega(V)$ under this identification. For $n=0$, we have (recalling (1.10))

$$
\begin{equation*}
d(\operatorname{tr}(f))=\left(\mathrm{m}\left(\frac{\partial f}{\partial x_{i}}\right)^{\sigma}\right)_{i=1}^{\ell} \tag{2.70}
\end{equation*}
$$

For $A=\left(A_{i_{1} \ldots i_{n}}\right)_{i_{1}, \ldots, i_{n}=1}^{\ell} \in \Sigma^{n}(V)$, where $n \geq 1$, we have, using notation (1.22) and recalling (1.26) and (1.27),

$$
\begin{equation*}
(d A)_{i_{1} \ldots i_{n+1}}=\frac{1}{n+1} \sum_{s=1}^{n+1} \sum_{t=1}^{n}(-1)^{s n+t-1}\left(\left(\frac{\partial}{\partial x_{i_{\sigma^{s}(t)}}}\right)_{(t)} A_{i_{\sigma^{s}(1)} \cdots i_{\sigma^{s}(n+1)}}\right)^{\sigma^{s}} \tag{2.71}
\end{equation*}
$$

where $\stackrel{t}{\square}$ means that we skip the index $i_{\sigma^{s}(t)}$. Thanks to Lemma 1.4, we can rewrite formula (2.71) as follows

$$
\begin{align*}
& (d A)_{i_{1} \ldots i_{n+1}} \\
& =\frac{n}{n+1}\left(\sum_{s=1}^{n}(-1)^{s-1}\left(\frac{\partial}{\partial x_{i_{s}}}\right)_{(s)} A_{i_{1} \ldots i_{n+1}}+(-1)^{k}\left(\left(\frac{\partial}{\partial x_{i_{n+1}}}\right)_{(1)} A_{i_{1} \ldots i_{n}}\right)^{\sigma^{n}}\right) \tag{2.72}
\end{align*}
$$

In particular, for $n=1$, we have, for $F=\left(F_{j}\right)_{j=1}^{\ell} \in V^{\oplus \ell}=\Sigma^{1}(V)$,

$$
\begin{equation*}
(d F)_{i j}=\frac{1}{2}\left(\frac{\partial F_{j}}{\partial x_{i}}-\left(\frac{\partial F_{i}}{\partial x_{j}}\right)^{\sigma}\right) . \tag{2.73}
\end{equation*}
$$

For $n=2$, let $A=\left(A_{i j}\right)_{i, j=1}^{\ell} \in \Sigma^{2}(V)$, namely $A_{i j} \in V \otimes V$ and $\left(A_{j i}\right)^{\sigma}=-A_{i j}$. We have

$$
\begin{equation*}
(d A)_{i j k}=\frac{2}{3}\left(\left(\frac{\partial}{\partial x_{i}}\right)_{L} A_{j k}-\left(\frac{\partial}{\partial x_{j}}\right)_{R} A_{i k}+\left(\left(\frac{\partial}{\partial x_{k}}\right)_{L} A_{i j}\right)^{\sigma^{2}}\right) \tag{2.74}
\end{equation*}
$$

As an application of (2.66), we get the following
Corollary 2.20. (a) A 0 -form $\operatorname{tr}(f) \in \Omega^{0}\left(R_{\ell}\right)$ is closed if and only if $f \in \mathbb{F}+$ [ $\left.R_{\ell}, R_{\ell}\right]$.
(b) A 1-form $F=\left(F_{i}\right)_{i=1}^{\ell} \in R_{\ell}^{\oplus \ell}=\Sigma^{1}\left(R_{\ell}\right)$ is closed if and only if there exists $\operatorname{tr}(f) \in R_{\ell} /\left[R_{\ell}, R_{\ell}\right]$ such that $F_{i}=\mathrm{m}\left(\frac{\partial f}{\partial x_{i}}\right)^{\sigma}$ for every $i=1, \ldots, \ell$.
(c) A 2-form $\alpha=\left(A_{i j}\right)_{i, j=1}^{\ell} \in \Sigma^{2}\left(R_{\ell}\right)$ is closed if and only if there exists $F=$ $\left(F_{i}\right)_{i=1}^{\ell} \in R_{\ell}^{\oplus \ell}$ such that

$$
A_{i j}=\frac{1}{2}\left(\frac{\partial F_{j}}{\partial x_{i}}-\left(\frac{\partial F_{i}}{\partial x_{j}}\right)^{\sigma}\right)
$$

for every $i, j=1, \ldots, \ell$.
Remark 2.21. For $F \in V^{\oplus \ell}=\Sigma^{1}(V)$, define the Jacobian

$$
J_{F}=\left(\frac{\partial F_{i}}{\partial x_{j}}\right)_{i, j=1}^{\ell} \in \operatorname{Mat}_{\ell \times \ell}(V \otimes V)
$$

Then equation (2.73) becomes:

$$
d F=\frac{1}{2}\left(J_{F}^{t}-J_{F}^{\sigma}\right)
$$

(Here, $t$ stands for transpose.) Therefore, recalling (1.7), we see that $d F=0$ if and only if $J_{F}=J_{F}^{\dagger}$.

## 3. Double Poisson vertex algebras and non-commutative Hamiltonian PDEs

3.1. Definition of double Poisson vertex algebra. By a differential algebra we mean a unital associative (not necessarily commutative) algebra $\mathcal{V}$ over the field $\mathbb{F}$, endowed with a derivation $\partial \in \operatorname{Der} \mathcal{V}$.

Definition 3.1. An $n$-fold $\lambda$-bracket on $\mathcal{V}$ is a linear map

$$
\left\{\left\{-\lambda_{1}-\cdots-_{\lambda_{n-1}}-\right\}: \mathcal{V}^{\otimes n} \rightarrow \mathcal{V}^{\otimes n}\left[\lambda_{1}, \ldots, \lambda_{n-1}\right]\right.
$$

which satisfies the following sesquilinearity conditions

$$
\begin{equation*}
\left\{\left\{a_{1 \lambda_{1}} \cdots \lambda_{\lambda_{i-1}}\left(\partial a_{i}\right)_{\lambda_{i}} \cdots a_{n-1 \lambda_{n-1}} a_{n}\right\}\right\}=-\lambda_{i}\left\{\left\{a_{1 \lambda_{1}} \cdots a_{n-1 \lambda_{n-1}} a_{n}\right\}\right\}, \tag{3.1}
\end{equation*}
$$

for all $i=1, \ldots, n-1$, and

$$
\begin{equation*}
\left\{\left\{a_{1 \lambda_{1}} \cdots a_{n-1 \lambda_{n-1}}\left(\partial a_{n}\right)\right\}\right\}=\left(\lambda_{1} \cdots+\lambda_{n-1}+\partial\right)\left\{\left\{a_{1 \lambda_{1}} \cdots a_{n-1 \lambda_{n-1}} a_{n}\right\}\right\}, \tag{3.2}
\end{equation*}
$$

and the following Leibniz rules:

$$
\begin{align*}
& \left\{\left\{a_{1 \lambda_{1}} \cdots b c_{\lambda_{i}} \ldots a_{n-1 \lambda_{n-1}} a_{n}\right\}\right\}=\left(e^{\partial \partial_{\lambda_{i}}} b\right) \star_{i}\left\{\left\{a_{1 \lambda_{1}} \cdots c_{\lambda_{i}} \ldots a_{n-1 \lambda_{n-1}} a_{n}\right\}\right. \\
& \left.+\left\{a_{1 \lambda_{1}} \cdots b_{\lambda_{i}+\partial} \ldots a_{n-1 \lambda_{n-1}} a_{n}\right\}\right\} \rightarrow \star_{n-i} c \tag{3.3}
\end{align*}
$$

for all $i=1, \ldots, n$, where $\star_{n}=\star_{0}$. The notation on the RHS of (3.3) needs some explanations. The $\star_{i}$-products were introduced in (1.34). Given a polynomial $P\left(\lambda_{1}, \ldots, \lambda_{n-1}\right)=\sum a_{1} \otimes \cdots \otimes a_{n} \lambda_{1}^{k_{1}} \ldots \lambda_{n-1}^{k_{n}-1} \in \mathcal{V}^{\otimes n}\left[\lambda_{1}, \ldots, \lambda_{n-1}\right]$, we denote

$$
\begin{aligned}
& \left(e^{\partial \partial_{\lambda_{i}}} f\right) \star_{i} P\left(\lambda_{1}, \ldots, \lambda_{i}, \ldots, \lambda_{n-1}\right) \\
& =\sum \sum_{j=0}^{k_{i}}\binom{k_{i}}{j} a_{1} \otimes \ldots a_{i} \otimes\left(\partial^{j} f\right) a_{i+1} \otimes \cdots \otimes a_{n} \lambda_{1}^{k_{1}} \ldots \lambda_{i}^{k_{i}-j} \ldots \lambda_{n-1}^{k_{n-1}},
\end{aligned}
$$

in other words, applying $e^{\partial \partial_{\lambda_{i}}}$ amounts to replacing $\lambda_{i}$ by $\lambda_{i}+\partial$, and the parentheses mean that $\partial$ should be applied to $f$. Moreover, we denote

$$
\begin{align*}
& P\left(\lambda_{1}, \ldots, \lambda_{i}+\partial, \ldots, \lambda_{n-1}\right)_{\rightarrow} \star_{n-i} f \\
& =\sum \sum_{j=0}^{k_{i}}\binom{k_{i}}{j} a_{1} \otimes \cdots \otimes a_{i}\left(\partial^{j} f\right) \otimes \cdots \otimes a_{n} \lambda_{1}^{k_{1}} \ldots \lambda_{i}^{k_{i}-j} \ldots \lambda_{n-1}^{k_{n-1}}, \tag{3.4}
\end{align*}
$$

in other words, the arrow means that $\partial$ is applied to $f$.
In the special case of 2 -fold $\lambda$-brackets, the sesquilinearity conditions (3.1) and (3.2) are

$$
\begin{equation*}
\left\{\left\{\partial a_{\lambda} b\right\}\right\}=-\lambda\left\{\left\{a_{\lambda} b\right\}, \quad\left\{\left\{a_{\lambda} \partial b\right\}\right\}=(\lambda+\partial)\left\{\left\{a_{\lambda} b\right\},\right.\right. \tag{3.5}
\end{equation*}
$$

and the Leibniz rules (3.3) are

$$
\begin{align*}
& \left\{\left\{a_{\lambda} b c\right\}=\left\{\left\{a_{\lambda} b\right\} c+b\left\{a_{\lambda} c\right\}\right.\right. \\
& \left\{a b_{\lambda} c\right\}=\left\{\left\{a_{\lambda+\partial} c\right\} \rightarrow_{\star_{1}} b+\left(e^{\partial \partial_{\lambda}} a\right) \star_{1}\left\{\left\{b_{\lambda} c\right\} .\right.\right. \tag{3.6}
\end{align*}
$$

In analogy with (1.34) and (2.4), we also let, for $a, b, c \in \mathcal{V}$,

$$
\begin{align*}
& \left\{a_{\lambda}(b \otimes c)\right\}_{L}:=\left\{a_{\lambda} b\right\} \otimes c, \quad\left\{a_{\lambda}(b \otimes c)\right\}_{R}:=b \otimes\left\{\left\{a_{\lambda} c\right\},\right. \\
& \left\{(a \otimes b)_{\lambda} c\right\}_{L}:=\left\{\left\{a_{\lambda+\partial} c\right\} \rightarrow \otimes_{1} b,\right.  \tag{3.7}\\
& \left.\left\{(a \otimes b)_{\lambda} c\right\}\right\}_{R}:=\left(e^{\partial \partial_{\lambda}} a\right) \otimes_{1}\left\{\left\{b_{\lambda} c\right\}\right\}\left(=\left\{\left\{b_{\lambda+\partial} c\right\}\right\}_{\otimes_{1}} a\right),
\end{align*}
$$

Definition 3.2. A double Poisson vertex algebra is a differential algebra $\mathcal{V}$, with derivation $\partial: \mathcal{V} \rightarrow \mathcal{V}$, endowed with a 2 -fold $\lambda$-bracket $\left\{-_{\lambda}-\right\}: \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{V}$ satisfying the following axioms:
(i) skewsymmetry:

$$
\begin{equation*}
\left\{\left\{a_{\lambda} b\right\}=-\left\{\left\{b_{-\lambda-\partial} a\right\}\right\}^{\sigma},\right. \tag{3.8}
\end{equation*}
$$

where $-\lambda-\partial$ in the RHS is moved to the left, acting on the coefficients.
(ii) Jacobi identity (cf. Remark 2.2)

$$
\begin{equation*}
\left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} c\right\}\right\}\right\}_{L}-\left\{\left\{b_{\mu}\left\{\left\{a_{\lambda} c\right\}\right\}\right\}\right\}_{R}=\left\{\left\{\left\{\left\{a_{\lambda} b\right\}\right\}_{\lambda+\mu} c\right\}\right\}_{L} .\right. \tag{3.9}
\end{equation*}
$$

Lemma 3.3. (a) The sesquilinearity relations

$$
\begin{aligned}
& \left\{\partial A_{\lambda} B\right\}_{L(\text { resp } . R)}=-\lambda\left\{A_{\lambda} B\right\}_{L(\text { resp } . R)}, \\
& \left\{\left\{A_{\lambda} \partial B\right\}_{L(\text { resp } . R)}=(\lambda+\partial)\left\{\left\{A_{\lambda} B\right\}_{L(\text { resp } . R)},\right.\right.
\end{aligned}
$$

hold if either $A$ or $B$ lies in $\mathcal{V} \otimes \mathcal{V}$, and the other one lies in $\mathcal{V}$.
(b) For $a, b \in \mathcal{V}$ and $A, B \in \mathcal{V} \otimes \mathcal{V}$, the following skewsymmetry relations hold

$$
\begin{aligned}
& \left\{\left\{a_{\lambda} B\right\}_{L(\text { resp } . R)}=-\left\{B_{-\lambda-\partial}^{\sigma} a\right\}_{R(\text { resp } p . L)}^{\sigma\left(\sigma^{2}\right)}\right. \\
& \left\{\left\{A_{\lambda} b\right\}_{L(\text { resp } . R)}=-\left\{b_{-\lambda-\partial} A^{\sigma}\right\}_{R(\text { resp } . L)}^{\sigma\left(\text { res. } \sigma^{2}\right)}\right.
\end{aligned}
$$

(c) For $a, b \in \mathcal{V}$ and $A, B \in \mathcal{V} \otimes \mathcal{V}$, we have

$$
\left\{\left\{a_{\lambda} B\right\}\right\}_{L}^{\sigma}=\left\{\left\{a_{\lambda} B^{\sigma}\right\}_{R}, \quad\left\{\left\{A_{\lambda} b\right\}_{L}=\left\{\left\{A_{\lambda}^{\sigma} b\right\}_{R} .\right.\right.\right.
$$

Proof. Straightforward.
Lemma 3.4. Let $\{\{-\lambda-\}$ be a 2 -fold $\lambda$-bracket on $\mathcal{V}$ satisfying the skewsymmetry axiom (3.8). Then the linear map $\left\{\left[{ }_{\lambda}-{ }_{\mu}-\right\}: \mathcal{V}^{\otimes 3} \rightarrow \mathcal{V}^{\otimes 3}[\lambda, \mu]\right.$ defined by $(a, b, c \in \mathcal{V}):$

$$
\begin{equation*}
\left\{\left\{a_{\lambda} b_{\mu} c\right\}:=\left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} c\right\}\right\}\right\}_{L}-\left\{\left\{b_{\mu}\left\{\left\{a_{\lambda} c\right\}\right\}\right\}_{R}-\left\{\left\{\left\{a_{\lambda} b\right\}\right\}_{\lambda+\mu} c\right\}\right\}_{L},\right.\right. \tag{3.10}
\end{equation*}
$$

is a 3-fold $\lambda$-bracket on $\mathcal{V}$.

Proof. Sesquilinearity properties (3.1) and (3.2) follow easily from Lemma 3.3(a) and (3.5). Let us prove that $\left\{\left[{ }_{\lambda}-_{\mu}-\right\}\right.$ satisfies the Leibniz rule (3.3) with $i=3$. By (3.6) and (3.7), we have

$$
\begin{align*}
& \left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} c d\right\}\right\}\right\}_{L}=\left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} c\right\}\right\}\right\}_{L} d+c\left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} d\right\}\right\}\right\}_{L}+\left\{\{ a _ { \lambda } c \} \left\{\left\{b_{\mu} d\right\},\right.\right.\right.\right.\right. \\
& \left\{\left\{b_{\mu}\left\{\left\{a_{\lambda} c d\right\}\right\}\right\}_{R}=\left\{\left\{b_{\mu}\left\{\left\{a_{\lambda} c\right\}\right\}\right\}_{R} d+c\left\{\left\{b_{\mu}\left\{\left\{a_{\lambda} d\right\}\right\}\right\}_{R}+\left\{\left\{a_{\lambda} c\right\}\right\}\left\{b_{\mu} d\right\},\right.\right.\right.  \tag{3.11}\\
& \left.\left\{\left\{\left\{a_{\lambda} b\right\}\right\}_{\lambda+\mu} c d\right\}\right\}_{L}=\left\{\left\{\left\{a_{\lambda} b\right\}\right\}_{\lambda+\mu} c\right\}_{L} d+c\left\{\left\{\left\{a_{\lambda} b\right\}_{\lambda+\mu} d\right\}\right\}_{L} .
\end{align*}
$$

Hence, equation (3.3) with $i=3$ holds.
Next we prove that

$$
\begin{equation*}
\left.\left\{\left\{a_{\lambda} b_{\mu} c\right\}\right\}=\left\{b_{\mu} c_{-\lambda-\mu-\partial} a\right\}\right\}^{\sigma} . \tag{3.12}
\end{equation*}
$$

Equation (3.3) with $i=1$ and 2 then follows from (3.12) and (3.3) with $i=3$. By Lemma 3.3(b) and (c), we have

$$
\begin{align*}
& \left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} c\right\}\right\}\right\}_{L}=\left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} c\right\}^{\sigma}\right\}_{R}^{\sigma^{2}}=-\left\{\left\{\left\{b_{\mu} c\right\}\right\}_{-\lambda-\partial} a\right\}_{L}^{\sigma},\right.\right.\right. \\
& \left\{b_{\mu}\left\{\left\{a_{\lambda} c\right\}\right\}\right\}_{R}=\left\{\left\{b_{\mu}\left\{\left\{a_{\lambda} c\right\}\right\}^{\sigma}\right\}\right\}_{L}^{\sigma}=-\left\{\left\{b_{\mu}\left\{\left\{c_{-\lambda-\partial} a\right\}\right\}\right\}\right\}_{L}^{\sigma},  \tag{3.13}\\
& \left.\left\{\left\{\left\{a_{\lambda} b\right\}\right\}_{\lambda+\mu} c\right\}\right\}_{L}=-\left\{\left\{\left\{b_{\mu} a\right\}\right\}_{\lambda+\mu}^{\sigma} c\right\}_{L}=\left\{\left\{c_{-\lambda-\mu-\partial}\left\{\left\{b_{\mu} a\right\}\right\}\right\}_{R}^{\sigma} .\right.
\end{align*}
$$

Combining the three equations (3.13) we get (3.12).
3.2. The trace map and connection to Lie conformal algebras. Given a (not necessarily commutative) differential algebra $\mathcal{V}$, we denote by $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ the commutator subspace. For $f \in \mathcal{V}$, we let $\operatorname{tr}(f) \in \mathcal{V} /[\mathcal{V}, \mathcal{V}]$ be the corresponding coset. Furthermore, for $f \in \mathcal{V}$, we also let $\int f$ be the coset of $f \in \mathcal{V}$ in the quotient space $\mathcal{V} /([\mathcal{V}, \mathcal{V}]+\partial \mathcal{V})$.

Given a 2 -fold $\lambda$-bracket $\left\{-_{\lambda}-\right\}$ on $\mathcal{V}$ we define the following map $\left\{-_{\lambda}-\right\}$ : $\mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}[\lambda]$ by

$$
\begin{equation*}
\left\{a_{\lambda} b\right\}=\mathrm{m}\left\{\left\{a_{\lambda} b\right\}\right\} \tag{3.14}
\end{equation*}
$$

(we extend the multiplication map on $\mathcal{V} \otimes \mathcal{V}$ to a multiplication map $\mathrm{m}:(\mathcal{V} \otimes$ $\mathcal{V})[\lambda] \rightarrow \mathcal{V}[\lambda]$ in the obvious way), and we also define the map $\{-,-\}: \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}$ by

$$
\begin{equation*}
\{a, b\}=\left.m\left\{\left\{a_{\lambda} b\right\}\right\}\right|_{\lambda=0} . \tag{3.15}
\end{equation*}
$$

Lemma 3.5. If $\left\{\left\{-_{\lambda}-\right\}\right.$ is a skewsymmetric 2 -fold $\lambda$-bracket on $\mathcal{V}$, then the following identity holds in $\mathcal{V}^{\otimes 2}[\lambda, \mu](a, b, c \in \mathcal{V})$ :

$$
\left\{a_{\lambda}\left\{\left\{b_{\mu} c\right\}\right\}-\left\{\left\{b_{\mu}\left\{a_{\lambda} c\right\}\right\}-\left\{\left\{a_{\lambda} b\right\}_{\lambda+\mu} c\right\}\right\}=(\mathrm{m} \otimes 1)\left\{\left\{a_{\lambda} b_{\mu} c\right\}-(1 \otimes \mathrm{~m})\left\{\left\{b_{\mu} a_{\lambda} c\right\} .\right.\right.\right.
$$

where we set $\left\{a_{\lambda} b \otimes c\right\}=\left\{a_{\lambda} b\right\} \otimes c+b \otimes\left\{a_{\lambda} c\right\}$.
Proof. Let us compute the three terms of the LHSof (3.16). Using (3.7), we get, after a straightforward computation,

$$
\begin{aligned}
& \left\{a_{\lambda}\left\{\left\{b_{\mu} c\right\}\right\}\right\}=(\mathrm{m} \otimes 1)\left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} c\right\}\right\}\right\}_{L}+(1 \otimes \mathrm{~m})\left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} c\right\}\right\}\right\}_{R},\right.\right. \\
& \left\{\left\{b_{\mu}\left\{a_{\lambda} c\right\}\right\}=(1 \otimes \mathrm{~m})\left\{\left\{b_{\mu}\left\{\left\{a_{\lambda} c\right\}\right\}\right\}_{L}+(\mathrm{m} \otimes 1)\left\{\left\{b_{\mu}\left\{\left\{a_{\lambda} c\right\}\right\}\right\}_{R},\right.\right.\right. \\
& \left\{\left\{a_{\lambda} b\right\}_{\lambda+\mu} c\right\}=(\mathrm{m} \otimes 1)\left\{\left\{\left\{a_{\lambda} b\right\}_{\lambda+\mu} c\right\}_{L}+(1 \otimes \mathrm{~m})\left\{\left\{\left\{a_{\lambda} b\right\}_{\lambda+\mu} c\right\}_{R} .\right.\right.
\end{aligned}
$$

By skewsymmetry and Lemma 3.3(c), we can replace the last term in the RHS of the third equation by $-(1 \otimes \mathrm{~m})\left\{\left\{\left\{b_{\mu} a\right\}_{\lambda+\mu} c\right\}_{L}\right.$. Hence, combining the three equations above, we get (3.16).

Theorem 3.6. Let $\mathcal{V}$ be a differential algebra, with derivation $\partial$, endowed with a 2 -fold $\lambda$-bracket $\{\{-\lambda-\}$, and let $\{-\lambda-\}$ and $\{-,-\}$ be defined as in (3.14) and (3.15).
(a) $\partial[\mathcal{V}, \mathcal{V}] \subset[\mathcal{V}, \mathcal{V}]$. Hence, we have a well defined induced map (denoted, by abuse of notation, by the same symbol) $\partial: \mathcal{V} /[\mathcal{V}, \mathcal{V}] \rightarrow \mathcal{V} /[\mathcal{V}, \mathcal{V}]$, given by $\partial(\operatorname{tr} f)=$ $\operatorname{tr}(\partial f)$.
(b) $\left\{[\mathcal{V}, \mathcal{V}]_{\lambda} \mathcal{V}\right\}=0$, and $\left\{\mathcal{V}_{\lambda}[\mathcal{V}, \mathcal{V}]\right\} \subset[\mathcal{V}, \mathcal{V}] \otimes \mathbb{F}[\lambda]$. Hence, we have well defined induced maps (denoted, by abuse of notation, by the same symbol)

$$
\left\{-_{\lambda}-\right\}: \mathcal{V} /[\mathcal{V}, \mathcal{V}] \times \mathcal{V} \rightarrow \mathcal{V}[\lambda]
$$

and

$$
\left\{-_{\lambda}-\right\}: \mathcal{V} /[\mathcal{V}, \mathcal{V}] \times \mathcal{V} /[\mathcal{V}, \mathcal{V}] \rightarrow \mathcal{V} /[\mathcal{V}, \mathcal{V}][\lambda]
$$

given, respectively, by

$$
\begin{equation*}
\left\{\operatorname{tr}(f)_{\lambda} g\right\}=\mathrm{m}\left\{\left\{f_{\lambda} g\right\}\right. \tag{3.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\{\operatorname{tr}(f)_{\lambda} \operatorname{tr}(g)\right\}=\operatorname{tr}\left(\mathrm{m}\left\{\left\{f_{\lambda} g\right\}\right)\right. \tag{3.18}
\end{equation*}
$$

(c) If the 2 -fold $\lambda$-bracket is skewsymmetric, then so is the $\lambda$-bracket (3.18):

$$
\left\{\operatorname{tr}(f)_{\lambda} \operatorname{tr}(g)\right\}=-\left\{\operatorname{tr}(g)_{-\lambda-\partial} \operatorname{tr}(f)\right\}
$$

(d) If the 2-fold $\lambda$-bracket defines a structure of a double Poisson vertex algebra on $\mathcal{V}$, then the $\lambda$-bracket (3.18) satisfies the Jacobi identity $(f, g, h \in \mathcal{V})$
$\left\{\operatorname{tr}(f)_{\lambda}\left\{\operatorname{tr}(g)_{\mu} \operatorname{tr}(h)\right\}\right\}-\left\{\operatorname{tr}(g)_{\mu}\left\{\operatorname{tr}(f)_{\lambda} \operatorname{tr}(h)\right\}\right\}=\left\{\left\{\operatorname{tr}(f)_{\lambda} \operatorname{tr}(g)\right\}_{\lambda+\mu} \operatorname{tr}(h)\right\}$,
thus defining a structure of a Lie conformal algebra on $\mathcal{V} /[\mathcal{V}, \mathcal{V}]$. Furthermore, the $\lambda$-action of $\mathcal{V} /[\mathcal{V}, \mathcal{V}]$ on $\mathcal{V}$, given by (3.17), defines a representation of the Lie conformal algebra $\mathcal{V} /[\mathcal{V}, \mathcal{V}]$ given by conformal derivations of $\mathcal{V}$.
(e) $\{[\mathcal{V}, \mathcal{V}]+\partial V, \mathcal{V}\}=0$, and $\{\mathcal{V},[\mathcal{V}, \mathcal{V}]+\partial V\} \subset([\mathcal{V}, \mathcal{V}]+\partial \mathcal{V}) \otimes \mathbb{F}[\lambda]$. Hence, we have well defined induced brackets (denoted, by abuse of notation, by the same symbol)

$$
\{-,-\}: \mathcal{V} /([\mathcal{V}, \mathcal{V}]+\partial \mathcal{V}) \times \mathcal{V} \rightarrow \mathcal{V}
$$

and

$$
\{-,-\}: \mathcal{V} /([\mathcal{V}, \mathcal{V}]+\partial \mathcal{V}) \times \mathcal{V} /([\mathcal{V}, \mathcal{V}]+\partial \mathcal{V}) \rightarrow \mathcal{V} /([\mathcal{V}, \mathcal{V}]+\partial \mathcal{V})
$$

given, respectively, by

$$
\begin{equation*}
\left\{\int f, g\right\}:=\mathrm{m}\left\{\left.\left\{f_{\lambda} g\right\}\right|_{\lambda=0},\right. \tag{3.19}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\left\{\int f, \int g\right\}:=\int \mathrm{m}\left\{f_{\lambda} g\right\}\right\}\left.\right|_{\lambda=0} \tag{3.20}
\end{equation*}
$$

(f) If the 2 -fold $\lambda$-bracket is skewsymmetric, then so is the bracket (3.20).
(g) If the 2-fold $\lambda$-bracket defines a structure of a double Poisson vertex algebra on $\mathcal{V}$, then the bracket (3.20) defines a structure of a Lie algebra on $\mathcal{V} /([\mathcal{V}, \mathcal{V}]+\partial \mathcal{V})$. Furthermore, the action of $\mathcal{V} /([\mathcal{V}, \mathcal{V}]+\partial)$ on $\mathcal{V}$, given by (3.19), defines a representation of the Lie algebra $\mathcal{V} /([\mathcal{V}, \mathcal{V}]+\partial \mathcal{V})$ by derivations of $\mathcal{V}$ commuting with $\partial$.
Proof. Part (a) is clear, since $\partial$ is a derivation of the associative product of $\mathcal{V}$. By the second Leibniz rule (3.6) we have, for $a, b, c \in \mathcal{V}$,

$$
\left\{a b_{\lambda} c\right\}=\mathrm{m}\left\{\left\{a b_{\lambda} c\right\}=\mathrm{m}\left(\left(e^{\partial \partial_{\lambda}} a\right) \otimes_{1}\left\{\left\{b_{\lambda} c\right\}\right\}+\left\{\left\{a_{\lambda+\partial} c\right\} \rightarrow \otimes_{1} b\right) .\right.\right.
$$

Note that the expression in parenthesis in the RHS above is unchanged if we switch $a$ and $b$ (cf. the last identity in (3.7)). Hence, $\left\{a b_{\lambda} c\right\}=\left\{b a_{\lambda} c\right\}$. Furthermore, we have

$$
\left\{a_{\lambda} b c\right\}=\mathrm{m}\left\{\left\{a_{\lambda} b c\right\}\right\}=\mathrm{m}\left(\left\{\left\{a_{\lambda} b\right\}\right\} c+b\{\{a \lambda c\}\}\right)=\left\{a_{\lambda} b\right\} c+b\left\{a_{\lambda} c\right\} .
$$

Namely, the $\lambda$-action $\left\{a_{\lambda}-\right\}$ is by derivations of the associative product of $\mathcal{V}$. But then

$$
\left\{a_{\lambda} b c-c b\right\}=\left[b,\left\{a_{\lambda} c\right\}\right]+\left[\left\{a_{\lambda} b\right\}, c\right] \in[\mathcal{V}, \mathcal{V}] \otimes \mathbb{F}[\lambda] .
$$

This proves part (b). Part (c) is immediate. Part (d) is an immediate consequence of Lemma 3.5. Finally, parts (e), (f) and (g) can be proved in the same way, or by using the usual construction that associates to a Lie conformal algebra $R$ the corresponding Lie algebra $R / \partial R$ and its representation on $R$.
3.3. Double Poisson vertex algebra structure on an algebra of (noncommutative) differential functions. Consider the algebra of non-commutative differential polynomials $\mathcal{R}_{\ell}$ in $\ell$ variables $u_{i}, i \in I=\{1, \ldots, \ell\}$. It is the algebra of non-commutative polynomials in the indeterminates $u_{i}^{(n)}$,

$$
\mathcal{R}_{\ell}=\mathbb{F}\left\langle u_{i}^{(n)} \mid i \in I, n \in \mathbb{Z}_{+}\right\rangle,
$$

endowed with a derivation $\partial$, defined on generators by $\partial u_{i}^{(n)}=u_{i}^{(n+1)}$, and partial derivatives $\frac{\partial}{\partial u_{i}^{(n)}}$, for every $i \in I$ and $n \in \mathbb{Z}_{+}$, defined on monomials by

$$
\begin{equation*}
\frac{\partial}{\partial u_{i}^{(n)}}\left(u_{i_{1}}^{\left(n_{1}\right)} \ldots u_{i_{s}}^{\left(n_{s}\right)}\right)=\sum_{k=1}^{s} \delta_{i_{k}, i} \delta_{n_{k}, n} u_{i_{1}}^{\left(n_{1}\right)} \ldots x_{i_{k-1}}^{\left(n_{k-1}\right)} \otimes u_{i_{k+1}}^{\left(n_{k+1}\right)} \ldots u_{i_{s}}^{\left(n_{s}\right)} \tag{3.21}
\end{equation*}
$$

which are commuting 2 -fold derivations of $\mathcal{R}_{\ell}$ such that

$$
\begin{equation*}
\left[\frac{\partial}{\partial u_{i}^{(n)}}, \partial\right]=\frac{\partial}{\partial u_{i}^{(n-1)}}, \tag{3.22}
\end{equation*}
$$

where the RHS is zero for $n=0$.
Lemma 3.7. For any non-commutative differential polynomial $f \in \mathcal{R}_{\ell}$, and $i, j \in I$ and $n, m \in \mathbb{Z}_{+}$, the partial derivatives strongly commute, i.e. we have

$$
\left(\frac{\partial}{\partial u_{i}^{(m)}}\right)_{L} \frac{\partial f}{\partial u_{j}^{(n)}}=\left(\frac{\partial}{\partial u_{j}^{(n)}}\right)_{R} \frac{\partial f}{\partial u_{i}^{(m)}}
$$

Proof. Same as the proof of Lemma 2.6.
Definition 3.8. An algebra of differential functions in $\ell$ variables is a unital associative differential algebra $\mathcal{V}$, with derivation $\partial$, endowed with strongly commuting 2-fold derivations $\frac{\partial}{\partial u_{i}^{(n)}}: \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{V}, i \in I=\{1, \ldots, \ell\}, n \in \mathbb{Z}_{+}$, such that (3.22) holds and, for every $f \in \mathcal{V}$, we have $\frac{\partial f}{\partial u_{i}^{(n)}}=0$ for all but finitely many choices of indices $(i, n) \in I \times \mathbb{Z}_{+}$.

An example of such an algebra is the algebra $\mathcal{R}_{\ell}$, endowed with the 2 -fold derivations defined in (3.21), or its localization by non-zero elements.
Lemma 3.9. Let $\left\{-_{\lambda}-\right\}$ be a 2 -fold $\lambda$-bracket on a differential algebra $\mathcal{V}$. For every $a \in \mathcal{V}$ and $B, C \in \mathcal{V}^{\otimes 2}$, we have

$$
\begin{align*}
& \left\{\left\{a_{\lambda} B \bullet C\right\}_{L}=B \bullet_{2}\left\{\left\{a_{\lambda} C\right\}_{L}+\left\{\left\{a_{\lambda} B\right\}_{L} \bullet_{1} C,\right.\right.\right. \\
& \left\{a_{\lambda} B \bullet C\right\}_{R}=B \bullet_{2}\left\{\left\{a_{\lambda} C\right\}_{R}+\left\{\left\{a_{\lambda} B\right\}_{R} \bullet_{3} C .\right.\right.  \tag{3.23}\\
& \left\{B \bullet C_{\lambda} a\right\}_{L}=\left\{\left\{B_{\lambda+\partial} a\right\}_{L \rightarrow} \bullet_{3} C+\left\{\left\{C_{\lambda+\partial} a\right\}_{L \rightarrow} \bullet_{1} B^{\sigma} .\right.\right.
\end{align*}
$$

In the last equation the arrow has a meaning similar to (3.4) (with • product in place of $\star$ product).

Proof. Straightforward.
Theorem 3.10. (a) Any 2 -fold $\lambda$-bracket on $\mathcal{R}_{\ell}$ has the form $\left(f, g \in \mathcal{R}_{\ell}\right)$ :

$$
\begin{align*}
& \left\{\left\{f_{\lambda} g\right\}=\sum_{\substack{i, j \in I \\
m, n \in \mathbb{Z}_{+}}} \frac{\partial g}{\partial u_{j}^{(n)}} \bullet(\lambda+\partial)^{n}\left\{\left\{u_{i \lambda+\partial} u_{j}\right\} \rightarrow(-\lambda-\partial)^{m} \bullet\left(\frac{\partial f}{\partial u_{i}^{(m)}}\right)^{\sigma}\right.\right.  \tag{3.24}\\
& \text { where } \bullet \text { is as in (1.1). }
\end{align*}
$$

(b) Let $\mathcal{V}$ be an algebra of differential functions in $\ell$ variables. Let $H(\lambda)$ be an $\ell \times \ell$ matrix with entries in $(\mathcal{V} \otimes \mathcal{V})[\lambda]$. We denote its entries by $H_{i j}(\lambda)=\left\{\left\{u_{j_{\lambda}} u_{i}\right\}\right.$, $i, j \in I$. Then formula (3.24) defines a 2 -fold $\lambda$-bracket on $\mathcal{V}$.
(c) Equation (3.24) defines a structure of double Poisson vertex algebra on $\mathcal{V}$ if and only if the skewsymmetry axiom (i) and the Jacobi identity (ii) hold on the $u_{i}$ 's, namely, for all $i, j, k \in I$, we have

$$
\begin{equation*}
\left\{\left\{u_{i \lambda} u_{j}\right\}\right\}=-\left\{\left\{u_{j_{-\lambda-\lambda}} u_{i}\right\}\right\}^{\sigma}, \tag{3.25}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\{\left\{u_{i \lambda}\left\{\left\{u_{j_{\mu}} u_{k}\right\}\right\}\right\}_{L}-\left\{\left\{u_{j_{\mu}}\left\{\left\{u_{i_{\lambda}} u_{k}\right\}\right\}\right\}\right\}_{R}=\left\{\left\{\left\{u_{i_{\lambda}} u_{j}\right\}_{\lambda+\mu} u_{k}\right\}_{L},\right.\right. \tag{3.26}
\end{equation*}
$$

where each term is understood, in terms of the matrix $H$, via (2.25), for example

$$
\begin{equation*}
\left\{u_{i \lambda}\left\{\left\{u_{j_{\mu}} u_{k}\right\}\right\}\right\}_{L}=\sum_{i \in I, n \in \mathbb{Z}_{+}}\left(\left(\frac{\partial}{\partial u_{h}^{(n)}}\right)_{L} H_{k j}(\mu)\right) \bullet \bullet_{3}(\lambda+\partial)^{n} H_{h i}(\lambda) . \tag{3.27}
\end{equation*}
$$

Proof. The proof is similar to the proof of Theorem 2.8 using Lemma 3.9.
Remark 3.11. For a matrix differential operator $H(\partial)=\left(H_{i j}(\partial)\right)_{i, j=1}^{\ell}$, with entries $H_{i j}(\partial) \in(\mathcal{V} \otimes \mathcal{V})[\partial]$, we define its adjoint operator $H^{\dagger}(\partial)$, with entries $\left(H^{\dagger}\right)_{i j}(\partial)=$ $\left(\left(H_{j i}\right)^{*}(\partial)\right)^{\sigma}$, where $A^{*}(\partial)$ denotes the formal adjoint of the differential operator $A(\partial)$. Then property (3.25) means that the matrix differential operator $H(\partial)$ is skewadjoint.
Definition 3.12. A matrix differential operator $H(\partial)=\left(H_{i j}(\partial)\right)_{i, j=1}^{\ell}$, with entries $H_{i j}(\partial) \in(\mathcal{V} \otimes \mathcal{V})[\partial]$, satisfying (3.25) and (3.26) is called a Poisson structure on the algebra of differential functions $\mathcal{V}$.

### 3.4. Evolution PDE and Hamiltonian PDE over an algebra of differential

 functions. An evolution PDE over the algebra of differential functions $\mathcal{V}$ has the form$$
\begin{equation*}
\frac{d u_{i}}{d t}=P_{i} \in \mathcal{V}, \quad i \in I=\{1, \ldots, \ell\} \tag{3.28}
\end{equation*}
$$

Assuming that time derivative commutes with space derivative, we have $\frac{d u_{i}^{(n)}}{d t}=$ $\partial^{n} P_{i}$, and, by the chain rule, a function $f \in \mathcal{V}$ evolves according to

$$
\begin{equation*}
\frac{d f}{d t}=\sum_{(i, n) \in I \times \mathbb{Z}_{+}} \mathrm{m}\left(\left(\partial^{n} P_{i}\right) \otimes_{1} \frac{\partial f}{\partial u_{i}^{(n)}}\right)=X_{P}(f) \tag{3.29}
\end{equation*}
$$

An integral of motion is a local functional $\int f \in \mathcal{V} / \partial \mathcal{V}+[\mathcal{V}, \mathcal{V}]$ constant in time:

$$
\begin{equation*}
\frac{d \int f}{d t}=\int \sum_{(i, n) \in I \times \mathbb{Z}_{+}} \mathrm{m}\left(\left(\partial^{n} P_{i}\right) \otimes_{1} \frac{\partial f}{\partial u_{i}^{(n)}}\right)=0 \tag{3.30}
\end{equation*}
$$

A vector field on $\mathcal{V}$ is a derivation $X: \mathcal{V} \rightarrow \mathcal{V}$ of the form (cf. (2.44))

$$
\begin{equation*}
X(f)=\sum_{(i, n) \in I \times \mathbb{Z}_{+}} \mathrm{m}\left(P_{i, n} \otimes_{1} \frac{\partial f}{\partial u_{i}^{(n)}}\right) \tag{3.31}
\end{equation*}
$$

where $P_{i, n} \in \mathcal{V}$ for all $i, n$. Note that the RHS of (3.31) is a finite sum since, by assumptions on $\mathcal{V}, \frac{\partial f}{\partial u_{i}^{(n)}}=0$ for all but finitely many choices of indices $(i, n)$.

An evolutionary vector field is a vector field commuting with $\partial$. By (3.22), it has the form (cf. (3.29))

$$
\begin{equation*}
X_{P}(f)=\sum_{(i, n) \in I \times \mathbb{Z}_{+}} \mathrm{m}\left(\left(\partial^{n} P_{i}\right) \otimes_{1} \frac{\partial f}{\partial u_{i}^{(n)}}\right) \tag{3.32}
\end{equation*}
$$

for $P=\left(P_{i}\right)_{i=1}^{\ell} \in \mathcal{V}^{\ell}$, called the characteristics of the evolutionary vector field $X_{P}$.
Vector fields form a Lie algebra, denoted by $\operatorname{Vect}(\mathcal{V})$, and evolutionary vector fields form a Lie subalgebra, denoted $\operatorname{Vect}^{\partial}(\mathcal{V})$. The Lie bracket is given by the same formulas as in the finite case, equation (2.45).

Equation (3.28) is called compatible with another evolution $\operatorname{PDE} \frac{d u_{i}}{d \tau}=Q_{i}, i \in I$, if the corresponding evolutionary vector fields commute.

The Hamiltonian equation on a double Poisson vertex algebra $\mathcal{V}$, associated to the Hamiltonian functional $\int h \in \mathcal{V} /([\mathcal{V}, \mathcal{V}]+\partial \mathcal{V})$ is

$$
\begin{equation*}
\frac{d u}{d t}=\left\{\int h, u\right\} \tag{3.33}
\end{equation*}
$$

for any $u \in \mathcal{V}$. An integral of motion for the Hamiltonian equation (2.46) is an element $\int f \in \mathcal{V} /([\mathcal{V}, \mathcal{V}]+\partial \mathcal{V})$ such that $\left\{\int h, \int f\right\}=0$. In this case, by Theorem $3.6(\mathrm{~g})$, the derivations $X^{h}=\left\{\int h,-\right\}$ and $X^{f}=\left\{\int f,-\right\}$ (called Hamiltonian vector fields) commute, and equations $\frac{d u}{d t}=\{\operatorname{tr} f, u\}$ and (3.33) are compatible.

In the special case when $\mathcal{V}$ is an algebra of differential functions and the 2-fold $\lambda$-bracket $\{\{-\lambda-\}$ is given by a Poisson structure $H(\partial)$ on $\mathcal{V}$ via (3.24), where $\left.\left\{u_{i \lambda} u_{j}\right\}\right\}=H_{j i}(\lambda)$, the Hamiltonian equation (3.33) becomes the following evolution equation

$$
\begin{equation*}
\frac{d u_{i}}{d t}=\mathrm{m} \sum_{j=1}^{\ell} H_{i j}(\partial) \bullet\left(\frac{\delta h}{\delta u_{j}}\right)^{\sigma} \tag{3.34}
\end{equation*}
$$

where $\frac{\delta h}{\delta u_{j}} \in \mathcal{V} \otimes \mathcal{V}$ denotes the variational derivative of $h$ :

$$
\begin{equation*}
\frac{\delta h}{\delta u_{j}}=\sum_{n \in \mathbb{Z}_{+}}(-\partial)^{n} \frac{\partial h}{\partial u_{j}^{(n)}} \tag{3.35}
\end{equation*}
$$

In equation (3.34) $\partial$ is moved to the right of the $\bullet$ product, acting on $\frac{\delta h}{\delta u_{j}}$. Furthermore the Lie algebra bracket $\{-,-\}$ on $\mathcal{V} /([\mathcal{V}, \mathcal{V}]+\partial \mathcal{V})$ defined by (3.20), becomes $(f, g \in \mathcal{V})$ :

$$
\begin{equation*}
\left\{\int f, \int g\right\}=\int \sum_{i, j \in I} \mathrm{~m}\left(\frac{\delta g}{\delta u_{j}}\right)^{\sigma} \mathrm{m}\left(H_{j i}(\partial) \star_{1} \mathrm{~m}\left(\frac{\delta f}{\delta u_{i}}\right)^{\sigma}\right) . \tag{3.36}
\end{equation*}
$$

In this case the notions of compatibility and of integrals of motion, are consistent with those for general evolution equations, due to Theorem 3.10.
Remark 3.13. For $H(\partial) \in \operatorname{Mat}_{\ell \times \ell}(\mathcal{V} \otimes \mathcal{V})[\partial]$ and $F \in \mathcal{V}^{\oplus \ell}$ let $H(\partial) F \in \mathcal{V}^{\ell}$ be defined by (where we write $H_{i j}(\partial)=\sum_{n \in \mathbb{Z}_{+}}\left(H_{i j, n}^{\prime} \otimes H_{i j, n}^{\prime \prime}\right) \partial^{n}$ )

$$
\begin{equation*}
(H(\partial) F)_{i}=\sum_{j \in I} \mathrm{~m}\left(H_{i j}(\partial) \star_{1} F_{j}\right)=\sum_{j i n I, n \in \mathbb{Z}_{+}} H_{i j, n}^{\prime}\left(\partial^{n} F_{j}\right) H_{i j, n}^{\prime \prime} . \tag{3.37}
\end{equation*}
$$

Then, formula (3.36) can be written in the more traditional form

$$
\begin{equation*}
\left\{\int f, \int g\right\}=\int_{30} \delta g \cdot(H(\partial) \delta f) \tag{3.38}
\end{equation*}
$$

where $\cdot$ denotes the usual dot product of vectors and $(\delta f)_{i}=\mathrm{m}\left(\frac{\delta f}{\delta u_{i}}\right)^{\sigma}$. The latter notation is compatible with the theory of the variational complex (see equation (3.46)).

Remark 3.14. The theory of the Lenard-Magri scheme discussed in Section 2.5 for double Poisson algebras $V$ holds verbatim for double Poisson vertex algebras $\mathcal{V}$ if we replace the matrix $H$ by the matrix differential operator $H(\partial)$ and the map $\operatorname{tr}$ by the map $\int$.
3.5. de Rham complex over an algebra of differential functions. Let $\mathcal{V}$ be an algebra differential functions. We define the de Rham complex $\widetilde{\Omega}(\mathcal{V})$ as the free product of the algebra $\mathcal{V}$ and the algebra $\mathbb{F}\left\langle\delta u_{i}^{(n)} \mid i \in I=\{1, \ldots, \ell\}, n \in \mathbb{Z}_{+}\right\rangle$of non-commutative polynomials in the variables $\delta u_{i}^{(n)}$. The action of $\partial$ is extended from $\mathcal{V}$ to a derivation of $\widetilde{\Omega}(\mathcal{V})$ by letting $\partial\left(\delta u_{i}^{(n)}\right)=\delta u_{i}^{(n+1)}$ for all $(i, n) \in I \times \mathbb{Z}_{+}$. This makes $\widetilde{\Omega}(\mathcal{V})$ a differential algebra. It has a $\mathbb{Z}_{+}$-grading, such that $f \in \mathcal{V}$ has degree 0 and the $\delta u_{i}^{(n)}$ 's have degree 1 . We consider it as a superalgebra, with superstructure compatible with the $\mathbb{Z}_{+}$-grading. The subspace $\widetilde{\Omega}^{k}(\mathcal{V})$ of degree $n$ consists of linear combinations of elements of the form

$$
\begin{equation*}
\widetilde{\omega}=f_{1} \delta u_{i_{1}}^{\left(m_{1}\right)} f_{2} \delta u_{i_{2}}^{\left(m_{2}\right)} \ldots f_{k} \delta u_{i_{k}}^{\left(m_{k}\right)} f_{k+1}, \text { where } f_{1}, \ldots, f_{k+1} \in \mathcal{V} \tag{3.39}
\end{equation*}
$$

In particular, $\widetilde{\Omega}^{0}(\mathcal{V})=\mathcal{V}$ and $\widetilde{\Omega}^{1}(\mathcal{V}) \simeq \oplus_{(i, n) \in I \times \mathbb{Z}_{+}} \mathcal{V} \delta u_{i}^{(n)} \mathcal{V}$.
Define the de Rham differential $\delta$ on $\widetilde{\Omega}(\mathcal{V})$ as the odd derivation of degree 1 on the superalgebra $\widetilde{\Omega}(\mathcal{V})$ by letting

$$
\begin{equation*}
\delta f=\sum_{(i, n) \in I \times \mathbb{Z}_{+}}\left(\frac{\partial f}{\partial u_{i}^{(n)}}\right)^{\prime} \delta u_{i}^{(n)}\left(\frac{\partial f}{\partial u_{i}^{(n)}}\right)^{\prime \prime} \in \widetilde{\Omega}^{1}(\mathcal{V}) \text { for } f \in \mathcal{V}, \text { and } \delta\left(\delta u_{i}^{(n)}\right)=0 \tag{3.40}
\end{equation*}
$$

The proof that $\delta^{2}=0$ is the same as for the algebra of ordinary differential functions (cf. Section 2.7), so we can consider the corresponding cohomology complex $(\widetilde{\Omega}(\mathcal{V}), \delta)$.

Given a vector field $X_{P}=\sum_{(i, n) \in I \times \mathbb{Z}_{+}} \mathrm{m} \circ\left(P_{i, n} \otimes_{1} \frac{\partial}{\partial u_{i}^{(n)}}\right) \in \operatorname{Vect}(\mathcal{V})$ (cf. (3.31)), we define the corresponding Lie derivative $L_{P}: \widetilde{\Omega}(\mathcal{V}) \rightarrow \widetilde{\Omega}(\mathcal{V})$ as the even derivation of degree 0 extending $X_{P}$ from $\mathcal{V}$, and such that $L_{P}\left(\delta u_{i}^{(n)}\right)=\delta P_{i, n}, i \in I, n \in$ $\mathbb{Z}_{+}$. Next, we define the corresponding contraction operator $\iota_{P}: \widetilde{\Omega}(\mathcal{V}) \rightarrow \widetilde{\Omega}(\mathcal{V})$ as the odd derivation of degree -1 defined on generators by $\iota_{P}(f)=0$, for $f \in \mathcal{V}$, and $\iota_{P}\left(\delta u_{i}^{(n)}\right)=P_{i, n}$. It is easy to show, in analogy to Proposition 2.17 and Theorem 2.18, that $\widetilde{\Omega}(\mathcal{V})$ is a $\operatorname{Vect}(\mathcal{V})$-complex and the complex $\left(\widetilde{\Omega}\left(\mathcal{R}_{\ell}\right), \delta\right)$ is acyclic: $H^{n}\left(\widetilde{\Omega}\left(\mathcal{R}_{\ell}\right), \delta\right)=\delta_{n, 0} \mathbb{F}$.

### 3.6. Variational complex over an algebra of differential functions.

Proposition 3.15. In the de Rham complex $(\widetilde{\Omega}(\mathcal{V}), \delta)$ we have:
(a) The commutator subspace $[\widetilde{\Omega}(\mathcal{V}), \widetilde{\Omega}(\mathcal{V})]$ is compatible with the $\mathbb{Z}_{+}$-grading and is preserved by $\delta$.
(b) $\delta$ and $\partial$ commute: $\partial \circ \delta=\delta \circ \partial$; hence, $\partial \widetilde{\Omega}(\mathcal{V})$ is compatible with the $\mathbb{Z}_{+}$-grading and is preserved by $\delta$.
(c) The Lie derivative $L_{P}$ and the contraction operator $\iota_{P}$, associated to an evolutionary vector field $X_{P_{\sim}}$ of characteristics $P=\left(P_{i}\right)_{i=1}^{\ell}$ (cf. (3.32)), commute with the action of $\partial$ on $\widetilde{\Omega}(\mathcal{V})$.

Proof. Part (a) is immediate, since $\delta$ is a derivation of the associative product in $\widetilde{\Omega}(\mathcal{V})$. For part (b) we just need to prove that $\delta(\partial \widetilde{\omega})=\partial(\delta \widetilde{\omega})$ for every $\widetilde{\omega} \in \widetilde{\Omega}(\mathcal{V})$. Since $[\delta, \partial]$ is a derivation of $\widetilde{\Omega}(\mathcal{V})$, and it is obviously zero on $\delta u_{i}^{(n)}$, it suffices to check that it is zero on $f \in \mathcal{V}$. We have, by (3.40),

$$
\begin{aligned}
& \partial(\delta f)=\sum_{(i, n) \in I \times \mathbb{Z}_{+}}\left(\partial \frac{\partial f}{\partial u_{i}^{(n)}}\right)^{\prime} \delta u_{i}^{(n)}\left(\partial \frac{\partial f}{\partial u_{i}^{(n)}}\right)^{\prime \prime} \\
& +\sum_{(i, n) \in I \times \mathbb{Z}_{+}}\left(\frac{\partial f}{\partial u_{i}^{(n)}}\right)^{\prime} \delta u_{i}^{(n+1)}\left(\frac{\partial f}{\partial u_{i}^{(n)}}\right)^{\prime \prime}
\end{aligned}
$$

and

$$
\delta(\partial f)=\sum_{(i, n) \in I \times \mathbb{Z}_{+}}\left(\frac{\partial(\partial f)}{\partial u_{i}^{(n)}}\right)^{\prime} \delta u_{i}^{(n)}\left(\frac{\partial(\partial f)}{\partial u_{i}^{(n)}}\right)^{\prime \prime} .
$$

Hence, $\partial(\delta f)=\delta(\partial f)$ by (3.22).
As for part (c), since $L_{P}, \iota_{P}$ and $\partial$ are derivations of the associative product of $\widetilde{\Omega}(\mathcal{V})$, it suffices to check that $\left[L_{P}, \partial\right]$ and $\left[\iota_{P}, \partial\right]$ act as zero on generators, i.e. on $f \in \mathcal{V}$ and on $\delta u_{i}^{(n)}$. This is straightforward.

Thanks to Proposition $3.15(\mathrm{a}-\mathrm{b})$, we can consider the $\mathbb{Z}_{+}$-graded variational complex

$$
\begin{equation*}
\Omega(\mathcal{V})=\widetilde{\Omega}(\mathcal{V}) /(\partial \widetilde{\Omega}(\mathcal{V})+[\widetilde{\Omega}(\mathcal{V}), \widetilde{\Omega}(\mathcal{V})])=\oplus_{n \in \mathbb{Z}_{+}} \Omega^{n}(\mathcal{V}) \tag{3.41}
\end{equation*}
$$

with the induced action of $\delta$. Furthermore, by Proposition 3.15(c) the Lie derivatives $L_{P}$ and contraction operators $\iota_{P}$, associated to the evolutionary vector field $X_{P}$ of characteristics $P \in \mathcal{V}^{\ell}$, induce well defined maps on the variational complex $\Omega(\mathcal{V})$, and they define a structure of $\operatorname{Vect}^{\partial}(\mathcal{V})$-complex on $\widetilde{\Omega}(\mathcal{V})$.

Since the total degree vector field $X_{\Delta}$, with characteristics $\Delta=\left(u_{i}\right)_{i=1}^{\ell}$, is an evolutionary vector field on $\mathcal{R}_{\ell}$, the same argument in the proof of Theorem 2.18 shows that the complex $\left(\Omega\left(\mathcal{R}_{\ell}\right), \delta\right)$ is acyclic, i.e.

$$
\begin{equation*}
H^{k}\left(\Omega\left(\mathcal{R}_{\ell}\right), \delta\right)=\delta_{k, 0} \mathbb{F} \tag{3.42}
\end{equation*}
$$

As we did in the finite case, Section 2.8, we give an explicit description of the complex $(\Omega(\mathcal{V}), \delta)$. We obviously have $\Omega^{0}(\mathcal{V})=\mathcal{V} /(\partial \mathcal{V}+[\mathcal{V}, \mathcal{V}])$. For $k \geq 1$, we identify $\Omega^{k}(\mathcal{V})$ with the space $\Sigma^{k}(\mathcal{V})$ of arrays $\left(A_{i_{1} \ldots i_{k}}\left(\lambda_{1}, \ldots, \lambda_{k-1}\right)_{i_{1}, \ldots, i_{k}=1}^{\ell}\right.$ with entries $A_{i_{1} \ldots i_{k}}\left(\lambda_{1}, \ldots, \lambda_{k-1}\right) \in \mathcal{V}^{\otimes k}\left[\lambda_{1}, \ldots, \lambda_{k-1}\right]$, satisfying the following skewadjointness condition $\left(i_{1}, \ldots, i_{k} \in I\right)$ :

$$
\begin{equation*}
A_{i_{1} \ldots i_{k}}\left(\lambda_{1}, \ldots, \lambda_{k-1}\right)=-(-1)^{k}\left(A_{i_{2} \ldots i_{k} i_{1}}\left(\lambda_{2}, \ldots, \lambda_{k-1},-\lambda_{1}-\cdots-\lambda_{k-1}-\overleftarrow{\overleftarrow{\partial}}\right)\right)^{\sigma} \tag{3.43}
\end{equation*}
$$

where $\sigma$ denotes the action of the cyclic permutation on $\mathcal{V}^{\otimes k}$ as in (1.25), and the arrow means that $\partial$ is acting on the coefficients of the polynomial. To prove the isomorphism $\Omega^{k}(\mathcal{V}) \simeq \Sigma^{k}(\mathcal{V})$ we write explicitly the maps in both directions. Given the $\operatorname{coset} \omega=[\widetilde{\omega}] \in \Omega^{k}(\mathcal{V})$, where $\widetilde{\omega}$ is as in (3.39), we map it to the array $A=\left(A_{j_{1} \ldots j_{k}}\left(\lambda_{1}, \ldots, \lambda_{k-1}\right)\right)_{i_{1}, \ldots, i_{k}=1}^{\ell} \in \Sigma^{k}(\mathcal{V})$, with entries $A_{j_{1} \ldots j_{k}}\left(\lambda_{1}, \ldots, \lambda_{k-1}\right)=$ 0 unless $\left(j_{1}, \ldots, j_{k}\right)$ is a cyclic permutation of $\left(i_{1}, \ldots, i_{k}\right)$, and

$$
\begin{align*}
& A_{j_{1} \ldots j_{k}}\left(\lambda_{1}, \ldots, \lambda_{k-1}\right)=\frac{1}{k}(-1)^{s(k-s)} \lambda_{1}^{n_{s+1}} \ldots \lambda_{k-s}^{n_{k}} \lambda_{k-s+1}^{n_{1}} \cdots \lambda_{k-1}^{n_{s-1}}  \tag{3.44}\\
& \left(-\lambda_{1}-\cdots-\lambda_{k-1}-\partial\right)^{n_{s}}\left(f_{s+1} \otimes \cdots \otimes f_{k} \otimes f_{k+1} f_{1} \otimes f_{2} \otimes \cdots \otimes f_{s}\right)
\end{align*}
$$

for $\left(j_{1}, \ldots, j_{k}\right)=\left(i_{\sigma^{s}(1)}, \ldots, i_{\sigma^{s}(k)}\right)$. The inverse map $\Sigma^{k}(V) \rightarrow \Omega^{k}(V)$ is given by (in Sweedler's notation):

$$
\begin{align*}
& \left(\sum_{n_{1}, \ldots, n_{k-1} \in \mathbb{Z}_{+}} A_{i_{1} \ldots i_{k}}^{n_{1} \ldots n_{k-1}} \lambda_{1}^{n_{1}} \ldots \lambda_{k-1}^{n_{k-1}}\right)_{i_{1}, \ldots, i_{k}=1}^{\ell} \mapsto \\
& \quad \sum_{\substack{i_{1}, \ldots, i_{k} \in I \\
n_{1}, \ldots, n_{k-1} \in \mathbb{Z}_{+}}}\left[\left(A_{i_{1} \ldots i_{k}}^{n_{1} \ldots n_{k-1}}\right)^{\prime} \delta u_{i_{1}}^{\left(n_{1}\right)} \ldots\left(A_{i_{1} \ldots i_{k}}^{n_{1} \ldots n_{k-1}}\right)^{\prime \ldots \prime} \delta u_{i_{k-1}}^{\left(n_{k-1}\right)}\left(A_{i_{1} \ldots i_{k}}^{n_{1} \ldots n_{k-1}}\right)^{\rho_{\cdots} \ldots \prime} \delta u_{i_{k}}\right] \tag{3.45}
\end{align*}
$$

It is easy to check that the maps (3.44) and (3.45) are well defined, and they are inverse to each other, thus proving that the space $\Omega^{k}(\mathcal{V})$ and the space of arrays $\Sigma^{k}(\mathcal{V})$ can be identified using these maps.

It is also not hard to find the formula for the differential $\delta: \Sigma^{k}(\mathcal{V}) \rightarrow \Sigma^{k+1}(\mathcal{V})$ corresponding to the differential $\delta$ of the reduced complex $\Omega(\mathcal{V})$ under this identification. For $k=0$, we have

$$
\begin{equation*}
\delta\left(\int f\right)=\left(\sum_{n \in \mathbb{Z}_{+}}(-\partial)^{n} \mathrm{~m}\left(\frac{\partial f}{\partial u_{i}^{(n)}}\right)^{\sigma}\right)_{i=1}^{\ell}\left(=\left(\mathrm{m}\left(\frac{\delta f}{\delta u_{i}}\right)^{\sigma}\right)_{i=1}^{\ell}\right) . \tag{3.46}
\end{equation*}
$$

For $A=\left(A_{i_{1} \ldots i_{k}}\left(\lambda_{1}, \ldots, \lambda_{k-1}\right)\right)_{i_{1}, \ldots, i_{k}=1}^{\ell} \in \Sigma^{k}(\mathcal{V})$, where $k \geq 1$, we have, using notation (1.22) and recalling (1.26) and (1.27),

$$
\begin{align*}
& (\delta A)_{i_{1} \ldots i_{k+1}}\left(\lambda_{1}, \ldots, \lambda_{k}\right)=\frac{1}{k+1} \sum_{s=1}^{k+1} \sum_{t=1}^{k}(-1)^{s k+t-1} \times \\
& \times \sum_{n \in \mathbb{Z}_{+}}\left(\left(\frac{\partial}{\partial u_{i_{\sigma^{s}(t)}^{(n)}}^{(n)}}\right)_{(t)} A_{i_{\sigma^{s}(1)} \stackrel{t}{\cdots} i_{\sigma^{s}(k+1)}}\left(\lambda_{\sigma^{s}(1)} \stackrel{t}{\therefore} \cdot \lambda_{\sigma^{s}(k)}\right) \lambda_{\sigma^{s}(t)}^{n}\right)^{\sigma^{s}}, \tag{3.47}
\end{align*}
$$

where $\stackrel{t}{\therefore}$. means that we skip the index $i_{\sigma^{s}(t)}$ and, as before, we need to replace $\lambda_{k+1}$, when it occurs, by $-\lambda_{1}-\cdots-\lambda_{k}-\partial$, with $\partial$ acting on the coefficients (here $\sigma$ denotes the action of the cyclic element $(1,2, \ldots, k+1)$ ). We can use Lemma 1.4 to rewrite formula (3.47) as follows

$$
\begin{align*}
& (\delta A)_{i_{1} \ldots i_{k+1}}\left(\lambda_{1}, \ldots, \lambda_{k}\right) \\
& =\frac{k}{k+1} \sum_{n \in \mathbb{Z}_{+}}\left(\sum_{s=1}^{k}(-1)^{s+1}\left(\frac{\partial}{\partial u_{i_{s}}^{(n)}}\right)_{(s)} A_{i_{1} \cdots i_{k+1}}\left(\lambda_{1}, \stackrel{s}{\cdots}, \lambda_{k}\right) \lambda_{s}^{n}\right.  \tag{3.48}\\
& \left.+(-1)^{k}\left(-\lambda_{1}-\cdots-\lambda_{k}-\partial\right)^{n}\left(\left(\frac{\partial}{\partial u_{i_{k+1}}^{(n)}}\right)_{(1)} A_{i_{1}, \ldots, i_{k}}\left(\lambda_{1}, \ldots, \lambda_{k-1}\right)\right)^{\sigma^{k}}\right) .
\end{align*}
$$

In particular, for $k=1$, we have, for $F=\left(F_{j}\right)_{j=1}^{\ell} \in \mathcal{V}^{\ell}=\Sigma^{1}(\mathcal{V})$,

$$
\begin{equation*}
(\delta F)_{i j}(\lambda)=\frac{1}{2} \sum_{n \in \mathbb{Z}_{+}}\left(\frac{\partial F_{j}}{\partial u_{i}^{(n)}} \lambda^{n}-(-\lambda-\partial)^{n}\left(\frac{\partial F_{i}}{\partial u_{j}^{(n)}}\right)^{\sigma}\right) \tag{3.49}
\end{equation*}
$$

For $k=2$, let $A=\left(A_{i j}(\lambda)\right)_{i, j=1}^{\ell} \in \Sigma^{2}(\mathcal{V})$, namely $A_{i j}(\lambda) \in \mathcal{V}^{\otimes 2}[\lambda]$ and $\left(A_{j i}(-\lambda-\right.$ $\partial))^{\sigma}=-A_{i j}(\lambda)$. We have

$$
\begin{align*}
& (\delta A)_{i j k}(\lambda, \mu)=\frac{2}{3} \sum_{n \in \mathbb{Z}_{+}}\left(\left(\frac{\partial}{\partial u_{i}^{(n)}}\right)_{L} A_{j k}(\mu) \lambda^{n}\right. \\
& \left.-\left(\frac{\partial}{\partial u_{j}^{(n)}}\right)_{R} A_{i k}(\lambda) \mu^{n}+(-\lambda-\mu-\partial)^{n}\left(\left(\frac{\partial}{\partial u_{k}^{(n)}}\right)_{L} A_{i j}(\lambda)\right)^{\sigma^{2}}\right) \tag{3.50}
\end{align*}
$$

Remark 3.16. In analogy with [BDSK09], we introduce the Beltrami 2 -fold $\lambda$ bracket $\left\{\left\{-_{\lambda}-\right\}\right\}^{B}$ using equation (3.24) where we let $\left\{\left\{u_{i \lambda} u_{j}\right\}\right\}^{B}=\delta_{i j}(1 \otimes 1)$, for $i, j \in I$ (note that this 2 -fold $\lambda$-bracket is commutative and does not define a double Poisson vertex algebra structure on $\mathcal{V}$ ). Then equation (3.50) can be rewritten as follows

$$
(\delta A)_{i j k}(\lambda, \mu)=\frac{2}{3}\left(\left\{\left\{u_{i \lambda} A_{j k}(\mu)\right\}_{L}^{B}-\left\{\left\{u_{j \mu} A_{i k}(\lambda)\right\}_{R}^{B}+\left\{\left\{A_{i j}(\lambda)_{\lambda+\mu} u_{k}\right\}_{L}^{B}\right) .\right.\right.\right.
$$

More generally, generalizing notation (3.7), we set $\left(s=1, \ldots, k\right.$ and $\left.a, b_{i} \in \mathcal{V}\right)$

$$
\left\{\left\{a_{\lambda} b_{1} \otimes \cdots \otimes b_{k}\right\}_{(s)}^{B}=b_{1} \otimes \cdots \otimes b_{s-1} \otimes\left\{\left\{a_{\lambda} b_{s}\right\}\right\}^{B} \otimes b_{s+1} \otimes \cdots \otimes b_{k},\right.
$$

and

$$
\left\{\left\{b_{1} \otimes \cdots \otimes b_{k \lambda} a\right\}\right\}_{(1)}^{B}=\left\{\left\{b_{1 \lambda+\partial} a\right\}\right\}_{\rightarrow}^{B} \otimes_{1}\left(b_{2} \otimes \cdots \otimes b_{k}\right) .
$$

Then we can rewrite equation (3.48) as follows:

$$
\begin{aligned}
& (\delta A)_{i_{1} \ldots i_{k+1}}\left(\lambda_{1}, \ldots, \lambda_{k}\right)=\frac{k}{k+1}\left(\sum_{s=1}^{k}(-1)^{s+1}\left\{\left\{u_{i_{s} \lambda_{s}} A_{i_{1} \stackrel{s}{s} i_{k+1}}\left(\lambda_{1}, . \stackrel{s}{\cdots}, \lambda_{k}\right)\right\}\right\}_{(s)}^{B}\right. \\
& +(-1)^{k}\left\{\left\{A_{i_{1}, \ldots, i_{k}}\left(\lambda_{1}, \ldots, \lambda_{k-1}\right)_{\lambda_{1}+\cdots+\lambda_{k}} u_{\left.i_{k+1}\right\}}\right\}(1)\right) .
\end{aligned}
$$

As an application of (3.42), we get the following
Corollary 3.17. (a) A 0-form $\int f \in \Omega^{0}\left(\mathcal{R}_{\ell}\right)$ is closed if and only if $f \in \mathbb{F}+$ $\left[\mathcal{R}_{\ell}, \mathcal{R}_{\ell}\right]+\partial \mathcal{R}_{\ell}$.
(b) A 1-form $F=\left(F_{i}\right)_{i=1}^{\ell} \in \mathcal{R}_{\ell}^{\oplus \ell}=\Sigma^{1}\left(\mathcal{R}_{\ell}\right)$ is closed if and only if there exists $\int f \in \mathcal{R}_{\ell} /\left(\left[\mathcal{R}_{\ell}, \mathcal{R}_{\ell}\right]+\partial \mathcal{R}_{\ell}\right)$ such that $F_{i}=\mathrm{m}\left(\frac{\delta f}{\delta u_{i}}\right)^{\sigma}$ for every $i=1, \ldots, \ell$.
(c) A 2-form $\alpha=\left(A_{i j}(\lambda)\right)_{i, j=1}^{\ell} \in \Sigma^{2}\left(\mathcal{R}_{\ell}\right)$ is closed if and only if there exists $F=\left(F_{i}\right)_{i=1}^{\ell} \in \mathcal{R}_{\ell}^{\oplus \ell}$ such that

$$
A_{i j}(\lambda)=\frac{1}{2} \sum_{n \in \mathbb{Z}_{+}}\left(\frac{\partial F_{j}}{\partial u_{i}^{(n)}} \lambda^{n}-(-\lambda-\partial)^{n}\left(\frac{\partial F_{i}}{\partial u_{j}^{(n)}}\right)^{\sigma}\right)
$$

for every $i, j=1, \ldots, \ell$.
Remark 3.18. For $F \in \mathcal{V}^{\oplus \ell}=\Sigma^{1}(\mathcal{V})$, define the Frechet derivative

$$
D_{F}(\lambda)=\left(\sum_{n \in \mathbb{Z}_{+}} \frac{\partial F_{i}}{\partial u_{j}^{(n)}} \lambda^{n}\right)_{i, j=1}^{\ell} \in \operatorname{Mat}_{\ell \times \ell}(\mathcal{V} \otimes \mathcal{V})[\lambda]
$$

Therefore, recalling Remark 3.11, we see that $\delta F=0$ if and only if $D_{F}(\partial)$ is selfadjoint.

Note also that, using the Beltrami 2-fold $\lambda$-bracket introduced in Remark 3.16 we have

$$
D_{F}(\lambda)_{i j}=\left\{\left\{u_{j \lambda} F_{i}\right\}\right\}^{B}
$$

Furthermore, for $f \in \mathcal{V}$, we have $(i \in I)$

$$
\left(\frac{\delta f}{\delta u_{i}}\right)^{\sigma}=\left.\left\{\left\{f_{\lambda} u_{i}\right\}\right\}^{B}\right|_{\lambda=0} .
$$

3.7. Poisson vertex algebra structure on $\mathcal{V}_{m}$. In this subsection for each double PVA $\mathcal{V}$ and a positive integer $m$ we construct a Poisson vertex algebra $\mathcal{V}_{m}$. This construction is similar to the construction of a Poisson algebra $V_{m}$, associated to a double Poisson algebra $V$ in [VdB08].

The construction of $V_{m}$ is motivated by the following considerations. Let $V$ be a unital finitely generated associative algebra, and let $Y_{m}$ be the affine algebraic variety of all homomorphisms from $V$ to $\operatorname{Mat}_{m \times m} \mathbb{F}$. Let $V_{m}:=\mathbb{F}\left[Y_{m}\right]$ be the algebra of polynomial functions on $Y_{m}$. We have a natural map $V \rightarrow \operatorname{Mat}_{m \times m} V_{m}$, which induces the map

$$
\varphi_{m}: V /[V, V] \rightarrow V_{m}
$$

It was shown in [VdB08] that if $V$ is a double Poisson algebra with a 2-fold bracket $\{-,-\}$, then $V_{m}$ is a Poisson algebra with the bracket

$$
\left\{a_{i j}, b_{h k}\right\}=\left\{\{ a , b \} _ { h j } ^ { \prime } \left\{\{a, b\}_{i k}^{\prime \prime} .\right.\right.
$$

It is easy to check that $\varphi_{m}$ is then a Lie algebra homomorphism.
Note that by the above interpretation of the algebra $V_{m}$, it follows that for $V=R_{N}=\mathbb{F}\left\langle x_{1}, \ldots, x_{N}\right\rangle$ the algebra $V_{m}$ is a polynomial algebra as well (on generators $\left.\left(x_{s}\right)_{i j}, s=1, \ldots, N, i, j=1, \ldots, m\right)$.

Given a differential algebra $\mathcal{V}$ we define for each positive integer $m$ a commutative differential algebra $\mathcal{V}_{m}$, as the commutative algebra generated by the symbols $a_{i j}$, for $a \in \mathcal{V}$ and $i, j=1, \ldots, m$, subject to the relations $(k \in \mathbb{F}, a, b \in \mathcal{V})$ :

$$
\begin{equation*}
(k a)_{i j}=k a_{i j}, \quad(a+b)_{i j}=a_{i j}+b_{i j}, \quad(a b)_{i j}=\sum_{k=1}^{m} a_{i k} b_{k j} \tag{3.51}
\end{equation*}
$$

The derivation (which we still denote by $\partial$ ) on $\mathcal{V}_{m}$ is defined by

$$
\partial\left(a_{i j}\right)=(\partial a)_{i j} .
$$

Example 3.19. Let $\mathcal{V}=\mathcal{R}_{I}=\mathbb{F}\left\langle u_{i}^{(n)} \mid i \in I, n \in \mathbb{Z}_{+}\right\rangle$be the algebra of noncommutative differential polynomials in the variables $u_{i}, i \in I$. Then, by the above discussion, for every $m \geq 1$,

$$
\mathcal{V}_{m}=\mathbb{F}\left[u_{i, a b}^{(n)} \mid i \in I, a, b \in\{1, \ldots, m\}, n \in \mathbb{Z}_{+}\right]
$$

where $u_{i, a b}^{(n)}=\left(u_{i}^{(n)}\right)_{a b}$, is the (commutative) algebra of differential polynomials in the variables $u_{i, a b}$.

Proposition 3.20. Let $\mathcal{V}$ be a differential algebra endowed with a 2 -fold $\lambda$-bracket $\left\{\{-\lambda-\}\right.$, written as $\left\{\left\{a_{\lambda} b\right\}\right\}=\sum_{n \in \mathbb{Z}_{+}}\left(\left(a_{n} b\right)^{\prime} \otimes\left(a_{n} b\right)^{\prime \prime}\right) \lambda^{n}$, for all $a, b \in \mathcal{V}$. Then, for every $m \geq 1$, we have a well-defined $\lambda$-bracket on $\mathcal{V}_{m}$, given by $\left(a_{i j}, b_{h k} \in \mathcal{V}_{m}\right)$

$$
\begin{equation*}
\left\{a_{i j \lambda} b_{h k}\right\}=\sum_{n \in \mathbb{Z}_{+}}\left(a_{n} b\right)_{h j}^{\prime}\left(a_{n} b\right)_{i k}^{\prime \prime} \lambda^{n}, \tag{3.52}
\end{equation*}
$$

and extended to a bilinear map $\{\cdot \lambda \cdot\}: \mathcal{V}_{m} \otimes \mathcal{V}_{m} \rightarrow \mathcal{V}_{m}[\lambda]$ by sesquilinearity (0.1) and the left and right Leibniz rules (0.2) and (0.3). This $\lambda$-bracket is skewsymmetric (i.e., it satisfies (0.4)) if the 2 -fold $\lambda$-bracket is.

Proof. First we need to verify that the map $\{\cdot \lambda \cdot\}: \mathcal{V}_{m} \otimes \mathcal{V}_{m} \rightarrow \mathcal{V}_{m}[\lambda]$ is well defined, that is, it is compatible with the defining relations (3.51) of $\mathcal{V}_{m}$. Clearly, the RHS of (3.52) is linear in $a$ and $b$ since $\left\{\left\{-_{\lambda}-\right\}\right.$ is a linear map. Hence, we are left to show that

$$
\begin{equation*}
\left\{a_{i j \lambda}(b c)_{h k}\right\}=\sum_{l=1}^{N}\left\{a_{i j \lambda} b_{h l} c_{l k}\right\} \tag{3.53}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\{(b c)_{h k \lambda} a_{i j}\right\}=\sum_{l=1}^{N}\left\{b_{h l} c_{l k \lambda} a_{i j}\right\} \tag{3.54}
\end{equation*}
$$

Using the definition of the $\lambda$-bracket (3.52) and (0.2), we have

$$
R H S(3.53)=\sum_{l=1}^{N} \sum_{n \in \mathbb{Z}_{+}}\left(\left(a_{n} b\right)_{h j}^{\prime}\left(a_{n} b\right)_{i l}^{\prime \prime} c_{l k}+\left(a_{n} c\right)_{l j}^{\prime}\left(a_{n} c\right)_{i k}^{\prime \prime} b_{h l}\right) \lambda^{n} .
$$

On the other hand, by the first formula in (3.6),

$$
\left\{\left\{a_{\lambda} b c\right\}\right\}=b\left\{\left\{a_{\lambda} c\right\}\right\}+\left\{\left\{a_{\lambda} b\right\}\right\} c=\sum_{n \in \mathbb{Z}_{+}}\left(b\left(a_{n} c\right)^{\prime} \otimes\left(a_{n} c\right)^{\prime \prime}+\left(a_{n} b\right)^{\prime} \otimes\left(a_{n} b\right)^{\prime \prime} c\right) \lambda^{n} .
$$

Hence, using (3.52), we have

$$
\begin{aligned}
\operatorname{LHS}(3.53) & =\sum_{n \in \mathbb{Z}_{+}}\left(\left(b\left(a_{n} c\right)^{\prime}\right)_{h j}\left(a_{n} c\right)_{i k}^{\prime \prime}+\left(a_{n} b\right)_{h j}^{\prime}\left(\left(a_{n} b\right)^{\prime \prime} c\right)_{i k}\right) \lambda^{n} \\
& =\sum_{l=1}^{N} \sum_{n \in \mathbb{Z}_{+}}\left(b_{h l}\left(a_{n} c\right)_{l j}^{\prime}\left(a_{n} c\right)_{i k}^{\prime \prime}+\left(a_{n} b\right)_{h j}^{\prime}\left(a_{n} b\right)_{i l}^{\prime \prime} c_{l k}\right) \lambda^{n}
\end{aligned}
$$

where in the second identity we used the third relation in (3.51). Hence, the identity (3.53) follows by the commutativity of $\mathcal{V}_{m}$. For (3.54), using the right Leibniz rule (0.3) and the definition of the $\lambda$-bracket (3.52) we have

$$
R H S(3.54)=\sum_{l=1}^{N} \sum_{n \in \mathbb{Z}_{+}}\left(\left(b_{n} a\right)_{i l}^{\prime}\left(b_{n} a\right)_{h j}^{\prime \prime}(\lambda+\partial)^{n} c_{l k}+\left(c_{n} a\right)_{i k}^{\prime}\left(c_{n} a\right)_{l j}^{\prime \prime}(\lambda+\partial)^{n} b_{h l}\right) .
$$

On the other hand, by the second formula in (3.6),

$$
\begin{aligned}
& \left\{\left\{b c_{\lambda} a\right\}=\left\{\left\{b_{\lambda+\partial} a\right\} \rightarrow_{\star_{1}} c+\left(e^{\partial \frac{d}{d \lambda}} b\right) \star_{1}\left\{c_{\lambda} a\right\}\right.\right. \\
& =\sum_{n \in \mathbb{Z}_{+}}\left(\left(b_{n} a\right)^{\prime}(\lambda+\partial)^{n} c \otimes\left(b_{n} a\right)^{\prime \prime}+\left(c_{n} a\right)^{\prime} \otimes\left((\lambda+\partial)^{n} b\right)\left(c_{n} a\right)^{\prime \prime}\right)
\end{aligned}
$$

Hence, using (3.52), we have

$$
\begin{aligned}
\operatorname{LHS}(3.54) & =\sum_{n \in \mathbb{Z}_{+}}\left(\left(\left(b_{n} a\right)^{\prime}(\lambda+\partial)^{n} c\right)_{i k}\left(b_{n} a\right)_{h j}^{\prime \prime}+\left(c_{n} a\right)_{i k}^{\prime}\left(\left((\lambda+\partial)^{n} b\right)\left(c_{n} a\right)^{\prime \prime}\right)_{h j}\right) \\
& =\sum_{n \in \mathbb{Z}_{+}}\left(\left(b_{n} a\right)_{i l}^{\prime}(\lambda+\partial)^{n} c_{l k}\left(b_{n} a\right)_{h j}^{\prime \prime}+\left(c_{n} a\right)_{i k}^{\prime}\left((\lambda+\partial)^{n} b_{h l}\right)\left(c_{n} a\right)_{l j}^{\prime \prime}\right),
\end{aligned}
$$

where in the second identity we used the third relation in (3.51). Again, the identity (3.54) follows by the commutativity of $\mathcal{V}_{m}$.

Let us prove skewsymmetry of the $\lambda$-bracket given by equation (3.52) provided that the skewsymmetry (3.8) of the 2 -fold $\lambda$-bracket holds. We have to show that

$$
\begin{equation*}
\left\{a_{i j \lambda} b_{h k}\right\}=-\left\{b_{h k-\lambda-\partial} a_{i j}\right\} \tag{3.55}
\end{equation*}
$$

for all $a_{i j}, b_{h k} \in \mathcal{V}_{m}$. We can rewrite (3.8) as the following identity

$$
\sum_{n \in \mathbb{Z}_{+}}\left(\left(a_{n} b\right)^{\prime} \otimes\left(a_{n} b\right)^{\prime \prime}\right) \lambda^{n}=-\sum_{n \in \mathbb{Z}_{+}}(-\lambda-\partial)^{n}\left(\left(b_{n} a\right)^{\prime \prime} \otimes\left(b_{n} a\right)^{\prime}\right)
$$

Hence, we have

$$
\begin{equation*}
\sum_{n \in \mathbb{Z}_{+}}\left(a_{n} b\right)_{h j}^{\prime}\left(a_{n} b\right)_{i k}^{\prime \prime} \lambda^{n}=-\sum_{n \in \mathbb{Z}_{+}}(-\lambda-\partial)^{n}\left(b_{n} a\right)_{h j}^{\prime \prime}\left(b_{n} a\right)_{i k}^{\prime}, \tag{3.56}
\end{equation*}
$$

for all $i, j, h, k=1, \ldots, m$. Using (3.52) and the commutativity of $\mathcal{V}_{m}$, it follows that (3.56) is equivalent to (3.55), thus concluding the proof.

Proposition 3.21. Let $\mathcal{V}$ be a differential algebra with a 2 -fold $\lambda$-bracket $\left\{-{ }_{\lambda}-\right\}$. Let

$$
\left.\left\{a_{\lambda} b_{\mu} c\right\}\right\}=\sum_{p, q \in \mathbb{Z}_{+}}\left(h_{p q}^{\prime} \otimes h_{p q}^{\prime \prime} \otimes h_{p q}^{\prime \prime \prime}\right) \lambda^{p} \mu^{q},
$$

and

$$
\left\{\left\{b_{\mu} a_{\lambda} c\right\}\right\}=\sum_{p, q \in \mathbb{Z}_{+}}\left(g_{p q}^{\prime} \otimes g_{p q}^{\prime \prime} \otimes g_{p q}^{\prime \prime \prime}\right) \lambda^{p} \mu^{q},
$$

be the associated, by (3.10), 3-fold $\lambda$-bracket. Then, for any $m \geq 1$, we have $\left(a_{i j}, b_{h k}, c_{l n} \in \mathcal{V}_{m}\right):$

$$
\begin{align*}
&\left\{a_{i j \lambda}\left\{b_{h k \mu} c_{l n}\right\}\right\}-\left\{b_{h k \mu}\left\{a_{i j \lambda} c_{l n}\right\}\right\}-\left\{\left\{a_{i j \lambda} b_{h k}\right\}_{\lambda+\mu} c_{l n}\right\} \\
&= \sum_{p, q \in \mathbb{Z}_{+}}\left(\left(h_{p q}^{\prime}\right)_{l j}\left(h_{p q}^{\prime \prime}\right)_{i k}\left(h_{p q}^{\prime \prime \prime}\right)_{h n}-\left(g_{p q}^{\prime}\right)_{l k}\left(g_{p q}^{\prime \prime}\right)_{h j}\left(g_{p q}^{\prime \prime \prime}\right)_{i n}\right) \lambda^{p} \mu^{q} \tag{3.57}
\end{align*}
$$

Proof. Expanding in powers of $\lambda$ and $\mu$ equation (3.10) using (3.7), we get

$$
\begin{aligned}
\left\{a_{\lambda} b_{\mu} c\right\} & =\sum_{p, q \in \mathbb{Z}_{+}}\left(\left(a_{p}\left(b_{q} c\right)^{\prime}\right)^{\prime} \otimes\left(a_{p}\left(b_{q} c\right)^{\prime}\right)^{\prime \prime} \otimes\left(b_{q} c\right)^{\prime \prime}\right. \\
& \left.-\left(a_{p} c\right)^{\prime} \otimes\left(b_{q}\left(a_{p} c\right)^{\prime \prime}\right)^{\prime} \otimes\left(b_{q}\left(a_{p} c\right)^{\prime \prime}\right)^{\prime \prime}\right) \lambda^{p} \mu^{q} \\
& -\sum_{p, q \in \mathbb{Z}_{+}}\left(\left(\left(a_{p} b\right)_{q}^{\prime} c\right)^{\prime} \otimes(\lambda+\mu+\partial)^{q}\left(a_{p} b\right)^{\prime \prime} \otimes\left(\left(a_{p} b\right)_{q}^{\prime} c\right)^{\prime \prime}\right) \lambda^{p} .
\end{aligned}
$$

Exchanging $a$ with $b$ and $\lambda$ with $\mu$, we get

$$
\begin{aligned}
\left\{b_{\mu} a_{\lambda} c\right\} & =\sum_{p, q \in \mathbb{Z}_{+}}\left(\left(b_{q}\left(a_{p} c\right)^{\prime}\right)^{\prime} \otimes\left(b_{q}\left(a_{p} c\right)^{\prime}\right)^{\prime \prime} \otimes\left(a_{p} c\right)^{\prime \prime}\right. \\
& \left.-\left(b_{q} c\right)^{\prime} \otimes\left(a_{p}\left(b_{q} c\right)^{\prime \prime}\right)^{\prime} \otimes\left(a_{p}\left(b_{q} c\right)^{\prime \prime}\right)^{\prime \prime}\right) \lambda^{p} \mu^{q} \\
& +\sum_{p, q \in \mathbb{Z}_{+}}\left(\left(\left(a_{p} b\right)_{q}^{\prime \prime} c\right)^{\prime} \otimes(\lambda+\mu+\partial)^{q}\left(a_{p} b\right)^{\prime} \otimes\left(\left(a_{p} b\right)_{q}^{\prime \prime} c\right)^{\prime \prime}\right) \lambda^{p}
\end{aligned}
$$

where in the last term we used skewsymmetry (3.8) and Lemma 3.3(a). From the above equations it follows that

$$
\begin{align*}
R H S(3.57) & =\sum_{p, q \in \mathbb{Z}_{+}}\left(\left(a_{p}\left(b_{q} c\right)^{\prime}\right)_{l j}^{\prime}\left(a_{p}\left(b_{q} c\right)^{\prime}\right)_{i k}^{\prime \prime}\left(b_{q} c\right)_{h n}^{\prime \prime}\right. \\
& \left.-\left(a_{p} c\right)_{l j}^{\prime}\left(b_{q}\left(a_{p} c\right)^{\prime \prime}\right)_{i k}^{\prime}\left(b_{q}\left(a_{p} c\right)^{\prime \prime}\right)_{h n}^{\prime \prime}\right) \lambda^{p} \mu^{q} \\
& -\sum_{p, q \in \mathbb{Z}_{+}}\left(\left(\left(a_{p} b\right)_{q}^{\prime} c\right)_{l j}^{\prime}\left((\lambda+\mu+\partial)^{q}\left(a_{p} b\right)_{i k}^{\prime \prime}\right)\left(\left(a_{p} b\right)_{q}^{\prime} c\right)_{h n}^{\prime \prime}\right) \lambda^{p} \\
& -\sum_{p, q \in \mathbb{Z}_{+}}\left(\left(b_{q}\left(a_{p} c\right)^{\prime}\right)_{l k}^{\prime}\left(b_{q}\left(a_{p} c\right)^{\prime}\right)_{h j}^{\prime \prime}\left(a_{p} c\right)_{i n}^{\prime \prime}\right.  \tag{3.58}\\
& \left.-\left(b_{q} c\right)_{l k}^{\prime}\left(a_{p}\left(b_{q} c\right)^{\prime \prime}\right)_{h j}^{\prime}\left(a_{p}\left(b_{q} c\right)^{\prime \prime}\right)_{i n}^{\prime \prime}\right) \lambda^{p} \mu^{q} \\
& -\sum_{p, q \in \mathbb{Z}_{+}}\left(\left(\left(a_{p} b\right)_{q}^{\prime \prime} c\right)_{l k}^{\prime}\left((\lambda+\mu+\partial)^{q}\left(a_{p} b\right)_{h j}^{\prime}\right)\left(\left(a_{p} b\right)_{q}^{\prime \prime} c\right)_{i n}^{\prime \prime}\right) \lambda^{p}
\end{align*}
$$

On the other hand, using (3.52) and left and right Leibniz rules (0.4) and (0.5), we get

$$
\begin{align*}
& \left\{a_{i j \lambda}\left\{b_{h k \mu} c_{l n}\right\}\right\}=\sum_{p, q \in \mathbb{Z}_{+}}\left\{a_{i j \lambda}\left(b_{q} c\right)_{l k}^{\prime}\left(b_{q} c\right)_{h n}^{\prime \prime}\right\} \mu^{q} \\
& =\sum_{p, q \in \mathbb{Z}_{+}}\left(\left(a_{p}\left(b_{q} c\right)^{\prime}\right)_{l j}^{\prime}\left(a_{p}\left(b_{q} c\right)^{\prime}\right)_{l k}^{\prime \prime}\left(b_{q} c\right)_{h n}^{\prime \prime}\right.  \tag{3.59}\\
& \left.\quad-\left(b_{q} c\right)_{l k}^{\prime}\left(a_{p}\left(b_{q} c\right)^{\prime \prime}\right)_{h j}^{\prime}\left(a_{p}\left(b_{q} c\right)^{\prime \prime}\right)_{i n}^{\prime \prime}\right) \lambda^{p} \mu^{q} ; \\
& -\left\{b_{h k \mu}\left\{a_{i j \lambda} c_{l n}\right\}\right\}=-\sum_{p, q \in \mathbb{Z}_{+}}\left\{b_{h k \mu}\left(a_{p} c l_{l j}^{\prime}\left(a_{p} c\right)_{i n}^{\prime \prime}\right\} \lambda^{p}\right. \\
& =-\sum_{p, q \in \mathbb{Z}_{+}}\left(\left(b_{q}\left(a_{p} c\right)^{\prime}\right)_{l k}^{\prime}\left(b_{q}\left(a_{p} c\right)^{\prime}\right)_{h j}^{\prime \prime}\left(a_{p} c\right)_{i n}^{\prime \prime}\right.  \tag{3.60}\\
& \left.\quad+\left(a_{p} c\right)_{l j}^{\prime}\left(b_{q}\left(a_{p} c\right)^{\prime \prime}\right)_{i k}^{\prime}\left(b_{q}\left(a_{p} c\right)^{\prime \prime}\right)_{h n}^{\prime \prime}\right) \lambda^{p} \mu^{q} ; \\
& -\left\{\left\{a_{i j \lambda} b_{h k}\right\}_{\lambda+\mu} c_{l n}\right\}=-\sum_{p, q \in \mathbb{Z}_{+}}\left\{\left(a_{p} b\right)_{h j}^{\prime}\left(a_{p} b\right)_{i k}^{\prime \prime} \lambda+\mu c_{l n}\right\} \lambda^{p} \\
& =-\sum_{p, q \in \mathbb{Z}_{+}}\left(\left(\left(a_{p} b\right)_{q}^{\prime} c\right)_{l j}^{\prime}\left(\left(a_{p} b\right)_{q}^{\prime} c\right)_{h n}^{\prime \prime}(\lambda+\mu+\partial)^{q}\left(a_{p} b\right)_{i k}^{\prime \prime}\right.  \tag{3.61}\\
& \left.\quad+\left(\left(a_{p} b\right)_{q}^{\prime \prime} c\right)_{l k}^{\prime}\left(\left(a_{p} b\right)_{q}^{\prime \prime} c\right)_{i n}^{\prime \prime}(\lambda+\mu+\partial)^{q}\left(a_{p} b\right)_{h j}^{\prime}\right) \lambda^{p} .
\end{align*}
$$

Adding up (3.59), (3.60) and (3.61) and using the commutativity of $\mathcal{V}_{m}$ we see that the RHS of (3.56) is equal to (3.58), thus concluding the proof.

Theorem 3.22. If $\mathcal{V}$ is a double Poisson vertex algebra, then, for any positive integer $m, \mathcal{V}_{m}$ is a Poisson vertex algebra with $\lambda$-bracket defined by (3.52).

Proof. By Proposition 3.20, formula (3.52) gives a well-defined skewsymmetric $\lambda$ bracket on $\mathcal{V}_{m}$. Since $\mathcal{V}$ is a double Poisson vertex algebra, by Definition 3.2, $\left\{\left\{a_{\lambda} b_{\mu} c\right\}\right\}=0$, for all $a, b, c \in \mathcal{V}$. Hence, by Proposition 3.21, the RHS of (3.57) is equal to zero, for all $a, b, c \in \mathcal{V}$, proving that the $\lambda$-bracket defined by (3.52) satisfies the Jacobi identity (0.5).

### 3.8. Examples: affine and AGD double PVA structures.

3.8.1. Affine Poisson vertex algebra for $\mathfrak{g}=\mathfrak{g l}_{m}$. Let us consider the differential algebra $\mathcal{V}=\mathcal{R}_{1}=\mathbb{F}\left\langle u, u^{\prime}, u^{\prime \prime}, \ldots\right\rangle$ and let

$$
\begin{equation*}
\left\{\left\{u_{\lambda} u\right\}\right\}=1 \otimes u-u \otimes 1+c(1 \otimes 1) \lambda \in \mathcal{V}^{\otimes 2}[\lambda] \tag{3.62}
\end{equation*}
$$

where $c \in \mathbb{F}$. It is obvious that the skewsymmetry (3.8) holds for the pair $u, u$, and it is easy to check that the Jacobi identity (3.9) holds for the triple $u, u, u$. Hence, by Proposition 3.10, we have a family of compatible double Poisson vertex algebra structures on $\mathcal{V}$, uniquely extending (3.62). By Theorem 3.22, we get Poisson vertex algebra structures on the commutative differential algebra $\mathcal{V}_{m}=\mathbb{F}\left[u_{i j}^{(n)} \mid\right.$ $i, j=1, \ldots, m, n \in \mathbb{Z}_{+}$] (see Example 3.19). Using (3.62) and (3.52), we get the explicit formula for the $\lambda$-bracket among the generators of $\mathcal{V}_{m}$ :

$$
\left\{u_{i j \lambda} u_{h k}\right\}=\delta_{j h} u_{i k}-\delta_{k i} u_{h j}+\delta_{j h} \delta_{i k} c \lambda,
$$

for all $i, j, h, k=1, \ldots, m$. This is the $\lambda$-bracket among the generators of the affine Poisson vertex algebra for $\mathfrak{g l}_{m}$ (where we fixed the nondegenerate invariant bilinear form to be a scalar multiple of the trace form).

Example 3.23. Consider the differential algebra $\mathcal{R}_{2}=\mathbb{F}\left\langle u, v, u^{\prime}, v^{\prime}, \ldots\right\rangle$ with the double lambda bracket $\left\{\left\{u_{\lambda} u\right\}\right.$, given by (3.62) with $v$ central. Then a similar computation as the one in Section 2.6 shows that $h_{0}=1, h_{n}=\frac{1}{n}(u+v)^{n}$ for $n>0$, are in involution, and the corresponding integrable hierarchy of Hamiltonian equations is:

$$
\frac{d u}{d t_{n}}=v(u+v)^{n} u-u(u+v)^{n} v+c \partial(u+v)^{n+1}, \quad d v / d t_{n}=0, n \in \mathbb{Z}_{+}
$$

3.8.2. Adler type non-commutative pseudodifferential operators. Let us recall from [Kac98] that the $\delta$-function is, by definition, the $\mathbb{F}$-valued formal distribution

$$
\delta(z-w)=\sum_{n \in \mathbb{Z}_{+}} z^{n} w^{-n-1} \in \mathbb{F}\left[\left[z, z^{-1}, w, w^{-1}\right]\right] .
$$

Let $a(z)=\sum_{i \in \mathbb{Z}} a_{i} z^{i}$ be a formal Laurent series in $z^{-1}$ with values in some vector space $\mathcal{V}$. Then, we have

$$
\begin{equation*}
a(z) \delta(z-w)=a(w) \delta(z-w) \tag{3.63}
\end{equation*}
$$

We denote by $i_{z}$ the power series expansion for large $|z|$. For example,

$$
i_{z}(z-w)^{-1}=\sum_{k \in \mathbb{Z}_{+}} z^{-k-1} w^{k}, \quad i_{z}(z-w-\lambda-\partial)^{-1}=\sum_{k \in \mathbb{Z}_{+}} z^{-k-1}(w+\lambda+\partial)^{k}
$$

Using this notation, the $\delta$-function can be rewritten as follows:

$$
\begin{equation*}
\delta(z-w)=i_{z}(z-w)^{-1}-i_{w}(z-w)^{-1} \tag{3.64}
\end{equation*}
$$

In the sequel, we use the following notation: if $a(z)=\sum_{n \in \mathbb{Z}} a_{n} z^{n}$ and $b(z)$ are two formal Laurent series in $z^{-1}$, then

$$
a(z+\lambda+\partial) \otimes b(z)=\sum_{n \in \mathbb{Z}} a_{n} \otimes i_{z}(z+\lambda+\partial)^{n} b(z)
$$

namely, for any $n \in \mathbb{Z}$, we expand $(z+\lambda+\partial)^{n}$ in non-negative powers of $\lambda+\partial$ and we let powers of $\partial$ act to the right, on the coefficients of $b(z)$. In order to emphasize that $\partial$ does not act on some factors, we enclose in parenthesis the terms on which $\partial$ does act.

We denote by $\operatorname{Res}_{z}$ the coefficient of $z^{-1}$. The following identity holds for every formal power series $a(z)=\sum_{n} \in \mathbb{Z} a_{n} z^{n}$,

$$
\begin{equation*}
\operatorname{Res}_{z} a(z) i_{z}(z-w)^{-1}=a(w)_{+}, \tag{3.65}
\end{equation*}
$$

where $a(w)_{+}=\sum_{n=0}^{\infty} a_{n} z^{n}$ is the positive part of $a(z)$. Furthermore, the following identity is a consequence of integration by parts:

$$
\begin{equation*}
\operatorname{Res}_{z}\left(\iota_{z} a(z+t) \otimes b(z)\right)=\operatorname{Res}_{z}\left(a(z) \otimes \iota_{z} b(z-t)\right) \tag{3.66}
\end{equation*}
$$

Definition 3.24. Let $\mathcal{V}$ be a differential algebra endowed with a 2 -fold $\lambda$-bracket $\left\{\{-\lambda-\}\right.$. We call a pseudodifferential operator $L(\partial) \in \mathcal{V}\left(\left(\partial^{-1}\right)\right)$ of Adler type for $\left\{\{-\lambda-\}\right.$ if the following identity holds in $\mathcal{V}^{\otimes 2}[\lambda, \mu]\left(\left(z^{-1}, w^{-1}\right)\right)$ (cf. [DSKV14a]):

$$
\begin{align*}
& \left\{\left[L(z)_{\lambda} L(w)\right\}\right\}=L(z) \otimes i_{z}(z-w-\lambda-\partial)^{-1} L(w) \\
& -L(w+\lambda+\partial) \otimes i_{z}(z-w-\lambda-\partial)^{-1} L^{*}(-z+\lambda) \tag{3.67}
\end{align*}
$$

where $L^{*}(\partial)$ is the formal adjoint of the pseudodifferential operator $L(\partial)$, and $L^{*}(z)$ is its symbol.
Proposition 3.25. Let $\mathcal{V}$ be a differential algebra, let $\{[-\lambda-\}$ be a 2 -fold $\lambda$-bracket on $\mathcal{V}$, and let $L(\partial) \in \mathcal{V}\left(\left(\partial^{-1}\right)\right)$ be an Adler type pseudodifferential operator. Then:
(a) The following identity holds in $\mathcal{V}^{\otimes 2}[\lambda]\left(\left(z^{-1}, w^{-1}\right)\right)$ :

$$
\left\{\left\{L(z)_{\lambda} L(w)\right\}\right\}=-\underset{39}{\left\{\left\{L(w)_{-\lambda-\partial} L(z)\right\}\right\}^{\sigma} .}
$$

(b) The following identity holds in $\mathcal{V}^{\otimes 3}[\lambda, \mu]\left(\left(z_{1}^{-1}, z_{2}^{-1}, z_{3}^{-1}\right)\right)$ :

$$
\begin{gathered}
\left\{\left\{\left(z_{1}\right)_{\lambda}\left\{\left\{L\left(z_{2}\right)_{\mu} L\left(z_{3}\right)\right\}\right\}\right\}_{L}-\left\{\left\{L\left(z_{2}\right)_{\mu}\left\{\left\{L\left(z_{1}\right)_{\lambda} L\left(z_{3}\right)\right\}\right\}\right\}_{R}\right.\right. \\
=\left\{\left\{\left\{L\left(z_{1}\right)_{\lambda} L\left(z_{2}\right)\right\}_{\lambda+\mu} L\left(z_{3}\right)\right\}\right\}_{L} .
\end{gathered}
$$

Proof. By (3.67) we get

$$
\begin{gathered}
\left\{\left[L(w)_{-\lambda-\partial} L(z)\right\}\right\}^{\sigma}=L(w+\lambda+\partial) \otimes i_{w}(z-w-\lambda-\partial)^{-1} L^{*}(-z+\lambda) \\
-L(z) \otimes i_{w}(z-w-\lambda-\partial)^{-1} L(w) .
\end{gathered}
$$

Hence, in order to prove (a) we are left to show that

$$
\begin{aligned}
& L(z) \otimes i_{z}(z-w-\lambda-\partial)^{-1} L(w) \\
& -L(w+\lambda+\partial) \otimes i_{z}(z-w-\lambda-\partial)^{-1} L^{*}(-z+\lambda) \\
& =L(z) \otimes i_{w}(z-w-\lambda-\partial)^{-1} L(w) \\
& -L(w+\lambda+\partial) \otimes i_{w}(z-w-\lambda-\partial)^{-1} L^{*}(-z+\lambda),
\end{aligned}
$$

which can be rewritten, using equation (3.64), as

$$
\begin{equation*}
L(z) \otimes \delta(z-w-\lambda-\partial) L(w)=L(w+\lambda+\partial) \otimes \delta(z-w-\lambda-\partial) L^{*}(-z+\lambda) \tag{3.68}
\end{equation*}
$$

Identity (3.68) holds by applying the property of the $\delta$-function given by equation (3.63).

In order to prove part (b) let us compute the three terms separately using (3.7) and (3.67). We have

$$
\begin{align*}
& \left\{L\left(z_{1}\right)_{\lambda}\left\{\left\{L\left(z_{2}\right)_{\mu} L\left(z_{3}\right)\right\}\right\}\right\} L \\
& =L\left(z_{1}\right) \otimes\left(i_{z_{1}}\left(z_{1}-z_{2}-\lambda-\partial\right)^{-1} L\left(z_{2}\right)\right) \otimes i_{z_{2}}\left(z_{2}-z_{3}-\mu-\partial\right)^{-1} L\left(z_{3}\right)  \tag{3.69}\\
& -\left(L\left(z_{2}+\lambda+\partial\right) \otimes i_{z_{1}}\left(z_{1}-z_{2}-\lambda-\partial\right)^{-1} L^{*}\left(-z_{1}+\lambda\right)\right) \\
& \quad \otimes i_{z_{2}}\left(z_{2}-z_{3}-\mu-\partial\right)^{-1} L\left(z_{3}\right)  \tag{3.70}\\
& -L\left(z_{1}\right) \otimes i_{z_{1}}\left(z_{1}-z_{3}-\lambda-\mu-\partial\right)^{-1} L\left(z_{3}+\mu+\partial\right) \\
& \quad \otimes i_{z_{2}}\left(z_{2}-z_{3}-\mu-\partial\right)^{-1} L^{*}\left(-z_{2}+\mu\right)  \tag{3.71}\\
& +L\left(z_{3}+\lambda+\mu+\partial\right) \otimes i_{z_{1}}\left(z_{1}-z_{3}-\lambda-\mu-\partial\right)^{-1} L^{*}\left(-z_{1}+\lambda\right) \\
& \quad \otimes i_{z_{2}}\left(z_{2}-z_{3}-\mu-\partial\right)^{-1} L^{*}\left(-z_{2}+\mu\right) \tag{3.72}
\end{align*}
$$

$$
\begin{align*}
& -\left\{\left[L\left(z_{2}\right)_{\mu}\left\{\left\{\left(z_{1}\right)_{\lambda} L\left(z_{3}\right)\right\}\right\}_{R}\right.\right. \\
& =-L\left(z_{1}\right) \otimes i_{z_{1}}\left(z_{1}-z_{3}-\lambda-\mu-\partial\right)^{-1} L\left(z_{2}\right) \otimes i_{z_{2}}\left(z_{2}-z_{3}-\mu-\partial\right)^{-1} L\left(z_{3}\right) \tag{3.73}
\end{align*}
$$

$$
\begin{equation*}
+L\left(z_{1}\right) \otimes i_{z_{1}}\left(z_{1}-z_{3}-\lambda-\mu-\partial\right)^{-1} L\left(z_{3}+\mu+\partial\right) \tag{3.74}
\end{equation*}
$$

$$
\otimes i_{z_{2}}\left(z_{2}-z_{3}-\mu-\partial\right)^{-1} L^{*}\left(-z_{2}+\mu\right)
$$

$$
-L\left(z_{3}+\lambda+\mu+\partial\right) \otimes i_{z_{1}}\left(z_{1}-z_{3}-\lambda-\mu-\partial\right)^{-1}
$$

$$
\begin{equation*}
\times\left(L^{*}\left(\lambda+\mu+\partial-z_{1}\right) \otimes i_{z_{2}}\left(z_{1}-z_{2}-\lambda-\partial\right)^{-1} L\left(z_{2}\right)\right)^{\sigma} \tag{3.75}
\end{equation*}
$$

$$
+L\left(z_{3}+\lambda+\mu+\partial\right) \otimes i_{z_{1}}\left(z_{1}-z_{3}-\lambda-\mu-\partial\right)^{-1}
$$

$$
\begin{equation*}
\times\left(L^{*}\left(-z_{2}+\mu\right) \otimes i_{z_{2}}\left(z_{1}-z_{2}-\lambda-\partial\right)^{-1} L^{*}\left(-z_{1}+\lambda\right)\right)^{\sigma} \tag{3.76}
\end{equation*}
$$

$$
\begin{align*}
& \left\{\left\{\left\{L\left(z_{1}\right)_{\lambda} L\left(z_{2}\right)\right\}_{\lambda+\mu} L\left(z_{3}\right)\right\}\right\}_{L} \\
& =L\left(z_{1}\right) \otimes i_{z_{1}}\left(z_{1}-z_{3}-\lambda-\mu-\partial\right)^{-1}\left(L\left(z_{3}\right) \otimes i_{z_{1}}\left(z_{1}-z_{2}-\lambda-\partial\right)^{-1} L\left(z_{2}\right)\right)^{\sigma}  \tag{3.77}\\
& -L\left(z_{3}+\lambda+\mu+\partial\right) \otimes i_{z_{1}}\left(z_{1}-z_{3}-\lambda-\mu-\partial\right)^{-1} \\
& \quad \times\left(L^{*}\left(\lambda+\mu+\partial-z_{1}\right) \otimes i_{z_{1}}\left(z_{1}-z_{2}-\lambda-\partial\right)^{-1} L\left(z_{2}\right)\right)^{\sigma}  \tag{3.78}\\
& -\left(L\left(z_{2}+\lambda+\partial\right) \otimes i_{z_{1}}\left(z_{1}-z_{2}-\lambda-\partial\right)^{-1} L^{*}\left(-z_{1}+\lambda\right)\right) \\
& \quad \otimes i_{z_{2}}\left(z_{2}-z_{3}-\mu-\partial\right)^{-1} L\left(z_{3}\right)  \tag{3.79}\\
& +L\left(z_{3}+\lambda+\mu+\partial\right) \otimes\left(i_{z_{1}}\left(z_{1}-z_{2}-\lambda-\partial\right)^{-1} L^{*}\left(-z_{1}+\lambda\right)\right) \\
& \quad \otimes i_{z_{2}}\left(z_{2}-z_{3}-\mu-\partial\right)^{-1} L^{*}\left(-z_{2}+\mu\right) \tag{3.80}
\end{align*}
$$

Note that

$$
(3.70)=(3.79) \quad \text { and } \quad(3.71)+(3.74)=0
$$

Next, we claim that

$$
(3.69)+(3.73)=(3.77)
$$

Indeed, we can rewrite the above identity as

$$
\begin{align*}
& i_{z_{1}}\left(z_{1}-z_{2}-x\right)^{-1} i_{z_{2}}\left(z_{2}-z_{3}-y\right)^{-1} L\left(z_{1}\right) \otimes\left(\left.\right|_{x=\lambda+\partial} L\left(z_{2}\right)\right) \otimes\left(\left.\right|_{y=\mu+\partial} L\left(z_{3}\right)\right) \\
& =i_{z_{1}}\left(z_{1}-z_{3}-x-y\right)^{-1}\left(i_{z_{1}}\left(z_{1}-z_{2}-x\right)^{-1}\right. \\
& \left.\quad+i_{z_{2}}\left(z_{2}-z_{3}-y\right)^{-1}\right) L\left(z_{1}\right) \otimes\left(\left.\right|_{x=\lambda+\partial} L\left(z_{2}\right)\right) \otimes\left(\left.\right|_{y=\mu+\partial} L\left(z_{3}\right)\right) . \tag{3.81}
\end{align*}
$$

(Here and further, for a Laurent series $P(z)=\sum_{n=-\infty}^{N} c_{n} z^{n} \in \mathcal{V}\left(\left(z^{-1}\right)\right)$ and elements $a, b \in \mathcal{V}$, we let

$$
\left.a\left(\left.\right|_{x=\nu+\partial} P(z+x) b\right)=\sum_{n=-\infty}^{N} a i_{z}(z+\nu+\partial)^{n}\left(c_{n} b\right) .\right)
$$

The identity (3.81) follows from the obvious identity

$$
a^{-1} b^{-1}=i_{a}(a+b)^{-1} b^{-1}+i_{a}(a+b)^{-1} a^{-1}
$$

In order to prove part b) we are left to show that

$$
\begin{equation*}
(3.72)+(3.75)+(3.76)=(3.78)+(3.80) . \tag{3.82}
\end{equation*}
$$

Note that, using the definition and the properties of the $\delta$-function we have

$$
\begin{align*}
& (3.75)-(3.78) \\
& =L\left(z_{3}+\lambda+\mu+\partial\right) \otimes i_{z_{1}}\left(z_{1}-z_{3}-\lambda-\mu-\partial\right)^{-1} \\
& \quad \times\left(L^{*}\left(\lambda+\mu+\partial-z_{1}\right) \otimes \delta\left(z_{1}-z_{2}-\lambda-\partial\right) L\left(z_{2}\right)\right)^{\sigma}  \tag{3.83}\\
& =L\left(z_{3}+\lambda+\mu+\partial\right) \otimes i_{z_{1}}\left(z_{1}-z_{3}-\lambda-\mu-\partial\right)^{-1} \\
& \quad \times\left(L^{*}\left(-z_{2}+\mu\right) \otimes \delta\left(z_{1}-z_{2}-\lambda-\partial\right) L^{*}\left(-z_{1}+\lambda\right)\right)^{\sigma}  \tag{3.76}\\
& (3.76) \\
& =L\left(z_{3}+\lambda+\mu+\partial\right) \otimes i_{z_{1}}\left(z_{1}-z_{3}-\lambda-\mu-\partial\right)^{-1}  \tag{3.84}\\
& \quad \times\left(L^{*}\left(-z_{2}+\mu\right) \otimes i_{z_{1}}\left(z_{1}-z_{2}-\lambda-\partial\right)^{-1} L^{*}\left(-z_{1}+\lambda\right)\right)^{\sigma} \\
& -L\left(z_{3}+\lambda+\mu+\partial\right) \otimes i_{z_{1}}\left(z_{1}-z_{3}-\lambda-\mu-\partial\right)^{-1} \\
& \quad \times\left(L^{*}\left(-z_{2}+\mu\right) \otimes \delta\left(z_{1}-z_{2}-\lambda-\partial\right) L^{*}\left(-z_{1}+\lambda\right)\right)^{\sigma}
\end{align*}
$$

Using (3.83) and (3.83), identity (3.82) is proved once we show that

$$
\begin{aligned}
& i_{z_{1}}\left(z_{1}-z_{2}-x\right)^{-1} i_{z_{2}}\left(z_{2}-z_{3}-y\right)^{-1} \\
& \quad \times L\left(z_{3}+x+y\right) \otimes\left(\left.\right|_{x=\lambda+\partial} L^{*}\left(-z_{1}+\lambda\right)\right) \otimes\left(\left.\right|_{y=\mu+\partial} L^{*}\left(-z_{2}+\mu\right)\right) \\
& =i_{z_{1}}\left(z_{1}-z_{3}-x-y\right)^{-1}\left(i_{z_{1}}\left(z_{1}-z_{2}-x\right)^{-1}+i_{z_{2}}\left(z_{2}-z_{3}-y\right)^{-1}\right) \times \\
& \times L\left(z_{3}+x+y\right) \otimes\left(\left.\right|_{x=\lambda+\partial} L^{*}\left(-z_{1}+\lambda\right)\right) \otimes\left(\left.\right|_{y=\mu+\partial} L^{*}\left(-z_{2}+\mu\right)\right),
\end{aligned}
$$

which can be proved as we did in the previous claim.
Theorem 3.26. Let $\mathcal{V}$ be a differential algebra, endowed with a 2 -fold $\lambda$-bracket $\left\{\left\{-_{\lambda}-\right\}\right.$. Let $L(\partial) \in \mathcal{V}\left(\left(\partial^{-1}\right)\right)$ be a pseudodifferential operator, and let $\mathcal{U} \subset \mathcal{V}$ be the differential subalgebra generated by the coefficients of $L(\partial)$. If $L(\partial)$ is of Adler type, then $\left\{-_{\lambda}-\right\}$ restricts to a double PVA $\lambda$-bracket on $\mathcal{U}$.
Proof. By (3.67) $\left\{\left\{-_{\lambda}-\right\}\right.$ restricts to a 2 -fold $\lambda$-bracket of $\mathcal{U}$, and by Proposition 3.25 , together with Proposition 3.10, we have that $\mathcal{U}$ is a double PVA.
3.8.3. AGD Poisson vertex algebra structures for generic matrix (pseudo) differential operators of order $N$. Let $\mathcal{V}$ be the algebra of non-commutative differential polynomials in the variables $u_{i}, i \in I$, namely

$$
\mathcal{V}=\mathcal{R}_{I}=\mathbb{F}\left\langle u_{i}^{(n)} \mid i \in I, n \in \mathbb{Z}_{+}\right\rangle
$$

The index set $I$ will be either $I=\{i \in \mathbb{Z} \mid i \geq-N\}$ or $I_{-}=\{-N,-N+1, \ldots,-1\}$. Let us collect the differential generators of $\mathcal{V}$ into the generating series

$$
L(z)=z^{N}+\sum_{i \in I} u_{i} z^{-i-1} \in \mathcal{V}\left(\left(z^{-1}\right)\right)
$$

By Example 3.19 we have that $\mathcal{V}_{m}=\mathbb{F}\left[u_{i, a b} \mid i \in I, a, b \in\{1, \ldots, m\}, n \in \mathbb{Z}_{+}\right]$, for every $m \geq 1$. Let us denote

$$
L_{a b}(z)=z^{N}+\sum_{i \in I} u_{i, a b} z^{-i-1} \in \mathcal{V}_{m}\left(\left(z^{-1}\right)\right)
$$

for every $a, b=1, \ldots, m$.
Corollary 3.27. $\mathcal{V}$ has two compatible double Poisson vertex algebra structures with 2 -fold $\lambda$-brackets defined on generators by the following generating series:

$$
\begin{gather*}
\left\{L(z)_{\lambda} L(w)\right\}_{H}=L(z) \otimes i_{z}(z-w-\lambda-\partial)^{-1} L(w)  \tag{3.85}\\
-L(w+\lambda+\partial) \otimes i_{z}(z-w-\lambda-\partial)^{-1} L^{*}(-z+\lambda)
\end{gather*}
$$

and

$$
\begin{gather*}
\left\{\left[L(z)_{\lambda} L(w)\right\}_{K}=i_{z}(z-w-\lambda)^{-1}(L(z)-L(w+\lambda)) \otimes 1\right.  \tag{3.86}\\
+1 \otimes i_{z}(z-w-\lambda-\partial)^{-1}\left(L(w)-L^{*}(-z+\lambda)\right) .
\end{gather*}
$$

Furthermore, for all $m \geq 1, \mathcal{V}_{m}$ has two compatible Poisson vertex algebra structures, with $\lambda$-brackets on generators given by the generating series

$$
\begin{gathered}
\left\{L_{a b}(z)_{\lambda} L_{c d}(w)\right\}=L_{c b}(z) i_{z}(z-w-\lambda-\partial)^{-1} L_{a d}(w) \\
-L_{c b}(w+\lambda+\partial) i_{z}(z-w-\lambda-\partial)^{-1} L_{a d}^{*}(\lambda-z)
\end{gathered}
$$

and

$$
\begin{gathered}
\left\{L_{a b}(z)_{\lambda} L_{c d}(w)\right\}_{K}=\delta_{a d} i_{z}(z-w-\lambda)^{-1}\left(L_{c b}(z)-L_{c b}(w+\lambda)\right) \\
+\delta_{c b} i_{z}(z-w-\lambda-\partial)^{-1}\left(L_{a d}(w)-L_{a d}^{*}(\lambda-z)\right)
\end{gathered}
$$

for all $a, b, c, d=1, \ldots, m$.
Proof. The fact that $\mathcal{V}$ has two compatible double Poisson vertex algebra structures follows by Theorem 3.26 applied to the pseudodifferential operator $L-c, c \in \mathbb{F}$. The second claim follows by Theorem 3.22.

Remark 3.28. The compatible PVA structures on $\mathcal{V}_{m}$ are the same as the one appearing in [DSKV14a, Eq. (4.7)].

The following results will be used in Section 4.1.
Lemma 3.29. In the double Poisson vertex algebra $\mathcal{V}$, with $\lambda$-bracket defined by (3.85), we have:
(a) $\left\{\left\{u_{-N \lambda} L(w)\right\}_{H}=1 \otimes L(w)-L(w+\lambda) \otimes 1\right.$;
(b) $\left\{\left[L(z)_{\lambda} u_{-N}\right\}_{H}=1 \otimes L^{*}(-z+\lambda)-L(z) \otimes 1\right.$;
(c) $\left\{\left\{u_{-N \lambda} u_{-N}\right\}_{H}=1 \otimes u_{-N}-u_{-N} \otimes 1-(1 \otimes 1) N \lambda\right.$.

Proof. Same proof as Lemma 2.17 in [DSKV14a].
The following results follow immediately from the previous lemma and equation (3.52).

Corollary 3.30. For $m \geq 1$, in the Poisson vertex algebra $\mathcal{V}_{m}$ we have $(a, b, c, d=$ $1, \ldots, m)$ :
(a) $\left\{u_{-N, a b \lambda} L_{c d}(w)\right\}=\delta_{c b} L_{a d}(w)-\delta_{a d} L_{c b}(w+\lambda)$;
(b) $\left\{L_{a b}(z)_{\lambda} u_{-N, c d}\right\}=\delta_{c b} L_{a d}^{*}(-z+\lambda)-\delta_{a d} L_{c b}(z)$;
(c) $\left\{u_{-N, a b \lambda} u_{-N, c d}\right\}=\delta_{c b} u_{-N, a d}-\delta_{a d} u_{-N, c b}-\delta_{a d} \delta_{c b} N \lambda$.

## 4. Non-local double Poisson vertex algebras and their Dirac REDUCTION

4.1. Non-local double Poisson vertex algebras. The notion of a non-local double PVA is similar to that of a non-local PVA, introduced in [DSK13]. As in that paper, given a vector space $V$, we denote

$$
V_{\lambda, \mu}:=V\left[\left[\lambda^{-1}, \mu^{-1},(\lambda+\mu)^{-1}\right]\right][\lambda, \mu],
$$

namely, the quotient of the $\mathbb{F}[\lambda, \mu, \nu]$-module $V\left[\left[\lambda^{-1}, \mu^{-1}, \nu^{-1}\right]\right][\lambda, \mu, \nu]$ by the submodule $(\nu-\lambda-\mu) V\left[\left[\lambda^{-1}, \mu^{-1}, \nu^{-1}\right]\right][\lambda, \mu, \nu]$. Recall that we have the natural embedding $i_{\mu}: V_{\lambda, \mu} \hookrightarrow V\left(\left(\lambda^{-1}\right)\right)\left(\left(\mu^{-1}\right)\right)$ (see the beginning of Section 3.8.2).

Let $\mathcal{V}$ be a differential algebra with a derivation $\partial$. A (non-local) 2-fold $\lambda$-bracket on $\mathcal{V}$ is a linear map $\left\{-_{\lambda}-\right\}: \mathcal{V}^{\otimes 2} \rightarrow \mathcal{V}^{\otimes 2}\left(\left(\lambda^{-1}\right)\right)$ satisfying the sesquilinearity conditions (3.5) and the left and right Leibniz rules (3.6). Here and further an expression $\left\{\left\{a_{\lambda+\partial} b\right\} \rightarrow \star_{1} c\right.$ is interpreted as follows: if $\left\{\left\{a_{\lambda} b\right\}\right\}=\sum_{n=-\infty}^{N}\left(h_{n}^{\prime} \otimes h_{n}^{\prime \prime}\right) \lambda^{n}$, then $\left\{\left\{a_{\lambda+\partial} b\right\} \rightarrow_{1} c=\sum_{n=-\infty}^{N} h_{n}^{\prime}\left((\lambda+\partial)^{n} c\right) \otimes h_{n}^{\prime \prime}\right.$, where we expand $(\lambda+\partial)^{n}$ in non-negative powers of $\partial$ acting on $c$. The (non-local) 2-fold $\lambda$-bracket $\{\{\cdot \lambda \cdot\}$ is called skewsymmetric if equation (3.8) holds. The RHS of the skewsymmetry condition should be interpreted as follows: we move $-\lambda-\partial$ to the left and we expand its powers in non-negative powers of $\partial$, acting on the coefficients on the 2 -fold $\lambda$-bracket. We say that the (non-local) 2 -fold $\lambda$-bracket $\{[-\lambda-\}$ is admissible if

$$
\begin{equation*}
\left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} c\right\}\right\}\right\}_{L} \in \mathcal{V}_{\lambda, \mu}^{\otimes 3} \quad \text { for all } a, b, c \in \mathcal{V}\right. \tag{4.1}
\end{equation*}
$$

Here we are identifying the space $\mathcal{V}_{\lambda, \mu}^{\otimes 3}$ with its image in $\mathcal{V}^{\otimes 3}\left(\left(\lambda^{-1}\right)\right)\left(\left(\mu^{-1}\right)\right)$ via the embedding $i_{\mu}$.
Remark 4.1. If $\left\{\left\{-_{\lambda}-\right\}\right.$ is a skewsymmetric admissible (non-local) 2 -fold $\lambda$-bracket on $\mathcal{V}$, by equation (3.13) we have also $\left\{\left\{b_{\mu}\left\{\left\{a_{\lambda} c\right\}\right\}\right\}_{R} \in \mathcal{V}_{\lambda, \mu}^{\otimes 3}\right.$, since $\mathcal{V}_{\lambda, \mu}^{\otimes 3}=\mathcal{V}_{\mu, \lambda}^{\otimes 3}$, and similarly $\left\{\left\{\left\{a_{\lambda} b\right\}_{\lambda+\mu} c\right\}\right\}_{L} \in \mathcal{V}_{\lambda, \mu}^{\otimes 3}$, for all $a, b, c \in \mathcal{V}$.
Definition 4.2. A non-local double Poisson vertex algebra is a differential algebra $\mathcal{V}$ endowed with a non-local 2-fold $\lambda$-bracket, $\left\{\left[{ }_{\lambda}-\right\}: \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}\left(\left(\lambda^{-1}\right)\right)\right.$ satisfying skewsymmetry (3.8), admissibility (4.1), and Jacobi identity (3.9), where the latter is understood as an identity in the space $\mathcal{V}_{\lambda, \mu}^{\otimes 3}$.
4.2. Dirac reduction for non-local double Poisson vertex algebras. Dirac reduction for a double PVA is similar to that of a PVA, constructed in [DSKV14a]. Let $\mathcal{V}$ be a non-local double Poisson vertex algebra with 2 -fold $\lambda$-bracket $\left\{\left\{-{ }_{\lambda}-\right\}\right.$. Let $\theta_{1}, \ldots, \theta_{m}$ be elements of $\mathcal{V}$ and let us consider the matrix pseudodifferential operator

$$
C(\partial)=\left(C_{\alpha \beta}(\partial)\right)_{\alpha, \beta=1}^{m} \in \operatorname{Mat}_{m \times m} \mathcal{V}^{\otimes 2}\left(\left(\partial^{-1}\right)\right),
$$

where the symbol of the pseudodifferential operator $C_{\alpha \beta}(\partial)$ is

$$
\begin{equation*}
C_{\alpha \beta}(\lambda)=\left\{\left\{\theta_{\beta \lambda} \theta_{\alpha}\right\}\right\} . \tag{4.2}
\end{equation*}
$$

Given a pseudodifferential operator $A(\partial)=\sum_{n \leq N} A_{n} \partial^{n} \in \operatorname{Mat}_{m \times m} \mathcal{V}^{\otimes 2}\left(\left(\partial^{-1}\right)\right)$, recall that we defined its adjoint by

$$
A(\partial)^{\dagger}=\sum_{n \leq N}(-\partial)^{n} \circ\left(A_{n}^{t}\right)^{\sigma}
$$

where $A_{n}^{t}$ denotes the transpose matrix of $A_{n}$. By the skewsymmetry condition (3.8), the pseudodifferential operator $C(\partial)$ is skewadjoint. We shall assume that the matrix pseudodifferential operator $C(\partial)$ is invertible with respect to the - product (1.1), and we denote its inverse by $C^{-1}(\partial)=\left(\left(C^{-1}\right)_{\alpha \beta}(\partial)\right)_{\alpha, \beta=1}^{m} \in$ $\operatorname{Mat}_{m \times m} \mathcal{V}^{\otimes 2}\left(\left(\partial^{-1}\right)\right)$.

Definition 4.3. The Dirac modification of the 2-fold $\lambda$-bracket $\left\{\left\{-_{\lambda}-\right\}\right.$, associated to the elements $\theta_{1}, \ldots, \theta_{m}$, is the map $\{[-\lambda-\}\}^{D}: \mathcal{V}^{\otimes 2} \rightarrow \mathcal{V}^{\otimes 2}\left(\left(\lambda^{-1}\right)\right)$ given by $(a, b \in \mathcal{V})$ :

$$
\begin{equation*}
\left\{\left\{a_{\lambda} b\right\}\right\}^{D}=\left\{\left\{a_{\lambda} b\right\}\right\}-\sum_{\alpha, \beta=1}^{m}\left\{\left\{\theta_{\beta_{\lambda+\partial}} b\right\} \rightarrow \bullet\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial) \bullet\left\{\left\{a_{\lambda} \theta_{\alpha}\right\}\right\} .\right. \tag{4.3}
\end{equation*}
$$

Theorem 4.4. Let $\mathcal{V}$ be a non-local double Poisson vertex algebra with 2 -fold $\lambda$ bracket $\left\{\left\{{ }_{\lambda}-\right\}\right.$. Let $\theta_{1}, \ldots, \theta_{m} \in \mathcal{V}$ be elements such that the corresponding matrix pseudodifferential operator $C(\partial)=\left(C_{\alpha \beta}(\partial)\right)_{\alpha, \beta=1}^{m} \in \operatorname{Mat}_{m \times m} \mathcal{V}^{\otimes 2}\left(\left(\partial^{-1}\right)\right)$ given by (4.2) is invertible.
(a) The Dirac modification $\left.\left\{-{ }_{\lambda}-\right\}\right\}^{D}$ given by equation (4.3) is a 2 -fold $\lambda$-bracket on $\mathcal{V}$, giving $\mathcal{V}$ a structure of a non-local double PVA.
(b) All the elements $\theta_{i}, i=1, \ldots, m$, are central with respect to the Dirac modified $\lambda$-bracket: $\left\{\left\{a_{\lambda} \theta_{i}\right\}^{D}=\left\{\left\{\theta_{i \lambda} a\right\}^{D}=0\right.\right.$ for all $i=1, \ldots, m$ and $a \in \mathcal{V}$.

Before proving the theorem we state some lemmas that we will use in the proof.
Lemma 4.5. Let $A(\lambda, \mu) \in \mathcal{V}_{\lambda, \mu}^{\otimes 3}$ and $B(\lambda, \mu) \in \mathcal{V}_{\lambda, \mu}^{\otimes 2}$, and let $S, T: \mathcal{V}^{\otimes 3} \rightarrow \mathcal{V}^{\otimes 3}$ be endomorphisms of $\mathcal{V}^{\otimes 3}$ (viewed as a vector space). Then

$$
\begin{aligned}
& A(\lambda+S, \mu+T) \bullet_{1} B(\lambda, \mu) \in \mathcal{V}_{\lambda, \mu}^{\otimes 3}, \\
& A(\lambda+S, \mu+T) \bullet_{2} B(\lambda, \mu) \in \mathcal{V}_{\lambda, \mu}^{\otimes 3}, \\
& A(\lambda+S, \mu+T) \bullet_{3} B(\lambda, \mu) \in \mathcal{V}_{\lambda, \mu}^{\otimes 3},
\end{aligned}
$$

where we expand the negative powers of $\lambda+S$ and $\mu+T$ in non-negative powers of $S$ and $T$, acting on the coefficients of $B$, and

$$
\begin{aligned}
& B(\lambda+S, \mu+T) \bullet_{1} A(\lambda, \mu) \in \mathcal{V}_{\lambda, \mu}^{\otimes 3}, \\
& B(\lambda+S, \mu+T) \bullet_{2} A(\lambda, \mu) \in \mathcal{V}_{\lambda, \mu}^{\otimes 3}, \\
& B(\lambda+S, \mu+T) \bullet_{3} A(\lambda, \mu) \in \mathcal{V}_{\lambda, \mu}^{\otimes 3},
\end{aligned}
$$

where now $S$ and $T$ act on the coefficients of $A$.
Proof. The proof follows the same lines as the proof of Lemma 2.3 in [DSK13] using the definition of the $\bullet$-products given by equation (1.18).

Lemma 4.6. Let $\left\{\{-\lambda-\}: \mathcal{V}^{\otimes 2} \times \mathcal{V}^{\otimes 2} \rightarrow \mathcal{V}^{\otimes 2}\left(\left(\lambda^{-1}\right)\right)\right.$ be a 2 -fold $\lambda$-bracket on the differential algebra $\mathcal{V}$. Suppose that $C(\partial)=\left(C_{i j}(\partial)\right)_{i, j=1}^{\ell} \in \operatorname{Mat}_{\ell \times \ell} \mathcal{V}^{\otimes 2}\left(\left(\partial^{-1}\right)\right)$ is an invertible $\ell \times \ell$ matrix pseudodifferential operator with coefficients in $\mathcal{V}^{\otimes 2}$, and let $C^{-1}(\partial)=\left(\left(C^{-1}\right)_{i j}(\partial)\right)_{i, j=1}^{\ell} \in \operatorname{Mat}_{\ell \times \ell} \mathcal{V}^{\otimes 2}\left(\left(\partial^{-1}\right)\right)$ be its inverse. The following identities hold for every $a \in \mathcal{V}$ and $i, j=1, \ldots, \ell$ :

$$
\begin{align*}
& \left\{a_{\lambda}\left(C^{-1}\right)_{i j}(\mu)\right\}_{L}=-\sum_{r, t=1}^{\ell} i_{\mu}\left(C^{-1}\right)_{i r}(\lambda+\mu+\partial)  \tag{4.4}\\
& \bullet_{2}\left\{\left\{a_{\lambda} C_{r t}(y)\right\}_{L} \bullet_{1}\left(\left.\right|_{y=\mu+\partial}\left(C^{-1}\right)_{t j}(\mu)\right) \in \mathcal{V}^{\otimes 3}\left(\left(\lambda^{-1}\right)\right)\left(\left(\mu^{-1}\right)\right),\right. \\
& \left\{a_{\lambda}\left(C^{-1}\right)_{i j}(\mu)\right\}_{R}=-\sum_{r, t=1}^{\ell} i_{\mu}\left(C^{-1}\right)_{i r}(\lambda+\mu+\partial)  \tag{4.5}\\
& \bullet_{2}\left\{\left\{a_{\lambda} C_{r t}(y)\right\}\right\}_{R} \bullet_{3}\left(\left.\right|_{y=\mu+\partial}\left(C^{-1}\right)_{t j}(\mu)\right) \in \mathcal{V}^{\otimes 3}\left(\left(\lambda^{-1}\right)\right)\left(\left(\mu^{-1}\right)\right),
\end{align*}
$$

and

$$
\begin{gather*}
\left.\left\{\left(C^{-1}\right)_{i j}(\lambda)_{\lambda+\mu} a\right\}\right\}_{L}^{\sigma^{2}}=-\sum_{r, t=1}^{\ell}\left\{\left\{C_{r t}(x)_{x+y} a\right\}_{L}^{\sigma^{2}} \bullet_{3}\left(\left.\right|_{x=\lambda+\partial}\left(C^{-1}\right)_{t j}(\lambda)\right)\right.  \tag{4.6}\\
\bullet_{1}\left(\left.\right|_{y=\mu+\partial} i_{\lambda}\left(C^{*-1}\right)_{r i}(\mu)\right) \in \mathcal{V}^{\otimes 3}\left(\left((\lambda+\mu)^{-1}\right)\right)\left(\left(\lambda^{-1}\right)\right),
\end{gather*}
$$

where $i_{\mu}: \mathcal{V}_{\lambda, \mu}^{\otimes 3} \rightarrow \mathcal{V}^{\otimes 3}\left(\left(\lambda^{-1}\right)\right)\left(\left(\mu^{-1}\right)\right)$ and $i_{\lambda}: \mathcal{V}_{\lambda, \mu}^{\otimes 3} \rightarrow \mathcal{V}^{\otimes 3}\left(\left((\lambda+\mu)^{-1}\right)\right)\left(\left(\lambda^{-1}\right)\right)$ are the natural embeddings defined above. In equations (4.4) and (4.6), $C(\lambda) \in$ $\mathrm{Mat}_{\ell \times \ell} \mathcal{V}\left(\left(\lambda^{-1}\right)\right)$ denotes the symbol of the matrix pseudodifferential operator $C$ and $C^{*}$ denotes its adjoint (its inverse being $\left(C^{-1}\right)^{*}$ ).
Proof. The identity $C \circ C^{-1}=1 \otimes 1$ becomes, in terms of symbols,

$$
\sum_{t=1}^{\ell} C_{r t}(\mu+\partial) \bullet\left(C^{-1}\right)_{t j}(\mu)=\delta_{r j}(1 \otimes 1)
$$

For all $a \in \mathcal{V}$, we have, by sesquilinearity and the left Leibniz rule (the first equation in (3.23)):

$$
\begin{aligned}
& 0=\sum_{t=1}^{\ell}\left\{a_{\lambda} C_{r t}(\mu+\partial) \bullet\left(C^{-1}\right)_{t j}(\mu)\right\}_{L} \\
& =\sum_{t=1}^{\ell}\left\{a_{\lambda} C_{r t}(y)\right\}_{L} \bullet_{1}\left(\left.\right|_{y=\mu+\partial}\left(C^{-1}\right)_{t j}(\mu)\right) \\
& +\sum_{t=1}^{\ell} i_{\mu} C_{r t}(\lambda+\mu+\partial) \bullet_{2}\left\{\left\{a_{\lambda}\left(C^{-1}\right)_{t j}(\mu)\right\} .\right.
\end{aligned}
$$

Note that $i_{\mu} C(\lambda+\mu+\partial)$ is invertible in Mat $\operatorname{Max\ell }\left(\mathcal{V}^{\otimes 2}[\partial]\left(\left(\lambda^{-1}\right)\right)\left(\left(\mu^{-1}\right)\right)\right)$, its inverse being $i_{\mu} C^{-1}(\lambda+\mu+\partial)$. We then apply $i_{\mu}\left(C^{-1}\right)_{i r}(\lambda+\mu+\partial) \bullet_{2}$ on the left to both sides of the above equation, and using Lemma 1.1(a) and summing over $r=1, \ldots, \ell$, we get

$$
\begin{aligned}
\sum_{t=1}^{\ell} \delta_{i t}\left\{\left\{a_{\lambda}\left(C^{-1}\right)_{t j}(\mu)\right\}_{L}=-\right. & \sum_{r, t=1}^{\ell} i_{\mu}\left(C^{-1}\right)_{i r}(\lambda+\mu+\partial) \bullet_{2} \\
& \left\{\left\{a_{\lambda} C_{r t}(y)\right\}_{L} \bullet_{1}\left(\left.\right|_{y=\mu+\partial}\left(C^{-1}\right)_{t j}(\mu)\right),\right.
\end{aligned}
$$

proving equation (4.4). The proof of the equation (4.5) is done in a similar way using the second equation in (3.23).

For the third equation we have, by the right Leibniz rule (the third equation in (3.23)):

$$
\begin{aligned}
& 0=\sum_{t=1}^{\ell}\left\{\left\{C_{r t}(\lambda+\partial) \bullet\left(C^{-1}\right)_{t j}(\lambda)_{\lambda+\mu} a\right\}_{L}\right. \\
& =\sum_{t=1}^{\ell}\left\{\left\{C_{r t}(x)_{\lambda+\mu+\partial} a\right\}_{L_{\rightarrow}} \bullet_{3}\left(\left.\right|_{x=\lambda+\partial}\left(C^{-1}\right)_{t j}(\lambda)\right)\right. \\
& +\sum_{t=1}^{\ell}\left\{\left(C^{-1}\right)_{t j}(\lambda)_{\lambda+\mu+\partial} a\right\}_{L_{\rightarrow}} \bullet_{1}\left(i_{\lambda} C_{t r}^{\dagger}(\mu)\right)
\end{aligned}
$$

We next replace in the above equation $\mu$ (placed at the right) by $\mu+\partial$, and we apply the resulting differential operator to $i_{\lambda}\left(C^{\dagger-1}\right)_{r i}(\mu)$. As a result we get, after summing over $r=1, \ldots, \ell$ :

$$
\begin{aligned}
& \sum_{t=1}^{\ell}\left\{\left\{\left(C^{-1}\right)_{t j}(\lambda)_{\lambda+\mu+\partial} a\right\}_{L_{\rightarrow}} \delta_{t i}\right.=-\sum_{r, t=1}^{\ell}\left\{C_{r t}(x)_{\lambda+\mu+\partial} a\right\}_{L_{\rightarrow}} \\
& \bullet_{3}\left(\left.\right|_{x=\lambda+\partial}\left(C^{-1}\right)_{t j}(\lambda)\right) \bullet_{1}\left(i_{\lambda}\left(C^{\dagger-1}\right)_{r i}(\mu)\right)
\end{aligned}
$$

proving equation (4.6).
Corollary 4.7. Let $\left\{\{-\lambda-\}: \mathcal{V}^{\otimes 2} \rightarrow \mathcal{V}^{\otimes 2}\left(\left(\lambda^{-1}\right)\right)\right.$ be a 2-fold $\lambda$-bracket on the differential algebra $\mathcal{V}$. Let $C(\partial)=\left(C_{i j}(\partial)\right)_{i, j=1}^{\ell} \in \operatorname{Mat}_{\ell \times \ell} \mathcal{V}^{\otimes 2}\left(\left(\partial^{-1}\right)\right)$ be an invertible $\ell \times \ell$ matrix pseudodifferential operator with coefficients in $\mathcal{V}^{\otimes 2}$, and let $C^{-1}(\partial)=\left(\left(C^{-1}\right)_{i j}(\partial)\right)_{i, j=1}^{\ell} \in \operatorname{Mat}_{\ell \times \ell} \mathcal{V}^{\otimes 2}\left(\left(\partial^{-1}\right)\right)$ be its inverse. Let $a \in \mathcal{V}$, and assume that

$$
\begin{equation*}
\left\{a_{\lambda} C_{i j}(\mu)\right\}_{L} \in \mathcal{V}_{\lambda, \mu}^{\otimes 3} \quad \text { for all } i, j=1, \ldots, \ell \tag{4.7}
\end{equation*}
$$

(As before, we identify $\mathcal{V}_{\lambda, \mu}^{\otimes 3}$ with its image $i_{\mu}\left(\mathcal{V}_{\lambda, \mu}^{\otimes 3}\right) \subset \mathcal{V}^{\otimes 3}\left(\left(\lambda^{-1}\right)\right)\left(\left(\mu^{-1}\right)\right)$.) Then, we have $\left\{\left\{a_{\lambda}\left(C^{-1}\right)_{i j}(\mu)\right\}_{L},\left\{\left\{\left(C^{-1}\right)_{i j}(\lambda)_{\lambda+\mu} a\right\}_{L} \in \mathcal{V}_{\lambda, \mu}^{\otimes 3}\right.\right.$. In fact, the following identities hold in the space $\mathcal{V}_{\lambda, \mu}^{\otimes 3}$ :

$$
\begin{align*}
& \left\{\left\{a_{\lambda}\left(C^{-1}\right)_{i j}(\mu)\right\}_{L}=\right. \\
& -\sum_{r, t=1}^{\ell}\left(C^{-1}\right)_{i r}(\lambda+\mu+\partial) \bullet_{2}\left\{\left\{a_{\lambda} C_{r t}(y)\right\}_{L} \bullet_{1}\left(\left.\right|_{y=\mu+\partial}\left(C^{-1}\right)_{t j}(\mu)\right),\right.  \tag{4.8}\\
& \left\{\left\{a_{\lambda}\left(C^{-1}\right)_{i j}(\mu)\right\}_{R}=\right. \\
& -\sum_{r, t=1}^{\ell}\left(C^{-1}\right)_{i r}(\lambda+\mu+\partial) \bullet_{2}\left\{\left\{a_{\lambda} C_{r t}(y)\right\}_{R} \bullet_{3}\left(\left.\right|_{y=\mu+\partial}\left(C^{-1}\right)_{t j}(\mu)\right),\right.  \tag{4.9}\\
& \left\{\left\{\left(C^{-1}\right)_{i j}(\lambda)_{\lambda+\mu} a\right\}_{L}=\right. \\
& -\sum_{r, t=1}^{\ell}\left\{C_{r t}(x)_{\lambda+\mu+\partial} a\right\}_{L_{\rightarrow}} \bullet_{3}\left(\left.\right|_{x=\lambda+\partial}\left(C^{-1}\right)_{t j}(\lambda)\right) \bullet_{1}\left(\left(C^{\dagger-1}\right)_{r i}(\mu)\right) . \tag{4.10}
\end{align*}
$$

Proof. It is an immediate corollary of Lemmas 4.5 and 4.6.
Remark 4.8. Lemmas 4.5 and 4.6 are the double PVA analogues of (respectively) [DSK13, Lemmas 2.3 and 3.9]. Corollary 4.7 is the double PVA analogue of [DSKV14b, Corollary 1.4]. Naturally, if we put $\lambda=0$ in (4.3), we obtain Dirac modification of the corresponding double Poisson algebra.

Proof of Theorem 4.4. Both sesquilinearity conditions (3.5) for the Dirac modified 2 -fold $\lambda$-bracket (4.3) are immediate to check. The skewsymmetry condition (3.8) for the Dirac modified 2 -fold $\lambda$-bracket (4.3) can also be easily proved: it follows by the skewsymmetry of the 2 -fold $\lambda$-bracket $\left\{\left\{-_{\lambda}-\right\}\right.$, by the fact that the matrix
$C(\partial)$ (hence $C^{-1}(\partial)$ ) is skewadjoint, and by equation (1.3). Moreover, by equation (1.16), it follows that the Dirac modified 2 -fold $\lambda$-bracket $\{\{-\lambda-\}\}^{D}$ satisfies the Leibniz rules (3.6).

Next, we prove that the Dirac modified 2-fold $\lambda$-bracket $\left\{\left\{-{ }_{\lambda}-\right\}^{D}\right.$ is admissible, in the sense of equation (4.1). For this, we compute $\left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} c\right\}\right\}^{D}\right\}\right\}_{L}^{D}$ using the definition (4.3), the sesquilinearity conditions (3.5), the Leibniz rule (3.6) and equation (3.23). We get

$$
\begin{align*}
& \left\{a_{\lambda}\left\{b_{\mu} c\right\}^{D}\right\}_{L}^{D}=\left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} c\right\}\right\}_{L}\right.\right.  \tag{4.11}\\
& -\sum_{\gamma, \delta=1}^{m}\left\{\left\{a_{\lambda}\left\{\left\{\theta_{\delta y} c\right\}\right\}\right\}_{L} \bullet_{1}\left(\left.\right|_{y=\mu+\partial}\left(C^{-1}\right)_{\delta \gamma}(\mu+\partial)\left\{b_{\mu} \theta_{\gamma}\right\}\right\}\right)  \tag{4.12}\\
& -\sum_{\gamma, \delta=1}^{m}\left\{\theta_{\delta \lambda+\mu+\partial} c\right\} \rightarrow \bullet_{2}\left\{\left\{a_{\lambda}\left(C^{-1}\right)_{\delta \gamma}(y)\right\}_{L} \bullet_{1}\left(\left.\right|_{y=\mu+\partial}\left\{b_{\mu} \theta_{\gamma}\right\}\right\}\right)  \tag{4.13}\\
& -\sum_{\gamma, \delta=1}^{m}\left(\left\{\theta_{\delta \lambda+\mu+\partial} c\right\} \rightarrow\left(C^{-1}\right)_{\delta \gamma}(\lambda+\mu+\partial)\right) \bullet_{2}\left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} \theta_{\gamma}\right\}\right\}\right\}_{L}\right.  \tag{4.14}\\
& -\sum_{\alpha, \beta=1}^{m}\left\{\theta_{\beta_{\lambda+\partial}}\left\{b_{\mu} c\right\}\right\}_{L_{\rightarrow}} \bullet_{3}\left(\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial)\left\{a_{\lambda} \theta_{\alpha}\right\}\right)  \tag{4.15}\\
& +\sum_{\alpha, \beta, \gamma, \delta=1}^{m}\left\{\left\{\theta_{\beta_{x}}\left\{\left\{\theta_{\delta y} c\right\}\right\}\right\}_{L} \bullet_{3}\left(\left.\right|_{x=\lambda+\partial}\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial)\left\{a_{\lambda} \theta_{\alpha}\right\}\right)\right. \\
& \left.\bullet_{1}\left(\left.\right|_{y=\mu+\partial}\left(C^{-1}\right)_{\delta \gamma}(\mu+\partial)\left\{b_{\mu} \theta_{\gamma}\right\}\right\}\right)  \tag{4.16}\\
& +\sum_{\alpha, \beta, \gamma, \delta=1}^{m}\left\{\theta_{\delta \lambda+\mu+\partial} c\right\} \rightarrow \bullet_{2}\left\{\left\{\theta_{\beta_{x}}\left(C^{-1}\right)_{\delta \gamma}(y)\right\}_{L}\right. \\
& \bullet_{3}\left(\left.\right|_{x=\lambda+\partial}\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial)\left\{a_{\lambda} \theta_{\alpha}\right\}\right) \bullet_{1}\left(\left.\right|_{y=\mu+\partial}\left\{b_{\mu} \theta_{\gamma}\right\}\right)  \tag{4.17}\\
& +\sum_{\alpha, \beta, \gamma, \delta=1}^{m}\left(\left\{\theta_{\delta \lambda+\mu+\partial} c\right\} \rightarrow\left(C^{-1}\right)_{\delta \gamma}(\lambda+\mu+\partial)\right) \bullet_{2}\left\{\left\{\theta_{\beta \lambda+\partial}\left\{\left\{b_{\mu} \theta_{\gamma}\right\}\right\}\right\}_{L_{-}}\right. \\
& \bullet_{3}\left(\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial)\left\{a_{\lambda} \theta_{\alpha}\right\}\right) . \tag{4.18}
\end{align*}
$$

All the terms $(4.11),(4.12),(4.14),(4.15),(4.16)$, and (4.18), lie in $\mathcal{V}_{\lambda, \mu}^{\otimes 3}$ by the admissibility assumption on $\left\{\left\{-_{\lambda}-\right\}\right.$ and Lemma 4.5. Moreover, by the admissibility of $\{\{-\lambda-\}$ and the definition (4.2) of the matrix $C(\partial)$, condition (4.7) holds. Hence, we can use Corollary 4.7 and Lemma 4.5 to deduce that the terms (4.13) and (4.17) lie in $\mathcal{V}_{\lambda, \mu}^{\otimes 3}$ as well. Therefore, $\left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} c\right\}\right\}^{D}\right\}\right\}_{L}^{D}$ lies in $\mathcal{V}_{\lambda, \mu}^{\otimes 3}$ for every $a, b, c \in \mathcal{V}$, i.e. the Dirac modification $\left\{\left\{-{ }_{\lambda}-\right\}^{D}\right.$ is admissible.

In order to complete the proof of part (a) we are left to check the Jacobi identity (3.9) for the Dirac modified 2-fold $\lambda$-bracket. We can use equation (4.8) in Corollary

$$
\begin{align*}
& \left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} c\right\}\right\}^{D}\right\}_{L}^{D}=\left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} c\right\}\right\}\right\}_{L}\right.\right.  \tag{4.19}\\
& -\sum_{\gamma, \delta=1}^{m}\left\{\left\{a_{\lambda}\left\{\left\{\theta_{\delta y} c\right\}\right\}\right\}\right\}_{L} \bullet_{1}\left(\left.\right|_{y=\mu+\partial}\left(C^{-1}\right)_{\delta \gamma}(\mu+\partial)\left\{\left\{b_{\mu} \theta_{\gamma}\right\}\right\}\right)  \tag{4.20}\\
& +\sum_{\gamma, \delta, \eta, \zeta=1}^{m}\left(\left\{\left\{\theta_{\delta \lambda+\mu+\partial} c\right\} \rightarrow\left(C^{-1}\right)_{\delta \zeta}(\lambda+\mu+\partial)\right)\right. \\
& \left.\bullet_{2}\left\{\left\{a_{\lambda}\left\{\left\{\theta_{\eta y} \theta_{\zeta}\right\}\right\}\right\}\right\}_{L} \bullet_{1}\left(\left.\right|_{y=\mu+\partial}\left(C^{-1}\right)_{\eta \gamma}(\mu+\partial)\left\{b_{\mu} \theta_{\gamma}\right\}\right\}\right)  \tag{4.21}\\
& -\sum_{\gamma, \delta=1}^{m}\left(\left\{\left\{\theta_{\delta \lambda+\mu+\partial} c\right\}\right\} \rightarrow\left(C^{-1}\right)_{\delta \gamma}(\lambda+\mu+\partial)\right) \bullet_{2}\left\{\left\{a_{\lambda}\left\{\left\{b_{\mu} \theta_{\gamma}\right\}\right\}\right\}\right\}_{L}  \tag{4.22}\\
& -\sum_{\alpha, \beta=1}^{m}\left\{\left\{\theta_{\beta_{\lambda+\partial}}\left\{\left\{b_{\mu} c\right\}\right\}\right\}\right\}_{L_{\rightarrow}} \bullet_{3}\left(\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial)\left\{\left\{a_{\lambda} \theta_{\alpha}\right\}\right)\right.  \tag{4.23}\\
& \left.\left.+\sum_{\alpha, \beta, \gamma, \delta=1}^{m}\left\{\theta_{\beta_{x}}\left\{\theta_{\delta y} c\right\}\right\}\right\}\right\}_{L} \bullet_{3}\left(\left.\right|_{x=\lambda+\partial}\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial)\left\{\left\{a_{\lambda} \theta_{\alpha}\right\}\right\}\right) \\
& \text { • } 1\left(\left.\right|_{y=\mu+\partial}\left(C^{-1}\right)_{\delta \gamma}(\mu+\partial)\left\{\left\{b_{\mu} \theta_{\gamma}\right\}\right\}\right)  \tag{4.24}\\
& \left.+\sum_{\alpha, \beta, \gamma, \delta, \eta, \zeta=1}^{m}\left(\left\{\theta_{\delta \lambda+\mu+\partial} c\right\}\right\}_{\rightarrow}\left(C^{-1}\right)_{\delta \zeta}(\lambda+\mu+\partial)\right) \\
& \bullet_{2}\left\{\left\{\theta_{\beta_{x}}\left\{\left\{\theta_{\eta y} \theta_{\zeta}\right\}\right\}\right\}\right\}_{L} \bullet_{3}\left(\left.\right|_{x=\lambda+\partial}\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial)\left\{\left\{a_{\lambda} \theta_{\alpha}\right\}\right\}\right) \\
& \bullet_{3}\left(\left.\right|_{y=\mu+\partial}\left(C^{-1}\right)_{\eta \gamma}(\mu+\partial)\left\{\left\{b_{\mu} \theta_{\gamma}\right\}\right\}\right)  \tag{4.25}\\
& +\sum_{\alpha, \beta, \gamma, \delta=1}^{m}\left(\{ \{ \theta _ { \delta \lambda + \mu + \partial } c \} \rightarrow ( C ^ { - 1 } ) _ { \delta \gamma } ( \lambda + \mu + \partial ) ) \bullet _ { 2 } \left\{\left\{\theta_{\beta_{\lambda+\partial}}\left\{\left\{b_{\mu} \theta_{\gamma}\right\}\right\}\right\}_{L_{\rightarrow}}\right.\right. \\
& \bullet_{3}\left(\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial)\left\{\left\{a_{\lambda} \theta_{\alpha}\right\}\right\}\right) . \tag{4.26}
\end{align*}
$$

Next, we compute the second term in the Jacobi identity using the definition (4.3), the sesquilinearity conditions (3.5), the Leibniz rules (3.6), equations (3.23) and

$$
\begin{align*}
& \left\{\left\{b_{\mu}\left\{\left\{a_{\lambda} c\right\}\right\}^{D}\right\}_{R}^{D}=\left\{\left\{b_{\mu}\left\{\left\{a_{\lambda} c\right\}\right\}\right\}_{R}\right.\right.  \tag{4.27}\\
& -\sum_{\alpha, \beta=1}^{m}\left\{\left\{b_{\mu}\left\{\left\{\theta_{\beta_{x}} c\right\}\right\}\right\}\right\}_{R} \bullet_{3}\left(\left.\right|_{x=\lambda+\partial}\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial)\left\{\left\{a_{\lambda} \theta_{\alpha}\right\}\right)\right.  \tag{4.28}\\
& +\sum_{\alpha, \beta, \gamma, \delta}^{m}\left(\left\{\left\{\theta_{\delta \lambda+\mu+\partial} c\right\}\right\} \rightarrow\left(C^{-1}\right)_{\delta \gamma}(\lambda+\mu+\partial)\right) \bullet_{2}\left\{\left\{b_{\mu}\left\{\left\{\theta_{\beta_{x}} \theta_{\gamma}\right\}\right\}\right\}\right\}_{R} \\
& \bullet_{3}\left(\left.\right|_{x=\lambda+\partial}\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial)\left\{\left\{a_{\lambda} \theta_{\alpha}\right\}\right\}\right)  \tag{4.29}\\
& -\sum_{\gamma, \delta=1}^{m}\left(\left\{\left\{\theta_{\delta \lambda+\mu+\partial} c\right\} \rightarrow_{\rightarrow}\left(C^{-1}\right)_{\delta \gamma}(\lambda+\mu+\partial)\right) \bullet_{2}\left\{\left\{b_{\mu}\left\{\left\{a_{\lambda} \theta_{\gamma}\right\}\right\}\right\}\right\}_{R}\right.  \tag{4.30}\\
& -\sum_{\gamma, \delta=1}^{m}\left\{\theta_{\delta \mu+\partial}\left\{\left\{a_{\lambda} c\right\}\right\}\right\}_{R_{\rightarrow}} \bullet_{1}\left(\left(C^{-1}\right)_{\delta \gamma}(\mu+\partial)\left\{\left\{b_{\mu} \theta_{\gamma}\right\}\right\}\right)  \tag{4.31}\\
& +\sum_{\alpha, \beta, \gamma, \delta=1}^{m}\left\{\left\{\theta_{\delta y}\left\{\left\{\theta_{\beta_{x}} c\right\}\right\}\right\}_{R} \bullet_{1}\left(\left.\right|_{y=\mu+\partial}\left(C^{-1}\right)_{\delta \gamma}(\mu+\partial)\left\{\left\{b_{\mu} \theta_{\gamma}\right\}\right\}\right)\right. \\
& \left.\bullet_{3}\left(\left.\right|_{x=\lambda+\partial}\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial)\left\{a_{\lambda} \theta_{\alpha}\right\}\right\}\right)  \tag{4.32}\\
& -\sum_{\alpha, \beta, \gamma, \delta, \eta, \zeta=1}^{m}\left(\left\{\left\{\theta_{\delta \lambda+\mu+\partial} c\right\}_{\rightarrow}\left(C^{-1}\right)_{\delta \zeta}(\lambda+\mu+\partial)\right)\right. \\
& \left.\bullet_{2}\left\{\left\{\theta_{\eta_{y}}\left\{\left\{\theta_{\beta_{x}} \theta_{\zeta}\right\}\right\}\right\}\right\}{ }_{R} \bullet_{1}\left(\left.\right|_{y=\mu+\partial}\left(C^{-1}\right)_{\eta \gamma}(\mu+\partial)\left\{b_{\lambda} \theta_{\gamma}\right\}\right\}\right) \\
& \bullet_{3}\left(\left.\right|_{x=\lambda+\partial}\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial)\left\{\left\{a_{\lambda} \theta_{\alpha}\right\}\right\}\right)  \tag{4.33}\\
& +\sum_{\gamma, \delta, \eta, \zeta=1}^{m}\left(\left\{\left\{\theta_{\delta \lambda+\mu+\partial} c\right\} \rightarrow\left(C^{-1}\right)_{\delta \zeta}(\lambda+\mu+\partial)\right)\right. \\
& \bullet_{2}\left\{\left\{\theta_{\eta_{\mu+\partial}}\left\{\left\{a_{\lambda} \theta_{\zeta}\right\}\right\}\right\} R_{\rightarrow} \bullet_{1}\left(\left(C^{-1}\right)_{\eta \gamma}(\mu+\partial)\left\{\left\{b_{\mu} \theta_{\gamma}\right\}\right\}\right) .\right. \tag{4.34}
\end{align*}
$$

In a similar way we compute the RHS of the Jacobi identity for the Dirac modified 2fold $\lambda$-bracket, using the definition (4.3), the sesquilinearity (3.5), the right Leibniz rule (3.6), equation (4.10) (recall that the matrix $C$ is skewadjoint), and Lemma

## 1.1(a,c). We get

$$
\begin{align*}
& \left.\left\{\left\{\left\{a_{\lambda} b\right\}\right\}^{D}{ }_{\lambda+\mu} c\right\}\right\}_{L}^{D}=\left\{\left\{\left\{a_{\lambda} b\right\}_{\lambda+\mu} c\right\}\right\}_{L}  \tag{4.35}\\
& -\sum_{\alpha, \beta=1}^{m}\left\{\left\{\left\{\theta_{\beta_{x}} b\right\}_{\lambda+\mu+\partial} c\right\}\right\}_{L_{\rightarrow}} \bullet_{3}\left(\left.\right|_{x=\lambda+\partial}\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial)\left\{\left\{a_{\lambda} \theta_{\alpha}\right\}\right\}\right)  \tag{4.36}\\
& -\sum_{\alpha, \beta, \gamma, \delta=1}^{m}\left\{\left\{\left\{\theta_{\beta_{x}} \theta_{\gamma}\right\}\right\}_{\lambda+\mu+\partial} c\right\}_{L} \bullet_{3}\left(\left.\right|_{x=\lambda+\partial}\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial)\left\{\left\{a_{\lambda} \theta_{\alpha}\right\}\right\}\right) \\
& \bullet_{1}\left(\left(C^{-1}\right)_{\delta \gamma}(\mu+\partial)\left\{\left\{\theta_{\gamma_{-\mu-\partial}} b\right\}^{\sigma}\right)\right.  \tag{4.37}\\
& \left.-\sum_{\gamma, \delta=1}^{m}\left\{\left\{a_{\lambda} \theta_{\delta}\right\}\right\}_{\lambda+\mu+\partial} c\right\}_{L} \bullet_{1}\left(\left(C^{\dagger^{-1}}\right)_{\delta \gamma}(\mu+\partial)\left\{\left\{\theta_{\gamma_{-\mu-\partial}} b\right\}\right\}^{\sigma}\right)  \tag{4.38}\\
& -\sum_{\gamma, \delta=1}^{m}\left(\left\{\left\{\theta_{\delta \lambda+\mu+\partial} c\right\}\right\}_{\rightarrow}\left(C^{-1}\right)_{\delta \gamma}(\lambda+\mu+\partial)\right) \bullet_{2}\left\{\left\{\left\{\left\{a_{\lambda} b\right\}_{\lambda+\mu} \theta_{\gamma}\right\}_{L}\right.\right.  \tag{4.39}\\
& +\sum_{\alpha, \beta, \gamma, \delta=1}^{m}\left(\left\{\theta_{\delta \lambda+\mu+\partial} c\right\} \rightarrow\left(C^{-1}\right)_{\delta \gamma}(\lambda+\mu+\partial)\right) \\
& \bullet_{2}\left\{\left\{\left\{\theta_{\beta_{x}} b\right\}_{\lambda+\mu+\partial} \theta_{\gamma}\right\}_{L_{\rightarrow}} \bullet_{3}\left(\left.\right|_{x=\lambda+\partial}\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial)\left\{\left\{a_{\lambda} \theta_{\alpha}\right\}\right\}\right)\right.  \tag{4.40}\\
& +\sum_{\alpha, \beta, \gamma, \delta, \eta, \zeta=1}^{m}\left(\left\{\left\{\theta_{\delta \lambda+\mu+\partial} c\right\} \rightarrow\left(C^{-1}\right)_{\delta \gamma}(\lambda+\mu+\partial)\right)\right. \\
& \bullet_{2}\left\{\left\{\left\{\theta_{\beta_{x}} \theta_{\eta}\right\}_{\lambda+\mu+\partial} \theta_{\zeta}\right\}\right\}_{L} \bullet_{3}\left(\left.\right|_{x=\lambda+\partial}\left(C^{-1}\right)_{\beta \alpha}(\lambda+\partial)\left\{\left\{a_{\lambda} \theta_{\alpha}\right\}\right\}\right) \\
& { }^{1}\left(\left(C^{-1}\right)_{\eta \gamma}(\mu+\partial)\left\{\left\{\theta_{\gamma-\mu-\partial} b\right\}\right\}^{\sigma}\right)  \tag{4.41}\\
& +\sum_{\gamma, \delta, \eta, \zeta=1}^{m}\left(\left\{\theta_{\delta \lambda+\mu+\partial} c\right\} \rightarrow\left(C^{-1}\right)_{\delta \zeta}(\lambda+\mu+\partial)\right) \\
& \bullet_{2}\left\{\left\{\left\{a_{\lambda} \zeta_{\alpha}\right\}\right\}_{\lambda+\mu+\partial} \theta_{\eta}\right\}_{L_{\rightarrow}} \bullet_{1}\left(\left(C^{\dagger^{-1}}\right)_{\eta \gamma}(\mu+\partial)\left\{\left\{\theta_{\gamma_{-\mu-\partial}} b\right\}^{\sigma}\right) .\right. \tag{4.42}
\end{align*}
$$

The following equations hold due to the skewsymmetry (3.8), the Jacobi identity (3.9), and the fact that the matrix $C$ is skewadjoint:

$$
\begin{aligned}
& \operatorname{RHS}(4.19)-\operatorname{RHS}(4.27)=\operatorname{RHS}(4.35), \quad(4.20)-(4.31)=(4.38), \\
& (4.23)-(4.28)=(4.36), \quad(4.22)-(4.30)=(4.39) \\
& (4.21)-(4.34)=(4.42), \quad(4.26)-(4.29)=(4.40)
\end{aligned}
$$

Moreover, using Lemma 1.1(c), the skewsymmetry (3.8) and the Jacobi identity (3.9), we also get

$$
(4.24)-(4.32)=(4.37), \quad(4.25)-(4.33)=(4.41)
$$

This concludes the proof of the Jacobi identity for the Dirac modified 2-fold $\lambda$ bracket, and of part (a).

Note that the identities $C(\partial) C^{-1}(\partial)=C^{-1}(\partial) C(\partial)=1 \otimes 1$ read, in terms of the symbols of the pseudodifferential operators $C(\partial)$ and $C^{-1}(\partial)$, as
$\sum_{\beta=1}^{m}\left\{\left\{\theta_{\beta_{\lambda+}} \theta_{\alpha}\right\} \rightarrow \rightarrow\left(C^{-1}\right)_{\beta \gamma}(\lambda)=\delta_{\alpha \gamma}(1 \otimes 1), \sum_{\beta=1}^{m}\left(C^{-1}\right)_{\alpha \beta}(\lambda+\partial) \bullet\left\{\theta_{\gamma_{\lambda}} \theta_{\beta}\right\}\right\}=\delta_{\alpha \gamma}(1 \otimes 1)$.
Part (b) is an immediate consequence of these identities and the definition (4.3) of the Dirac modified 2 -fold $\lambda$-bracket.

Definition 4.9. Let $\mathcal{I} \subset \mathcal{V}$ be a two-sided differential ideal. We say that $\mathcal{I}$ is a double Poisson vertex algebra ideal if

$$
\left\{\left\{\mathcal{V}_{\lambda} \mathcal{I}\right\},\left\{\left\{\mathcal{I}_{\lambda} \mathcal{V}\right\} \in(\mathcal{V} \otimes \mathcal{I}+\mathcal{I} \otimes \mathcal{V})[\lambda] .\right.\right.
$$

If $\mathcal{I} \subset \mathcal{V}$ is a double PVA ideal then we have an induced double PVA structure on the quotient differential algebra $\mathcal{V} / \mathcal{I}$.

Corollary 4.10. The two-sided differential ideal $\mathcal{I}=\left\langle\theta_{1}, \ldots, \theta_{m}\right\rangle_{\mathcal{V}} \subset \mathcal{V}$, generated by $\theta_{1}, \ldots, \theta_{m}$, is a double PVA ideal with respect to the Dirac modified double $\lambda$ bracket $\{[\cdot \lambda \cdot\}\}^{D}$. Hence, the quotient space $\mathcal{V} / \mathcal{I}$ is a (non-local) double PVA, with 2-fold $\lambda$-bracket induced by $\{[-\lambda-\}\}^{D}$, which we call the double Dirac reduction of $\mathcal{V}$ by the constraints $\theta_{1}, \ldots, \theta_{m}$.
Proof. The statement follows by the sesquilinearity conditions and the left and right Leibniz rules for the Dirac modified 2 -fold $\lambda$-bracket.

Example 4.11. Let us consider the non-commutative algebra of differential polynomials $\mathcal{V}=\mathcal{R}_{I}=\mathbb{F}<u_{i}^{(n)} \mid i \in I, n \in \mathbb{Z}_{+}>$(the index set may be either $I=\{i \in \mathbb{Z} \mid i \geq-N\}$ or $I_{-}=\{-N,-N+1, \ldots,-1\}$ ) with the double Poisson vertex algebra structure defined by equation (3.85). Let us denote $C(\lambda)=\left\{\left\{u_{-N \lambda} u_{-N}\right\}_{H}\right.$. By Lemma 3.29 we have that

$$
C(\lambda)=1 \otimes u_{-N}-u_{-N} \otimes 1-(1 \otimes 1) N \lambda
$$

thus $C(\partial) \in \mathcal{V}^{\otimes 2}\left(\left(\partial^{-1}\right)\right)$ is an invertible pseudodifferential operator. Denote by $\mathcal{I}$ the two-sided differential ideal of $\mathcal{V}$ generated by $u_{-N}$. Note that $\mathcal{V} / \mathcal{I} \simeq \mathbb{F}<u_{i}^{(n)} \mid$ $i \in I \backslash\{-N\}, n \in \mathbb{Z}_{+}>$. Using the explicit expression for $C(\lambda)$, the formula for the Dirac reduction given by (4.3), and the expression for the 2-fold $\lambda$-bracket $\left\{\left\{-_{\lambda}-\right\}_{H}\right.$ given by (3.85) and Lemma 3.29, the Dirac reduced 2 -fold $\lambda$-bracket on the quotient space $\mathbb{F}<u_{i}^{(n)} \mid i \in I \backslash\{-N\}, n \in \mathbb{Z}_{+}>$, which we denote by $\left\{\left\{-_{\lambda}-\right\}_{H^{D}}\right.$, becomes

$$
\begin{align*}
& \left\{L(z)_{\lambda} L(w)\right\}_{H^{D}}=L(z) \otimes i_{z}(z-w-\lambda-\partial)^{-1} L(w) \\
& -L(w+\lambda+\partial) \otimes i_{z}(z-w-\lambda-\partial)^{-1} L^{*}(-z+\lambda) \\
& -\frac{1}{N} L(w+\lambda+\partial) \otimes(\lambda+\partial)^{-1} L^{*}(-z+\lambda)-\frac{1}{N}\left((\lambda+\partial)^{-1} L(z)\right) \otimes L(w) \\
& +\frac{1}{N} L(w+\lambda+\partial)(\lambda+\partial)^{-1} L(z) \otimes 1+\frac{1}{N} \otimes\left((\lambda+\partial)^{-1} L^{*}(-z+\lambda)\right) L(w) . \tag{4.43}
\end{align*}
$$

Using equations (4.43) and (3.52), the corresponding $\lambda$-bracket on $(\mathcal{V} / \mathcal{I})_{m}=\mathbb{F}<$ $u_{a b, i}^{n} \mid i \in I \backslash\{-N\}, a, b=1, \ldots, m, n \in \mathbb{Z}_{+}>$is given by the following generating series

$$
\begin{align*}
& \left\{L_{a b}(z)_{\lambda} L_{c d}(w)\right\}_{H^{D}}=L_{c b}(z) i_{z}(z-w-\lambda-\partial)^{-1} L_{a d}(w) \\
& -L_{c b}(w+\lambda+\partial) i_{z}(z-w-\lambda-\partial)^{-1} L_{a d}^{*}(-z+\lambda) \\
& -\frac{1}{N} L_{c b}(w+\lambda+\partial)(\lambda+\partial)^{-1} L_{a d}^{*}(-z+\lambda)-\frac{1}{N} L_{a d}(w)(\lambda+\partial)^{-1} L_{c b}(z) \\
& +\frac{1}{N} \sum_{k=1}^{m} \delta_{a d} L_{c k}(w+\lambda+\partial)(\lambda+\partial)^{-1} L_{k b}(z)  \tag{4.44}\\
& +\frac{1}{N} \sum_{k=1}^{m} \delta_{c b} L_{k d}(w)(\lambda+\partial)^{-1} L_{a k}^{*}(-z+\lambda) .
\end{align*}
$$

This is the same as equation (4.11) in [DSKV14a].
Example 4.12. For $N=2$, we have $\mathcal{R}_{2} / \mathcal{I}=\mathbb{F}\left\langle u^{(n)} \mid n \in \mathbb{Z}_{+}\right\rangle$(where $u$ is the image of $u_{-1}$ ). The two compatible double PVA structures ( $H^{D}, K$ ) given by the
equations (4.44) and (3.86) (using that $L(z)=z^{2}+u$ ) are

$$
\begin{aligned}
& \left\{\left\{u_{\lambda} u\right\}_{H^{D}}=(1 \otimes 1) \frac{\lambda^{3}}{2}+\frac{1}{2}(2 \lambda+\partial)(u \otimes 1+1 \otimes u)-\frac{1}{2} u \otimes(\lambda+\partial)^{-1} u\right. \\
& -\frac{1}{2}\left((\lambda+\partial)^{-1} u\right) \otimes u+\frac{1}{2} u(\lambda+\partial)^{-1} u \otimes 1+\frac{1}{2} \otimes\left((\lambda+\partial)^{-1} u\right) u \\
& \left\{\left\{u_{\lambda} u\right\}_{K}=2(1 \otimes 1) \lambda .\right.
\end{aligned}
$$

The above double Poisson structures are (up to a scalar factor) the same as that appearing in formula (6.15) in [OS98]. Note that the Olver and Sokolov formula is given in terms of operators on $\mathcal{R}_{2} / \mathcal{I}$, rather than as an element of $\left(\mathcal{R}_{2} / \mathcal{I} \otimes \mathcal{R}_{2} / \mathcal{I}\right)[\lambda]$. The rule to recover their formula is the following: we should replace an element $p \otimes q \in \mathcal{R}_{2} / \mathcal{I} \otimes \mathcal{R}_{2} / \mathcal{I}$ by the operator $\mathcal{L}_{p} \circ \mathcal{R}_{q}$, where $\mathcal{L}_{p}$ (respectively $\mathcal{R}_{q}$ ) denotes the left (respectively right) multiplication by $p$ (respectively $q$ ).

Example 4.13. For $N=3$, we have $\mathcal{R}_{3} / \mathcal{I}=\mathbb{F}\left\langle u^{(n)}, v^{(n)} \mid n \in \mathbb{Z}_{+}\right\rangle$(where $u$ is the image of $u_{-2}$ and $v$ is the image of $u_{-1}$ ). The two compatible double PVA structures $\left(H^{D}, K\right)$ given by the equations (4.44) and (3.86) (using that $L(z)=z^{3}+u z+v$ ) are

$$
\begin{aligned}
& \left\{u_{\lambda} u\right\}_{H^{D}}=\frac{1}{3}\left(u(\lambda+\partial)^{-1} u \otimes 1+1 \otimes\left((\lambda+\partial)^{-1} u\right) u\right) \\
& -\frac{1}{3}\left(u \otimes(\lambda+\partial)^{-1} u+\left((\lambda+\partial)^{-1} u\right) \otimes u\right)+1 \otimes v-v \otimes 1 \\
& +(1 \otimes u) \lambda+(\lambda+\partial) u \otimes 1+2(1 \otimes 1) \lambda^{3}, \\
& \left\{\left\{u_{\lambda} v\right\}_{H^{D}}=\frac{1}{3}\left(v(\lambda+\partial)^{-1} u \otimes 1+1 \otimes\left((\lambda+\partial)^{-1} u\right) v\right)\right. \\
& -\frac{1}{3}\left(v \otimes(\lambda+\partial)^{-1} u+\left((\lambda+\partial)^{-1} u\right) \otimes v\right)+\frac{1}{3}\left(u^{2} \otimes 1-u \otimes u\right) \\
& +1 \otimes(2 \lambda+\partial) v+(v \otimes 1) \lambda+\frac{1}{3}(\lambda+\partial)^{2}(u \otimes 1-1 \otimes u)+(u \otimes 1) \lambda^{2} \\
& +(1 \otimes 1) \lambda^{4}, \\
& \left\{v_{\lambda} v\right\}_{H^{D}}=\frac{1}{3}\left(v(\lambda+\partial)^{-1} v \otimes 1+1 \otimes\left((\lambda+\partial)^{-1} v\right) v\right)+\frac{2}{3}(u \otimes v-v \otimes u) \\
& +\frac{1}{3}(u v \otimes 1-1 \otimes u v)-\frac{2}{3}(u \otimes(\lambda+\partial) u)+\frac{2}{3}\left((\lambda+\partial)^{2}(1 \otimes v)-(v \otimes 1) \lambda^{2}\right) \\
& +\frac{1}{3}\left((\lambda+\partial)^{2}(v \otimes 1)-(1 \otimes v) \lambda^{2}\right)-\frac{2}{3}\left((\lambda+\partial)^{3}(1 \otimes u)-(u \otimes 1) \lambda^{3}\right) \\
& -\frac{2}{5}(1 \otimes 1) \lambda^{5}, \quad\left\{\begin{array}{l}
\left\{1 \otimes v_{\lambda}\right. \\
\left\{u_{\lambda} u\right\}_{K}=0,
\end{array}\left\{u_{\lambda} v\right\}_{K}=3(1 \otimes 1) \lambda, \quad\{\otimes 1-1 \otimes u .\right.
\end{aligned}
$$

## 5. Adler-Gelfand-Dickey non-Commutative integrable hierarchies

In this section we want to show how to apply the Lenard-Magri scheme of integrability (see Section 2.5) in order to obtain integrable hierarchies for the compatible pair of PVAs we constructed in Section 3.8.2.

First we state a technical lemma.
Lemma 5.1. Let $\mathcal{V}$ be an arbitrary differential algebra endowed with a 2 -fold $\lambda$ bracket $\left\{\{-\lambda-\}\right.$. Let $L(\partial) \in \mathcal{V}\left(\left(\partial^{-1}\right)\right)$ be a monic pseudodifferential operator of order $N>0$. Then, for all $k \geq 1$, the following identity holds in $\mathcal{V}\left(\left(w^{-1}\right)\right)$ :

$$
\begin{align*}
& \operatorname{Res}_{z} \mathrm{~m}\left(\left\{\left.\left\{L^{\frac{k}{N}}(z)_{\lambda} L(w)\right\}\right|_{\lambda=0}\right)\right. \\
& =\frac{k}{N} \operatorname{Res}_{z} \mathrm{~m}\left(\left\{\left\{L(z+x)_{x} L(w)\right\} \star_{1}\left(\left.\right|_{x=\partial} L^{\frac{k}{N}-1}(z)\right)\right) .\right. \tag{5.1}
\end{align*}
$$

Proof. Since, $L^{\frac{k}{N}}(z)=L^{\frac{1}{N}}(z+\partial) L^{\frac{1}{N}}(z+\partial) \ldots L^{\frac{1}{N}}(z)$ ( $k$ times), we have, by sesquilinearity and the right Leibniz rule,

$$
\begin{align*}
& \left\{\left\{L^{\frac{k}{N}}(z)_{\lambda} L(w)\right\}=\sum_{l=1}^{k}\left(\left.\right|_{y=\partial}\left(L^{*}\right)^{\frac{l-1}{N}}(-z+\lambda)\right)\right.  \tag{5.2}\\
& \star_{1}\left\{\left\{L^{\frac{1}{N}}(z+x)_{\lambda+x+y} L(w)\right\} \star_{1}\left(\left.\right|_{x=\partial} L^{\frac{k-l}{N}}(z)\right) .\right.
\end{align*}
$$

Since $\mathrm{m}\left(a \star_{1} X \star_{1} b\right)=\mathrm{m}\left(X \star_{1} b a\right)$, for all $a, b \in \mathcal{V}$ and $X \in \mathcal{V}^{\otimes 2}$, we have

$$
\begin{align*}
& \mathrm{m}\left\{\left\{L^{\frac{k}{N}}(z)_{\lambda} L(w)\right\}\right\}=\sum_{l=1}^{k} \mathrm{~m}\left(\left\{\left\{L^{\frac{1}{N}}(z+x)_{\lambda+x+y} L(w)\right\}\right\}\right.  \tag{5.3}\\
& \left.\star_{1}\left(\left.\right|_{x=\partial} L^{\frac{k-l}{N}}(z)\right)\left(\left.\right|_{y=\partial}\left(L^{*}\right)^{\frac{l-1}{N}}(-z+\lambda)\right)\right) .
\end{align*}
$$

Taking the residue of both sides of equation (5.3) and using (3.66) with $t=\lambda+y$, we get

$$
\begin{aligned}
& \operatorname{Res}_{z} \mathrm{~m}\left\{\left\{L^{\frac{k}{N}}(z)_{\lambda} L(w)\right\}=\operatorname{Res}_{z} \sum_{l=1}^{k} \mathrm{~m}\left(\left\{\left\{L^{\frac{1}{N}}(z+\lambda+x+y)_{\lambda+x+y} L(w)\right\}\right.\right.\right. \\
& \left.\star_{1}\left(\left.\right|_{x=\partial} L^{\frac{k-l}{N}}(z+\lambda+y)\right)\left(\left.\right|_{y=\partial} ^{L^{\frac{l-1}{N}}}(z)\right)\right),
\end{aligned}
$$

and setting $\lambda=0$ we get

$$
\begin{align*}
& \operatorname{Res}_{z} \mathrm{~m}\left(\left\{\left.\left\{L^{\frac{k}{N}}(z)_{\lambda} L(w)\right\}\right|_{\lambda=0}\right)=\right.  \tag{5.4}\\
& k \operatorname{Res}_{z} \mathrm{~m}\left(\left\{\left\{L^{\frac{1}{N}}(z+x)_{x} L(w)\right\} \star_{1}\left(\left.\right|_{x=\partial} L^{\frac{k-1}{N}}(z)\right)\right) .\right.
\end{align*}
$$

On the other hand, letting $k=N$ in (5.2), we have

$$
\begin{align*}
& \left.\left\{L(z)_{\lambda} L(w)\right\}\right\}=\sum_{l=1}^{N}\left(\left.\right|_{x=\partial} L^{\frac{N-l}{N}}(z)\right)  \tag{5.5}\\
& \star_{1}\left\{\left\{L^{\frac{1}{N}}(z+x)_{\lambda+x+y} L(w)\right\} \star_{1}\left(\left.\right|_{y=\partial}\left(L^{*}\right)^{\frac{l-1}{N}}(-z+\lambda)\right) .\right.
\end{align*}
$$

If we replace in equation (5.5), $z$ by $z+\partial$ and $\lambda$ by $\lambda+\partial$ acting on $L^{\frac{k}{N}-1}(z)$, and we apply the multiplication map $m$ we get

$$
\begin{align*}
& \mathrm{m}\left(\left\{\{ L ( z + x ) _ { \lambda + x } L ( w ) \} \star _ { 1 } \left(\left.\right|_{x=\partial} ^{\left.\left.L^{\frac{k}{N}-1}(z)\right)\right)}\right.\right.\right. \\
& =\sum_{l=1}^{N} \mathrm{~m}\left(\left\{\left\{L^{\frac{1}{N}}(z+x)_{\lambda+x+y} L(w)\right\} \star_{1}\left(\left.\right|_{x=\partial^{L}} L^{\frac{k-l}{N}}(z)\right)\left(\left.\right|_{y=\partial}\left(L^{*}\right)^{\frac{l-1}{N}}(-z+\lambda)\right)\right) .\right. \tag{5.6}
\end{align*}
$$

Taking, as before, residues of both sides of equation (5.6) and using (3.66) with $t=\lambda+y$, we get, after setting $\lambda=0$,

$$
\begin{align*}
& \operatorname{Res}_{z} \mathrm{~m}\left(\left\{\left\{L(z+x)_{x} L(w)\right\} \star_{1}\left(\left.\right|_{x=\partial} L^{\frac{k}{N}-1}(z)\right)\right)\right. \\
& =N \operatorname{Res}_{z} \mathrm{~m}\left(\left\{\left\{L^{\frac{1}{N}}(z+x)_{x} L(w)\right\}\left(\left.\right|_{x=\partial} L^{\frac{k-1}{N}}(z)\right)\right) .\right. \tag{5.7}
\end{align*}
$$

Equation (5.1) follows from equations (5.4) and (5.7).
As in Section 3.8.2, let $\mathcal{V}$ be the non-commutative algebra of differential polynomials in the variables $u_{i}, i \in I$, namely

$$
\mathcal{V}=\mathcal{R}_{I}=\mathbb{F}\left\langle u_{i}^{(n)} \mid i \in I, n \in \mathbb{Z}_{+}\right\rangle
$$

(The index set $I$ may be either $I=\{i \in \mathbb{Z} \mid i \geq-N\}$ or $I_{-}=\{-N,-N+$ $1, \ldots,-1\})$. Let us collect the differential generators of $\mathcal{V}$ into the generating series

$$
L(z)=z^{N}+\sum_{i \in I} u_{i} z^{-i-1} \in \mathcal{V}\left(\left(z^{-1}\right)\right)
$$

and let us consider the compatible double PVA structures $(H, K)$ on $\mathcal{V}$ defined by Corollary 3.27. Let, for $k \geq 1$,

$$
\begin{equation*}
h_{k}=\frac{N}{k} \operatorname{Res}_{z} L^{\frac{k}{N}}(z) \in \mathcal{V} \tag{5.8}
\end{equation*}
$$

where $L^{\frac{1}{N}}(\partial) \in \mathcal{V}\left(\left(\partial^{-1}\right)\right)$ is uniquely defined (see Proposition 1.1 in [DSKV14a]).
Lemma 5.2. For the compatible pair of double PVA structures $(H, K)$ on $\mathcal{V}$, we have for $k \geq 1$ :
(a) $\mathrm{m}\left\{\left.\left\{h_{k \lambda} L(w)\right\}_{H}\right|_{\lambda=0}=L^{\frac{k}{N}}(w+\partial)_{+} L(w)-L(w+\partial) L^{\frac{k}{N}}(w)_{+}\right.$;
(b) $\left.\mathrm{m}\left\{h_{k \lambda} L(w)\right\}\right\}_{\lambda=0}=L^{\frac{k}{N}-1}(w+\partial)_{+} L(w)-L(w+\partial) L^{\frac{k}{N}-1}(w)_{+}$.

Proof. By Lemma 5.1 and equation (3.85) we have

$$
\begin{align*}
& \mathrm{m}\left\{\left.\left\{h_{k \lambda} L(w)\right\}_{H}\right|_{\lambda=0}=\operatorname{Res}_{z} L^{\frac{k}{N}}(z) i_{z}(z-w-\partial)^{-1} L(w)\right.  \tag{5.9}\\
& -L(w+\partial) \operatorname{Res}_{z} L^{\frac{k}{N}-1}(z) i_{z}(z-w-\partial)^{-1} L^{*}(-z)
\end{align*}
$$

Using (3.65), we have

$$
\begin{equation*}
\operatorname{Res}_{z} L^{\frac{k}{N}}(z) i_{z}(z-w-\partial)^{-1}=L^{\frac{k}{N}}(w+\partial)_{+} \tag{5.10}
\end{equation*}
$$

while, using (3.66) and equation (3.65), we have

$$
\begin{align*}
& \operatorname{Res}_{z} L^{\frac{k}{N}-1}(z) i_{z}(z-w-\partial)^{-1} L^{*}(-z)=\operatorname{Res}_{z} L^{\frac{k}{N}-1}(z+\partial) i_{z}(z-w)^{-1} L(z) \\
& =\operatorname{Res}_{z} L^{\frac{k}{N}}(z) i_{z}(z-w)^{-1}=L^{\frac{k}{N}}(w)_{+} \tag{5.11}
\end{align*}
$$

Combining equations (5.9), (5.10) and (5.11), we get part (a). Similarly, for part (b), we use Lemma 5.1 and equation (3.86) to get

$$
\begin{align*}
& \left.\mathrm{m}\left\{h_{k \lambda} L(w)\right\}_{K}\right|_{\lambda=0}=\operatorname{Res}_{z} i_{z}(z-w)^{-1}(L(z+\partial)-L(w+\partial)) L^{\frac{k}{N}-1}(z) \\
& +\operatorname{Res}_{z} L^{\frac{k}{N}-1}(z) i_{z}(z-w-\partial)^{-1}\left(L(w)-L^{*}(-z)\right) \tag{5.12}
\end{align*}
$$

By equations (3.65) and (3.66), we have

$$
\begin{align*}
& \operatorname{Res}_{z} i_{z}(z-w)^{-1} L(z+\partial) L^{\frac{k}{N}-1}(z)=L^{\frac{k}{N}}(w)_{+} \\
& =\operatorname{Res}_{z} L^{\frac{k}{N}-1}(z) i_{z}(z-w-\partial)^{-1} L^{*}(-z) \tag{5.13}
\end{align*}
$$

Moreover, by equation (3.65) we also have

$$
\begin{equation*}
\operatorname{Res}_{z} i_{z}(z-w)^{-1} L(w+\partial) L^{\frac{k}{N}-1}(z)=L(w+\partial) L^{\frac{k}{N}-1}(w)_{+}, \tag{5.14}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Res}_{z} L^{\frac{k}{N}-1}(z) i_{z}(z-w-\partial)^{-1} L(w)=L^{\frac{k}{N}-1}(w+\partial)_{+} L(w) \tag{5.15}
\end{equation*}
$$

Combining equations (5.12), (5.13), (5.14), and (5.15), we get the claim.
The following Theorem says that the Lenard-Magri scheme of integrability works for the bi-Poisson structure ( $H, K$ ), see Remark 3.14:

Theorem 5.3. (a) For every $\varepsilon \in\{1, \ldots, N\}$, we have

$$
\begin{equation*}
\left\{\int h_{\varepsilon}, u\right\}_{K}=0, \quad \text { for all } u \in \mathcal{V} \tag{5.16}
\end{equation*}
$$

(b) For every $k \geq 1$, we have the Lenard-Magri recursion

$$
\begin{equation*}
\left\{\int h_{k}, u\right\}_{H}=\left\{\int h_{k+N}, u\right\}_{K}, \quad \text { for all } u \in \mathcal{V} \tag{5.17}
\end{equation*}
$$

Proof. For $1 \leq \varepsilon<N$, we have $L^{\frac{\varepsilon}{N}-1}(w)_{+}=0$, and therefore, recalling (3.19), equation (5.16) holds by Lemma 5.2(b). Moreover, $\left.\mathrm{m}\left\{h_{N \lambda} L(w)\right\}_{K}\right|_{\lambda=0}=L(w)-$ $L(w+\partial) \cdot 1=0$. This proves part (a). For part (b), by Lemma 5.2, the recursion (5.17) holds for $u=L(w)$, the generating series of the generators of $\mathcal{V}$. Hence, (5.17) holds for all $u \in \mathcal{V}$ by the left Leibniz rule.

Remark 5.4. It follows from Lemma 5.2 that the Hamiltonian equation corresponding to the Hamiltonian functional $\int h_{k}, k \geq 1$, can be written as (in terms of generating series)

$$
\begin{equation*}
\frac{d L(w)}{d t_{k}}=\left[\left(L^{\frac{k}{N}}\right)_{+}, L\right](w) \tag{5.18}
\end{equation*}
$$

where on the RHS we have to take the symbol of the usual commutator of pseudodifferential operators. This equation is the symbol of the usual Lax pair representation of the AGD hierarchies of Hamiltonian equations.
Example 5.5. On $\mathbb{F}\left\langle u_{i}^{(n)} \mid i, n \in \mathbb{Z}_{+}\right\rangle$, we have $L(\partial)=\partial+\sum_{i \in \mathbb{Z}_{+}} u_{i} \partial^{-i-1}$. By an explicit computation we get

$$
\begin{aligned}
& L^{2}(\partial)=\partial^{2}+2 u_{0}+\left(2 u_{1}-u_{0}^{\prime}\right) \partial^{-1}+\left(2 u_{2}+u_{1}^{\prime}+u_{0}^{2}\right) \partial^{-2}+\ldots, \\
& L^{3}(\partial)=\partial^{3}+3 u_{0} \partial+3\left(u_{1}+u_{0}^{\prime}\right)+\left(3 u_{2}+3 u_{1}^{\prime}+3 u_{0}^{2}+u_{0}^{\prime \prime}\right) \partial^{-2}+\ldots .
\end{aligned}
$$

Hence, the first few integrals of motion are

$$
\int h_{1}=\int \operatorname{tr}\left(u_{0}\right), \int h_{2}=\int \operatorname{tr}\left(u_{1}\right), \int h_{3}=\int \operatorname{tr}\left(u_{2}+u_{0}^{2}\right), \ldots
$$

To find the corresponding bi-Hamiltonian equations, we use Lemma 5.2. We have $L(w)_{+}=w, L^{2}(w)_{+}=w^{2}+2 u_{0}, L^{3}(w)_{+}=w^{3}+3 u_{0} w+3\left(u_{0}^{\prime}+u_{1}\right)$. Hence,

$$
\begin{align*}
\frac{d L(w)}{d t_{1}}=\partial L(w), & \frac{d L(w)}{d t_{2}}=\partial^{2} L(w)+2 w \partial L(w)+2\left(u_{0} L(w)-L(w+\partial) u_{0}\right) \\
\frac{d L(w)}{d t_{3}}=\partial^{3} L(w) & +3 w \partial^{2} L(w)+3 w^{2} \partial L(w)+3 u_{0} \partial L(w) \\
& +3\left(\left((w+\partial) u_{0}+u_{1}\right) L(w)-L(w+\partial)\left((w+\partial) u_{0}+u_{1}\right)\right) \ldots \tag{5.19}
\end{align*}
$$

Consider the first two equations in the second system of the hierarchy (5.19), and the first equation in the third system of (5.19). Namely,

$$
\left\{\begin{array}{l}
\frac{d u_{0}}{d t_{2}}=u_{0}^{\prime \prime}+2 u_{1}^{\prime} \\
\frac{d u_{1}}{d t_{2}}=u_{1}^{\prime \prime}+2 u_{2}^{\prime}+2 u_{0} u_{0}^{\prime}+2 u_{0} u_{1}-2 u_{1} u_{0} \\
\frac{d u_{0}}{d t_{3}}=u_{0}^{\prime \prime \prime}+3 u_{1}^{\prime \prime}+3 u_{2}^{\prime}+3 u_{0} u_{0}^{\prime}+3 u_{0}^{\prime} u_{0}
\end{array}\right.
$$

We can eliminate the variable $u_{2}$ from this system. After relabeling $t_{2}=y, t_{3}=t$, $u=2 u_{0}$ and $w=4 u_{1}+2 u_{0}^{\prime}$, we get the system

$$
\left\{\begin{array}{l}
u_{y}=w^{\prime}  \tag{5.20}\\
3 w_{y}=4 u_{t}-u^{\prime \prime \prime}-3\left(u^{2}\right)^{\prime}+3[u, w]
\end{array}\right.
$$

We call the system (5.20) the non-commutative Kadomtsev-Petviashvili (KP) equation.

Example 5.6. On $\mathbb{F}\left\langle u^{(n)} \mid n \in \mathbb{Z}_{+}\right\rangle$, we have $L(\partial)=\partial^{2}+u$. By an explicit computation we get

$$
\begin{aligned}
L^{\frac{1}{2}}(\partial) & =\partial+\frac{u}{2} \partial^{-1}-\frac{u^{\prime}}{4} \partial^{-2}+\frac{1}{8}\left(u^{\prime \prime}-u^{2}\right) \partial^{-3} \ldots \\
L^{\frac{3}{2}}(\partial) & =\partial^{3}+\frac{3}{2} u \partial+\frac{3}{4} u^{\prime}+\frac{1}{8}\left(3 u^{2}+u^{\prime \prime}\right) \partial^{-1}+\ldots
\end{aligned}
$$

Hence, the first two integrals of motion are $\int h_{1}=\int \operatorname{tr}(u)$ and $\int h_{3}=\int \frac{1}{4} \operatorname{tr}\left(u^{2}\right)$. To find the corresponding bi-Hamiltonian equations, we use Lemma 5.2. We have $L^{\frac{1}{2}}(w)_{+}=w$ and $L^{\frac{3}{2}}(w)_{+}=w^{3}+\frac{3}{2} u w+\frac{3}{4} u^{\prime}$. Hence, $\frac{d u}{d t_{1}}=u^{\prime}$ and

$$
\begin{equation*}
\frac{d u}{d t_{3}}=\frac{1}{4}\left(u^{\prime \prime \prime}+3 u u^{\prime}+3 u^{\prime} u\right) \tag{5.21}
\end{equation*}
$$

The equation (5.21) is the non-commutative KdV equation.
Example 5.7. On $\mathbb{F}\left\langle u^{(n)}, v^{(n)} \mid n \in \mathbb{Z}_{+}\right\rangle$, we have $L(\partial)=\partial^{3}+u \partial+v$. By an explicit computation we get

$$
\begin{aligned}
& L^{\frac{1}{3}} \\
&(\partial)=\partial+\frac{u}{3} \partial^{-1}+\frac{1}{3}\left(v-u^{\prime}\right) \partial^{-2}+\frac{1}{9}\left(2 u^{\prime \prime}-u^{2}-3 v^{\prime}\right) \partial^{-3} \ldots \\
& L^{\frac{2}{3}}(\partial)=\partial^{2}+\frac{2}{3} u+\frac{1}{3}\left(2 v-u^{\prime}\right) \partial^{-1}+\frac{1}{9}\left(3 u^{\prime \prime}-u^{2}-3 v^{\prime}\right) \partial^{-2}+\ldots
\end{aligned}
$$

Hence, the first two integrals of motion are $\int h_{1}=\int \operatorname{tr}(u)$ and $\int h_{2}=\int \operatorname{tr}(v)$. To find the corresponding bi-Hamiltonian equations, we use Lemma 5.2. We have $L^{\frac{1}{3}}(w)_{+}=w$ and $L^{\frac{2}{3}}(w)_{+}=w^{2}+\frac{2}{3} u$. Hence, the corresponding Hamiltonian equations are $\frac{d u}{d t_{1}}=u^{\prime}, \frac{d v}{d t_{1}}=v^{\prime}$ and

$$
\left\{\begin{array}{l}
\frac{d u}{d t_{2}}=2 v^{\prime}-u^{\prime \prime}  \tag{5.22}\\
\frac{d v}{d t_{2}}=v^{\prime \prime}-\frac{2}{3} u^{\prime \prime \prime}-\frac{2}{3} u u^{\prime}+\frac{2}{3}[u, v] .
\end{array}\right.
$$

We call the equation (5.22) the non-commutative Boussinesq equation (since it reduces to the classical Boussinesq equation if $[u, v]=0$ ).

## References

[BDSK09] Barakat A., De Sole A., Kac V.G., Poisson vertex algebras in the theory of Hamiltonian equations, Jpn. J. Math. 4 (2009), 141-252.
[CBEG07] Crawley-Boevey W., Etingof P., Ginzburg V., Non-commutative geometry and quiver algebras, Adv. Math. 209 (2007), 274-336.
[DSK13] De Sole A., Kac V.G., Non-local Poisson structures and applications to the theory of integrable systems, Jpn. J. Math. 8 (2013), no. 2, 233-347.
[DSKV14a] De Sole A., Kac V.G., Valeri D., Adler-Gelfand-Dickey approach to classical $\mathcal{W}$ algebras within the theory of Poisson vertex algebras, IMRN 2015; doi: 10.1093/imrn/rnv017
[DSKV14b] De Sole A., Kac V.G., Valeri D., Dirac reduction for Poisson vertex algebras, Comm. Math. Phys. 331 (2014), no. 3, 1155-1190.
[DF92] Dorfman I.Y., Fokas A.S., Hamiltonian theory over non-commutative rings and integrability in multidimensions, J. Math. Phys. 33 (1992), no.7, 2504-2514.
[FT86] Faddeev L.D., Takhtajan L.A., Hamiltonian approach in soliton theory, Nauka, 1986.
[Kac98] Kac V.G., Vertex algebras for beginners, University Lecture Series 10 (1998). Second edition. American Mathematical Society, Providence, RI.
[Kup00] Kupershmidt B.A., KP or mKP. Noncommutative mathematics of Lagrangian, Hamiltonian, and integrable systems, Mathematical Surveys and Monographs 78. American Mathematical Society, Providence, RI, 2000.
[Mag78] Magri F., A simple model of the integrable Hamiltonian equation, J. Math. Phys. 19 (1978), no. 5, 1156-1162.
[MS00] Mikhailov A.V., Sokolov V.V., Integrable ODEs on associative algebras, Comm. Math. Phys. 211 (2000), 231-251.
[ORS13] Odesskii A., Rubtsov V., Sokolov V., Double Poisson brackets on free associative algebras, in Noncommutative birational geometry, representations and combinatorics, Contemp. Math. 592 (2013), 225-239, Amer. Math. Soc., Providence, RI.
[Olv93] Olver P.J., Applications of Lie groups to differential equations, Second edition, Graduate Texts in Mathematics, 107, Springer-Verlag, New York, 1993.
[OS98] Olver P.J., Sokolov V.V., Integrable evolution equations on associative algebras, Comm. Math. Phys. 193 (1998), no. 2, 245-268.
[VdB08] Van den Bergh M., Double Poisson algebras, Trans. Amer. Math. Soc. 360 (2008), no. 11, 5711-5769.

Dipartimento di Matematica, Sapienza Università di Roma, P.le Aldo Moro 2, 00185 Rome, Italy.

E-mail address: desole@mat.uniroma1.it
Department of Mathematics, Mit, 77 Massachusetts Ave, Cambridge, MA 02139, USA.

E-mail address: kac@math.mit.edu

Yau Mathematical Sciences Center, Tsinghua University, 100084 Beijing, China. E-mail address: daniele@math.tsinghua.edu.cn

