Homogenization of elliptic problems involving interfaces and singular data

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Abstract

We prove existence and homogenization results for a family of elliptic problems involving interfaces and a singular lower order term. These problems model heat or electrical conduction in composite media.

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1. Introduction

We consider a family (depending on a small parameter $\varepsilon > 0$ and on a parameter $\alpha \geq -1$) of elliptic problems involving a singular lower order term and representing the Euler equations of energy functionals, which describe the equilibrium for the heat conduction in composite materials with two finely mixed phases having a microscopic periodic structure (for details on the related physical models see for instance [18, 19, 24] and the reference quoted therein). The same kind of energies can be also useful to study the electrical conduction in biological tissues (see for instance [6]–[9], where the related evolutive problems without singular source are considered). Similar models in the framework of electrical or thermal conduction in composite materials are treated in [5, 10, 11].

We assume that the domain $\Omega \subseteq \mathbb{R}^N$, which models the region occupied by the material, is made by two phases separated by an active interface. The parameter ε, which will be sent to 0, is related to the period of the microstructure (for more details on the geometrical setting, see the next section). The mathematical description of our model in the microscopic setting is given by two non-homogeneous elliptic equations in each phase, complemented with the assumption that the flux of the solution u_ε is continuous across the interface and proportional to the jump of u_ε. Moreover, we assume that in both phases the rate of heat generation is given by a singular source of the form $\frac{f}{u_\varepsilon^\theta}$, with $0 < \theta < 1$ and $f \in L^{\frac{2}{1+\theta}}$. The restriction on θ is required in order to get suitable a priori estimates, although the source term is singular.

Our main results concern the study of the limit (as $\varepsilon \to 0$) of the solutions u_ε, focusing our attention on the differences of the limit equations (characterizing the properties of the material from the macroscopic point of view) with respect to the parameter α (appearing in the interface condition). We confine our study to the case $\alpha \geq -1$, where a suitable Poincare's inequality for general geometries is available.

In order to get the homogenized problem, we use the *two-scale convergence* technique (see for instance [2, 3, 4, 26]). In particular, we obtain four different behaviours:

$$\alpha > 1, \quad \alpha = 1, \quad \alpha \in (-1, 1), \quad \alpha = -1 \,.$$

In the first three cases, we get in the limit a second order elliptic equation with singular source, whose homogenized matrix is different in each case. Instead, for $\alpha = -1$, we get a bidomain governed by a system of two coupled elliptic equations. Moreover, we remark that, when $\alpha > 1$ or $\alpha \in (-1, 1)$, the homogenized problem loses memory of the physical properties of the interfaces, thus suggesting that the main models are those with $\alpha = \pm 1$.

In order to handle with the singular term, we follow some ideas already present in [18] and in some previous papers (see, in particular, [20]), but our different geometrical setting gives rise to technical difficulties due to the interaction between jumps and singularities, which can be overcome by means of a new strategy (see, for instance, the proof of theorem 4.1).

Another crucial point, in order to get the homogenized problem, is the proof of the strict positivity of the limit solution, which is a non trivial result, at least when $\alpha = -1$. In this case, our geometry does not allow to follow the arguments in [18], but it requires a new idea (see Lemma 5.7). To get this result, it should be possible to use the so-called *two-scale decomposition* introduced in [29] in order to prove the lower semicontinuity of a suitable functional, which implies as a by-product, the requested positivity of the limit solution. However, due to the special structure of our model, we prefer to follow a more direct approach, appealing to the *unfolding* technique introduced by Cioranescu, Damlamian and Griso in 2002 (see for instance [16, 17]).

The paper is organized as follows: in Section 2 we recall notations and preliminary results and we set our problems; in Section 3 we state the necessary estimates for the compactness results; in Section 4 we state and prove our main homogenization theorems. Finally, the paper contains an Appendix divided into two parts: in the first one, we prove the well-posedness of our microscopic problem (10), while in the second one we recall some tools from the unfolding technique and we prove the strict positivity of the homogenized solution for $\alpha = -1$.

2. Preliminaries

2.1. The geometrical setting

For $N \geq 3$, let $\Omega \subset \mathbb{R}^N$ be an open, connected and bounded set. Let E be a periodic open subset of $\mathbb{R}^N$, so that $E + z = E$ for all $z \in \mathbb{Z}^N$. For all $\varepsilon > 0$ we define the two open sets

$$\Omega_1^\varepsilon = \Omega \cap \varepsilon E, \qquad \Omega_2^\varepsilon = \Omega \setminus \overline{\varepsilon E}.$$

We assume that Ω and E have Lipschitz continuous boundary and that Ω_2^ε is connected. We set

$$\Gamma^\varepsilon = \partial\Omega_1^\varepsilon \cap \Omega = \partial\Omega_2^\varepsilon \cap \Omega,$$

so that we have $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$. We also employ the notation $Y = (0,1)^N$, and $E_1 = E \cap Y$, $E_2 = Y \setminus \overline{E}$, $\Gamma = \partial E \cap \overline{Y}$ and we assume that $|\Gamma \cap \partial Y|_{N-1} = 0$ and that E_2 is connected.

In the following, we will consider two different situations.

- We will name the *connected/disconnected geometry* the case where $\Gamma \cap \partial Y = \emptyset$, and in this case we will assume that $\mathrm{dist}(\Gamma^\varepsilon, \partial\Omega) \geq \gamma_0 \varepsilon$, for a suitable $\gamma_0 > 0$. To this purpose, for each ε, we are ready to remove the inclusions in all the cells which are not completely contained in Ω. In this case, the sets Ω_1^ε and Ω_2^ε are usually called the *inner* and the *outer domain*, respectively (see Figure 1).

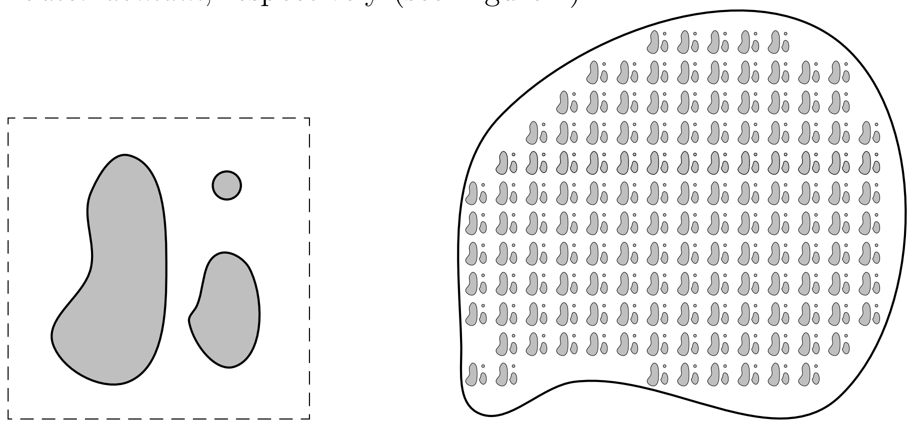

Figure 1: *Left*: the periodic cell Y. E_1 is the shaded region and E_2 is the white region. *Right*: the region Ω.

- We will name the *connected/connected geometry* the case where E_1, E_2, Ω_1^ε, Ω_2^ε are connected. In this case, we will assume that both ∂E_1 and ∂E_2 have Lipschitz regularity and, moreover, we will need that Ω, E_1 and E_2 are such that $\partial\Omega_1^\varepsilon$ and $\partial\Omega_2^\varepsilon$ are still Lipschitz regular at each ε-step, at least for a suitable choice of a subsequence ε_n tending to zero. For instance, this is the case when Ω is a rectangular domain with $\varepsilon_n = |\Omega|/n$; indeed, this choice implies that Ω always contains an integer number of ε-cells. In the following, that regularity assumption will be always implicit; however, we will omit the subindex n, even in the case in which it should be necessary.

We denote by ν_ε the normal unit vector to Γ^ε pointing into Ω_2^ε and by ν the normal unit vector to Γ pointing into E_2.

For a function u defined on Ω, we denote by $u^{(1)}$ and $u^{(2)}$ the restriction of u to Ω_1^ε and Ω_2^ε, respectively. On Γ^ε we define

$$[u] := u^{(2)} - u^{(1)}\,.$$

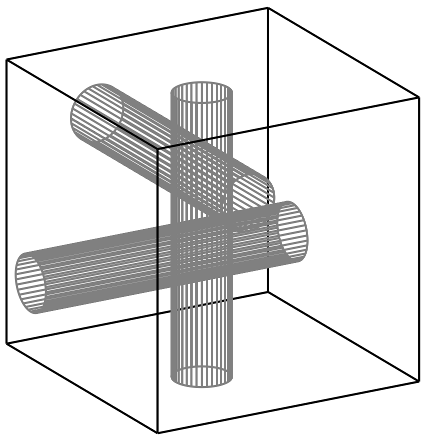

Figure 2: the periodic cell Y. E_1 is the shaded region and E_2 is the white region.

We use the same notation for functions defined in the unit cell Y, where $u^{(1)}$ and $u^{(2)}$ stands here for the restriction of u to E_1 and E_2, respectively.

In the following x and y will denote the macro and micro-variable, respectively, so that, for a function $u(x, y)$ defined on $\Omega \times Y$, we denote by $\nabla_x u$, $\nabla_y u$ and $\mathrm{div}_x u$, $\mathrm{div}_y u$ the gradient and the divergence of u computed with respect to the variables x and y, respectively. When no confusion is possible, we write ∇u for $\nabla_x u$ and $\mathrm{div} u$ for $\mathrm{div}_x u$.

Given $\xi, \eta \in \mathbb{R}^N$, $\xi \otimes \eta$ will denote the matrix whose entries are $(\xi \otimes \eta)_{ij} = \xi_i \eta_j$. We denote by $\mathbf{e}_1, \ldots, \mathbf{e}_N$ the euclidian basis of $\mathbb{R}^N$. In the sequel C will denote a positive constant, which may vary from line to line.

2.2. Functional spaces

We set

$$V_0^\varepsilon(\Omega) = \{u = (u^{(1)}, u^{(2)}), u^{(1)} \in H^1(\Omega_1^\varepsilon),\ u^{(2)} \in H^1(\Omega_2^\varepsilon),\ u = 0 \text{ on } \partial\Omega\},$$

and

$$\mathfrak{L}_0^\varepsilon(\Omega) = \{u = (u^{(1)}, u^{(2)}), u^{(1)} \in \mathrm{Lip}(\overline{\Omega_1^\varepsilon}), u^{(2)} \in \mathrm{Lip}(\overline{\Omega_2^\varepsilon}),\ u = 0 \text{ on } \partial\Omega\}.$$

Analogously, we define the following space

$$V_\#(Y) = \{v = (v^{(1)}, v^{(2)}),\ v \text{ is } Y\text{-periodic},\ v^{(1)} \in H^1_\#(E_1),\ v^{(2)} \in H^1_\#(E_2)\},$$

and

$$\mathfrak{L}_\#(Y) = \{v = (v^{(1)}, v^{(2)}),\ v \text{ is } Y\text{-periodic},\ v^{(1)} \in \mathrm{Lip}(\overline{E_1}), v^{(2)} \in \mathrm{Lip}(\overline{E_2})\}.$$

Here Y is identified with the flat torus in $\mathbb{R}^N$.

Remark 2.1. *Notice that, if $u \in V_0^\varepsilon(\Omega)$, then $[u] \in L^2(\Gamma^\varepsilon)$ and, analogously, if $v \in V_\#(Y)$, then $[v] \in L^2(\Gamma)$.*

We recall the following Poincaré's inequality (see [23, Lemma 6]).

Theorem 2.2. *There exists $C > 0$, independent of ε, such that*

$$\int_\Omega v^2\, \mathrm{d}x \le C \left\{ \int_\Omega |\nabla v|^2\, \mathrm{d}x + \varepsilon \int_{\Gamma^\varepsilon} [v]^2\, \mathrm{d}\sigma \right\} \qquad \forall v \in V_0^\varepsilon(\Omega). \tag{1}$$

Remark 2.3. *Notice that (1) holds in this form (i.e., with ε in front of the integral over the interface Γ^ε), since we have assumed that Ω_2^ε is connected.*

We also recall the following technical lemma proved in [2, Lemma 2.10], which will be useful in the sequel.

Lemma 2.4. *For any vector function $\Phi \in L^2(\Omega; \mathbb{R}^N)$, there exists a vector function $\Psi \in L^2(\Omega; H^1_\#(E_2; \mathbb{R}^N))$ such that*

$$\begin{aligned} &div_y \Psi(x,y) = 0\,, && \text{in } E_2\,;\\ &\Psi(x,y) = 0\,, && \text{on } \Gamma\,;\\ &\int_{E_2} \Psi(x,y)\, \mathrm{d}y = \Phi(x)\,. && \end{aligned} \tag{2}$$

Moreover, $||\Psi||_{L^2(\Omega; H^1_\#(E_2;\mathbb{R}^N))} \le C||\Phi||_{L^2(\Omega;\mathbb{R}^N)}$.

Clearly, in the connected/connected case, an analogous result holds with E_2 replaced by E_1.

2.3. Two-scale convergence

We recall some basic definitions and properties of the *two-scale convergence* technique. For more details see, for instance, [2, 3, 4, 9, 22] and the references therein.

Definition 2.5. *A function $\varphi \in L^2(\Omega \times Y)$ is said an admissible test function if φ is Y-periodic with respect to the second variable and satisfies:*

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi^2\left(x,\frac{x}{\varepsilon}\right)\,\mathrm{d}x = \int_{\Omega\times Y}\varphi^2(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Remark 2.6. *If $\varphi \in \mathcal{C}^0(\overline{\Omega};\mathcal{C}^0_\#(Y))$ or, more in general, if $\varphi \in L^2(\Omega;\mathcal{C}^0_\#(Y))$ or $\varphi \in L^2_\#(Y;\mathcal{C}^0(\overline{\Omega}))$, then φ is an admissible test function. Moreover, if $\varphi(x,y) = \varphi_1(x)\varphi_2(y)$ with $\varphi_1 \in L^2(\Omega)$ and $\varphi_2 \in L^2_\#(Y)$, then φ is an admissible test function.*

Definition 2.7 (Two-scale convergence)**.** *For $\{u_\varepsilon\} \subset L^2(\Omega)$ and $u_0 \in L^2(\Omega\times Y)$, we say that $\{u_\varepsilon\}$ two-scale converges to u_0 in $L^2(\Omega\times Y)$ as $\varepsilon\to 0$ (and we write $u_\varepsilon \stackrel{2-sc}{\longrightarrow} u_0$) if*

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\varphi\left(x,\frac{x}{\varepsilon}\right)\,\mathrm{d}x = \int_{\Omega\times Y} u_0(x,y)\varphi(x,y)\,\mathrm{d}x\,\mathrm{d}y\,,$$

for every admissible test function φ.

Definition 2.8 (Two-scale convergence on surfaces)**.** *For $\{w_\varepsilon\} \subset L^2(\Gamma^\varepsilon)$ and $w_0 \in L^2(\Omega\times\Gamma)$, we say that $\{w_\varepsilon\}$ two-scale converges to w_0 in $L^2(\Omega\times\Gamma)$ as $\varepsilon\to 0$ (and, as above, we use the notation $w_\varepsilon \stackrel{2-sc}{\longrightarrow} w_0$) if*

$$\lim_{\varepsilon\to 0}\varepsilon\int_{\Gamma^\varepsilon} w_\varepsilon(x)\varphi\left(x,\frac{x}{\varepsilon}\right)\,\mathrm{d}\sigma = \int_{\Omega\times\Gamma} w_0(x,y)\varphi(x,y)\,\mathrm{d}x\;\mathrm{d}\sigma(y)\,,$$

for every $\varphi \in \mathcal{C}^0(\overline{\Omega};\mathcal{C}^0_\#(Y))$.

Theorem 2.9. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$. Then there exist a subsequence of $\{u_\varepsilon\}$ (still denoted by $\{u_\varepsilon\}$) and a function $u_0 \in L^2(\Omega\times Y)$ such that $u_\varepsilon \stackrel{2-sc}{\longrightarrow} u_0$ in $L^2(\Omega\times Y)$.*

Proposition 2.10. *Let $\{u_\varepsilon\}$ be a sequence of functions in $L^2(\Omega)$, which two-scale converges to a limit $u_0(x,y) \in L^2(\Omega\times Y)$. Then, u_ε converges weakly to $u(x) = \int_Y u_0(x,y)dy$ in $L^2(\Omega)$. Furthermore, we have*

$$\liminf_{\varepsilon\to 0}||u_\varepsilon||_{L^2(\Omega)} \geq ||u_0||_{L^2(\Omega\times Y)} \geq ||u||_{L^2(\Omega)}.$$

Theorem 2.11. *Let $\{w_\varepsilon\} \subset L^2(\Gamma^\varepsilon)$. Assume that there exists $C > 0$, independent of ε, such that*

$$\varepsilon \int_{\Gamma^\varepsilon} |w_\varepsilon|^2 \,\mathrm{d}\sigma \le C\,, \qquad \forall \varepsilon > 0.$$

Then, there exist a subsequence of $\{w_\varepsilon\}$ (still denoted by $\{w_\varepsilon\}$) and a function $w_0 \in L^2(\Omega \times \Gamma)$ such that $w_\varepsilon \xrightarrow{2-sc} w_0$ in $L^2(\Omega \times \Gamma)$.

Theorem 2.12. *Let $\{u_\varepsilon\} \subset V_0^\varepsilon(\Omega)$. Assume that there exists $C > 0$ (independent of ε) such that*

$$\int_\Omega |u_\varepsilon|^2 \,\mathrm{d}x + \int_\Omega |\nabla u_\varepsilon|^2 \,\mathrm{d}x \le C\,, \qquad \forall \varepsilon > 0. \tag{3}$$

Then, there exists $u \in L^2(\Omega; V_\#(Y))$, whose restrictions to E_1 and E_2 satisfy

$$u(x,y) = u^{(1)}(x) \in H^1(\Omega)\,, \quad \text{in } E_1\,, \qquad u(x,y) = u^{(2)}(x) \in H^1(\Omega)\,, \quad \text{in } E_2\,;$$

and there exists $u_1 \in L^2(\Omega; V_\#(Y))$ such that, up to subsequence, as $\varepsilon \to 0$ we have

$$\chi_{\Omega_1^\varepsilon} u_\varepsilon^{(1)} \xrightarrow{2-sc} \chi_{E_1} u^{(1)} \quad \text{and} \quad \chi_{\Omega_2^\varepsilon} u_\varepsilon^{(2)} \xrightarrow{2-sc} \chi_{E_2} u^{(2)}\,, \qquad \text{in } L^2(\Omega \times Y)\,; \tag{4}$$

$$\chi_{\Omega_1^\varepsilon} \nabla u_\varepsilon^{(1)} \xrightarrow{2-sc} \chi_{E_1} \left(\nabla u^{(1)} + \nabla_y u_1^{(1)} \right), \qquad \text{in } L^2(\Omega \times Y)\,; \tag{5}$$

$$\chi_{\Omega_2^\varepsilon} \nabla u_\varepsilon^{(2)} \xrightarrow{2-sc} \chi_{E_2} \left(\nabla u^{(2)} + \nabla_y u_1^{(2)} \right), \qquad \text{in } L^2(\Omega \times Y)\,. \tag{6}$$

where, for $\mathcal{O} \subseteq \mathbb{R}^N$, $\chi_{\mathcal{O}}$ denotes the characteristic function of $\mathcal{O}$. Moreover, we have also

$$\varepsilon \int_\Gamma [u_\varepsilon]^2 \,\mathrm{d}\sigma \le C\,, \qquad \forall \varepsilon > 0\,,$$

with C independent of ε, and

$$[u_\varepsilon] \xrightarrow{2-sc} [u]\,, \qquad \text{in } L^2(\Omega \times \Gamma)\,. \tag{7}$$

We refer to [2, Theorem 2.9] (see also [3, Theorem 4.6]) for the proof of (4)–(6) and to [4, Proposition 2.6], which must be applied separately in Ω_1^ε and Ω_2^ε, in order to prove (7).

2.4. Extension result

In this subsection, we recall an extension result (see [1, Theorem 2.1]), which will be used in the proof of Theorems 4.7 and 4.10. This result permits to extend a function from the connected set Ω_2^ε to Ω, without any assumption on the connection of the set Ω_1^ε. Actually, when we are in the connected/disconnected geometry, we could apply a more classical extension theorem due to Tartar (see [15, 28]), but this is not the case in the connected/connected geometry.

We state below the version proposed in [25, Lemma 1]; to this purpose, let us define

$$V_{2,0}^\varepsilon = \{w \in H^1(\Omega_2^\varepsilon) \ : \ w \mid_{\partial\Omega\cap\partial\Omega_2^\varepsilon} = 0\}\,.$$

Theorem 2.13. *For every $\varepsilon > 0$, there exist a continuous linear operator $T_\varepsilon^2 : V_{2,0}^\varepsilon \to H^1(\Omega)$ and a constant $C > 0$ (independent on ε) such that $T_\varepsilon^2 w = w$ a.e. in Ω_2^ε and*

$$\|T_\varepsilon^2 w\|_{L^2(\Omega)} \le C\|w\|_{L^2(\Omega_2^\varepsilon)}\,, \tag{8}$$

$$\|\nabla T_\varepsilon^2 w\|_{L^2(\Omega)} \le C\|\nabla w\|_{L^2(\Omega_2^\varepsilon)}\,. \tag{9}$$

Notice that, in the connected/connected case, where the role of Ω_2^ε and Ω_1^ε can be interchanged, the previous theorem can be applied also to extend from Ω_1^ε into Ω, defining an operator T_ε^1, in an analogous way as done for T_ε^2.

2.5. Statement of the problem

Let $\lambda_1, \lambda_2, \beta$ be positive constants and $\theta \in (0,1)$. In the following, we will assume that $f \in L^{\frac{2}{1+\theta}}(\Omega)$ is a nonnegative function a.e. in Ω, not identically equal to zero in Ω_1^ε nor in Ω_2^ε, for every $\varepsilon > 0$. Let us define the functions $\lambda_\varepsilon : \Omega \to \mathbb{R}$ and $\lambda : Y \to \mathbb{R}$ as

$$\lambda_\varepsilon(x) = \begin{cases} \lambda_1, & \text{if } x \in \Omega_1^\varepsilon \\ \lambda_2, & \text{if } x \in \Omega_2^\varepsilon \end{cases} \qquad \text{and} \qquad \lambda(y) = \begin{cases} \lambda_1, & \text{if } y \in E_1 \\ \lambda_2, & \text{if } y \in E_2 \end{cases}$$

and set $\lambda_0 = \lambda_1|E_1| + \lambda_2|E_2|$. For $\alpha \ge -1$, we consider the problem

$$\begin{array}{rll} -\mathrm{div}(\lambda_\varepsilon \nabla u_\varepsilon) & = \frac{f}{u_\varepsilon^\theta}, & \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon; \\ [\lambda_\varepsilon \nabla u_\varepsilon \cdot \nu] & = 0, & \text{on } \Gamma^\varepsilon; \\ \frac{\beta}{\varepsilon^\alpha}[u_\varepsilon] & = \lambda_2 \nabla u_\varepsilon^{(2)} \cdot \nu, & \text{on } \Gamma^\varepsilon; \\ u_\varepsilon & > 0, & \text{in } \Omega; \\ u_\varepsilon & = 0, & \text{on } \partial\Omega\,. \end{array} \tag{10}$$

Definition 2.14. *We say that* $u \in V_0^\varepsilon(\Omega)$ *is a weak solution of* (10) *if* $u_\varepsilon > 0$ *a.e. in* Ω *and it satisfies*

$$\left| \int_\Omega \frac{f}{u_\varepsilon^\theta} \psi \, \mathrm{d}x \right| < +\infty \,, \tag{11}$$

$$\int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla \psi \, \mathrm{d}x + \frac{\beta}{\varepsilon^\alpha} \int_{\Gamma^\varepsilon} [u_\varepsilon][\psi] \, \mathrm{d}\sigma = \int_\Omega \frac{f}{u_\varepsilon^\theta} \psi \, \mathrm{d}x \tag{12}$$

for every $\psi \in V_0^\varepsilon(\Omega)$.

Remark 2.15. *Note that the assumption* (11) *is indeed contained in* (12), *since it is a consequence of the finiteness of the left-hand side of* (12)*; nevertheless we prefer to require it explicitly, being crucial in the proof of existence and homogenization results. Moreover, taking into account that* u_ε *and* f *are positive and recalling the decomposition* $\psi = \psi^+ - \psi^-$, (11) *can be rewritten for* $\psi > 0$ *and without the absolute value, or even in the apparently stronger form*

$$\int_\Omega \frac{f}{u_\varepsilon^\theta} |\psi| \, \mathrm{d}x < +\infty \,.$$

We will prove in the Appendix (Theorem 5.1) that, for every $\varepsilon > 0$ fixed, the problem (10) admits a unique solution $u_\varepsilon \in V_0^\varepsilon(\Omega)$.

Notice that, for the sake of simplicity, in the problem (10) we have considered only the model case, where the singular term is given by $\frac{f(x)}{s^\theta}$; however, all the proofs also work if we take into account a more general singularity of the form $f(x) \cdot g(s)$, with a non increasing function g such that $0 \le g(s) \le \frac{1}{s^\theta}$.

3. Estimates

The aim of this section is to prove that the solution u_ε satisfies some uniform (with respect to ε) estimates.

Proposition 3.1. *Let* u_ε *be the weak solution of problem* (10). *Then there exists* $C > 0$, *independent of* ε *(and* α*), such that*

$$\int_\Omega |\nabla u_\varepsilon|^2 \, \mathrm{d}x + \frac{1}{\varepsilon^\alpha} \int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \, \mathrm{d}\sigma \le C \|f\|_{L^{\frac{2}{1+\theta}}(\Omega)}^{\frac{2}{1+\theta}} \,, \qquad \forall \varepsilon > 0 \,, \tag{13}$$

$$\int_\Omega u_\varepsilon^2 \, \mathrm{d}x \le C ||f||_{L^{\frac{2}{1+\theta}}(\Omega)}^{\frac{2}{1+\theta}} \,, \qquad \forall \varepsilon > 0 \,. \tag{14}$$

Proof. Taking $\psi = u_\varepsilon$ in (12), we get

$$\int_\Omega \lambda_\varepsilon |\nabla u_\varepsilon|^2 \, \mathrm{d}x + \frac{\beta}{\varepsilon^\alpha} \int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \, \mathrm{d}\sigma = \int_\Omega f u_\varepsilon^{1-\theta} \, \mathrm{d}x \leq ||f||_{L^{\frac{2}{1+\theta}}(\Omega)} ||u_\varepsilon||^{1-\theta}_{L^2(\Omega)}. \quad (15)$$

By Theorem 2.2 ($\alpha \geq -1$), it follows

$$\begin{aligned} ||u_\varepsilon||^{1-\theta}_{L^2(\Omega)} &\leq C \left[\int_\Omega |\nabla u_\varepsilon|^2 \, \mathrm{d}x + \varepsilon \int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \, \mathrm{d}\sigma \right]^{\frac{1-\theta}{2}} \quad (16) \\ &\leq C \left[\int_\Omega |\nabla u_\varepsilon|^2 \, \mathrm{d}x + \frac{1}{\varepsilon^\alpha} \int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \, \mathrm{d}\sigma \right]^{\frac{1-\theta}{2}} . \end{aligned}$$

Hence, (13) follows by (15) and (16), and by (16) and (13), we get (14). □

Proposition 3.2. *Let u_ε be the weak solution of problem* (10)*. Then, for every $\psi \in H^1_0(\Omega)$, we have*

$$\left| \int_\Omega \frac{f}{u_\varepsilon^\theta} \psi(x) dx \right| \leq C ||\nabla \psi||_{L^2(\Omega)} ||\nabla u_\varepsilon||_{L^2(\Omega)} \quad (17)$$

with $C = \max(\lambda_1, \lambda_2)$.

Proof. Taking in (12) a testing function $\psi \in H^1_0(\Omega)$, and applying Holder's inequality, we find that

$$\begin{aligned} \left| \int_\Omega \frac{f}{u_\varepsilon^\theta} \psi \, \mathrm{d}x \right| &= \left| \int_\Omega \lambda \nabla u_\varepsilon \cdot \nabla \psi \, \mathrm{d}x \right| \\ &\leq \max(\lambda_1, \lambda_2) \left(\int_\Omega |\nabla u_\varepsilon|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla \psi|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} . \end{aligned}$$

□

4. Homogenization

4.1. The case $\alpha = 1$

In this subsection we will assume to be in anyone of the geometrical settings described in Section 2. We will see that the homogenized problem will depend on the physical properties of the bulk regions (i.e., λ_1, λ_2) as well as the physical properties of the interfaces (i.e. β).

Theorem 4.1. *For $\varepsilon > 0$, let $u_\varepsilon \in V_0^\varepsilon(\Omega)$ be the weak solution of the problem* (10)*. Then, there exist $u \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega; V_\#(Y))$ with $\int_Y u_1(x,y)\,\mathrm{d}y = 0$ a.e. in Ω, such that, as $\varepsilon \to 0$, we have*

$$u_\varepsilon \to u\,, \qquad \textit{strongly in } L^2(\Omega)\,; \tag{18}$$

$$u_\varepsilon \xrightarrow{2-sc} u\,, \qquad \textit{in } L^2(\Omega \times Y)\,; \tag{19}$$

$$\chi_{\Omega\setminus\Gamma^\varepsilon} \nabla u_\varepsilon \xrightarrow{2-sc} \nabla u + \nabla_y u_1\,, \qquad \textit{in } L^2(\Omega \times Y)\ : \tag{20}$$

$$\frac{1}{\varepsilon}[u_\varepsilon] \xrightarrow{2-sc} [u_1]\,, \qquad \textit{in } L^2(\Omega; L^2(\Gamma))\,. \tag{21}$$

Moreover,

$$\left| \int_\Omega \frac{f}{u^\theta} \varphi \,\mathrm{d}x \right| < +\infty\,, \qquad \forall \varphi \in H_0^1(\Omega)\,, \tag{22}$$

and the pair (u, u_1) solve

$$-\,\mathit{div}\left(\lambda_0 \nabla u + \int_Y \lambda \nabla_y u_1 \,\mathrm{d}y \right) = \frac{f}{u^\theta}\,, \qquad \textit{in } \Omega\,; \tag{23}$$

$$-\,\mathit{div}_y\left(\lambda(\nabla u + \nabla_y u_1) \right) = 0\,, \qquad \textit{in } E_1 \cup E_2\,; \tag{24}$$

$$[\lambda(\nabla u + \nabla_y u_1) \cdot \nu] = 0\,, \qquad \textit{on } \Omega \times \Gamma\,; \tag{25}$$

$$\beta[u_1] = \lambda_2(\nabla u + \nabla_y u_1) \cdot \nu\,, \qquad \textit{on } \Omega \times \Gamma\,; \tag{26}$$

$$u > 0\,, \qquad \textit{in } \Omega\,; \tag{27}$$

$$u = 0\,, \qquad \textit{on } \partial\Omega\,, \tag{28}$$

where λ_0 and λ are defined at the beginning of Subsection 2.5.

Remark 4.2. *As usual, from* (24)*–*(26)*, we can factorize u_1 as*

$$u_1(x,y) = \chi(y) \cdot \nabla u(x), \tag{29}$$

with $\chi = (\chi_1, \dots, \chi_N)$ and $\chi_j \in V_\#(Y)$ with $\int_Y \chi_j \,\mathrm{d}y = 0$, for each $j \in \{1, \dots, N\}$, satisfying

$$\begin{aligned} &-\mathit{div}_y(\lambda(\nabla_y \chi_j + \mathbf{e}_j)) = 0\,, && \textit{in } E_1 \cup E_2\,;\\ &[\lambda(\nabla_y \chi_j + \mathbf{e}_j) \cdot \nu] = 0\,, && \textit{on } \Gamma\,;\\ &\beta\,[\chi_j] = \lambda_2(\nabla_y \chi_j + \mathbf{e}_j) \cdot \nu\,, && \textit{on } \Gamma\,. \end{aligned} \tag{30}$$

By [23, Theorem 2] the previous problem (30) *admits a unique solution. Replacing* (29) *in* (23)*, it follows that u solves*

$$\begin{aligned} &-div(A_{hom}\nabla u) = \frac{f}{u^\theta}\,, && in\ \Omega\,; \\ &u = 0\,, && on\ \partial\Omega\,, \end{aligned} \tag{31}$$

where the matrix A_{hom} is defined as

$$A_{hom} = \lambda_0 I + \int_Y \lambda (\nabla_y \chi)^T \,\mathrm{d}y. \tag{32}$$

Here, $\nabla_y\chi$ is the matrix whose entries are $(\nabla_y\chi)_{ij} = \frac{\partial \chi_i}{\partial y_j}$ and $(\nabla_y\chi)^T$ denotes its transposed. Therefore, we have

$$\begin{aligned} \left(\int_Y \lambda(\nabla\chi)^T \,\mathrm{d}y\right)_{ij} &= \int_Y \lambda \frac{\partial \chi_j}{\partial y_i}\,\mathrm{d}y = \int_{E_1} \lambda_1 \frac{\partial \chi_j}{\partial y_i}\,\mathrm{d}y + \int_{E_2} \lambda_2 \frac{\partial \chi_j}{\partial y_i}\,\mathrm{d}y \\ &= \int_\Gamma \lambda_1 \chi_j \nu_i \,\mathrm{d}\sigma - \int_\Gamma \lambda_2 \chi_j \nu_i \,\mathrm{d}\sigma = -\int_\Gamma [\lambda\chi_j]\nu_i \,\mathrm{d}\sigma \end{aligned}$$

and hence we may write

$$A_{hom} = \lambda_0 I + \int_Y \lambda (\nabla_y\chi)^T \,\mathrm{d}y = \lambda_0 I - \int_\Gamma \nu \otimes [\lambda\chi] \,\mathrm{d}\sigma. \tag{33}$$

We can prove that the factorization (29) *is unique. Indeed, as we have recalled above, the problem* (30) *is well posed. Moreover, the homogenized matrix A_{hom} is symmetric and positive definite, as proved in [23, end of Section 3.2]. Therefore, by [13, Theorem 5.2 and Remark 5.4] we obtain the existence and uniqueness of a solution of* (31)*.*

Remark 4.3. *Notice that the problem* (23)*–*(28) *admits at most one pair of solutions (u, u_1). Indeed, assume by contradiction that (u^i, u_1^i), for $i = 1, 2$ are two pair of solutions and denote by $U = u^1 - u^2$ and $U_1 = u_1^1 - u_1^2$. Using U as test function in* (23) *written for u^1 and U_1 as test function in* (24) *written for u_1^1, adding the two equations, integrating by parts and using* (25)*–*(26)*, we get*

$$\begin{aligned} \int_\Omega\int_Y \lambda(\nabla u^1 + \nabla_y u_1^1)\cdot\nabla U \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega\int_Y \lambda(\nabla u^1 + \nabla_y u_1^1)\cdot\nabla_y U_1 \,\mathrm{d}x\,\mathrm{d}y \\ + \beta\int_\Omega\int_\Gamma [u_1^1][U_1]\,\mathrm{d}x\,\mathrm{d}\sigma(y) = \int_\Omega \frac{f}{(u^1)^\theta} U \,\mathrm{d}x\,. \end{aligned}$$

Repeating the same procedure for (u^2, u_1^2) and subtracting the equation for (u^1, u_1^1) from the equation for (u^2, u_1^2), it follows

$$\begin{aligned}&\int_\Omega\int_Y \lambda|\nabla U + \nabla_y U_1|^2 \,\mathrm{d}x\,\mathrm{d}y \\ +\quad & \beta\int_\Omega\int_\Gamma [U_1]^2 \,\mathrm{d}x\,\mathrm{d}\sigma(y) = \int_\Omega f\left(\frac{1}{(u^1)^\theta} - \frac{1}{(u^2)^\theta}\right)(u^1-u^2)\,\mathrm{d}x\,.\end{aligned}$$

Taking into account that the function $s \mapsto \frac{1}{s^\theta}$ is decreasing, it follows that the right-hand side in the last equality is non positive, which implies $[U_1] = 0$. Moreover,

$$\begin{aligned}\int_\Omega |\nabla U|^2\,\mathrm{d}x + \int_\Omega\int_Y |\nabla_y U_1|^2\,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega |\nabla U|^2\,\mathrm{d}x + \int_\Omega\int_Y |\nabla_y U_1|^2\,\mathrm{d}x\,\mathrm{d}y \\ +2\int_\Omega \nabla u\cdot\left(\int_Y \nabla_y U_1\,\mathrm{d}y\right)\mathrm{d}x &= \int_\Omega\int_Y |\nabla U + \nabla_y U_1|^2\,\mathrm{d}x\,\mathrm{d}y = 0\,,\end{aligned}$$

where we have taken into account that $\int_Y \nabla_y U_1\,\mathrm{d}y = 0$, because of the Y-periodicity of U_1 and the fact that $[U_1] = 0$. Thus, $\nabla U = \nabla_y U_1 = 0$, which implies $U = 0$ in Ω, since it satisfies the homogeneous boundary condition, and $U_1 = 0$, since it has null mean average on Y.

As a consequence of Remarks 4.2 and 4.3, we get that the homogenized problem (23)–(28) admits a unique solution and that such a solution can be factorized as in (29).

Proof of Theorem 4.1. By Proposition 3.1 and [22, Proposition 5.5] we get that (18)–(21) hold. Hence, taking into account (13) and (18) and passing to the limit in (17), when $\varepsilon \to 0$, by Fatou's Lemma we get (22) which also implies that u is not identically zero in Ω.

In order to pass to the two-scale limit in (12), with $\alpha = 1$, we choose as test function $\psi(x) = \varphi(x) + \varepsilon\Phi\left(x, \frac{x}{\varepsilon}\right)$ with $\varphi \in \mathcal{C}_c^1(\Omega)$ and $\Phi \in \mathcal{C}_c^1(\Omega; \mathfrak{L}_\#(Y))$. Then, we get

$$\begin{aligned}\int_\Omega \lambda_\varepsilon \nabla u_\varepsilon\cdot\nabla\varphi\,\mathrm{d}x + \varepsilon\int_\Omega \lambda_\varepsilon\nabla u_\varepsilon\cdot\nabla_x\Phi\,\mathrm{d}x + \int_\Omega \lambda_\varepsilon\nabla u_\varepsilon\cdot\nabla_y\Phi\,\mathrm{d}x \\ + \beta\int_{\Gamma^\varepsilon}[u_\varepsilon][\Phi]\,\mathrm{d}\sigma = \int_\Omega \frac{f}{u_\varepsilon^\theta}\varphi\,\mathrm{d}x + \varepsilon\int_\Omega \frac{f}{u_\varepsilon^\theta}\Phi\,\mathrm{d}x. \quad (34)\end{aligned}$$

By (20) and (21), as $\varepsilon \to 0$, the left-hand side of (34) converges to

$$\int_\Omega\int_Y \lambda(\nabla u + \nabla_y u_1)\cdot\nabla\varphi \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega\int_Y \lambda(\nabla u + \nabla_y u_1)\cdot\nabla_y\Phi \,\mathrm{d}x\,\mathrm{d}y \\ + \beta\int_\Omega\int_\Gamma [u_1][\Phi]\,\mathrm{d}x\,\mathrm{d}\sigma(y). \quad (35)$$

We now focus our attention on the right-hand side of (34) and we set

$$I_\varepsilon := \int_\Omega \frac{f}{u_\varepsilon^\theta}\varphi\,\mathrm{d}x\,, \qquad J_\varepsilon := \varepsilon\int_\Omega \frac{f}{u_\varepsilon^\theta}\Phi\,\mathrm{d}x\,. \quad (36)$$

In order to deal with the term J_ε, we rewrite the function $\Phi\left(x, \frac{x}{\epsilon}\right) = \varphi_1(x)\varphi_2\left(\frac{x}{\epsilon}\right)$; moreover, by the decomposition $\varphi_1 = \varphi_1^+ - \varphi_1^-$ and $\varphi_2 = \varphi_2^+ - \varphi_2^-$, we can assume $\varphi_1, \varphi_2 \geq 0$ (notice that the Lipschitz continuity of φ_1 is enough for our purposes). We have that

$$0 \leq J_\varepsilon = \varepsilon\int_\Omega \frac{f}{u_\varepsilon^\theta}\varphi_1(x)\varphi_2\left(\frac{x}{\varepsilon}\right)\,\mathrm{d}x \leq \varepsilon||\varphi_2||_{L^\infty(Y)}\int_\Omega \frac{f}{u_\varepsilon^\theta}\varphi_1(x)\,\mathrm{d}x \\ \leq C\varepsilon||\varphi_2||_{L^\infty(Y)}||\nabla\varphi_1||_{L^2(\Omega)}||f||_{L^{\frac{2}{1+\theta}}(\Omega)}^{\frac{1}{1+\theta}}, \quad (37)$$

where we used (17) and (13). Since C is independent of ε, as $\varepsilon \to 0$, also $J_\varepsilon \to 0$. In order to study the limit of I_ε, having in mind the decomposition $\varphi = \varphi^+ - \varphi^-$ (notice again that the Lipschitz continuuity of φ is enough for our purposes), we may assume $\varphi \geq 0$. Moreover, we have to split the behaviour of the singular term into the part near to and far away from the singularity. To this purpose, we write

$$I_\varepsilon = \int_{\Omega\cap\{0<u_\varepsilon\leq\delta\}} \frac{f}{u_\varepsilon^\theta}\varphi\,\mathrm{d}x + \int_{\Omega\cap\{u_\varepsilon>\delta\}} \frac{f}{u_\varepsilon^\theta}\varphi\,\mathrm{d}x := I^1_{\varepsilon,\delta} + I^2_{\varepsilon,\delta}\,. \quad (38)$$

where, by the Lebesgue dominated convergence theorem and taking into account that $0 \leq \frac{f}{u_\varepsilon^\theta}\varphi \leq \frac{f}{\delta^\theta}\varphi \in L^1(\Omega)$ in the set $\{u_\varepsilon > \delta\}$ (here it is crucial that φ is bounded), we get

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I^2_{\varepsilon,\delta} = \int_{\Omega\cap\{u>0\}} \frac{f}{u^\theta}\varphi\,\mathrm{d}x\,, \quad (39)$$

once we have taken $\delta \notin \mathcal{C} = \{\delta > 0 : |\{u(x) = \delta\}| > 0\}$, which is at most countable (exactly as in [18, Proof of Theorem 3.6]).

Moreover, introducing the function $Z_\delta : \mathbb{R} \to [0, +\infty)$ defined by

$$Z_\delta(s) = \begin{cases} 1\,, & \text{if } 0 \le s \le \delta\,; \\ -\frac{s}{\delta} + 2\,, & \text{if } \delta \le s \le 2\delta\,; \\ 0\,, & \text{if } s \ge 2\delta\,, \end{cases} \tag{40}$$

using as test function in (12) (with $\alpha = 1$) the function $Z_\delta(u_\varepsilon)\varphi$, with φ as above, and recalling that $s \mapsto Z_\delta(s)$ is decreasing, we arrive at

$$\begin{aligned} I^1_{\varepsilon,\delta} &\le \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi\, Z_\delta(u_\varepsilon)\,\mathrm{d}x \\ &= \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi [Z_\delta(u_\varepsilon) - Z_\delta(u)]\,\mathrm{d}x + \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi\, Z_\delta(u)\,\mathrm{d}x \end{aligned} \tag{41}$$

since

$$\frac{\beta}{\varepsilon}\int_{\Gamma^\varepsilon} [u_\varepsilon]\left(Z_\delta(u^{(2)}_\varepsilon) - Z_\delta(u^{(1)}_\varepsilon)\right)\varphi\,\mathrm{d}x \le 0$$

and

$$\int_{\Omega\cap\{\delta\le u_\varepsilon\le 2\delta\}} \frac{f}{u_\varepsilon^\theta} Z_\delta(u_\varepsilon)\varphi\,\mathrm{d}x \ge 0.$$

In order to pass to the two-scale limit in (41), we have to take into account that $\lambda_\varepsilon \nabla u_\varepsilon$ is bounded in $L^2(\Omega)$ and $Z_\delta(u_\varepsilon) - Z_\delta(u) \to 0$ strongly in $L^2(\Omega)$ (since $s \mapsto Z_\delta(s)$ is continuous and (18) holds), so that the first integral in (41) vanishes, while in the second integral, thanks to Remark 2.6, we can take $\lambda_\varepsilon \nabla\varphi Z_\delta(u)$ as admissible test function for the two-scale convergence. Therefore, we get

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I^1_{\varepsilon,\delta} \le \int_{\Omega\cap\{u=0\}}\int_Y |\lambda(\nabla u + \nabla_y u_1)|\;|\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}y. \tag{42}$$

In order to prove that the right-hand side of (42) is zero, we notice that, choosing $\varphi \equiv 0$ in (34) and letting $\varepsilon \to 0$, we obtain

$$\int_\Omega\int_Y \lambda(\nabla u + \nabla_y u_1)\cdot\nabla_y\Phi\,\mathrm{d}x\,\mathrm{d}y + \beta\int_\Omega\int_\Gamma [u_1][\Phi]\,\mathrm{d}x\,\mathrm{d}\sigma(y) = 0\,,$$

which is the problem in the micro variable y (i.e. (24)–(26)); therefore, we get the factorization (29) for u_1. This implies that

$$\begin{aligned} &\int_{\Omega\cap\{u=0\}}\int_Y |\lambda(\nabla u + \nabla_y u_1)|\;|\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}y = \\ &= \int_{\Omega\cap\{u=0\}}\int_Y |\lambda(I + \nabla_y\chi)\nabla u||\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}y = 0, \end{aligned} \tag{43}$$

where, in the last equality, we used that u is a Sobolev function and hence its gradient vanishes on the level sets of u. Then, passing to the limit for $\varepsilon \to 0$ in (34), by (35), (39), (42), (43) and taking into account the density of our test functions in $H_0^1(\Omega) \times L^2(\Omega; V_\#(Y))$, we obtain

$$\int_{\Omega\times Y} \lambda(\nabla u + \nabla_y u_1)\cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times Y} \lambda(\nabla u + \nabla_y u_1)\cdot \nabla_y \Phi \,\mathrm{d}x\,\mathrm{d}y \\ + \beta \int_\Omega \int_\Gamma [u_1][\Phi]\,\mathrm{d}x\,\mathrm{d}\sigma(y) = \int_\Omega \frac{f}{u^\theta}\varphi\chi_{\{u>0\}}\,\mathrm{d}x\,. \quad (44)$$

Taking first $\varphi = 0$ and then $\Phi = 0$ in (44), it follows that u is a nonnegative (being the limit of the sequence of positive solutions u_ε) weak solution of the problem (23)–(26) and (28), with $\frac{f}{u^\theta}$ replaced by $\frac{f}{u^\theta}\chi_{\{u>0\}}$. In order to conclude the proof, it remains to show that $u > 0$ a.e. in Ω. To this purpose, we recall again the factorization given in (29), where u solves the problem (31) with the new nonnegative source $\frac{f}{u^\theta}\chi_{\{u>0\}}$ and the matrix A_{hom} defined in (33) is positive definite. Therefore, taking into account (22), we can apply the strong maximum principle to deduce that $u > 0$ a.e. in Ω. Finally, by Remark 4.3, it follows that the whole sequence $\{u_\varepsilon\}$ converges and the thesis is accomplished. □

4.2. The case $\alpha > 1$

As in the previous subsection, we will assume to be in anyone of the geometrical setting described in Section 2. Moreover, we will see that, due to the particular scaling $\varepsilon^{-\alpha}$ in front of the interface term, the homogenized problem will not take memory of β, as pointed out in Remark 4.6.

Theorem 4.4. *For $\varepsilon > 0$, let $u_\varepsilon \in V_0^\varepsilon(\Omega)$ be the weak solution of the problem* (10)*. Then, there exist $u \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega; H_\#^1(Y))$ with $\int_Y u_1(x,y)\,\mathrm{d}y = 0$ a.e. in Ω, such that, as $\varepsilon \to 0$,* (18)*–*(22) *hold. Moreover, the pair (u, u_1) solve*

$$-\,div\left(\lambda_0\nabla u + \int_Y \lambda\nabla_y u_1\,\mathrm{d}y\right) = \frac{f}{u^\theta}\,, \qquad in\ \Omega\,; \quad (45)$$

$$-\,div_y\left(\lambda(\nabla u + \nabla_y u_1)\right) = 0\,, \qquad in\ Y\,; \quad (46)$$

$$u > 0\,, \qquad in\ \Omega\,; \quad (47)$$

$$u = 0\,, \qquad on\ \partial\Omega\,, \quad (48)$$

where λ_0 and λ are defined at the beginning of Subsection 2.5.

Remark 4.5. *Notice that, similarly as in Remark 4.3, it is possible to prove that the problem* (45)*–*(48) *admits at most one pair of solutions* (u, u_1)*. Moreover, we can factorize* u_1 *as in* (29) *with* $\chi = (\chi_1, \dots, \chi_N)$ *and* $\chi_j \in H^1_\#(Y)$ *such that* $\int_Y \chi_j \,\mathrm{d}y = 0$*, for each* $j = 1, \dots, n$*,* χ_j*, we get that* χ_j *must solve*

$$-div_y(\lambda(\nabla_y \chi_j + \mathbf{e}_j)) = 0\,, \qquad \text{in } Y. \tag{49}$$

Replacing the factorization of u_1 *in* (45)*, it follows that* u *solves*

$$\begin{aligned} &- div(A_{hom}\nabla u) = \frac{f}{u^\theta}\,, && \text{in } \Omega\,;\\ &u > 0\,, && \text{in } \Omega;\\ &u = 0\,, && \text{on } \partial\Omega; \end{aligned} \tag{50}$$

where the matrix A_{hom} *is defined as*

$$A_{hom} = \lambda_0 I - \int_Y \lambda(\nabla_y \chi)^T \,\mathrm{d}y = \lambda_0 I - \int_\Gamma [\lambda]\nu \otimes \chi \,\mathrm{d}\sigma. \tag{51}$$

We recall that by standard arguments equation (49) *admits a unique solution. Moreover by Proposition 4.1 in [6] we know that* A_{hom} *is symmetric and positive definite and therefore, by [13, Theorem 5.2 and Remark 5.4], also the solution of equation* (50) *is unique.*

Remark 4.6. *Notice that, from the definition* (49)*, the cell functions do not depend on the coefficient* β*. Therefore, the homogenized matrix and, hence, the macroscopic function* u *lose any memory of the physical properties of the interfaces.*

Proof of Theorem 4.4. By Proposition 3.1, [22, Proposition 5.5] and Fatou's Lemma, we get that (18)–(21) and (22) hold (as in the proof of Theorem 4.1).

Moreover, by (13) we also know that

$$\frac{1}{\varepsilon^\alpha}\int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}x = \varepsilon \int_{\Gamma^\varepsilon} \left(\frac{[u_\varepsilon]}{\varepsilon^{\frac{\alpha+1}{2}}}\right)^2 \mathrm{d}x \le C\,,$$

uniformly with respect to ε. Hence, as ε tends to 0, by Theorem 2.11 it follows that there exists $v \in L^2(\Omega \times \Gamma)$ such that, up to subsequence, $v_\varepsilon := \frac{[u_\varepsilon]}{\varepsilon^{\frac{\alpha+1}{2}}} \xrightarrow{2-sc} v$ in $L^2(\Omega^\varepsilon)$. However, by (21) we already know that $\frac{[u_\varepsilon]}{\varepsilon} \xrightarrow{2-sc} [u_1]$;

therefore, taking into account that $\frac{[u_\varepsilon]}{\varepsilon} = \varepsilon^{\frac{\alpha-1}{2}} v_\varepsilon$, with $\frac{\alpha-1}{2} > 0$, we infer that $[u_1] = 0$, so that $u_1 \in L^2(\Omega; H^1_\#(Y))$.

In order to pass to the two-scale limit in (12), with $\alpha > 1$, we choose as test function $\psi(x) = \varphi(x) + \varepsilon\Phi\left(x, \frac{x}{\varepsilon}\right)$ with $\varphi \in \mathcal{C}^1_c(\Omega)$ and $\Phi \in \mathcal{C}^1_c(\Omega; \mathcal{C}^1_\#(Y))$ (i.e., we can take $[\Phi] = 0$, since $[u_1] = 0$) and we get

$$\begin{aligned}\int_\Omega \lambda \nabla u_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x + \varepsilon \int_\Omega \lambda \nabla u_\varepsilon \cdot \nabla_x \Phi \,\mathrm{d}x + \int_\Omega \lambda \nabla u_\varepsilon \cdot \nabla_y \Phi \,\mathrm{d}x = \\ = \int_\Omega \frac{f}{u_\varepsilon^\theta} \varphi \,\mathrm{d}x + \varepsilon \int_\Omega \frac{f}{u_\varepsilon^\theta} \Phi \,\mathrm{d}x. \qquad (52)\end{aligned}$$

By (20), we obtain that the left-hand side of (52) converges to

$$\int_{\Omega\times Y} \lambda(\nabla u + \nabla_y u_1) \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times Y} \lambda(\nabla u + \nabla_y u_1) \cdot \nabla_y \Phi \,\mathrm{d}x\,\mathrm{d}y\,.$$

Moreover, by (17) and reasoning as in (37), the second term in the right-hand side tends to 0. Finally, arguing as in the proof of Theorem 4.1 for the study of the first integral in the right-hand side of (52), as ε goes to 0, we have

$$\int_\Omega \frac{f}{u_\varepsilon^\theta} \varphi \,\mathrm{d}x \to \int_\Omega \frac{f}{u^\theta} \varphi \chi_{\{u>0\}} \,\mathrm{d}x. \qquad (53)$$

The proof that $u > 0$ a.e. in Ω follows, as usual, from the strong maximum principle, taking into account (22), so that we can replace the source $\frac{f}{u^\theta}\varphi\chi_{\{u>0\}}$ with $\frac{f}{u^\theta}\varphi$. Finally, recalling the density of our test functions in $H^1_0(\Omega) \times L^2(\Omega; H^1_\#(Y))$, taking alternatively $\varphi = 0$ and $\Phi = 0$ in (53) and integrating by parts, we deduce (45) and (46). Therefore, by the uniqueness of the problem (45)–(48) (see Remark 4.5), it follows that the whole sequence $\{u_\varepsilon\}$ converges and the thesis is accomplished.

□

4.3. The case $\alpha \in (-1, 1)$

As in the previous subsections, we will assume to be in anyone of the geometrical settings described in Section 2. Moreover, analogously to the case $\alpha > 1$, we will see that also in this case, due to the particular scaling $\varepsilon^{-\alpha}$ in front of the interface term, the homogenized problem will not take memory of β (see the end of Remark 4.8).

Theorem 4.7. *For $\varepsilon > 0$, let $u_\varepsilon \in V_0^\varepsilon(\Omega)$ be the weak solution of the problem* (10). *Then, there exist $u \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega; V_\#(Y))$ with $u_1 = (u_1^{(1)}, u_1^{(2)})$, $\int_{E_1} u_1^{(1)}(x,y)\,\mathrm{d}y = 0 = \int_{E_2} u_1^{(2)}(x,y)\,\mathrm{d}y$ a.e. in Y, such that, as $\varepsilon \to 0$, we have*

$$u_\varepsilon \xrightarrow{2-sc} u\,, \qquad \text{in } L^2(\Omega \times Y)\,; \tag{54}$$

$$\chi_{\Omega \setminus \Gamma^\varepsilon} \nabla u_\varepsilon \xrightarrow{2-sc} \nabla u + \nabla_y u_1\,, \qquad \text{in } L^2(\Omega \times Y)\,; \tag{55}$$

$$[u_\varepsilon] \xrightarrow{2-sc} 0\,, \qquad \text{in } L^2(\Omega; L^2(\Gamma))\,. \tag{56}$$

Moreover, (18) *and* (22) *hold and the pair (u, u_1) solve*

$$-\,div\left(\lambda_0 \nabla u + \int_Y \lambda \nabla_y u_1\,\mathrm{d}y\right) = \frac{f}{u^\theta}\,, \quad \text{in } \Omega\,; \tag{57}$$

$$-\,div_y(\lambda(\nabla u + \nabla_y u_1)) = 0\,, \qquad \text{in } E_1 \cup E_2\,; \tag{58}$$

$$[\lambda(\nabla u + \nabla_y u_1) \cdot \nu] = 0\,, \qquad \text{on } \Omega \times \Gamma\,; \tag{59}$$

$$\lambda_2(\nabla u^{(2)} + \nabla_y u_1^{(2)}) \cdot \nu = 0 \qquad \text{on } \Omega \times \Gamma\,; \tag{60}$$

$$u > 0\,, \qquad \text{in } \Omega\,; \tag{61}$$

$$u = 0\,, \qquad \text{on } \partial\Omega\,, \tag{62}$$

where λ_0 and λ_1 are defined in Subsection 2.5.

Remark 4.8. *Following the same ideas as in Remarks 4.2 and 4.3, we have that the problem* (57)–(62) *admits at most one pair of solutions (u, u_1) and that u_1 can be factorized as in* (29) *where, in this case, the cell function $\chi = (\chi_1, \dots, \chi_N)$ is such that $\chi_j \in V_\#(Y)$ with $\int_{E_1} \chi_j^{(1)}\,\mathrm{d}y = 0 = \int_{E_2} \chi_j^{(2)}\,\mathrm{d}y$ for each $j \in \{1, \dots, N\}$ and satisfies the cell problem*

$$\begin{aligned} &-div_y(\lambda(\nabla_y \chi_j + \mathbf{e}_j)) = 0\,, && \text{in } E_1 \cup E_2\,;\\ &[\lambda(\nabla_y \chi_j + \mathbf{e}_j) \cdot \nu] = 0\,, && \text{on } \Gamma\,;\\ &\lambda_2(\nabla_y \chi_j^{(2)} + \mathbf{e}_j) \cdot \nu = 0\,, && \text{on } \Gamma\,, \end{aligned} \tag{63}$$

which admits a unique solution. Replacing the factorization of u_1 in (57), *we still obtain that u solves an elliptic problem analogous to* (31), *where the new matrix A_{hom} is defined as in* (32) *and* (33) *in terms of the cell functions given in* (63). *Following [14, Proposition 5.1 and Remark 5.2], as done in [24, Section 7], we obtain that the matrix A_{hom} is symmetric and positive definite.*

As in the case $\alpha > 1$, from the definition (63), *we see that the cell functions do not depend on the coefficient β.*

Proof of Theorem 4.7. As a consequence of Theorem 2.2 and Proposition 3.1, we can apply Theorem 2.12, obtaining that (4)–(7) hold. Moreover, by (13), it follows

$$\varepsilon \int_{\Gamma^\varepsilon} \left(\frac{[u_\varepsilon]}{\varepsilon^{\frac{\alpha+1}{2}}} \right)^2 \mathrm{d}\sigma = \frac{1}{\varepsilon^\alpha} \int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}\sigma \leq C\,, \tag{64}$$

with C independent of ε. Hence, after setting $v_\varepsilon := \frac{[u_\varepsilon]}{\varepsilon^{\frac{\alpha+1}{2}}}$, as in the proof of Theorem 4.4, we can apply Theorem 2.11 to $\{v_\varepsilon\}$, obtaining that, up to subsequence, v_ε two-scale converges in $L^2(\Omega \times \Gamma)$ to some $v \in L^2(\Omega \times \Gamma)$, so that

$$0 \overset{2-sc}{\longleftarrow} v_\varepsilon \varepsilon^{\frac{\alpha+1}{2}} = [u_\varepsilon] \overset{2-sc}{\longrightarrow} [u]\,, \tag{65}$$

where we have taken into account that $\alpha + 1 > 0$. Therefore, (56) holds and $[u] = 0$. Taking into account (65), (4)–(6) become (54)–(55).

Now, let us choose $\psi(x) = \varphi(x) + \varepsilon\Phi\left(x, \frac{x}{\varepsilon}\right)$, with $\varphi \in \mathcal{C}^1_c(\Omega)$ and $\Phi \in \mathcal{C}^1_c(\Omega; \mathfrak{L}_\#(Y))$, as test function in (12). Then, we get

$$\begin{aligned} &\int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x + \varepsilon \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla_x \Phi \,\mathrm{d}x + \\ + &\int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla_y \Phi \,\mathrm{d}x + \beta\varepsilon^{1-\alpha} \int_{\Gamma^\varepsilon} [u_\varepsilon][\Phi] \,\mathrm{d}\sigma = \\ = &\int_\Omega \frac{f}{u_\varepsilon^\theta} \varphi \,\mathrm{d}x + \varepsilon \int_\Omega \frac{f}{u_\varepsilon^\theta} \Phi \,\mathrm{d}x := I_\varepsilon + J_\varepsilon. \end{aligned} \tag{66}$$

By (55), as $\varepsilon \to 0$, we obtain

$$\begin{aligned} &\int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x + \varepsilon \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla_x \Phi \,\mathrm{d}x + \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla_y \Phi \,\mathrm{d}x \\ &\to \int_{\Omega \times Y} \lambda(\nabla u + \nabla_y u_1) \cdot \nabla\varphi \,\mathrm{d}x \,\mathrm{d}y + \int_{\Omega \times Y} \lambda(\nabla u + \nabla_y u_1) \cdot \nabla_y \Phi \,\mathrm{d}x \,\mathrm{d}y. \end{aligned} \tag{67}$$

Moreover, we can write

$$\beta\varepsilon^{1-\alpha} \int_{\Gamma^\varepsilon} [u_\varepsilon][\Phi] \,\mathrm{d}\sigma = \beta\varepsilon^{\frac{1-\alpha}{2}} \varepsilon \int_{\Gamma^\varepsilon} \frac{[u_\varepsilon]}{\varepsilon^{\frac{\alpha+1}{2}}} [\Phi] \,\mathrm{d}\sigma \to 0\,, \tag{68}$$

as a consequence of (64) and the fact that $1 - \alpha > 0$.

In order to pass to the limit in the right-hand side of (66), i.e. to deal with the singular term, we consider the extension of u_ε from Ω_2^ε to Ω_1^ε as in Theorem 2.13, and for the sake of simplicity, let us denote by $T(u_\varepsilon)$ this extension, i.e. $T(u_\varepsilon) \in H^1(\Omega), T(u_\varepsilon) = u_\varepsilon$ in Ω_2^ε, $\|T(u_\varepsilon)\|_{L^2(\Omega)} \leq C\|u_\varepsilon\|_{L^2(\Omega_2^\varepsilon)}$ and $\|\nabla T(u_\varepsilon)\|_{L^2(\Omega)} \leq C\|\nabla u_\varepsilon\|_{L^2(\Omega_2^\varepsilon)}$, with C independent of ε. Then, by (13), it follows that there exists $\upsilon \in H^1(\Omega)$ such that, up to a subsequence, $T(u_\varepsilon) \rightharpoonup \upsilon$ weakly in $H^1(\Omega)$ and $T(u_\varepsilon) \to \upsilon$ strongly in $L^2(\Omega)$. Moreover, recalling [2, Proposition 1.14 (i)] we have also $T(u_\varepsilon) \stackrel{2-sc}{\longrightarrow} \upsilon$ in $L^2(\Omega \times Y)$. By Lemma 6 of [23] applied to function $u_\varepsilon - T(u_\varepsilon)$, we have that

$$
\begin{aligned}
\|u_\varepsilon - \upsilon\|_{L^2(\Omega)}^2 &= \|(u_\varepsilon - T(u_\varepsilon)) + (T(u_\varepsilon) - \upsilon)\|_{L^2(\Omega)}^2 \\
&\leq 2\left(\|u_\varepsilon - T(u_\varepsilon)\|_{L^2(\Omega)}^2 + \|T(u_\varepsilon) - \upsilon\|_{L^2(\Omega)}^2\right) \\
&\leq C\Big(\|u_\varepsilon - T(u_\varepsilon)\|_{L^2(\Omega_2^\varepsilon)}^2 + \varepsilon \int_{\Gamma^\varepsilon} [u_\varepsilon - T(u_\varepsilon)]^2 \,\mathrm{d}\sigma \\
&\qquad\quad + \varepsilon^2 \|\nabla u_\varepsilon - \nabla T(u_\varepsilon)\|_{L^2(\Omega)}^2 + \|T(u_\varepsilon) - \upsilon\|_{L^2(\Omega)}^2\Big) \\
&\leq C\Big(\varepsilon^{1+\alpha} \frac{1}{\varepsilon^\alpha} \int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}\sigma + \varepsilon^2 \|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 \\
&\qquad\quad + \varepsilon^2 \|\nabla T(u_\varepsilon)\|_{L^2(\Omega)}^2 + \|T(u_\varepsilon) - \upsilon\|_{L^2(\Omega)}^2\Big) \to 0\,,
\end{aligned}
$$

where we have taken into account again that $\alpha + 1 > 0$. It remains to prove that $\upsilon = u$, but this is a direct consequence of the fact that $T(u_\varepsilon) = u_\varepsilon$ in Ω_2^ε, indeed taking a test function $\phi\left(x, \frac{x}{\varepsilon}\right) = \phi_1(x)\phi_2\left(\frac{x}{\varepsilon}\right)$, with $\phi_1 \in \mathcal{C}_c^0(\Omega)$ and $\phi_2 \in \mathcal{C}_\#^0(Y)$ with compact support in E_2, it follows

$$
\begin{aligned}
\left(\int_\Omega u(x)\phi_1(x)\,\mathrm{d}x\right)\left(\int_{E_2} \phi_2(y)\,\mathrm{d}y\right) &\leftarrow \int_\Omega u_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right)\,\mathrm{d}x \\
= \int_\Omega T(u_\varepsilon)(x)\phi\left(x, \frac{x}{\varepsilon}\right)\,\mathrm{d}x &\to \left(\int_\Omega \upsilon(x)\phi_1(x)\,\mathrm{d}x\right)\left(\int_{E_2} \phi_2(y)\,\mathrm{d}y\right).
\end{aligned}
$$

Therefore, $u_\varepsilon \to u$ strongly in $L^2(\Omega)$, i.e. (18) holds. In order to get the homogenous boundary condition (62), we proceed as in [9, Proof of Theorem 2.2]. Let $\Phi \in L^2(\Omega; \mathbb{R}^N)$ and let Ψ be the function associated to Φ on E_2 by

Lemma 2.4. Integrating by parts and passing to the two-scale limit, we have

$$\int_\Omega \int_{E_2} (\nabla u(x) + \nabla_y u_1(x,y))\, \Psi(x,y)\, \mathrm{d}y\, \mathrm{d}x \leftarrow \int_{\Omega_2^\varepsilon} \nabla u_\varepsilon(x) \cdot \Psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x$$

$$= - \int_{\Omega_2^\varepsilon} u_\varepsilon(x) \mathrm{div}_x \Psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x \to - \int_\Omega \int_{E_2} u(x) \mathrm{div}_x \Psi(x,y)\, \mathrm{d}y\, \mathrm{d}x$$

$$= - \int_\Omega u(x) \mathrm{div} \Phi(x)\, \mathrm{d}x\,. \quad (69)$$

Moreover, by (2) there holds

$$\int_{E_2} \nabla_y u_1(x,y) \Psi(x,y)\, \mathrm{d}y = - \int_\Gamma u_1(x,y) \Psi \cdot \nu\, \mathrm{d}\sigma - \int_{E_2} u_1(x,y) \mathrm{div}_y \Psi(x,y)\, \mathrm{d}y = 0\,. \quad (70)$$

By (69) and (70), we conclude

$$\int_\Omega \nabla u(x) \Phi(x)\, \mathrm{d}x = \int_\Omega \nabla u(x) \left(\int_{E_2} \Psi(x,y)\, \mathrm{d}y \right) \mathrm{d}x = - \int_\Omega u(x) \mathrm{div} \Phi(x)\, \mathrm{d}x\,, \quad (71)$$

and hence $u = 0$ on $\partial\Omega$. Then, we can repeat the argument in the proof of Theorem 4.1 in order to obtain (22) and

$$I_\varepsilon \to \int_\Omega \frac{f}{u^\theta} \varphi \chi_{\{u>0\}}\, \mathrm{d}x\,, \qquad J_\varepsilon \to 0\,, \qquad \text{for } \varepsilon \to 0. \quad (72)$$

Moreover, using the strong maximum principle as in Theorem 4.1, we obtain $u > 0$ a.e. in Ω, so that we can drop the characteristic function $\chi_{\{u>0\}}$ in (72). Finally, taking first $\varphi = 0$ and then $\Phi = 0$, we deduce the strong formulation (57)–(62). □

Remark 4.9. *Notice that, when we are in the connected/disconnected case, as already pointed out in Subsection 2.4, we can refer to the more classical extension theorem in [15, 28], where the extension is found directly in* $H_0^1(\Omega)$. *Thus the proof of Theorem 4.7 can be achieved in a simpler way, avoiding steps* (69)–(71).

4.4. The case $\alpha = -1$

In this subsection we will assume to be in the connected/connected geometry. Moreover, we stipulate that the source $f \in L^{\frac{2}{1+\theta}}(\Omega)$ is strictly positive a.e. in Ω. We will see that the homogenized problem will take into account the physical properties of the bulk regions (i.e., λ_1, λ_2) as well as the physical properties of the interfaces (i.e. β).

Theorem 4.10. *For $\varepsilon > 0$, let $u_\varepsilon \in V_0^\varepsilon(\Omega)$ be the weak solution of the problem* (10)*. Then, there exist $u = \left(u^{(1)}, u^{(2)}\right) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $u_1 = \left(u_1^{(1)}, u_1^{(2)}\right) \in L^2(\Omega; V_\#(Y))$ with $\int_{E_1} u_1^{(1)}\,\mathrm{d}y = 0 = \int_{E_2} u_1^{(2)}\,\mathrm{d}y$, such that*

$$\chi_{\Omega_1^\varepsilon} u_\varepsilon^{(1)} \xrightarrow{2-sc} \chi_{E_1} u^{(1)}, \quad \chi_{\Omega_2^\varepsilon} u_\varepsilon^{(2)} \xrightarrow{2-sc} \chi_{E_2} u^{(2)}, \qquad \text{in } L^2(\Omega \times Y)\,; \tag{73}$$

$$\chi_{\Omega_1^\varepsilon} \nabla u_\varepsilon^{(1)} \xrightarrow{2-sc} \chi_{E_1}\left(\nabla u^{(1)} + \nabla_y u_1^{(1)}\right), \qquad \text{in } L^2(\Omega \times Y)\,; \tag{74}$$

$$\chi_{\Omega_2^\varepsilon} \nabla u_\varepsilon^{(2)} \xrightarrow{2-sc} \chi_{E_2}\left(\nabla u^{(2)} + \nabla_y u_1^{(2)}\right), \qquad \text{in } L^2(\Omega \times Y)\,; \tag{75}$$

$$[u_\varepsilon] \xrightarrow{2-sc} [u]\,, \qquad \text{in } L^2(\Omega; L^2(\Gamma))\,. \tag{76}$$

Moreover,

$$\left|\int_\Omega \frac{f}{(u^{(i)})^\theta}\varphi\,\mathrm{d}x\right| < +\infty\,, \qquad \forall \varphi \in H_0^1(\Omega)\,, \quad i = 1, 2\,, \tag{77}$$

and the pair (u, u_1) solve

$$-\,div\left(\lambda_1 |E_1| \nabla u^{(1)} + \int_{E_1} \lambda_1 \nabla_y u_1^{(1)}\,\mathrm{d}y\right) = |E_1| \frac{f}{(u^{(1)})^\theta} + |\Gamma|\beta[u], \text{ in } \Omega\,; \tag{78}$$

$$-\,div\left(\lambda_2 |E_2| \nabla u^{(2)} + \int_{E_2} \lambda_2 \nabla_y u_1^{(2)}\,\mathrm{d}y\right) = |E_2| \frac{f}{(u^{(2)})^\theta} - |\Gamma|\beta[u], \text{ in } \Omega\,; \tag{79}$$

$$-\,div_y(\lambda(\nabla u + \nabla_y u_1)) = 0\,, \text{ in } E_1 \cup E_2\,; \tag{80}$$

$$\lambda_1(\nabla u^{(1)} + \nabla_y u_1^{(1)}) \cdot \nu = 0\,, \text{ on } \Omega \times \Gamma\,; \tag{81}$$

$$\lambda_2(\nabla u^{(2)} + \nabla_y u_1^{(2)}) \cdot \nu = 0\,, \text{ on } \Omega \times \Gamma\,; \tag{82}$$

$$u^{(1)}, u^{(2)} > 0\,, \text{ in } \Omega\,, \tag{83}$$

$$u^{(1)} = u^{(2)} = 0\,, \text{ on } \partial\Omega\,, \tag{84}$$

where, with a slight abuse of notation, we set $[u] = u^{(2)} - u^{(1)}$.

Remark 4.11. *Following the same ideas as in Remark 4.3, we obtain that problem* (78) *admits at most one pair of solutions* (u, u_1). *Moreover, we can factorize* u_1 *as*

$$u_1^{(1)}(x,y) = \chi^{(1)}(y)\nabla u^{(1)}(x)\,, \qquad u_1^{(2)}(x,y) = \chi^{(2)}(y)\nabla u^{(2)}(x)\,, \tag{85}$$

where $\chi^{(k)} = (\chi_1^{(k)}, \dots, \chi_N^{(k)})$, *for* $k = 1,2$, $\int_{E_1} \chi_j^{(1)}\,\mathrm{d}y = 0 = \int_{E_2} \chi_j^{(2)}\,\mathrm{d}y$, *for each* $j \in \{1, \dots, N\}$, *and, recalling the usual notation, we set* $\chi = (\chi^{(1)}, \chi^{(2)}) \in \big(V_\#(Y)\big)^N$. *Then by* (78) *we obtain that, for each* $j \in \{1, \dots, N\}$, χ_j *satisfies* (63) *and* $u^{(1)}, u^{(2)}$ *solve the following system*

$$\begin{aligned} &-div(A_{hom}^{(1)}\nabla u^{(1)}) = |E_1|\tfrac{f}{(u^{(1)})^\theta} + |\Gamma|\beta\left(u^{(2)} - u^{(1)}\right)\,, && in\ \Omega\,;\\ &-div(A_{hom}^{(2)}\nabla u^{(2)}) = |E_2|\tfrac{f}{(u^{(2)})^\theta} - |\Gamma|\beta\left(u^{(2)} - u^{(1)}\right)\,, && in\ \Omega\,;\\ &u^{(1)} = u^{(2)} = 0\,, && on\ \partial\Omega\,, \end{aligned} \tag{86}$$

where, for $k = 1,2$, *the matrix* $A_{hom}^{(k)}$ *is defined as*

$$A_{hom}^{(k)} = \lambda_k|E_k|I + \lambda_k\int_{E_k}(\nabla_y\chi^{(k)})^T\,\mathrm{d}y.$$

Since

$$\left(\int_{E_k}(\nabla\chi^{(k)})^T\,\mathrm{d}y\right)_{ij} = \int_{E_k}\frac{\partial\chi_j^{(k)}}{\partial y_i}\,\mathrm{d}y = -(-1)^k\int_\Gamma \chi_j^{(k)}\nu_i\,\mathrm{d}y,$$

we also have

$$A_{hom}^{(k)} = \lambda_k|E_k|I - (-1)^k\lambda_k\int_\Gamma \nu\otimes\chi^{(k)}\,\mathrm{d}\sigma. \tag{87}$$

Following the same ideas as in [12, Remark 2.6], it is not difficult to prove that the matrices $A_{hom}^{(k)}$ *are symmetric and positive definite. Therefore, the solution* $u = (u^{(1)}, u^{(2)})$ *of* (86) *is unique. In fact, if* $\hat u = (\hat u^{(1)}, \hat u^{(2)})$ *and* $\bar u = (\bar u^{(1)}, \bar u^{(2)})$ *are two different solutions, then for* $\varphi \in H_0^1(\Omega)$ *and* $k = 1,2$, *we have*

$$\int_\Omega A_{hom}^{(k)}\nabla\hat u^{(k)}\cdot\nabla\varphi\,\mathrm{d}x = |E_k|\int_\Omega\frac{f}{(\hat u^{(k)})^\theta}\varphi\,\mathrm{d}x - (-1)^k|\Gamma|\beta\int_\Omega[\hat u]\varphi\,\mathrm{d}x\,, \tag{88}$$

$$\int_\Omega A_{hom}^{(k)}\nabla\bar u^{(k)}\cdot\nabla\varphi\,\mathrm{d}x = |E_k|\int_\Omega\frac{f}{(\bar u^{(k)})^\theta}\varphi\,\mathrm{d}x - (-1)^k|\Gamma|\beta\int_\Omega[\bar u]\varphi\,\mathrm{d}x\,. \tag{89}$$

Subtracting (89) *from* (88) *and taking* $\varphi = \hat{u}^{(k)} - \bar{u}^{(k)}$*, separately for* $k = 1, 2$*, we have*

$$\begin{aligned}
&\int_{\Omega} A^{(k)}_{hom} \nabla \left(\hat{u}^{(k)} - \bar{u}^{(k)}\right) \cdot \nabla \left(\hat{u}^{(k)} - \bar{u}^{(k)}\right) \,\mathrm{d}x \\
=&|E_k| \int_{\Omega} \left(\frac{f}{(\hat{u}^{(k)})^{\theta}} - \frac{f}{(\bar{u}^{(k)})^{\theta}} \right) \left(\hat{u}^{(k)} - \bar{u}^{(k)}\right) \,\mathrm{d}x - |\mathit{\Gamma}|\beta \int_{\Omega} \left(\hat{u}^{(k)} - \bar{u}^{(k)}\right)^2 \,\mathrm{d}x \\
&\quad + |\mathit{\Gamma}|\beta \int_{\Omega} \left(\hat{u}^{(1)} - \bar{u}^{(1)}\right) \left(\hat{u}^{(2)} - \bar{u}^{(2)}\right) \,\mathrm{d}x.
\end{aligned} \tag{90}$$

Summing (90) *for* $k = 1, 2$*, we get*

$$\begin{aligned}
& \int_{\Omega} A^{(1)}_{hom} \nabla \left(\hat{u}^{(1)} - \bar{u}^{(1)}\right) \cdot \nabla \left(\hat{u}^{(1)} - \bar{u}^{(1)}\right) \,\mathrm{d}x \\
+ & \int_{\Omega} A^{(2)}_{hom} \nabla \left(\hat{u}^{(2)} - \bar{u}^{(2)}\right) \cdot \nabla \left(\hat{u}^{(2)} - \bar{u}^{(2)}\right) \,\mathrm{d}x \\
= & \ |E_1| \int_{\Omega} \left(\frac{f}{(\hat{u}^{(1)})^{\theta}} - \frac{f}{(\bar{u}^{(1)})^{\theta}} \right) \left(\hat{u}^{(1)} - \bar{u}^{(1)}\right) \,\mathrm{d}x \\
+ & \ |E_2| \int_{\Omega} \left(\frac{f}{(\hat{u}^{(2)})^{\theta}} - \frac{f}{(\bar{u}^{(2)})^{\theta}} \right) \left(\hat{u}^{(2)} - \bar{u}^{(2)}\right) \,\mathrm{d}x \\
- & \ |\Gamma|\beta \int_{\Omega} \left(\left(\hat{u}^{(1)} - \bar{u}^{(1)}\right) - \left(\hat{u}^{(2)} - \bar{u}^{(2)}\right)\right)^2 \mathrm{d}x.
\end{aligned} \tag{91}$$

Recalling that $A^{(1)}_{hom}$ *and* $A^{(2)}_{hom}$ *are positive definite and taking into account that the function* $s \mapsto \frac{1}{s^{\theta}}$ *is decreasing, by* (91) *we infer*

$$\int_{\Omega} \left|\nabla \left(\hat{u}^{(1)} - \bar{u}^{(1)}\right)\right|^2 + \int_{\Omega} \left|\nabla \left(\hat{u}^{(2)} - \bar{u}^{(2)}\right)\right|^2 \leq 0\,,$$

which implies $\hat{u}^{(1)} = \bar{u}^{(1)}$ *and* $\hat{u}^{(2)} = \bar{u}^{(2)}$*.*

Proof. First we note that (73)–(76) follow by Proposition 3.1 and Theorem 2.12. In order to proceed with the homogenization, we choose $\psi = (\psi^{(1)}, \psi^{(2)})$, $\psi^{(i)}(x) = \varphi_i(x) + \varepsilon \Phi_i\left(x, \frac{x}{\varepsilon}\right)$ in $\mathit{\Omega}^{\varepsilon}_i \times E_i$, with $\varphi_i \in \mathcal{C}^1_c(\mathit{\Omega})$ and $\Phi_i \in \mathcal{C}^1_c(\mathit{\Omega}; \mathfrak{L}_{\#}(Y))$, for $i = 1, 2$, as test function in (12), with $\alpha = -1$. We

363 get

$$
\begin{aligned}
& \int_{\Omega_1^\varepsilon} \lambda_1 \nabla u_\varepsilon \cdot \nabla\varphi_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} \lambda_2 \nabla u_\varepsilon \cdot \nabla\varphi_2 \,\mathrm{d}x + \varepsilon \int_{\Omega_1^\varepsilon} \lambda_1 \nabla u_\varepsilon \cdot \nabla_x \Phi_1 \,\mathrm{d}x \\
+ \;\; & \varepsilon \int_{\Omega_2^\varepsilon} \lambda_2 \nabla u_\varepsilon \cdot \nabla_x \Phi_2 \,\mathrm{d}x + \int_{\Omega_1^\varepsilon} \lambda_1 \nabla u_\varepsilon \nabla_y \Phi_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} \lambda_2 \nabla u_\varepsilon \nabla_y \Phi_2 \,\mathrm{d}x \\
+ \;\; & \beta\varepsilon \int_{\Gamma^\varepsilon} [u_\varepsilon][\psi] \,\mathrm{d}\sigma \\
= \;\; & \int_{\Omega_1^\varepsilon} \frac{f}{u_\varepsilon^\theta}\varphi_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} \frac{f}{u_\varepsilon^\theta}\varphi_2 \,\mathrm{d}x + \varepsilon \int_{\Omega_1^\varepsilon} \frac{f}{u_\varepsilon^\theta}\Phi_1 \,\mathrm{d}x + \varepsilon \int_{\Omega_2^\varepsilon} \frac{f}{u_\varepsilon^\theta}\Phi_2 \,\mathrm{d}x \\
=: \;\; & I_\varepsilon^1 + I_\varepsilon^2 + J_\varepsilon^1 + J_\varepsilon^2. \qquad (92)
\end{aligned}
$$

Hence, taking into account (73)–(76), as $\varepsilon \to 0$, the left-hand side converges to

$$
\begin{aligned}
& \int_{\Omega\times E_1} \lambda_1(\nabla u^{(1)} + \nabla_y u_1^{(1)}) \cdot \nabla\varphi_1 \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times E_2} \lambda_2(\nabla u^{(2)} + \nabla_y u_1^{(2)}) \cdot \nabla\varphi_2 \,\mathrm{d}x\,\mathrm{d}y \\
+ & \int_{\Omega\times E_1} \lambda_1(\nabla u^{(1)} + \nabla_y u_1^{(1)}) \cdot \nabla_y\Phi_1 \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times E_2} \lambda_2(\nabla u^{(2)} + \nabla_y u_1^{(2)}) \cdot \nabla_y\Phi_2 \,\mathrm{d}x\,\mathrm{d}y \\
& + \beta \int_{\Omega\times\Gamma} [u][\varphi] \,\mathrm{d}x\,\mathrm{d}\sigma(y). \qquad (93)
\end{aligned}
$$

In order to treat the right-hand side of (92), we will need to making use of the extension operator introduced in Subsection 2.4. More precisely, we consider the extensions of $u_\varepsilon^{(1)}$ and $u_\varepsilon^{(2)}$, which can be obtained applying Theorem 2.13 both in Ω_1^ε and Ω_2^ε. In the sequel, for the sake of simplicity, we set $T(u_\varepsilon^{(i)}) = T_\varepsilon^i u_\varepsilon^{(i)}$, $i = 1, 2$. We recall that $u_\varepsilon^{(1)}$ and $u_\varepsilon^{(2)}$ are positive and, without loss of generality, we can assume that also $T(u_\varepsilon^{(1)})$ and $T(u_\varepsilon^{(2)})$ are positive (in fact, if the extension given by Theorem 2.13 would not be positive, we could replace it with its positive part). Moreover, by Theorem 2.13 and (73) we get

$$
\begin{aligned}
& T(u_\varepsilon^{(1)})\chi_{\Omega_1^\varepsilon} = u_\varepsilon^{(1)}\chi_{\Omega_1^\varepsilon}, \qquad T(u_\varepsilon^{(2)})\chi_{\Omega_2^\varepsilon} = u_\varepsilon^{(2)}\chi_{\Omega_2^\varepsilon}, \qquad (94) \\
& u_\varepsilon^{(1)}\chi_{\Omega_1^\varepsilon} \xrightarrow{2-sc} u^{(1)}\chi_{E_1}, \qquad u_\varepsilon^{(2)}\chi_{\Omega_2^\varepsilon} \xrightarrow{2-sc} u^{(2)}\chi_{E_2}, \qquad (95)
\end{aligned}
$$

364 and, by (8), (9), (13) and (14), it follows that there exist v_1, v_2 such that

$$
T(u_\varepsilon^{(1)}) \to v_1\,, \qquad T(u_\varepsilon^{(2)}) \to v_2 \quad \text{strongly in } L^2(\Omega). \qquad (96)
$$

Finally, we obtain

$$\upsilon_1(x) = u^{(1)}(x), \qquad \upsilon_2(x) = u^{(2)}(x), \qquad \text{for a.e. } x \in \Omega. \tag{97}$$

In fact, for $i = 1, 2$, we have that $T(u_\varepsilon^{(i)})\chi_{\Omega_i^\varepsilon} \xrightarrow{2-sc} \upsilon_i \chi_{E_i}$, since $T(u_\varepsilon^{(i)}) \to \upsilon_k$ strongly in $L^2(\Omega)$ on compact sets contained in Ω. Hence, by (95), it follows

$$\int_\Omega u^{(i)}|E_i|\varphi \,\mathrm{d}x \leftarrow \int_\Omega u_\varepsilon^{(i)}\chi_{\Omega_i^\varepsilon}\varphi \,\mathrm{d}x = \int_\Omega T(u_\varepsilon^{(i)})\chi_{\Omega_i^\varepsilon}\varphi \,\mathrm{d}x \to \int_\Omega \upsilon^{(i)}|E_i|\varphi \,\mathrm{d}x\,,$$

for every $\varphi \in \mathcal{C}_c^1(\Omega)$. Therefore, we have proved that

$$T(u_\varepsilon^{(1)}) \to u^{(1)}\,, \qquad T(u_\varepsilon^{(2)}) \to u^{(2)} \quad \text{strongly in } L^2(\Omega). \tag{98}$$

We remark also that, arguing as in (69)–(71), both for $u^{(1)}$ and $u^{(2)}$, we get $u^{(1)} = u^{(2)} = 0$ on $\partial\Omega$.

We are now ready to deal with the right hand side of (92). Taking into account that the integrands in J_ε^1 and J_ε^2 can be assumed positive, we can estimate from above each J_ε^i, $i = 1, 2$, with the integral over the whole Ω. Therefore, reasoning as in (37), we obtain that, as $\varepsilon \to 0$,

$$J_\varepsilon^1 \to 0 \qquad \text{and} \qquad J_\varepsilon^2 \to 0.$$

On the other hand, we rewrite I_ε^i, $i = 1, 2$, in the following way

$$I_\varepsilon^i = \int_{\Omega_i^\varepsilon \cap \{0 \le u_\varepsilon < \delta\}} \frac{f}{u_\varepsilon^\theta}\varphi_i \,\mathrm{d}x + \int_{\Omega_i^\varepsilon \cap \{u_\varepsilon \ge \delta\}} \frac{f}{u_\varepsilon^\theta}\varphi_i \,\mathrm{d}x := I_{\varepsilon,\delta}^{i,1} + I_{\varepsilon,\delta}^{i,2}\,.$$

We can adapt the same argument used for the term $I_{\varepsilon,\delta}^{i,1}$ in the case $\alpha = 1$. In particular, as in the proof of Theorem 4.1, we take $Z_\delta(u_\varepsilon)\varphi_i$ as test function in (12) with Z_δ defined in (40) and we assume $\varphi_i \ge 0$, obtaining

$$\begin{aligned}
I_{\varepsilon,\delta}^{i,1} &\le \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi_i Z_\delta(u_\varepsilon)\,\mathrm{d}x = \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi_i Z_\delta(u_\varepsilon)\big(\chi_{\Omega_1^\varepsilon} + \chi_{\Omega_2^\varepsilon}\big)\,\mathrm{d}x \\
&= \sum_{k=1}^2 \int_\Omega \lambda_\varepsilon^k \nabla u_\varepsilon^{(k)} \cdot \nabla\varphi_i Z_\delta(u_\varepsilon^{(k)})\chi_{\Omega_k^\varepsilon}\,\mathrm{d}x \\
&= \sum_{k=1}^2 \int_\Omega \lambda_\varepsilon^k \nabla T(u_\varepsilon^{(k)}) \cdot \nabla\varphi_i Z_\delta(T(u_\varepsilon^{(k)}))\chi_{\Omega_k^\varepsilon}\,\mathrm{d}x \\
&= \sum_{k=1}^2 \int_\Omega \lambda_\varepsilon^k \nabla T(u_\varepsilon^{(k)}) \cdot \nabla\varphi_i \big(Z_\delta(T(u_\varepsilon^{(k)})) - Z_\delta(u^{(k)})\big)\chi_{\Omega_k^\varepsilon}\,\mathrm{d}x \\
&+ \sum_{k=1}^2 \int_\Omega \lambda_\varepsilon^k \nabla T(u_\varepsilon^{(k)}) \cdot \nabla\varphi_i Z_\delta(u^{(k)})\chi_{\Omega_k^\varepsilon}\,\mathrm{d}x\,.
\end{aligned} \tag{99}$$

Recalling that $\lambda_\varepsilon \nabla T(u_\varepsilon^{(k)})\chi_{\Omega_k^\varepsilon}$ is equi-bounded in $L^2(\Omega)$, using (98) in order to obtain that $Z_\delta(T(u_\varepsilon^{(k)})) \to Z_\delta(u^{(k)})$ strongly in $L^2(\Omega)$, we get

$$\lim_{\varepsilon\to 0}\int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi_i Z_\delta(u_\varepsilon)\,\mathrm{d}x =$$
$$= \sum_{k=1}^{2}\int_{\Omega\times E^k} \lambda^k\left(\nabla u^{(k)} + \nabla_y u_1^{(k)}\right)\nabla\varphi_i Z_\delta(u^{(k)})\,\mathrm{d}x\,\mathrm{d}y\,,$$

where we have taken into account (74), (74) and (94). Hence,

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I_{\varepsilon,\delta}^{i,1} \le \sum_{k=1}^{2}\int_{(\Omega\cap\{u^{(k)}=0\})\times E^k}\left|\lambda^k(\nabla u^{(k)} + \nabla_y u_1^{(k)})\right||\nabla\varphi_i|. \tag{100}$$

By Remark (4.11), for $k = 1, 2$, we may rewrite

$$\int_{(\Omega\cap\{u^{(k)}=0\})\times E^k}\left|\lambda^k(\nabla u^{(k)} + \nabla_y u_1^{(k)})\right||\nabla\varphi_i| =$$
$$= \int_{(\Omega\cap\{u^{(k)}=0\})\times E^k}\left|\lambda^k(I + \nabla_y\chi^{(k)})\nabla u^{(k)}\right||\nabla\varphi_i| = 0\,,$$

because $\nabla u^{(k)}$ vanishes on $\{u^{(k)} = 0\}$. Therefore, we conclude

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I_{\varepsilon,\delta}^{i,1} = 0\,. \tag{101}$$

We now focus our attention on the term $I_{\varepsilon,\delta}^{i,2}$. We have

$$I_{\varepsilon,\delta}^{i,2} = \int_{\Omega_i^\varepsilon\cap\{u_\varepsilon^{(i)}\ge\delta\}} \frac{f}{(u_\varepsilon^{(i)})^\theta}\varphi_i\,\mathrm{d}x = \int_\Omega \frac{f}{(T(u_\varepsilon^{(i)}))^\theta}\chi_{\Omega_i^\varepsilon}\chi_{\{T(u_\varepsilon^{(i)})\ge\delta\}}\varphi_i\,\mathrm{d}x\,. \tag{102}$$

Since $0 \le \frac{f}{(T(u_\varepsilon^{(i)}))^\theta}\varphi_i \le \frac{f}{\delta^\theta}\varphi_i \in L^1(\Omega)$ in the set $\{T(u_\varepsilon^{(i)}) \ge \delta\}$ and $\chi_{\Omega_i^\varepsilon} \rightharpoonup |E^i|$ weakly* in $L^\infty(\Omega)$, we can argue as in (39), once we have taken $\delta \notin \mathcal{C} = \bigcup_{k=1}^2\{\delta > 0\ :\ |\{u^{(k)}(x) = \delta\}| > 0\}$, which is at most countable. Thus we obtain

$$\lim_{\varepsilon\to 0} I_{\varepsilon,\delta}^{i,2} = |E_i|\int_\Omega \frac{f}{(u^{(i)})^\theta}\chi_{\{u^{(i)}>\delta\}}\varphi_i\,\mathrm{d}x\,. \tag{103}$$

Finally, by (92), (93), (101) and (103), we arrive at

$$\begin{aligned} & \int_{\Omega\times E_1} \lambda_1(\nabla u^{(1)} + \nabla_y u_1^{(1)}) \cdot \nabla\varphi_1 \,\mathrm{d}x\,\mathrm{d}y \\ + & \int_{\Omega\times E_2} \lambda_2(\nabla u^{(2)} + \nabla_y u_1^{(2)}) \cdot \nabla\varphi_2 \,\mathrm{d}x\,\mathrm{d}y \\ + & \int_{\Omega\times E_1} \lambda_1(\nabla u^{(1)} + \nabla_y u_1^{(1)}) \cdot \nabla_y\Phi_1 \,\mathrm{d}x\,\mathrm{d}y \\ + & \int_{\Omega\times E_2} \lambda_2(\nabla u^{(2)} + \nabla_y u_1^{(2)}) \cdot \nabla_y\Phi_2 \,\mathrm{d}x\,\mathrm{d}y + \beta \int_{\Omega\times\Gamma} [u][\varphi] \,\mathrm{d}x\,\mathrm{d}\sigma(y) \\ = & \; |E_1| \int_\Omega \frac{f}{(u^{(1)})^\theta} \chi_{\{u^{(1)}>0\}} \varphi_1 \,\mathrm{d}x + |E_2| \int_\Omega \frac{f}{(u^{(2)})^\theta} \chi_{\{u^{(2)}>0\}} \varphi_2 \,\mathrm{d}x\,. \end{aligned} \tag{104}$$

Choosing $\varphi_1, \varphi_2, \Phi_1, \Phi_2$ respectively equal to 0 in (104), we obtain (78)–(82) and (84) with $\frac{f}{(u^{(i)})^\theta}$ replaced by $\frac{f}{(u^{(i)})^\theta}\chi_{\{u^{(i)}>0\}}$, $i = 1, 2$. Moreover, using the factorization of $u_1^{(1)}$ and $u_1^{(2)}$ given in Remark 4.11, we obtain that $(u^{(1)}, u^{(2)})$ solve the system (86), with the new sources $\frac{f}{(u^{(i)})^\theta}\chi_{\{u^{(i)}>0\}}$, $i = 1, 2$. In order to conclude the proof, we have to show that (77) and (83) hold so that we can drop $\chi_{\{u^{(i)}>0\}}$ in (104)). These properties will be proved in Lemma 5.7 in Section 5.2. □

5. Appendix

5.1. Existence and uniqueness for the ε-problem

We devote this subsection to prove the existence and uniqueness for problem (10), following the ideas in [13] as done in [18, Theorem 3.1]. The main difference in the present case is the underline geometrical setting, which requires different a-priori estimates. For this reason and for convenience of the reader, we will give a sketch of the proof.

Since here ε is fixed, we will omit it so that, similarly to Section 2, we rewrite $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ and

$$V_0(\Omega) = \{u = (u^{(1)}, u^{(2)}), u^{(1)} \in H^1(\Omega_1),\, u^{(2)} \in H^1(\Omega_2),\, u = 0 \text{ on } \partial\Omega\},$$

endowed with the norm defined by

$$\|u\|_{V_0(\Omega)} := \|\nabla u\|_{L^2(\Omega_1\cup\Omega_2)} + \|[u]\|_{L^2(\Gamma)}\,.$$

Moreover, we denote by

$$\mathfrak{L}_0(\Omega) = \{\varphi = (\varphi^{(1)}, \varphi^{(2)}) \ : \ \varphi^{(1)} \in \mathrm{Lip}(\overline{\Omega}_1), \varphi^{(2)} \in \mathrm{Lip}(\overline{\Omega}_2), \varphi = 0 \text{ on } \partial\Omega\}.$$

Finally, we set $\lambda(x) = \lambda_1$ a.e. in Ω_1 and $\lambda(x) = \lambda_2$ a.e. in Ω_2.

Theorem 5.1. *Assume that* $f \in L^{\frac{2}{1+\theta}}(\Omega)$, $\theta \in (0,1)$, *and* $f \geq 0$ *a.e. in* Ω, *with* f *not identically zero in* Ω_1 *nor in* Ω_2. *Then, the problem*

$$\begin{aligned} &\left|\int_\Omega \frac{f}{u^\theta}\psi \,\mathrm{d}x\right| < +\infty\,, \\ &\int_\Omega \lambda\nabla u \cdot \nabla\psi \,\mathrm{d}x + \beta\int_\Gamma [u][\psi]\,\mathrm{d}\sigma = \int_\Omega \frac{f}{u^\theta}\psi\,\mathrm{d}x\,, \qquad \forall \psi \in V_0(\Omega)\,, \end{aligned} \tag{105}$$

admits a unique solution $u \in V_0(\Omega)$, *with* $u > 0$ *a.e. in* Ω.

In order to prove the previous result, we first need a preliminary existence result for a sequence of approximating problems. To this purpose, for $n \in \mathbb{N}$, we set

$$f_n(x) = \min\{f(x), n\} \tag{106}$$

and we consider the problem to find $u_n \in V_0(\Omega)$ satisfying the system

$$\begin{aligned} -\mathrm{div}(\lambda\nabla u_n) &= \frac{f_n}{\left(u_n+\frac{1}{n}\right)^\theta}, && \text{in } \Omega_1 \cup \Omega_2; \\ [\lambda\nabla u_n \cdot \nu] &= 0, && \text{on } \Gamma; \\ \beta[u_n] &= \lambda\nabla u_n^{(2)} \cdot \nu, && \text{on } \Gamma; \\ u_n &\geq 0, && \text{in } \Omega; \\ u_n &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{107}$$

whose weak formulation is

$$\int_\Omega \lambda\nabla u_n \cdot \nabla\psi \,\mathrm{d}x + \beta\int_\Gamma [u_n][\psi]\,\mathrm{d}\sigma = \int_\Omega \frac{f_n}{\left(u_n + \frac{1}{n}\right)^\theta}\psi\,\mathrm{d}x\,, \qquad \forall\psi \in V_0(\Omega). \tag{108}$$

Theorem 5.2. *The problem* (107) *admits a unique nonnegative solution* $u_n \in V_0(\Omega)$.

Proof. Let $w \in L^2(\Omega)$ be fixed. For any $n \in \mathbb{N}$ we consider the following nonsingular linear problem

$$\begin{cases} -\mathrm{div}(\lambda \nabla u_n) &= \frac{f_n}{\left(|w|+\frac{1}{n}\right)^\theta}\,, & \text{in } \Omega_1 \cup \Omega_2\,; \\ [\lambda \nabla u_n \cdot \nu] &= 0\,, & \text{on } \Gamma\,; \\ \beta[u_n] &= \lambda_2 \nabla u_n^{(2)} \cdot \nu\,, & \text{on } \Gamma\,; \\ u_n &= 0\,, & \text{on } \partial\Omega, \end{cases} \tag{109}$$

whose weak formulation is

$$\int_\Omega \lambda \nabla u_n \cdot \nabla \psi \,\mathrm{d}x + \beta \int_\Gamma [u_n][\psi]\,\mathrm{d}\sigma = \int_\Omega \frac{f_n}{\left(|w|+\frac{1}{n}\right)^\theta} \psi \,\mathrm{d}x\,, \qquad \forall \psi \in V_0(\Omega). \tag{110}$$

Since the datum $(|w|+\frac{1}{n})^{-\theta} f_n$ is bounded by $n^{1+\theta}$, there exists a unique solution $u_n \in V_0(\Omega)$, as a consequence of the well-known Lax-Milgram Lemma. Moreover, by standard energy estimates and by Poincaré's inequality (2.2), there exists a positive constant C, depending on n but not on w, such that

$$||u_n||_{L^2(\Omega)} \le C||u_n||_{V_0(\Omega)} \le C. \tag{111}$$

In order to prove the existence of a solution to problem (107), we will use Schauder's Theorem. To this purpose we introduce the map $F : L^2(\Omega) \to L^2(\Omega)$ defined by $F(w) = u_n$, where u_n is the solution of (109). Let B be the ball in $L^2(\Omega)$ of radius C, where C is the constant appearing in (111). Clearly $F(B) \subseteq B$. In order to apply the Schauder's Theorem, we need to prove that F is continuous and compact on B. The compactness of F follows by the fact that the inclusion of V_0 in $L^2(\Omega)$ is compact. In order to prove that F is continuous we proceed as follows. Let $\{w_r\} \subset B$ be a sequence in $L^2(\Omega)$ strongly converging to a function $w \in L^2(\Omega)$. We want to prove that $u_{n,r} := F(w_r)$ strongly converges in $L^2(\Omega)$ to $u_n = F(w)$, for $r \to +\infty$. Since w_r is strongly convergent in $L^2(\Omega)$ to w, we have also that, up to a subsequence, $w_r(x) \to w(x)$ for a.e. $x \in \Omega$ and therefore also $\left(|w_r|+\frac{1}{n}\right)^{-\theta} f_n$ converges to $\left(|w|+\frac{1}{n}\right)^{-\theta} f_n$ a.e. in Ω, which implies the strong convergence in $L^q(\Omega)$ for every $q \ge 1$. By (111) with u_n replaced by $u_{n,r}$ and the compactness of the inclusion of V_0 in $L^2(\Omega)$, it follows that there exists $u_n \in V_0$ such that, up to a subsequence,

$$\begin{aligned} &u_{n,r} \to u_n\,, && \text{strongly in } L^2(\Omega), \\ &\nabla u_{n,r} \rightharpoonup \nabla u_n\,, && \text{weakly in } L^2(\Omega), \\ &[u_{n,r}] \rightharpoonup [u_n]\,, && \text{weakly in } L^2(\Gamma). \end{aligned}$$

Passing to the limit in (110) written for $u_{n,r}$ and w_r, it follows that $u_n = F(w)$ and by the uniqueness of the solution of problem (109)–(110) we have that the whole sequence $F(w_{n,r}) = u_{n,r} \to u_n = F(w)$, strongly in $L^2(\Omega)$, for $r \to +\infty$. Hence F is continuous and therefore there exists a fixed point u_n which is a solution of the problem

$$\int_\Omega \lambda \nabla u_n \cdot \nabla \psi \,\mathrm{d}x + \beta \int_\Gamma [u_n][\psi] \,\mathrm{d}\sigma = \int_\Omega \frac{f_n}{\left(|u_n| + \frac{1}{n}\right)^\theta} \psi \,\mathrm{d}x\,, \qquad \forall \psi \in V_0(\Omega).$$

The proof that u_n is nonnegative can be obtained following the same computations at page 15 of [18, Proof of Theorem 3.1], as well as the proof that the solution u_n is unique follows by [18, Proof of Theorem 3.5]. □

Proof of Theorem 5.1. Taking u_n as test function in (108) and using the Poincaré inequality (1), we obtain

$$\begin{aligned} \int_\Omega u_n^2 \,\mathrm{d}x &\le C \left(\int_\Omega |\nabla u_n|^2 \,\mathrm{d}x + \int_\Gamma [u_n]^2 \,\mathrm{d}\sigma \right) \le C \int_\Omega \frac{f_n}{\left(u_n + \frac{1}{n}\right)^\theta} u_n \,\mathrm{d}x \\ &\le C ||f||_{L^{\frac{2}{1+\theta}}(\Omega)} ||u_n||^{1-\theta}_{L^2(\Omega)} \end{aligned}$$

and hence

$$||u_n||_{L^2(\Omega)} \le C ||u_n||_{V_0} \le C ||f||^{\frac{1}{1+\theta}}_{L^{\frac{2}{1+\theta}}(\Omega)}, \tag{112}$$

where C is independent of n. By (112) and the compactness of the inclusion of V_0 in $L^2(\Omega)$, we infer that there exists $u \in V_0$, $u \ge 0$ a.e. in Ω, such that, up to a subsequence,

$$\begin{aligned} &u_n \to u\,, && \text{strongly in } L^2(\Omega); \\ &\nabla u_n \rightharpoonup \nabla u\,, && \text{weakly in } L^2(\Omega); \\ &[u_n] \rightharpoonup [u]\,, && \text{weakly in } L^2(\Gamma). \end{aligned} \tag{113}$$

Moreover, by (108), with $\psi \in V_0(\Omega)$, and (112), we obtain

$$\left| \int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^\theta} \psi \,\mathrm{d}x \right| \le C\,,$$

so that, when $n \to +\infty$, by Fatou's Lemma it follows

$$\left| \int_\Omega \frac{f}{u^\theta} \psi \,\mathrm{d}x \right| \le C\,, \tag{114}$$

which also implies that u is not identically zero in Ω (nor in Ω_1 neither in Ω_2). Now, we can pass to the limit in the weak formulation (108). Clearly, the left-hand side converges to the left-hand side of (105). In order to pass to the limit in the right-hand side, we proceed again as in [18, Proof of Theorem 3.1], assuming that ψ is a nonnegative function belonging to $\mathfrak{L}_0(\Omega)$. As in (38), we can write

$$I_n = \int_{\Omega\cap\{0\le u_n\le\delta\}} \frac{f_n}{\left(u_n+\frac{1}{n}\right)^\theta}\psi\,\mathrm{d}x + \int_{\Omega\cap\{u_n>\delta\}} \frac{f_n}{\left(u_n+\frac{1}{n}\right)^\theta}\psi\,\mathrm{d}x := I^1_{n,\delta} + I^2_{n,\delta}\,, \tag{115}$$

where

$$\lim_{\delta\to 0}\lim_{n\to+\infty} I^2_{n,\delta} = \int_{\Omega\cap\{u>0\}} \frac{f}{u^\theta}\psi\,\mathrm{d}x\,. \tag{116}$$

Moreover, using as test function in (108) the function $Z_\delta(u_n)\psi$, with Z_δ defined in (40) and ψ as above, we arrive at

$$I^1_{n,\delta} \le \int_\Omega \lambda\nabla u_n\cdot\nabla\psi Z_\delta(u_n)\,\mathrm{d}x + 2\beta\delta||\psi^{(2)}+\psi^{(1)}||_{L^1(\Gamma)}, \tag{117}$$

as in [18, Proposition 4.4]. Therefore,

$$\lim_{n\to+\infty} I^1_{n,\delta} \le \int_\Omega \lambda\nabla u\cdot\nabla\psi Z_\delta(u)\,\mathrm{d}x + 2\beta\delta||\psi^{(2)}+\psi^{(1)}||_{L^1(\Gamma)},$$

where we have taken into account that $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$ and $Z_\delta(u_n)\to Z_\delta(u)$ strongly in $L^2(\Omega)$, since $s\mapsto Z_\delta(s)$ is continuous and $u_n\to u$ strongly in $L^2(\Omega)$. Then, passing to the limit as $\delta\to 0$, we get

$$\lim_{\delta\to 0}\lim_{n\to+\infty} I^1_{n,\delta} \le \int_{\Omega\cap\{u=0\}} \lambda\nabla u\cdot\nabla\psi\,\mathrm{d}x = 0\,, \tag{118}$$

where we have taken into account that $\nabla u = 0$ a.e. on the level set $\{u=0\}$. Clearly, as done before, we have paid attention to choose $\delta\notin\mathcal{C}=\{\delta>0 : |\{u(x)=\delta\}|>0\}$, which is at most countable.

From (115), (116), (118), the density of $\mathfrak{L}_0(\Omega)$ in $V_0(\Omega)$ and the standard decomposition of $\psi\in V_0(\Omega)$ as $\psi=\psi^+-\psi^-$, it follows that u satisfies

$$\int_\Omega \lambda\nabla u\cdot\nabla\psi\,\mathrm{d}x + \beta\int_\Gamma [u][\psi]\,\mathrm{d}\sigma = \int_\Omega \frac{f}{u^\theta}\chi_{\{u>0\}}\psi\,\mathrm{d}x\,,$$

for every $\psi \in V_0(\Omega)$. It remains to prove that $u > 0$ a.e. in Ω, in order to replace $\frac{f}{u^\theta}\chi_{\{u>0\}}$ with $\frac{f}{u^\theta}$. This is a direct consequence of the maximum principle (see [21, Theorem 8.19] and also [20, Proposition 3.5]) applied to u in Ω_1 and Ω_2, separately, recalling that (114) implies that u is not identically zero in Ω_1 nor in Ω_2. Indeed, in the connected/connected geometry the maximum principle can be applied since $\inf u = 0$ in each Ω_i, $i = 1, 2$ (being $u = 0$ in $\partial\Omega \cap \partial\Omega_i \neq \emptyset$, $i = 1, 2$). The same approach can be followed in the connected/disconnected geometry for the outer domain Ω_2, where we have $u = 0$ on $\partial\Omega \cap \partial\Omega_2 \neq \emptyset$. On the contrary in Ω_1, taking into account that u is nonnegative (being the strong L^2-limit of the sequence of positive function u_n) we should distinguish two different situations: or $\inf u > 0$ in Ω_1 and, therefore, there is nothing to prove, or $\inf u = 0$ in Ω_1 and in this case we can appeal again to the maximum principle. □

5.2. Positivity of the bidomain homogenized solution

We devote this subsection to the proof of the strict positivity a.e. in Ω of the solution of the bidomain problem (78)–(84) obtained from the homogenization of the system (10) in the case $\alpha = -1$ (Lemma 5.7 below). Notice that this result can be obtained from (17), by using the so-called *two-scale decomposition* introduced in [29] and following the approach used in [30, Section 1]. However, due to the special factorized form of the integral in the left-hand side of (17), we prefer to give a direct proof based on the unfolding homogenization technique which, in this case, essentially corresponds to the *two-scale decomposition*. To this purpose, we recall the definition and those properties of the unfolding operator which are necessary in order to achieve our result (see [16, 17]).

Let us set

$$\Xi_\varepsilon = \left\{\xi \in \mathbb{Z}^N , \quad \varepsilon(\xi + Y) \subset \Omega\right\} , \quad \widehat{\Omega}_\varepsilon = \text{interior}\left\{\bigcup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi + \overline{Y})\right\} .$$

Denoting by $[r]$ the integer part of $r \in \mathbb{R}$, we define for $x \in \mathbb{R}^N$

$$\left[\frac{x}{\varepsilon}\right]_Y = \left(\left[\frac{x_1}{\varepsilon}\right], \dots, \left[\frac{x_N}{\varepsilon}\right] \right), \quad \text{so that} \qquad x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right) .$$

Definition 5.3. *For w Lebesgue-measurable on Ω the periodic unfolding operator $\mathcal{T}_\varepsilon$ is defined as*

$$\mathcal{T}_\varepsilon(w)(x,y) = \begin{cases} w\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon y\right), & (x,y) \in \widehat{\Omega}_\varepsilon \times Y\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

Clearly, $\mathcal{T}_\varepsilon$ is linear and for w_1, w_2 as in Definition 5.3

$$\mathcal{T}_\varepsilon(w_1 w_2) = \mathcal{T}_\varepsilon(w_1)\mathcal{T}_\varepsilon(w_2)\,. \tag{119}$$

Proposition 5.4. *Let $w \in L^1(\Omega)$, then*

$$\int_{\Omega\times Y} |\mathcal{T}_\varepsilon(w)|\,\mathrm{d}x\,\mathrm{d}y \le \int_\Omega |w|\,\mathrm{d}x\,. \tag{120}$$

Proposition 5.5. *Let $\{w_\varepsilon\}$ be a sequence of functions in $L^p(\Omega)$, $p > 1$. If $w_\varepsilon \to w$ strongly in $L^p(\Omega)$ as $\varepsilon \to 0$, then*

$$\mathcal{T}_\varepsilon(w_\varepsilon) \to w\,, \quad \text{strongly in } L^p(\Omega \times Y)\,. \tag{121}$$

Proposition 5.6. *Let $\phi : Y \to \mathbb{R}$ be a function extended by Y-periodicity to the whole of $\mathbb{R}^N$ and define the sequence*

$$\phi^\varepsilon(x) = \phi\left(\frac{x}{\varepsilon}\right)\,, \qquad x \in \mathbb{R}^N\,. \tag{122}$$

If ϕ is measurable on Y, then

$$\mathcal{T}_\varepsilon(\phi^\varepsilon)(x,y) = \begin{cases} \phi(y)\,, & (x,y) \in \widehat{\Omega}_\varepsilon \times Y\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{123}$$

Moreover, if $\phi \in L^p(Y)$, $p > 1$, as $\varepsilon \to 0$

$$\mathcal{T}_\varepsilon(\phi^\varepsilon) \to \phi\,, \qquad \text{strongly in } L^p(\Omega \times Y)\,. \tag{124}$$

Lemma 5.7. *Under the assumption of Theorem 4.10,*

$$\left|\int_\Omega \frac{f}{(u^{(i)})^\theta}\varphi\,\mathrm{d}x\right| < +\infty\,, \qquad \forall \varphi \in H_0^1(\Omega)\,, \quad i = 1, 2\,, \tag{125}$$

holds and the functions $u^{(1)}$ and $u^{(2)}$ are strictly positive a.e. in Ω.

Proof. As in the proof of Theorem 4.10, let T denotes the extension operator. Recalling that, for a.e. $x \in \Omega$, $\chi_{\Omega_1^\varepsilon}(x) = \chi_{E_1}(\varepsilon^{-1}x)$ and $\chi_{\Omega_2^\varepsilon}(x) = \chi_{E_2}(\varepsilon^{-1}x)$, extended by periodicity from Y to the whole of $\mathbb{R}^N$, and taking into account (98) and the properties of the unfolding operator (119), (121) and (124), we have that

$$\begin{aligned}\mathcal{T}_\varepsilon(u_\varepsilon) = \mathcal{T}_\varepsilon(u_\varepsilon\chi_{\Omega_1^\varepsilon} + u_\varepsilon\chi_{\Omega_2^\varepsilon}) &= \mathcal{T}_\varepsilon\left(T(u_\varepsilon^{(1)})\chi_{\Omega_1^\varepsilon} + T(u_\varepsilon^{(2)})\chi_{\Omega_2^\varepsilon}\right)\\ &= \mathcal{T}_\varepsilon\left(T(u_\varepsilon^{(1)})\right)\mathcal{T}_\varepsilon\left(\chi_{\Omega_1^\varepsilon}\right) + \mathcal{T}_\varepsilon\left(T(u_\varepsilon^{(2)})\right)\mathcal{T}_\varepsilon\left(\chi_{\Omega_2^\varepsilon}\right)\\ &\longrightarrow u^{(1)}\chi_{E_1} + u^{(2)}\chi_{E_2}\,, \qquad \text{strongly in } L^1(\Omega\times Y).\end{aligned}$$

Therefore, there exists a set $\mathcal{N} \subset \Omega \times Y$, with $|\mathcal{N}| = 0$, such that

$$\mathcal{T}_\varepsilon(u_\varepsilon)(x,y) \to u^{(1)}(x)\chi_{E_1}(y) + u^{(2)}(x)\chi_{E_2}(y)$$

for every $(x,y) \in (\Omega\times Y)\setminus\mathcal{N}$. Then, by (17) with $\psi \in \mathcal{C}_c^1(\Omega)$, $\psi \ge 0$, (13) and applying Fatou's Lemma, we get

$$\begin{aligned}\int_{\Omega\times Y}\frac{f}{(u^{(1)}\chi_{E_1} + u^{(2)}\chi_{E_2})^\theta}\psi\,\mathrm{d}x\,\mathrm{d}y &\le \liminf_{\varepsilon\to 0}\int_{\Omega\times Y}\frac{\mathcal{T}_\varepsilon(f)}{\mathcal{T}_\varepsilon(u_\varepsilon)^\theta}\mathcal{T}_\varepsilon(\psi)\,\mathrm{d}x\,\mathrm{d}y\\ = \liminf_{\varepsilon\to 0}\int_{\Omega\times Y}\mathcal{T}_\varepsilon\left(\frac{f}{u_\varepsilon^\theta}\psi\right)\mathrm{d}x\,\mathrm{d}y &\le \liminf_{\varepsilon\to 0}\int_\Omega\frac{f}{u_\varepsilon^\theta}\psi\,\mathrm{d}x \le C\,, \end{aligned} \tag{126}$$

where we used also (120). Inequality (126) implies, in particular,

$$|E_i|\int_\Omega\frac{f}{(u^{(i)})^\theta}\psi\,\mathrm{d}x = \int_{\Omega\times E_i}\frac{f}{(u^{(i)}\chi_{E_i})^\theta}\psi\,\mathrm{d}x\,\mathrm{d}y \le C\,, \qquad i = 1,2\,; \tag{127}$$

thus, (125) is proved and hence, taking into account that $f > 0$ a.e. in Ω, (127) implies $u^{(i)} > 0$ a.e. in Ω, $i = 1, 2$. ☐

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