Differentiability properties of the symbol of a generalized Riesz potential with homogeneous characteristic

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Dedicated to Nina Uraltseva

Abstract. Let f be a positive homogeneous function of degree 0 defined on the sphere Σ of the space $\mathbb{R}^n$ and let Φ_α be the symbol of the integral operator

$$\int_{\mathbb{R}^n} \frac{f((\mathbf{x}-\mathbf{y})/|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|^{n-\alpha}} u(\mathbf{y}) d\mathbf{y}, \quad u \in C_0^\infty(\mathbb{R}^n)$$

with $0<\alpha<n$. We study differentiability properties of the restriction of Φ_α to the unit sphere Σ in the spaces $H_p^l(\Sigma)$ for $p \in (1,\infty)$. Here $H_p^l(\Sigma)$ denotes the space of Bessel potentials with the norm $\|f\|_{H_p^l(\Sigma)} = \|(\delta+I)^{l/2} f\|_{L_p(\Sigma)}$, δ being the Beltrami operator on the sphere. We prove that, if $f \in L_p(\Sigma)$ then $\Phi_\alpha|_\Sigma \in H_p^\ell(\Sigma)$ for any $\ell \leq n/2-\alpha-|p^{-1}-2^{-1}|(n-2)$. Conversely, if $\Phi_\alpha|_\Sigma \in H_p^\ell(\Sigma)$, with $\ell \geq n/2-\alpha+|p^{-1}-2^{-1}|(n-2)$, then $f \in L_p(\Sigma)$. The results are sharp.

1 Introduction

Let Σ be the unit sphere in the space $\mathbb{R}^n$ centered at the origin and let f be a positive homogeneous function of degree zero defined through the space $\mathbb{R}^n \setminus 0$ and suppose that $f \in L_p(\Sigma)$ with $p>1$. Let us consider the integral operator

$$\mathcal{K}_\alpha u(\mathbf{x}) = \int_{\mathbb{R}^n} K_\alpha(\mathbf{x}-\mathbf{y}) u(\mathbf{y}) d\mathbf{y}, \quad u \in C_0^\infty(\mathbb{R}^n) \tag{1.1}$$

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where the kernel has the form

$$K_\alpha(\mathbf{x}) = \frac{f(\vartheta)}{|\mathbf{x}|^{n-\alpha}}, \quad \mathbf{x} \in \mathbb{R}^n \setminus 0, \quad \vartheta = \frac{\mathbf{x}}{|\mathbf{x}|}$$

with $0 < \alpha < n$, $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ are points in $\mathbb{R}^n$. The integral (1.1) is called a generalized Riesz potential. The function $f(\vartheta)$ is the *characteristic* of the $n-$ dimensional integral operator (1.1) with kernel $K_\alpha(\mathbf{x})$. If $\alpha = 0$ then (1.1) is a singular integral ([16]) and the function K_0 exists as a generalized function if ([9, p.310])

$$\int_\Sigma f(\vartheta) d\sigma_\vartheta = 0\,. \tag{1.2}$$

We denote by $\mathcal{F}$ the Fourier transform of functions given on $\mathbb{R}^n$

$$\widehat{f}(\mathbf{x}) = (\mathcal{F}f)(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{y}) \mathrm{e}^{i\mathbf{x}\cdot\mathbf{y}} d\mathbf{y}, \qquad \mathbf{x}\cdot\mathbf{y} = x_1 y_1 + ... + x_n y_n.$$

The Fourier transform of the kernel $K_\alpha(\mathbf{x})$, understood in the sense of generalized functions ([9], cf. also [12]), is called the *symbol* of the integral operator. We denote the symbol by $\Phi_\alpha(\mathbf{y}) = \mathcal{F}_{\mathbf{x}\to\mathbf{y}} K_\alpha = (A_\alpha f)(\mathbf{y})$. Since the kernel $K_\alpha(\mathbf{x})$ is a positive homogeneous (generalized) function of degree $-n+\alpha$, then the symbol is a homogeneous function of degree $-\alpha$. We remark that, when $\alpha = n/2$ and $K_{n/2}(\mathbf{x}) = f(\vartheta)|\mathbf{x}|^{-n/2}$ is an eigenfunction of the Fourier transform with eigenvalue λ, then

$$A_{n/2} f(\omega) = \lambda f(\omega), \qquad \lambda^4 = (2\pi)^{2n}\,.$$

Eigenfunctions of the Fourier transform in the sense of generalized functions are studied in [12, 13].

If $\alpha = 0$ the following integral representation for the symbol Φ_0 by its characteristic f was obtained by Calderón and Zygmund ([16, p.249])

$$\Phi_0(\omega) = A_0 f(\omega) = \int_\Sigma f(\vartheta) \left(\log \frac{1}{|\cos\gamma|} - \frac{i\pi}{2} \mathrm{sign}\,(\cos\gamma) \right) d\sigma_\vartheta\,, \omega \in \Sigma,$$

γ denoting the angle between the vectors ϑ and ω. The symbol Φ_0, as well as the characteristic f, is a homogeneous function of degree 0 with zero mean on Σ. The singular kernel $|\mathbf{x}|^{-n} f(\vartheta)$, which is homogeneous of degree $-n$, can be uniquely recovered by its Fourier transform Φ_0 ([17, Theorem 2.16, p.116]).

We denote by $H^l_p(\Sigma)$ the space of Bessel potentials on the sphere (cf., e.g., [1]). If $1 < p < \infty$ and $-\infty < l < \infty$ the space $H^l_p(\Sigma)$ consists of functions f defined on Σ such that $(\delta + I)^{l/2} f \in L_p(\Sigma)$, with the norm

$$\|f\|_{H^l_p(\Sigma)} = \|(\delta + I)^{l/2} f\|_{L_p(\Sigma)}$$

([14, Proposition 2.3.2]). Here δ denotes the Beltrami operator on the sphere (the spherical part of the Laplace operator), I the identity operator and $||\cdot||_{L_p(\Sigma)}$ is the norm in $L_p(\Sigma)$. The space $C^\infty(\Sigma)$ is dense in $H^l_p(\Sigma)$.

The differentiability properties of the symbol Φ_0 of the singular integral

$$\int_{\mathbb{R}^n} \frac{f((\mathbf{x}-\mathbf{y})/(|\mathbf{x}-\mathbf{y}|))}{|\mathbf{x}-\mathbf{y}|^n} u(\mathbf{y}) d\mathbf{y}$$

in the space $W^l_2(\Sigma) = H^l_2(\Sigma), l > 0$, were studied by Mikhlin [16], Agranovich [3] and Mikhailova-Gubenko [15] and are expressed in the following theorem.

Theorem 1.1. *([16, Theorem 7.1, p. 266]) The symbol of a singular integral satisfies the relation $\Phi_0 \in H^{n/2}_2(\Sigma)$ if and only if the characteristics $f \in L_2(\Sigma)$.*

Gadzjiev in [7, 8] described the smoothness of $\Phi_0 = A_0 f$ with $f \in L_p(\Sigma)$ in terms of the space $H^l_p(\Sigma)$ with $p \in (1, \infty)$. Gadjiev's results can be formulated as follows.

Theorem 1.2. *([7, 8]) Let $1 < p < \infty$ and $\ell_0 = (n-2)\left|p^{-1} - 2^{-1}\right|$. Then*

$$f \in L_p(\Sigma) \Rightarrow \Phi_0 \in H^{n/2-\ell_0}_p(\Sigma) \tag{1.3}$$

$$\Phi_0 \in H^{n/2+\ell_0}_p(\Sigma) \Rightarrow f \in L_p(\Sigma)\,. \tag{1.4}$$

The implication given are sharp.

The imbedding (1.3) means that if the characteristic f belongs to $L_p(\Sigma)$ and is orthogonal to 1 on Σ, then the corresponding symbol Φ_0 belongs to $H^{n/2-\ell_0}_p(\Sigma)$ and

$$||\Phi_0||_{H^{n/2-\ell_0}_p(\Sigma)} \le C\,||f||_{L_p(\Sigma)}$$

where the constant C does not depend on f. The optimality of (1.3) means that there exists a function $f \in L_p(\Sigma)$ such that the corresponding symbol Φ_0 does not belong to $H^\ell_p(\Sigma)$ for any $\ell > n/2 - \ell_0$.

The imbedding (1.4) means that if Φ_0 belongs to $H_p^{n/2+\ell_0}(\Sigma)$ then there exists a function $f \in L_p(\Sigma)$ with zero mean value on the sphere such that $\Phi_0 = A_0 f$ and

$$||f||_{L_p(\Sigma)} \leq C\, ||\Phi_0||_{H_p^{n/2+\ell_0}(\Sigma)}\,.$$

Moreover, for any $\ell < n/2+\ell_0$ there exists a symbol $\Phi_0 \in H_p^\ell(\Sigma)$ such that the corresponding characteristic f does not belong to $L_p(\Sigma)$.

Kryuchkov in [10, 11] extended the description of $A_0 L_p(\Sigma)$ given by Gadjiev by including spaces $H_q^l(\Sigma)$ for $q \neq p$.

Questions about the connection between the smoothness of the characteristic f and of the symbol Φ_α have been studied by Samko ([19]) in the space $C^\lambda(\Sigma)$ and by Plamenevskii and Judovin ([18]) in the space $H_2^l(\Sigma)$.

The aim of this paper is to study the differentiability properties of the restriction of the symbol Φ_α to the unit sphere, with $0 < \alpha < n$, in terms of the spaces $H_p^l(\Sigma)$ with $1 < p < \infty$. This problem consists in finding conditions on the indices ℓ and s such that

$$f \in L_p(\Sigma) \Rightarrow \Phi_\alpha \in H_p^\ell(\Sigma)\,, \qquad \Phi_\alpha \in L_p(\Sigma) \Rightarrow f \in H_p^s(\Sigma)\,.$$

The main tool for obtaining our results is the use of the multipliers on the sphere.

The article is organized as follows. In Section 2 we introduce an integral representation over the sphere of the symbol Φ_α by means of the characteristic f and a representation in the form of a series of spherical functions. The last representation is employed to study the differentiability properties of the symbol Φ_α. In Section 3 we prove that, if $f \in L_p(\Sigma)$ then $\Phi_\alpha \in H_p^\ell(\Sigma)$ with $\ell \leq n/2-\alpha-|p^{-1}-2^{-1}|(n-2)$, while $\Phi_\alpha \notin H_p^\ell(\Sigma)$ for any $\ell > n/2-\alpha-|p^{-1}-2^{-1}|(n-2)$. In Section 4 we prove that if $\Phi_\alpha \in H_p^l(\Sigma)$ with $\ell \geq n/2-\alpha+|p^{-1}-2^{-1}|(n-2)$ then there exists $f \in L_p(\Sigma)$ such that $A_\alpha f = \Phi_\alpha$, while the assertion fails for any $\ell < n/2-\alpha+|p^{-1}-2^{-1}|(n-2)$.

2 Analysis of the symbol Φ_α

The symbol Φ_α is homogeneous of degree $-\alpha$ (i.e. $\Phi_\alpha(tx) = t^{-\alpha}\Phi(x), t > 0$) and can be viewed as an operator applied to the characteristic. Indeed, we have

$$\begin{aligned}\Phi_\alpha(\mathbf{y}) &= \int_{\mathbb{R}^n} \frac{f(\frac{\mathbf{x}}{|\mathbf{x}|})}{|\mathbf{x}|^{n-\alpha}} \mathrm{e}^{i\mathbf{x}\cdot\mathbf{y}} d\mathbf{x} = \int_\Sigma f(\vartheta) d\sigma_\vartheta \int_0^\infty R^{\alpha-1} \mathrm{e}^{iR\rho\omega\cdot\vartheta} dR \\ &= |\mathbf{y}|^{-\alpha} \int_\Sigma f(\vartheta) d\sigma_\vartheta \int_0^\infty R^{\alpha-1} \mathrm{e}^{iR\omega\cdot\vartheta} dR\,.\end{aligned}$$

Here ω, ϑ are unit vectors, $\rho = |\mathbf{y}|$, $R = |\mathbf{x}|$, $\mathbf{x} = (R, \vartheta)$, $\mathbf{y} = (\rho, \omega)$, $\omega \cdot \vartheta = \omega_1\vartheta_1 + ... + \omega_n\vartheta_n$. For all $0 < \alpha < n$ ([9, p.171])

$$\int_0^\infty R^{\alpha-1} \mathrm{e}^{iR\sigma} dR = \mathrm{e}^{i\alpha\frac{\pi}{2}}\Gamma(\alpha)(\sigma + i0)^{-\alpha}\,.$$

Then, for $\omega \in \Sigma$,

$$\Phi_\alpha(\omega) = \mathrm{e}^{i\alpha\frac{\pi}{2}}\Gamma(\alpha)\int_\Sigma (\omega\cdot\vartheta + i0)^{-\alpha} f(\vartheta) d\sigma_\vartheta, \qquad 0 < \alpha < n\,.$$

The expression $(x + i0)^{-\alpha}$ with a real variable x and a complex exponent α is understood in the distributional sense ([9, p.60]), namely

$$\begin{aligned} (x + i0)^{-\alpha} &= x_+^{-\alpha} + \mathrm{e}^{-i\,\alpha\,\pi} x_-^{-\alpha}, && \alpha \neq 1, 2, ... \\ (x + i0)^{-m} &= x^{-m} - i\pi\frac{(-1)^{m-1}}{(m-1)!}\delta^{(m-1)}(x), && m = 1, 2, ... \end{aligned}$$

Here we used the standard notation

$$x_+^\alpha = \begin{cases} x^\alpha & x > 0 \\ 0 & x < 0 \end{cases} \qquad x_-^\alpha = \begin{cases} 0 & x > 0 \\ |x|^\alpha & x < 0 \end{cases}$$

with δ being the Dirac distribution. $(x + i0)^\alpha$ is an entire function in the parameter α.

We denote the operator taking the characteristic into the symbol by A_α that is $\Phi_\alpha|_\Sigma = A_\alpha f$. Summarizing, the operator A_α can be expressed in terms of f by the formula

$$\begin{aligned} (A_0 f)(\omega) = &\quad \int_\Sigma \left(\log\frac{1}{|\cos(\omega\cdot\vartheta)|} - \frac{i\pi}{2}\mathrm{sign}\,(\cos(\omega\cdot\vartheta))\right) f(\vartheta) d\sigma_\vartheta, \\ (A_\alpha f)(\omega) = &\quad \mathrm{e}^{i\alpha\frac{\pi}{2}}\Gamma(\alpha)\int_\Sigma ((\omega\cdot\vartheta)_+^{-\alpha} + \mathrm{e}^{-i\,\alpha\,\pi}(\omega\cdot\vartheta)_-^{-\alpha}) f(\vartheta) d\sigma_\vartheta, \\ &\quad \alpha \neq 0, 1, 2, 3, ... \\ (A_m f)(\omega) = &\; i^m (m-1)!\int_\Sigma ((\omega\cdot\vartheta)^{-m} - \frac{i\pi(-1)^{m-1}}{(m-1)!}\delta^{(m-1)}(\omega\cdot\vartheta)) f(\vartheta) d\sigma_\vartheta, \\ &\quad m = 1, 2, 3, ... \end{aligned}$$

We denote by $Y_{m,n}^{(k)}(\omega)$ the spherical functions of order m in the n dimensional space, ω is a point of the unit sphere Σ. The upper index k

numbers the linearly independent spherical functions of the same order m and it varies between the bounds

$$1 \leq k \leq k_{m,n} = (2m+n-2)\frac{(m+n-3)!}{(n-2)!m!}\,.$$

The functions $Y_{m,n}^{(k)}(\omega)$ are supposed to be orthonormal in $L_2(\Sigma)$. The spherical functions are eigenfunctions of the Beltrami operator δ and the corresponding eigenvalues are $\lambda_{m,n} = m(m+n-2)$ ([16, p.215]).

We expand the characteristic f in a series of spherical functions (Fourier - Laplace series)

$$f(\theta) = \sum_{m=0}^{\infty} \sum_{k=1}^{k_{m,n}} f_m^{(k)} Y_{m,n}^{(k)}(\theta)\,, \quad \theta \in \Sigma \tag{2.5}$$

where

$$f_m^{(k)} = \int_{\Sigma} f(\vartheta) Y_{m,n}^{(k)}(\vartheta) d\sigma_{\vartheta}.$$

If $\alpha = 0$, by the assumption (1.2) f is ortogonal to 1 on Σ, then $f_0^{(1)} = 0$ and the series (2.5) starts from $m = 1$. For $f \in L^p(\Sigma)$, $1 \leq p \leq \infty$, the convergence of (2.5) can be understood in generalized sense ([6, p.42]). If $f \in C^\infty(\Sigma)$, then (2.5) converges absolutely and uniformly.

Definition 2.1. *Any operator acting on functions f in* (2.5) *by the formula*

$$Tf = \sum_{m=0}^{\infty} t_m \sum_{k=1}^{k_{m,n}} f_m^{(k)} Y_{m,n}^{(k)}(\theta)$$

is called an operator with multipliers $\{t_m\}$. The numbers $\{t_m\}$ are called $(p,q)-$multipliers on the sphere Σ if

$$||Tf||_{L_q(\Sigma)} \leq C\,||f||_{L_p(\Sigma)}\,.$$

An operator acting boundedly from $L_p(\Sigma)$ to $L_q(\Sigma)$ is called an operator of strong type (p,q). We henceforth denote this as follows: $\{t_m\} \in M_{pq}$ or $\{t_m\} \in M_p$ if $p = q$.

With the notations $\omega = \mathbf{y}/|\mathbf{y}|$ and $\vartheta = \mathbf{x}/|\mathbf{x}|$, as a consequence of the Bochner formula ([5, p.807]), we have

$$\int_{\mathbb{R}^n} \frac{Y_{m,n}^{(k)}(\vartheta)}{|\mathbf{x}|^{n-\alpha}} \mathrm{e}^{i\mathbf{y}\cdot\mathbf{x}} d\mathbf{x} = \mu_m(\alpha) \frac{Y_{m,n}^{(k)}(\omega)}{|\mathbf{y}|^{\alpha}}$$

with

$$\mu_m(\alpha) = i^m\,\pi^{n/2} 2^\alpha \frac{\Gamma\left(\frac{m+\alpha}{2}\right)}{\Gamma\left(\frac{m+n-\alpha}{2}\right)}\,.$$

It follows that, for $\omega \in \Sigma$,

$$A_\alpha Y^{(k)}_{m,n}(\omega) = \mathcal{F}_{\mathbf{x}\to\mathbf{y}}(\frac{Y^{(k)}_{m,n}(\vartheta)}{|\mathbf{x}|^{n-\alpha}})(\omega) = \int_{\mathbb{R}^n} \frac{Y^{(k)}_{m,n}(\vartheta)}{|\mathbf{x}|^{n-\alpha}} \mathrm{e}^{i\omega\cdot\mathbf{x}} d\mathbf{x} = \mu_m(\alpha) Y^{(k)}_{m,n}(\omega).$$

For functions f given in (2.5), the restriction of the symbol Φ_α on the sphere is defined by the series ([18, p.210])

$$\Phi_\alpha(\omega) = \sum_{m=0}^{\infty} \mu_m(\alpha) \sum_{k=1}^{k_{m,n}} f^{(k)}_m Y^{(k)}_{m,n}(\omega) \tag{2.6}$$

and, according to Definition 2.1, the symbol Φ_α is an operator with the multipliers $\{\mu_m(\alpha)\}$.
Since the symbol Φ_α is the Fourier transform of the kernel $f(\vartheta)|\mathbf{x}|^{-n+\alpha}$, by applying the inverse Fourier transform (understood in the sense of generalized functions) we get

$$f(\vartheta) = |\mathbf{x}|^{n-\alpha}(\mathcal{F}^{-1}_{\mathbf{y}\to\mathbf{x}}\Phi_\alpha)\,, \quad \vartheta = \frac{\mathbf{x}}{|\mathbf{x}|}, \quad \omega = \frac{\mathbf{y}}{|\mathbf{y}|}\,.$$

Hence the function f defines an operator whose symbol on the sphere coincides with Φ_α and we denote $f = A^{-1}_\alpha \Phi_\alpha$. The multipliers on the sphere associated to the operator A^{-1}_α are $\{(\mu_m(\alpha))^{-1}\}$.

Theorem 1.1 is based on the following theorem, proved by Mikhlin ([16]) for integer values of l and improved indipendently by Agranovich ([3]) and Mikhailova-Gubenko ([15]).

Theorem 2.1. *Let l be a real number. Assume that a function f admits the expansion* (2.5)*. Then $f \in H^l_2(\Sigma)$ if and only if*

$$\sum_{m=0}^{\infty} \sum_{k=1}^{k_{m,n}} m^{2l} |f^{(k)}_m|^2 < \infty\,. \tag{2.7}$$

For the integral operator (1.1) and $p = 2$ the following result, based on Theorem 2.1, holds.

Theorem 2.2. *Let $0 < \alpha < n$. Then $f \in L_2(\Sigma)$ if and only if $\Phi_\alpha|_\Sigma \in H_2^{\frac{n}{2}-\alpha}(\Sigma)$.*

Proof. By Stirling's formula [2, 6.1.39]

$$\Gamma(p/2) \approx \sqrt{2\pi}\, e^{-p/2}(p/2)^{(p-1)/2} \quad p \to \infty$$

we obtain

$$\mu_m(\alpha) \approx (2\,\pi)^{n/2} m^{\alpha-n/2} \quad m \to \infty\,. \tag{2.8}$$

Let $f \in L_2(\Sigma)$. Therefore, since by Theorem 2.1 the series (2.7) converges, we deduce that

$$\sum_{m=0}^{\infty}\sum_{k=1}^{k_{m,n}} m^{n-2\alpha}(\mu_m(\alpha))^2|f_m^{(k)}|^2 < +\infty.$$

Keeping in mind (2.6) and Theorem 2.1 we conclude that $\Phi_\alpha|_\Sigma \in H_2^{\frac{n}{2}-\alpha}(\Sigma)$.

Conversely, let $\Phi_\alpha|_\Sigma \in H_2^{\frac{n}{2}-\alpha}(\Sigma)$. Then $g = (\delta + I)^{\frac{n}{2}-\alpha}\Phi_\alpha|_\Sigma \in L_2(\Sigma)$ and $||g||_{L_2(\Sigma)} = ||\Phi_\alpha||_{H_2^{\frac{n}{2}-\alpha}(\Sigma)}$. Without loss of generality we assume that $\Phi_\alpha \in C^\infty(\Sigma)$, it follows that $g \in C^\infty(\Sigma)$ and, denoting by

$$g(\omega) = \sum_{m=0}^{\infty}\sum_{k=0}^{k_{m,n}} g_m^{(k)} Y_{m,n}^{(k)}(\omega), \quad \omega \in \Sigma$$

the Fourier decomposition of g, we have

$$\Phi_\alpha = (\delta + I)^{\alpha-\frac{n}{2}} g = \sum_{m=0}^{\infty}\sum_{k=0}^{k_{m,n}} (1 + m(m+n-2))^{\alpha-\frac{n}{2}} g_m^{(k)} Y_{m,n}^{(k)}\,.$$

Since $\Phi_\alpha \in H_2^{\frac{n}{2}-\alpha}(\Sigma)$, from Theorem 2.1 we have

$$\sum_{m=0}^{\infty} m^{n-2\alpha}(1 + m(m+n-2))^{2\alpha-n} \sum_{k=0}^{k_{m,n}} |g_m^{(k)}|^2 < \infty$$

and, for (2.8), also

$$\sum_{m=0}^{\infty} (\mu_m(\alpha))^{-2}(1 + m(m+n-2))^{2\alpha-n} \sum_{k=0}^{k_{m,n}} |g_m^{(k)}|^2 < \infty\,.$$

We infer that

$$A_\alpha^{-1}\Phi_\alpha(\omega) = \sum_{m=0}^{\infty} (\mu_m(\alpha))^{-1}(1 + m(m+n-2))^{\alpha-\frac{n}{2}} \sum_{k=0}^{k_{m,n}} g_m^{(k)} Y_{m,n}^{(k)}(\omega)$$

belongs to $L_2(\Sigma)$. □

Remark 2.1. *Theorem 2.2 states that, if the domain of definition of the operator A_α is $L_2(\Sigma)$, then the range is $H_2^{n/2-\alpha}(\Sigma)$ that is*

$$A_\alpha L_2(\Sigma) = H_2^{n/2-\alpha}(\Sigma).$$

In the particular case $\alpha = n/2$ it is clear that $A_{n/2}L_2(\Sigma) = L_2(\Sigma)$.

The case $p \neq 2$ will be considered in the following sections.

3 Differentiability properties of the symbol Φ_α

A sufficient condition for an operator on the sphere to be bounded in $L_p(\Sigma)$ is contained in the next theorem by Strichartz.

Theorem 3.1. *([20]) Let $t(x)$ be a function of a single variable such that for some constant C*

$$|x^k t^{(k)}(x)| \leq C, \qquad k = 0, 1, ..., s.$$

If $t_m = t(m)$, $m = 0, 1, ...$ then $\{t_m\} \in M_p$ for all $p \in (1, \infty)$ satisfying the condition $|p^{-1} - 2^{-1}| < s(n-1)^{-1}$.

Remark 3.1. *If $s = [n/2]$ is the integer part of $n/2$ then $\{t_m\} \in M_p$ for all $p \in (1, \infty)$. Indeed, suppose that $n = 2r$. Then, for any $1 < p < 2$ we have*

$$\frac{1}{2} < \frac{1}{p} < 1 < 1 + \frac{1}{2n-2} = \frac{1}{2} + \frac{n/2}{n-1} \Rightarrow 0 < \frac{1}{p} - \frac{1}{2} < \frac{n/2}{n-1} = \frac{s}{n-1};$$

for any $p \geq 2$ we have

$$\frac{1}{2} \geq \frac{1}{p} > 0 > \frac{1}{2} - \frac{n/2}{n-1} \Rightarrow 0 \leq \frac{1}{2} - \frac{1}{p} < \frac{n/2}{n-1} = \frac{s}{n-1}.$$

If $n = 2r + 1$ and $s = [n/2] = r$ then the condition

$$|p^{-1} - 2^{-1}| < \frac{s}{n-1} = \frac{1}{2} \Leftrightarrow 0 < \frac{1}{p} < 1$$

is satisfied for any $p > 1$.

We use Theorem 3.1 to study the multipliers

$$\tau_m = \tau_m(\alpha) = \frac{\Gamma\left(\dfrac{m+\alpha}{2}\right)}{\Gamma\left(\dfrac{m+n-\alpha}{2}\right)} m^{n/2-\alpha}, \quad m = 1, 2, ..., \quad \tau_0 = 1. \tag{3.9}$$

The following Lemma can be proved by induction.

Lemma 3.1. *([11, Lemma 7, p. 173]) Let $g(x)$ be solution of the equation $g'(x) = a(x)g(x)$ in (x_0, ∞), $x_0 > 0$, where $a(x) \in C^\infty((x_0, \infty))$. Then*

$$g^{(k)}(x) = g(x) \sum A_{j_0,j_1,...,j_{k-1}} (a(x))^{j_0} (a^{(1)}(x))^{j_1} .... (a^{(k-1)}(x))^{j_{k-1}},$$

where the $A_{j_0,j_1,...,j_{k-1}}$ are constants, and the summation is over nonnegative $j_0, ..., j_{k-1}$ such that $j_0 + 2j_1 + ... + kj_{k-1} = k$.

Theorem 3.2. *Let $0 < \alpha < n$. Consider the sequence $\{\tau_m(\alpha)\}$ defined in (3.9). Then $\{\tau_m(\alpha)\}$ and $\{(\tau_m(\alpha))^{-1}\}$ belong to M_p for any $p \in (1, \infty)$.*

Proof. If $\alpha = n/2$ then $\tau_m(\alpha) = 1$, for any $m \geq 1$. Suppose that $\alpha \neq n/2$. The functions

$$g_1(x) = \frac{\Gamma\left(\dfrac{x+\alpha}{2}\right)}{\Gamma\left(\dfrac{x+n-\alpha}{2}\right)} x^{n/2-\alpha}, \qquad g_2(x) = \frac{\Gamma\left(\dfrac{x+n-\alpha}{2}\right)}{\Gamma\left(\dfrac{x+\alpha}{2}\right)} \frac{1}{x^{n/2-\alpha}}$$

satisfy, respectively, the equations

$$g_1'(x) = a_\alpha(x) g_1(x), \qquad g_2'(x) = -a_\alpha(x) g_2(x)$$

for $x \geq x_0 > 0$, where

$$a_\alpha(x) = \frac{1}{2} b_\alpha(x) + \left(\frac{n}{2} - \alpha\right) \frac{1}{x}, \quad b_\alpha(x) = \psi\left(\frac{x+\alpha}{2}\right) - \psi\left(\frac{x+n-\alpha}{2}\right).$$

Here ψ denotes the Digamma function [2, 6.3.1]

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

We denote by $[\alpha]$ the greatest integer less than or equal to α and denote by $\beta = \alpha - [\alpha]$. It is clear that $0 \leq \beta < 1$.

For all $\alpha : n - \alpha > \alpha$ we have

$$\begin{aligned} b_\alpha(x) = \sum_{s=0}^{n-2[\alpha]-1} \left(\psi\left(\frac{x+\alpha+s}{2}\right) - \psi\left(\frac{x+\alpha+s+1}{2}\right) \right) \\ + \psi\left(\frac{x+n-\alpha}{2} + \beta\right) - \psi\left(\frac{x+n-\alpha}{2}\right) \end{aligned} \tag{3.10}$$

and, for $\alpha : n - \alpha < \alpha$ we have

$$b_\alpha(x) = \sum_{s=0}^{2[\alpha]-n+1} \left(\psi\left(\frac{x+n-\alpha+s+1}{2}\right) - \psi\left(\frac{x+n-\alpha+s}{2}\right)\right)$$
$$+ \psi\left(\frac{x+\alpha}{2}\right) - \psi\left(\frac{x+\alpha}{2}+1-\beta\right). \quad (3.11)$$

We prove that

$$y\,|(\psi(y)-\psi(y+\xi))| \le C_0 \quad y \ge y_0 > 0, \quad 0 < \xi \le 1$$

and, more generally, for $k \ge 0$,

$$y^{k+1}|\psi_k(y)-\psi_k(y+\xi)| \le C_k \quad y \ge y_0 > 0, \quad 0 < \xi \le 1 \qquad (3.12)$$

where

$$\psi_k(y) = \frac{d^k}{dy^k}\psi(y), \quad \psi_0(y) = \psi(y).$$

We use the asymptotic formula [2, 6.3.16]

$$\psi_0(1+y) = -\gamma + \sum_{p=1}^{\infty}\left(\frac{1}{p} - \frac{1}{y+p}\right), \qquad y \ne -1, -2, ...$$

and, for $k \ge 1$ [2, 6.4.10]

$$\psi_k(y) = (-1)^{k+1}k!\sum_{p=0}^{\infty}\frac{1}{(y+p)^{k+1}}, \qquad y \ne 0, -1, -2, ...$$

Hence,

$$\psi_0(y+\xi) - \psi_0(y) = \sum_{p=0}^{\infty}\left(\frac{1}{y+p} - \frac{1}{y+p+\xi}\right).$$

and, keeping in mind that $0 < \xi \le 1$, we have

$$0 < \psi_0(y+\xi) - \psi_0(y) \le \psi_0(y+1) - \psi_0(y).$$

We have

$$\psi_k(y+\xi) - \psi_k(y) = (-1)^{k+1}k!\sum_{p=0}^{\infty}\left(\frac{1}{(y+p+\xi)^{k+1}} - \frac{1}{(y+p)^{k+1}}\right)$$

If k is even,

$$0 \le \psi_k(y+\xi) - \psi_k(y) = k! \sum_{p=0}^{\infty} \left(\frac{1}{(y+p)^{k+1}} - \frac{1}{(y+p+\xi)^{k+1}} \right)$$
$$\le k! \sum_{p=0}^{\infty} \left(\frac{1}{(y+p)^{k+1}} - \frac{1}{(y+p+1)^{k+1}} \right) = \psi_k(y+1) - \psi_k(y)$$

Similarly, if k is odd,

$$0 \le \psi_k(y) - \psi_k(y+\xi) = k! \sum_{p=0}^{\infty} \left(\frac{1}{(y+p)^{k+1}} - \frac{1}{(y+p+\xi)^{k+1}} \right)$$
$$\le k! \sum_{p=0}^{\infty} \left(\frac{1}{(y+p)^{k+1}} - \frac{1}{(y+p+1)^{k+1}} \right) = \psi_k(y) - \psi_k(y+1).$$

Hence, by using the recurrence formula [2, 6.4.6]

$$\psi_k(y+1) = \psi_k(y) + (-1)^k k! y^{-k-1}$$

we get (3.12). From (3.12), (3.10) and (3.11) it follows that

$$x^{k+1} |a_\alpha^{(k)}(x)| \le C_k, \quad k \ge 0, \quad x \ge x_0 > 0\,.$$

Hence, applying Lemma 3.1, we obtain

$$|x^k g_1^{(k)}(x)| \le c_1, \quad |x^k g_2^{(k)}(x)| \le c_2 \quad k = 0, 1, 2....$$

It follows from Theorem 3.1 that the multipliers $\tau_m = g_1(m)$ and $\tau_m^{-1} = g_2(m)$ belong to M_p for any $p \in (1, \infty)$. □

We make use of the following theorem of Askey and Wainger regarding p-multipliers on the sphere.

Theorem 3.3. *([4, Theorem 4]) Let* $\ell_0 = |p^{-1} - 2^{-1}|(n-2)$ *and*

$$a_m(\beta) = i^m m^{-\beta}, \qquad m = 1, 2, ..., \quad a_0(\beta) = 0.$$

Then

$$a_m(\beta) \in M_p \qquad if \qquad \beta > \ell_0$$

and

$$a_m(\beta) \notin M_p \qquad if \qquad \beta < \ell_0\,.$$

A refinement of Theorem 3.3 is obtained by Gadjiev.

Theorem 3.4. *([8, Theorem 2]) Let* $\ell_0 = |p^{-1} - 2^{-1}|(n-2)$ *and* $a_m(\ell_0) = i^m m^{-\ell_0}$. *Then* $a_m(\ell_0)$ *is a* (p,p)*-multiplier for any* $p \in (1,\infty)$.

The following assertion will be used to obtain the main result of the section.

Lemma 3.2. *([10, Lemma 9, p. 178]) If* $w_m = z_m t_m$ *and* $\{t_m\}, \{t_m^{-1}\} \in M_p$, $p \in (1,\infty)$, *then* $\{w_m\} \in M_{p,q}$ *if and only if* $\{z_m\} \in M_{p,q}$.

We are in a position to prove the main theorem.

Theorem 3.5. *Let* $1 < p < \infty$, $0 < \alpha < n$ *and* $\ell_0 = |p^{-1} - 2^{-1}|(n-2)$. *Then the operator* A_α *is bounded from* $L_p(\Sigma)$ *to* $H_p^\ell(\Sigma)$ *for* $\ell \le n/2 - \alpha - \ell_0$. *The result is sharp.*

Proof. We show that $(I+\delta)^{\ell/2} A_\alpha$ is an operator of strong type (p,p) for $\ell \le n/2 - \alpha - \ell_0$ and is not such an operator for $\ell > n/2 - \alpha - \ell_0$.
We recall that to the operator $(I+\delta)^{\ell/2}$ there corresponds the multipliers $\{(1 + m(m+n-2))^{\ell/2}\}$ ([16, p.262]). Then, we have

$$(I+\delta)^{\ell/2} A_\alpha f = \pi^{n/2} 2^\alpha \sum_{m=0}^{\infty} a_m(\ell,\alpha) \sum_{k=1}^{k_{m,n}} f_m^{(k)} Y_{m,n}^{(k)}$$

where $\{a_m(\ell,\alpha)\}$ are the multipliers corresponding to the operator $(I+\delta)^{\ell/2} A_\alpha$. They have the form

$$a_m(\ell,\alpha) = \mu_m(\alpha)(1+m(m+n-2))^{\ell/2} = i^m \frac{\Gamma\left(\frac{m+\alpha}{2}\right)}{\Gamma\left(\frac{m+n-\alpha}{2}\right)} (1+m(m+n-2))^{\ell/2}.$$

We represent $a_m(\ell,\alpha) = t_m z_m$ where

$$z_m = i^m m^{-(n/2-\ell-\alpha)}$$

$$t_m = \tau_m(\alpha)(1 + m(m+n-2))^{\ell/2} \, m^{-\ell}$$

and $\{\tau_m(\alpha)\}$ is defined in (3.9).
We have $(1 + m(m+n-2))^{\ell/2} \, m^{-\ell} \in M_p$ for any ℓ. Indeed, we can write

$$(1 + m(m+n-2))^{\ell/2} \, m^{-\ell} = \big(\frac{m+a}{m}\big)^{\ell/2} \big(\frac{m+b}{m}\big)^{\ell/2}, \quad a, b \in \mathbb{R}$$

and, applying Theorem 3.1, we prove that each factor belongs to M_p for any ℓ. Keeping in mind Theorem 3.9 and Lemma 3.2 we get that $t_m \in M_p$, for any $p \in (1,\infty)$. Theorem 3.3 ensures that

$$a_m(\ell,\alpha) \in M_p \quad \text{if} \quad \ell < n/2 - \alpha - |p^{-1} - 2^{-1}|(n-2) \tag{3.13}$$

$$a_m(\ell,\alpha) \notin M_p \quad \text{if} \quad \ell > n/2 - \alpha - |p^{-1} - 2^{-1}|(n-2)\,.$$

If we apply Theorem 3.4 we can insert the equality sign in (3.13). □

We can reformulate Theorem 3.5 as follows.

Theorem 3.6. *Let $1 < p < \infty$, $0 < \alpha < n$ and $\ell_0 = |p^{-1} - 2^{-1}|(n-2)$. There are continuous embeddings*

$$A_\alpha L_p(\Sigma) \subset H_p^\ell(\Sigma) \tag{3.14}$$

for $\ell \leq n/2 - \alpha - \ell_0$. The embeddings (3.14) are the best possible.

4 Differentiability properties of the characteristic

In this section we prove a theorem characterizing the influence of the symbol Φ_α on the properties of the corresponding characteristic. Namely, we are looking for the values of the index ℓ such that the condition $\Phi_\alpha \in H_p^\ell(\Sigma)$ ensures that $f \in L_p(\Sigma)$.

Theorem 4.1. *Let $1 < p < \infty$ and $\ell_0 = |p^{-1} - 2^{-1}|(n-2)$. Let $\Phi_\alpha \in H_p^\ell(\Sigma)$ with $\ell \geq n/2 - \alpha + \ell_0$. Then there exists a function $f \in L_p(\Sigma)$ such that $\Phi_\alpha = A_\alpha f$ and*

$$||f||_{L_p(\Sigma)} \leq C\,||\Phi_\alpha||_{H_p^\ell(\Sigma)}\,.$$

Equivalently, if $\ell \geq n/2 - \alpha + \ell_0$ then

$$H_p^\ell(\Sigma) \subset A_\alpha L_p(\Sigma).$$

These embeddings are optimal.

Proof. Let $\Phi_\alpha \in H_p^\ell(\Sigma)$. Suppose that $\Phi_\alpha \in C^\infty(\Sigma)$ and let

$$\Phi_\alpha(\omega) = \sum_{m=0}^{\infty} \sum_{k=1}^{k_{m,n}} \phi_m^{(k)} Y_m^{(k)}(\omega), \quad \omega \in \Sigma.$$

Then

$$A_\alpha^{-1}\Phi_\alpha(\omega) = \sum_{m=0}^{\infty}(\mu_m(\alpha))^{-1}\sum_{k=1}^{k_{m,n}}\phi_m^{(k)}Y_m^{(k)}(\omega)\,.$$

By definition of the space $H_p^\ell(\Sigma)$ we have $g := (I+\delta)^{\ell/2}\Phi_\alpha \in L_p(\Sigma)$ and $||\Phi_\alpha||_{H_p^\ell} = ||g||_{L_p}$. Since $\Phi_\alpha \in C^\infty(\Sigma)$ then $g \in C^\infty(\Sigma)$. Let

$$g(\omega) = \sum_{m=0}^{\infty}\sum_{k=1}^{k_{m,n}} g_m^{(k)}Y_m^{(k)}(\omega) \Rightarrow g_1(\omega) = g(-\omega) = \sum_{m=0}^{\infty}\sum_{k=1}^{k_{m,n}}(-1)^m g_m^{(k)}Y_m^{(k)}(\omega).$$

We deduce that

$$A_\alpha^{-1}\Phi_\alpha(\omega) = A_\alpha^{-1}(I+\delta)^{-\ell/2}g(\omega) = \sum_{m=0}^{\infty} b_m(\ell,\alpha)\sum_{k=1}^{k_{m,n}}(-1)^m g_m^{(k)}Y_{m,n}^{(k)}(\omega)$$

where the multipliers $\{b_m(\ell,\alpha)\}$ have the form

$$b_m(\ell,\alpha) = i^m\,\pi^{-n/2}2^{-\alpha}\frac{\Gamma\left(\frac{m+n-\alpha}{2}\right)}{\Gamma\left(\frac{m+\alpha}{2}\right)}(1+m(m+n-2))^{-\ell/2}\,.$$

Let us represent the multiplier $b_m(\ell,\alpha)$ in the form $b_m(\ell,\alpha) = t_m z_m$, with $z_m = i^m m^{n/2-\ell-\alpha}$,

$$t_m = \pi^{-n/2}2^{\alpha}(\tau_m(\alpha))^{-1}(1+m(m+n-2))^{-\ell/2}\,m^\ell$$

and $\tau_m(\alpha)$ given in (3.9). It was shown in Theorem 3.2 that $(\tau_m(\alpha))^{-1} \in M_p$ for any $p \in (1,\infty)$. If we apply Theorems 3.3 and 3.4 to the multipliers $\{z_m\}$ we get $\{z_m\} \in M_p$ if $\ell \geq n/2-\alpha+\ell_0$ and $\{z_m\} \notin M_p$ if $\ell < n/2-\alpha+\ell_0$. Hence $\{b_m(\ell,\alpha)\} \in M_p$ if $\ell \geq n/2-\alpha+\ell_0$, and

$$||A_\alpha^{-1}\Phi_\alpha||_{L_p(\Sigma)} \leq C\,||g_1||_{L_p(\Sigma)} = C\,||\Phi_\alpha||_{H_p^\ell(\Sigma)},$$

and $\{b_m(\ell,\alpha)\} \notin M_p$ if $\ell < n/2-\alpha+\ell_0$. □

It follows from Theorems 3.5 and 4.1 that the range $R(A_\alpha)$ of the operator A_α, defined on $L_p(\Sigma)$, satisfies the relations

$$H_p^{n/2-\alpha+\ell_0}(\Sigma) \subset R(A_\alpha) \subset H_p^{n/2-\alpha-\ell_0}(\Sigma)\,, \quad \ell_0 = |p^{-1}-2^{-1}|(n-2) \quad (4.15)$$

and the embeddings (4.15) are best possible.

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