

ABP and global Hölder estimates for fully nonlinear elliptic equations in unbounded domains

I. Birindelli, I. Capuzzo Dolcetta, A. Vitolo

1 Introduction

This paper is focused on two goals about fully nonlinear degenerate elliptic equations in unbounded domains of $\Omega \subset \mathbb{R}^n$. The first one is to provide some generalization of a well-known result of X. Cabré who proved in [5] that the Alexandrov-Bakelman-Pucci (ABP, in short) holds for linear second order uniformly elliptic operators under the following measure theoretic condition on Ω : *there exist positive real numbers σ and R_0 such that for each $y \in \Omega$ there is a ball B_{r_y} of radius $r_y \leq R_0$ such that*

$$y \in B_{r_y} \quad \text{and} \quad |B_{r_y} \setminus \Omega_y| \geq \sigma |B_{r_y}| \quad (G_y)$$

where $|\cdot|$ denotes the n -dimensional Lebesgue measure and Ω_y is the (connected) component of $\Omega \cap B_{r_y}$ containing y . We will refer to this condition as to (G_Ω) and to Ω satisfying this conditions as to a cylindrical domain. The reader will notice a slight simplification with respect to the original condition [5], see also [22] in this respect.

The second goal is to prove global Hölder estimates for the same class of operators. This will require that the unbounded domain satisfies the slightly stronger condition (G_Ω^d) , namely: *there exist real numbers $0 < \sigma < 1$, $d_0 > 0$ and $K_0 > 1$ such that, for each $y \in \Omega$, condition (G_y) above holds for radii*

$$r_y \leq K_0 d(y) \quad (d_y)$$

and $d(y) \leq d_0$. Here, $d(y)$ denotes the distance from point y to $\partial\Omega$. As mentioned above, this is a slightly stronger condition than (G_Ω) : indeed (G_Ω^d) implies (G_Ω) with $R_0 \leq K_0 d_0$.

Observe that the converse is not true. In fact, it is immediate to realize that, if we set $R_k^\pm = \{x = (x_1, x_2) \in \mathbb{R}^2 : \pm x_1 \geq 1, x_2 = k\}$ and

$$C = \bigcup_{k \in \mathbb{Z}} (R_k^+ \cup R_k^-),$$

then $\Omega = \mathbb{R}^2 \setminus C$ satisfies condition (G_Ω) but not (G_Ω^d) .

In [9] S. Cho and M. Safonov have proved global Hölder estimates for solutions of the Dirichlet problem, with homogeneous boundary condition, for linear second-order uniformly elliptic equations in non-divergence form. Their assumption on $\partial\Omega$ is the following ‘exterior measure’ condition:

there exists a constant $\sigma > 0$ such that for each $y \in \partial\Omega$ and $r > 0$

$$|B_r(y) \setminus \Omega| \geq \sigma |B_r| \tag{A_y}$$

where $B_r(y)$ is the ball of radius $r > 0$, centered at y . We now describe more precisely the class of operators that will be considered in the present paper. Let \mathcal{S}_n be the space of $n \times n$ real symmetric matrices and recall the definition of the maximal and minimal Pucci operators, acting on \mathcal{S}^n as

$$\begin{aligned} \mathcal{P}_{\lambda, \Lambda}^+(X) &= \sup_{\lambda I \leq A \leq \Lambda I} \text{Tr}(AX) \equiv \Lambda \text{Tr}(X^+) - \lambda \text{Tr}(X^-), \\ \mathcal{P}_{\lambda, \Lambda}^-(X) &= \inf_{\lambda I \leq A \leq \Lambda I} \text{Tr}(AX) \equiv \lambda \text{Tr}(X^+) - \Lambda \text{Tr}(X^-) \end{aligned}$$

where $\Lambda \geq \lambda > 0$, I is the n -dimensional identity matrix and $\text{Tr}(X)$ is the trace of X .

The structural condition that we assume on the scalar mapping F defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$: *there exists $\delta \in (0, 1)$ such that*

$$\mathcal{P}_{\lambda, \Lambda}^-(X) - b^-(x)|\xi| \leq F(x, u, \xi, X) \leq \mathcal{P}_{\lambda, \Lambda}^+(X) + b^+(x)|\xi|, \tag{1.1}$$

for all $x \in \Omega$, $u \geq 0$, $X \in \mathcal{S}^n$ and all ξ such that $\delta \leq |\xi| \leq \frac{1}{\delta}$, and $b^\pm(x) = \max(\pm b(x), 0)$ are continuous functions.

The above condition has been introduced by C. Imbert in [14], where he proved the ABP maximum principle and Harnack inequality in bounded domains. If condition (1.1) holds

with $\delta = 0$, then F is in between two uniformly elliptic operators, and therefore we are essentially in the framework of uniform ellipticity. On the other hand, should (1.1) hold with $\delta > 0$, the uniform elliptic bound from above and from below is required only when the gradient term is away from zero and from infinity. As an example, consider

$$F(\xi, X) = |\xi|^\beta \mathcal{P}_{\lambda, \Lambda}^+(X), \text{ with } \beta \geq 0, \quad (1.2)$$

or, more generally, $G(x, \xi, X) = |\xi|^\beta F(x, X)$ with F uniformly elliptic.

Consider now the Dirichlet problem

$$\begin{cases} F(x, u, Du, D^2u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

The first result of this paper is the following uniform estimate of (ABP) type for viscosity solutions of (1.3).

Theorem 1. *Assume that Ω and F satisfy, respectively conditions (G_Ω^d) and (1.1) with $b \in C(\Omega)$ such that $|b(x)| \leq b_0$ and $f \in C(\Omega)$. Let $u \in C(\Omega)$ be a bounded viscosity solution of (1.3). Then there exists a positive constant $\alpha' = \alpha'(n, \lambda, \Lambda, K_0, b_0 d_0, \sigma) \in (0, 1)$ such that*

$$\sup_{\Omega} |u(y)| d^{-\alpha}(y) \leq C_0 K_0 d_0^\alpha \delta_{\alpha, f} \quad (1.4)$$

where

$$\delta_{\alpha, f} = \max \left[d_0^{1-\alpha} \delta; \sup_{y \in \Omega} d^{1-\alpha}(y) \|f\|_{L^n(\Omega \cap B_{r_y})} \right] \quad (1.5)$$

and B_{r_y} is a ball provided by condition (G_y) .

For $\alpha = 0$ the above result generalizes the ABP estimate by Cabré [5] concerning linear uniformly elliptic operators in cylindrical domains. The same result has been proved by Cafagna and Vitolo in [6] in more general domains satisfying condition (G_Ω) with no condition on the radii r_y , for pure second order operators, and in [20] with unbounded radii r_y of at most linear growth, for operators with lower order terms. In the uniformly elliptic fully nonlinear case, the ABP estimate in unbounded domains have been proved first by Capuzzo Dolcetta, Leoni and Vitolo in [11].

As a consequence of Theorem 1, using known interior regularity results (see [8],[16],[1]), we can derive the global Hölder continuity of viscosity solutions of the Dirichlet problem (1.3) in a domain satisfying property (G_Ω^d) . Consider, for $\alpha \in (0, 1)$, the Hölder norm

$$\|u\|_{\alpha, \Omega} = \|u\|_{L^\infty(\Omega)} + [u]_{\alpha, \Omega} ,$$

where $[u]_{\alpha, \Omega} = \sup_{x, y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$.

Theorem 2. *Assume that Ω satisfies condition (G_Ω^d) for some positive real numbers $\sigma < 1$, $K_0 > 1$ and d_0 , and that F satisfies (1.1) with $b \in C(\Omega)$ such that $|b(x)| \leq b_0$. Assume, moreover, that $f \in C(\Omega)$ is such that, for some $\bar{\alpha} > 0$,*

$$\sup_{y \in \Omega} d^{1-\bar{\alpha}}(y) \|f\|_{L^n(\Omega \cap B_{r_y})} < \infty . \quad (1.6)$$

Let $u \in C(\bar{\Omega})$ be a viscosity solution of (1.3) such that $|u(x)| \leq M$. Then there exists $\alpha'' = \alpha''(n, \lambda, \Lambda, K_0, b_0, d_0, \sigma) \in (0, 1)$ such that

$$\|u\|_{\alpha, \Omega} \leq C \delta_{\alpha, f} \quad (1.7)$$

for some $\alpha \in (0, \alpha'']$ where C is a positive constant depending on $n, \lambda, \Lambda, K_0, b_0, d_0, \sigma, \alpha$ and $\delta_{\alpha, f}$ is as in (1.5).

Operators like the one in (1.2) fit into our framework. Nonetheless, the results obtained here in the case $\beta \in (0, 1)$ can be generalized to the singular case $-\beta$ with $\beta \in (0, 1)$, using the fact that solutions of singular equations are solutions of uniformly elliptic equations with lower order terms. In order to be more precise on this point, consider the equation

$$G(x, \nabla u, D^2 u) := |\nabla u|^{-\beta} F(x, D^2 u) = f(x) \text{ in } \Omega, \text{ with } \beta \in (0, 1). \quad (1.8)$$

According to Birindelli-Demengel [2] a lower semicontinuous function u is a supersolution of (1.8) with $\beta \in (0, 1)$ if, for each $x_o \in \Omega$, either there exists an open ball $B(x_o, \delta) \subset \Omega$, $\delta > 0$, on which u is constant and $f \geq 0$,

or for all $\varphi \in \mathcal{C}^2(\Omega)$ such that $u - \varphi$ has a local minimum at x_o and $\nabla \varphi(x_o) \neq 0$ the following inequality holds

$$|\nabla \varphi(x_o)|^{-\beta} F(x_o, D^2 \varphi(x_o)) \leq f(x_o).$$

Similar definition can be given for subsolutions. It can be proved, see Lemma 4 in the next Section, that if u is a solution of (1.8) in the sense of the above definition then it is a solution of

$$F(x, D^2u) - |\nabla u|^\beta f(x) = 0 \text{ in } \Omega$$

in the standard viscosity sense, see [12].

We recall that for the whole range $\beta \in (-1, 1)$, regularity results and Harnack inequalities have been proved by Birindelli and Demengel in [2, 3] in the case of bounded smooth domains. See also [4] for domains where the boundary may contain conical points.

As already said, global Hölder results for solutions of second order linear uniformly elliptic equations in bounded domains have been proved by Cho and Safonov [9]. Interior Hölder continuity estimates in the fully nonlinear setting are due to Trudinger [19] and Caffarelli [7]; see also [8], where an extension up to the boundary for regular bounded domains is reported on. Further local results have been proved by Świech [18] and by Sirakov [17], where also a global C^α -regularity result is proved in bounded domains with an uniform exterior cone property.

The paper is organized as follows: in the next Section we establish a Growth Lemma for subsolutions, based on a boundary weak Harnack inequality and prove Lemma 4; in the Section 3 we prove our main result concerning the uniform estimate of the velocity of solutions approaching the boundary value zero and consequently, combining with interior C^α -estimates, our global Hölder regularity results.

2 Preliminaries

We start with a growth lemma (see [9]). The argument of the proof goes back to [5] and can be also found in [6], [20] for the linear case, and in [11] considering fully nonlinear operators. The basic tool is the Krylov-Safonov weak Harnack inequality proved by C.

Imbert in Theorem 2 of [14], which we will use in the rescaled version:

$$\left(\frac{1}{|B_r|} \int_{B_r} (v)^{p_0} dx \right)^{1/p_0} \leq C(\inf_{B_r} v + r \max(\delta, \|g^+\|_{L^n(B_{\frac{r}{\tau}})}) \quad (2.1)$$

for positive constants p_0 and $C > 1$ depending on $n, \lambda, \Lambda, b_0 r, \tau$. Here v is a non-negative solution of a second-order degenerate elliptic equation

$$G(x, v, Dv, D^2v) \leq g(x) \quad (2.2)$$

in $B_{\frac{r}{\tau}}$, where

$$\mathcal{P}_{\lambda, \Lambda}^-(X) - b^-(x)|\xi| \leq G(x, t, \xi, X) \quad \text{for all } \xi \in \mathbb{R}^n : |\xi| \geq \delta, \quad (2.3)$$

for some $\delta \in \mathbb{R}_+$ and for all $x \in \Omega$, $u \geq 0$, $X \in \mathcal{S}_n$. We also notice that $b^-(x)$ and $g^+(x)$ are continuous functions.

By a standard viscosity argument, if v satisfies (2.2) in D , then inequality (2.1) can be extended up to the boundary (see, for instance, [13] in the linear case and [8] in the fully nonlinear viscosity setting) introducing the supersolution (defined in all $B_{\frac{r}{\tau}}$)

$$v_m^-(x) = \begin{cases} \inf(v(x), m) & \text{if } x \in B_{\frac{r}{\tau}} \cap D \\ m & \text{if } x \in B_{\frac{r}{\tau}} \setminus D, \end{cases}$$

where $m = \inf_{\partial D \cap B_{\frac{r}{\tau}}} v$. In this case (2.1) yields

$$\left(\frac{1}{|B_r|} \int_{B_r} (v_m^-)^{p_0} dx \right)^{1/p_0} \leq C(\inf_{D \cap B_r} v + r \max(\delta, \|g^+\|_{L^n(D \cap B_{\frac{r}{\tau}})})) \quad (2.4)$$

with p_0 and C as before.

Lemma 3. *Let D be a domain of \mathbb{R}^n and B_r be a ball such that $B_r \cap D \neq \emptyset$ and $|B_r \setminus D| \geq \sigma |B_r|$ for $\sigma \in (0, 1)$. Let also $B_{\frac{r}{\tau}}$ be the concentric ball of radius $\frac{r}{\tau}$ with $\tau \in (0, 1)$. Let also $\delta \geq 0$ and F be such that the right-hand side of the structure condition (1.1) holds true. Suppose furthermore that $b^+(x)$ and $f^-(x)$ are continuous functions such that*

$$\sup_{\Omega} b^+(x) \leq b_0 < \infty. \quad (2.5)$$

If $u \in usc(\overline{D})$ is a viscosity subsolution, bounded above, of equation

$$F(x, u, Du, D^2u) = f(x) \quad (2.6)$$

in D . Then there exists a positive constant $\theta_0 = \theta_0(n, \lambda, \Lambda, b_0r, \sigma, \tau) < 1$ such that

$$\sup_{\Omega \cap B_r} u \leq \theta_0 \sup_{D \cap B_{\frac{r}{\tau}}} u^+ + (1 - \theta_0) \sup_{\partial D \cap B_{\frac{r}{\tau}}} u^+ + r \delta_f^-(D \cap B_{\frac{r}{\tau}}), \quad (2.7)$$

where

$$\delta_f^-(\Omega) = \max(\delta, \|f^-\|_{L^n(\Omega)}). \quad (2.8)$$

Proof. Passing to u^+ , which is in turn a viscosity solution of the differential inequality

$$F(x, 0, Du^+, D^2u^+) \geq -f^-(x),$$

we set $M_r \equiv \sup_{D \cap B_r} u^+$ and observe that

$$v(x) = M_{\frac{r}{\tau}} - u^+(x)$$

satisfies the differential inequality (2.2) with

$$G(x, t, \xi, X) = -F(x, 0, -\xi, -X), \quad g(x) = f^-(x).$$

Then we apply the above inequality (2.4) noting that

$$\begin{aligned} m &= \inf_{\partial D \cap B_{\frac{r}{\tau}}} (M_{\frac{r}{\tau}} - u^+) \geq M_{\frac{r}{\tau}} - \sup_{\partial D \cap B_{\frac{r}{\tau}}} u^+, \\ \inf_{D \cap B_r} v &= \inf_{D \cap B_r} (M_{\frac{r}{\tau}} - u^+) = M_{\frac{r}{\tau}} - M_r. \end{aligned}$$

We get

$$\begin{aligned} \sigma^{\frac{1}{p_0}} (M_{\frac{r}{\tau}} - \sup_{\partial D \cap B_{\frac{r}{\tau}}} u^+) &\leq \left(\frac{1}{|B_r|} \int_{B_r} (v_m^-)^{p_0} dx \right)^{1/p_0} \\ &\leq C \left(\inf_{D \cap B_r} v + r \delta_f^-(D \cap B_{\frac{r}{\tau}}) \right) \\ &\leq C (M_{\frac{r}{\tau}} - M_r + r \delta_f^-(D \cap B_{\frac{r}{\tau}})) \end{aligned}$$

from which

$$M_r \leq \left(1 - \frac{\sigma^{p_0}}{C}\right)M_{\frac{r}{\tau}} + \frac{\sigma^{p_0}}{C} \sup_{\partial D \cap B_{\frac{r}{\tau}}} u^+ + r \delta_f^-(D \cap B_{\frac{r}{\tau}})$$

and therefore the assert follows with $\theta_0 = 1 - \frac{\sigma^{p_0}}{C}$. \square

We end this section with the following lemma concerning solutions of singular equations:

Lemma 4. *If u is a solution of (1.8) with $\beta \in [0, 1)$ then it is a solution of*

$$F(x, D^2u) - |\nabla u|^\beta f(x) = 0 \text{ in } \Omega$$

in the standard viscosity sense.

Note that, for $\beta \in [0, 1)$, the operator $\tilde{G}(x, \xi, M) = F(x, M) - f(x)|\xi|^\beta$ does satisfy (1.1) for $|\xi| \geq \delta$, with $b^\pm(x) = f^\mp(x)\delta^{\beta-1}$.

Proof. Let u be a super solution of (1.8) and let $\varphi \in \mathcal{C}^2(\Omega)$, such that $u(x) \geq \varphi(x)$ and $u(x_o) = \varphi(x_o)$. If $\nabla\varphi(x_o) \neq 0$, there is clearly nothing to prove. So we will suppose that $\nabla\varphi(x_o) = 0$, and for simplicity we will suppose that $x_o = 0$ and $u(0) = \varphi(0) = 0$

Without loss of generality we will take $\varphi(x) = \frac{1}{2}\langle Ax, x \rangle$ and suppose that $u(x) > \varphi(x)$ in a neighbourhood of 0. We want to prove that

$$F(0, A) \leq f(0)|\nabla\varphi(0)|^\beta = 0.$$

We suppose by contradiction that $F(0, A) > 0$. By the ellipticity hypothesis on F this implies that V^+ , the space of eigenvectors corresponding to positive eigenvalues of A , has at least dimension one. Let $e \in V^+$ be a unitary eigenvector for A .

And for $\varepsilon > 0$ we introduce $\psi(x) = \varphi(x) + \varepsilon|\langle x, e \rangle|$. Let $x_o^\varepsilon \in B_r(0)$ such that

$$u(x_o^\varepsilon) - \psi(x_o^\varepsilon) = \inf_{x \in B_r(0)} (u(x) - \psi(x)).$$

Observe first that, for ε sufficiently small, the minimum is achieved inside i.e. $|x_o^\varepsilon| < r$. Indeed $u(0) - \psi(0) = 0$, while, for $0 < k := \min_{x \in \partial B_r(0)} (u(x) - \phi(x))$ and $\varepsilon < \frac{K}{r}$,

$$\min_{x \in \partial B_r(0)} (u(x) - \psi(x)) \geq K - \varepsilon r > 0.$$

Remark also that $\langle x_o^\varepsilon, e \rangle \neq 0$. Indeed suppose that it is zero, then, we would have that $\tilde{\psi}(x) = \varphi(x) + \varepsilon \langle x, e \rangle + u(x_o^\varepsilon) - \psi(x_o^\varepsilon)$

$$u(x) \geq \psi(x) + u(x_o^\varepsilon) - \psi(x_o^\varepsilon) \geq \tilde{\psi}(x), \quad u(x_o^\varepsilon) = \tilde{\psi}(x_o^\varepsilon).$$

Furthermore, $\nabla \tilde{\psi}(x_o^\varepsilon) = Ax_o^\varepsilon + \varepsilon e \neq 0$ since

$$\langle Ax_o^\varepsilon + \varepsilon e, e \rangle = \varepsilon.$$

So from the equation we get

$$F(x_o^\varepsilon, A) \leq f(x_o^\varepsilon) |Ax_o^\varepsilon + \varepsilon e|^\beta.$$

Observe that for $\varepsilon \rightarrow 0$, $x_o^\varepsilon \rightarrow 0$, so passing to the limit we get $F(0, A) \leq 0$ a contradiction.

We have obtained that ψ is a test function for u at x_o^ε , with $\nabla \psi(x_o^\varepsilon) = Ax_o^\varepsilon + \varepsilon e_\varepsilon e \neq 0$ where $e_\varepsilon = \frac{\langle x_o^\varepsilon, e \rangle}{|\langle x_o^\varepsilon, e \rangle|} = \pm 1$.

We choose a sequence ε_k such that $e_{\varepsilon_k} = 1$ and $\langle Ax_o^{\varepsilon_k} + \varepsilon_k e, e \rangle = \mu \langle x_o^{\varepsilon_k}, e \rangle + \varepsilon_k > 0$, μ being the eigenvalue corresponding to e .

Finally we can use ψ as a test function:

$$F(x_o^{\varepsilon_n}, A + \varepsilon_n B) \leq |Ax_o^{\varepsilon_n} + \varepsilon_n B|^\beta f(x_o^{\varepsilon_n}).$$

Passing to the limit we obtain that $F(0, A) \leq 0$ a contradiction.

3 Geometric conditions and boundary inequalities

We recall that $y \in \Omega$ satisfies condition (G_y) in Ω with parameter $\sigma \in (0, 1)$ if there exists a ball B_{r_y} of radius r_y such that

$$y \in B_{r_y}, \quad |B_{r_y} \setminus \Omega_y| \geq \sigma |B_{r_y}|$$

where Ω_y is the (connected) component of $\Omega \cap B_{r_y}$ containing y .

Here we use Lemma 3 to obtain a pointwise estimate for viscosity solutions of second-order uniformly elliptic equations in a point $y \in \Omega$ satisfying condition (G_y) in Ω .

Lemma 5. *Let Ω be a domain of \mathbb{R}^n and suppose that y satisfies condition (G_y) in Ω with parameter $\sigma \in (0, 1)$. Let also B_{r_y} be a ball of radius $r_y \leq d_0$ realizing condition (G_y) for a positive constant d_0 , and $B_{\frac{r_y}{\tau}}$ be the concentric ball of radius $\frac{r_y}{\tau}$ with $\tau \in (0, 1)$.*

Suppose that $u \in usc(\overline{\Omega})$ is a viscosity solution, bounded above, of the differential inequality

$$F(x, u, Du, D^2u) \geq f(x) \tag{3.1}$$

in Ω , where F and f satisfy the assumptions of Lemma 3. There exists a positive constant $\theta_1 = \theta_1(n, \lambda, \Lambda, b_0 d_0, \sigma, \tau) < 1$ such that

$$u(y) \leq \theta_1 \sup_{\Omega \cap B_{\frac{r_y}{\tau}}} u^+ + (1 - \theta_1) \sup_{\partial\Omega \cap B_{\frac{r_y}{\tau}}} u^+ + 2r_y \delta_f^-(\Omega \cap B_{\frac{r_y}{\tau}}). \tag{3.2}$$

where $\delta_f^-(\Omega \cap B_{\frac{r_y}{\tau}}) = \max\left(\delta, \|f^-\|_{L^n(\Omega \cap B_{\frac{r_y}{\tau}})}\right)$, according to (2.8).

Proof. We set $\tau_0 = (1 - \frac{\sigma}{4})^{1/n}$, noticing that $\frac{3}{4} < \tau_0 < 1$ and considering two different cases according that (i) $y \in B_{\tau_0 r_y}$ or (ii) $y \notin B_{\tau_0 r_y}$.

Case (i). Recall that Ω_y is the component of B_{r_y} containing y . Since

$$\begin{aligned} |B_{\tau_0 r_y} \setminus \Omega_y| &\geq |B_{r_y} \setminus \Omega_y| - |B_{r_y} \setminus B_{\tau_0 r_y}| \\ &\geq \sigma |B_{r_y}| - (1 - \tau_0^n) |B_{r_y}| \geq \frac{3}{4} \sigma |B_{r_y}| \end{aligned}$$

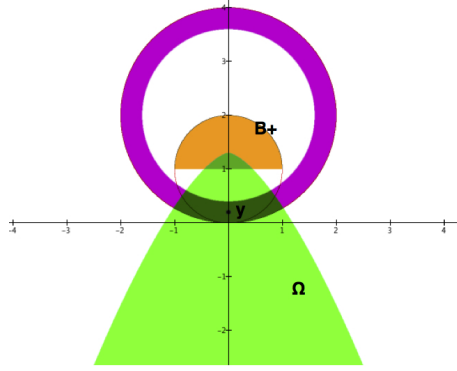


Figure 1: Case (ii)

Therefore we can apply Lemma 3 in $D = \Omega_y$ with $r = \tau_0 r_y$ and $\tau = \tau_0$ to get

$$\begin{aligned} \sup_{x \in \Omega_y \cap B_{\tau_0 r_y}} u(x) &\leq \theta_0 \sup_{\Omega_y \cap B_{r_y}} u^+ + (1 - \theta_0) \sup_{\partial(\Omega_y \cap B_{r_y})} u^+ + \tau_0 r_y \delta_f^-(\Omega \cap B_{r_y}) \\ &\leq \theta_0 \sup_{\Omega \cap B_{r_y}} u^+ + (1 - \theta_0) \sup_{\partial\Omega \cap B_{r_y}} u^+ + r_y \delta_f^-(\Omega \cap B_{r_y}) \end{aligned} \quad (3.3)$$

and this concludes the proof of (3.2) with $\theta_1 = \theta_0$ in Case (A), since $y \in B_{\tau_0 r_y}$.

Case (ii). Without loss of generality, suppose that $B_{r_y} = B_r(P_r)$ i.e it is the ball of radius r centered at the point $P_r = (0, r)$. In this case $y \in \Omega \cap (B_r(P_r) \setminus B_{\tau_0 r}(P_r))$ and we may suppose that $y = (0, y_n)$ with $0 \leq y_n < (1 - \tau_0)r_y$, because $y_n = (1 - \tau_0)r$ would mean $y \in \overline{B_{\tau_0 r}(P_r)}$ and in this case the result follows from (3.3) with $\theta_1 = \theta_0$ by continuity. Let $y^o = (0, \frac{r}{2})$. Since $\tau_0 > \frac{3}{4}$, then the upper half-ball

$$B^+ = \{x \in B_{\frac{r}{2}}(y^o) / x_n \geq \frac{r}{2}\} \subset B_{\tau_0 r}(P_r)$$

lies in the complement of $\Omega_y \setminus \overline{B_{\tau_0 r}(P_r)}$. Consequently, for all $\tau \in (0, 1)$, denoting by D the component of the open set $(\Omega_y \setminus \overline{B_{\tau_0 r}(P_r)}) \cap B_{\frac{r}{\tau}}(y^o)$ containing y , satisfies the measure condition

$$|B_{\frac{r}{2}}(y^o) \setminus D| \geq |B^+| \geq \frac{1}{2} |B_{\frac{r}{2}}(y^o)|.$$

So that we can apply Lemma 3 to get

$$\begin{aligned}
u(y) &\leq \theta_\tau \sup_{D \cap B_{\frac{r}{2}}(y^o)} u^+ + (1 - \theta_\tau) \sup_{\partial D \cap B_{\frac{r}{2}}(y^o)} u^+ + r \delta_f^-(D \cap B_{\frac{r}{2}}(y^o)) \\
&\leq \theta_\tau \sup_{\Omega \cap B_{\frac{r_y}{\tau}}(y^o)} u^+ + (1 - \theta_\tau) \max \left(\sup_{\partial \Omega \cap B_{\frac{r_y}{\tau}}(y^o)} u^+, \sup_{\Omega_y \cap \partial B_{\tau_0 r_y}} u^+ \right) \\
&\quad + \frac{1}{2} r_y \delta_f^-(\Omega \cap B_{\frac{r_y}{\tau}}).
\end{aligned} \tag{3.4}$$

If $\sup_{\partial \Omega \cap B_{\frac{r}{2}}(y^o)} u^+ \geq \sup_{\Omega_y \cap \partial B_{\tau_0 r_y}} u^+$, we are done and we get (3.2) with $\theta_1 = \theta_\tau$.

Otherwise, we again use (3.3) obtaining from (3.4)

$$\begin{aligned}
u(y) &\leq \theta_\tau \sup_{\Omega \cap B_{\frac{r_y}{\tau}}(y^o)} u^+ + (1 - \theta_\tau) (\theta_0 \sup_{\Omega \cap B_{r_y}} u^+ + (1 - \theta_0) \sup_{\partial \Omega \cap B_{r_y}} u^+) \\
&\quad + (1 - \theta_\tau) r_y \delta_f^-(\Omega \cap B_{\frac{r_y}{\tau}}) + \frac{1}{2} r_y \delta_f^-(\Omega \cap B_{\frac{r_y}{\tau}}) \\
&\leq (\theta_\tau + (1 - \theta_\tau) \theta_0) \sup_{\Omega \cap B_{\frac{r_y}{\tau}}(y^o)} u^+ + (1 - \theta_\tau) (1 - \theta_0) \sup_{\partial \Omega \cap B_{r_y}} u^+ \\
&\quad + \frac{3}{2} r_y \delta_f^-(\Omega \cap B_{\frac{r_y}{\tau}})
\end{aligned} \tag{3.5}$$

which yields (3.2) with $\theta_1 = \theta_\tau + (1 - \theta_\tau) \theta_0$ and concludes the proof. \square

Comment. Taking the supremum over $y \in \Omega$ in Lemma 5, if Ω satisfies condition G_Ω , being $r_y \leq R_0$, we get

$$\sup_{\Omega} u \leq \sup_{\partial \Omega} u^+ + CR_0 \max(\delta, \sup_{y \in \Omega} \|f^-\|_{L^n(\Omega \cap B_{r_y})}) \tag{3.6}$$

for solutions, bounded above, of the differential inequality (3.1) in domains Ω satisfying condition (G_Ω) . This provides a generalization of the above quoted ABP estimate of Cabré, which corresponds to $\delta = 0$.

If Ω satisfies condition (G_Ω^d) , being $r_y \leq K_0 d(y)$ and $d(y) \leq d_0$, from Lemma 5 we also get for all $\alpha \in (0, 1)$ the estimate

$$\sup_{\Omega} u \leq \sup_{\partial \Omega} u^+ + CK_0 d_0^\alpha \delta_{\alpha, f}^- \tag{3.7}$$

where $\delta_{\alpha, f}^-$ is given by

$$\delta_{\alpha, f}^- = \max \left(d_0^{1-\alpha} \delta, \sup_{y \in \Omega} d^{1-\alpha}(y) \|f^-\|_{L^n(\Omega \cap B_{r_y})} \right), \tag{3.8}$$

and $C = C(n, \lambda, \Lambda, b_0, d_0, \sigma, \tau, \alpha)$ is a positive constant. \square

Theorem 6. *Let Ω be a domain of \mathbb{R}^n satisfying condition (G_Ω^d) for some positive real numbers $\sigma < 1$, $K_0 > \max(1, d_0)$. As in Lemma 3 we suppose that the right-hand side of the structure condition (1.1) is satisfied with some $\delta > 0$, $b^+(x)$ and $f^-(x)$ are continuous functions, $b^+(x) \leq b_0$ in Ω for a positive real number and recall the definition (3.8),*

$$\delta_{\alpha, f}^- = \max \left(d_0^{1-\alpha} \delta, \sup_{y \in \Omega} d^{1-\alpha}(y) \|f^-\|_{L^n(\Omega \cap B_{r_y})} \right)$$

where B_{r_y} is a ball provided by condition (G_y) .

Let $u \in \text{usc}(\bar{\Omega})$ be a viscosity solution, bounded above, of the degenerate elliptic differential inequality (3.1).

There exists a positive constant $\alpha' = \alpha'(n, \lambda, \Lambda, K_0, b_0 d_0, \sigma) \in (0, 1)$ such that, if $u \leq 0$ on $\partial\Omega$, then for $\alpha \in (0, \alpha']$ we have

$$\sup_{\Omega} u(y) d^{-\alpha}(y) \leq C_0 K_0 d_0^\alpha \delta_{\alpha, f}^- \quad (3.9)$$

where C_0 is a positive constant depending on $n, \lambda, \Lambda, K_0 b_0 d_0, \sigma, \alpha$.

Proof. Let us consider $\alpha > 0$ to be chosen in the sequel (3.11) and an arbitrary point $y \in \Omega$, which by assumption satisfies condition (G_y) with parameter $\sigma \in (0, 1)$ and a ball B_{r_y} containing y such that $r_y \leq K_0 d(y)$.

Since u is a subsolution of equation (2.6), then

$$F(x, u^+, Du^+, D^2u^+) \geq -f^-(x). \quad (3.10)$$

Moreover, since u is supposed to be bounded above, then $u^+(x)(d(x) + \frac{1}{j})^{-\alpha}$ is bounded for all $j \in \mathbb{N}$, and we set

$$N_j \equiv \sup_{x \in \Omega} \frac{u^+(x)}{(d(x) + \frac{1}{j})^\alpha} < \infty.$$

From inequality (3.2) of Lemma 5, since $u = 0$ on $\partial\Omega$ there exists $y^* \in \Omega \cap \overline{B_{\frac{r_y}{\tau}}}$ such that

$$\begin{aligned} u^+(y) &\leq \theta_1 \sup_{\Omega \cap B_{\frac{r_y}{\tau}}} u^+ + 2r_y \delta_f^-(\Omega \cap B_{\frac{r_y}{\tau}}) \\ &= \frac{\theta_1 u^+(y^*)}{(d(y^*) + \frac{1}{j})^\alpha} (d(y^*) + \frac{1}{j})^\alpha + 2r_y \delta_f^-(\Omega \cap B_{\frac{r_y}{\tau}}) \\ &\leq \theta_1 N_j (d(y^*) + \frac{1}{j})^\alpha + 2r_y \delta_f^-(\Omega \cap B_{\frac{r_y}{\tau}}) \\ &\leq \theta_1 N_j ((1 + \frac{1}{\tau})r_y + \frac{1}{j})^\alpha + 2r_y \delta_f^-(\Omega \cap B_{\frac{r_y}{\tau}}), \end{aligned}$$

where $\tau \in (0, 1)$ and we have used the fact that $d(x) = \text{dist}(x; \partial\Omega) \leq (1 + \frac{1}{\tau})r_y$ for $x \in B_{\frac{r_y}{\tau}}$ in the last inequality.

Since $d(y) \geq \frac{r_y}{K_0}$, we may divide by $(d(y) + \frac{1}{j})^\alpha$ the above inequality and get

$$\begin{aligned} \frac{u^+(y)}{(d(y) + \frac{1}{j})^\alpha} &\leq \theta_1 N_j \frac{((1 + \frac{1}{\tau})r_y + \frac{1}{j})^\alpha}{(\frac{r_y}{K_0} + \frac{1}{j})^\alpha} + \frac{K_0 d(y)}{(d(y) + \frac{1}{j})^\alpha} \delta_f^-(\Omega \cap B_{\frac{r_y}{\tau}}) \\ &\leq \theta_1 N_j ((1 + \frac{1}{\tau})K_0)^\alpha + K_0 d^{1-\alpha}(y) \delta_f^-(\Omega \cap B_{\frac{r_y}{\tau}}). \end{aligned}$$

Hence, taking the sup over $y \in \Omega$ we obtain

$$N_j \leq N_j \theta_1 ((1 + \frac{1}{\tau})K_0)^\alpha + K_0 d_0^\alpha \delta_{\alpha, f}^-$$

from which, for any positive number

$$\alpha < \frac{\log \theta_1^{-1}}{\log((1 + \frac{1}{\tau})K_0)}, \quad (3.11)$$

we deduce

$$N_j \leq C_0 K_0 d_0^\alpha \delta_{\alpha, f}^-$$

where $C_0 = C_0(n, \lambda, \Lambda, b_0 K_0 d_0, \sigma, \tau, \alpha)$. Finally, letting $j \rightarrow \infty$, we get the result. \square

Suppose that Ω satisfies condition (G_Ω^d) as in Theorem 6. Let

$$F(x, u, \xi, X) \geq \mathcal{P}_{\lambda, \Lambda}^-(X) - b^-(x)|\xi| \quad \text{for all } \xi \in \mathbb{R}^n : |\xi| \geq \delta, \quad (3.12)$$

and for all $x \in \Omega$, $u \geq 0$, $X \in \mathcal{S}_n$, where $b^-(x) \leq b_0$ is a continuous function. Changing u with $-u$, a similar estimate

$$\inf_{\Omega} u(y)d^{-\alpha}(y) \geq -C_0K_0d_0^\alpha \delta_{\alpha,f}^+, \quad (3.13)$$

where

$$\delta_{\alpha,f}^+ = \max \left(d_0^{1-\alpha} \delta, \sup_{y \in \Omega} d^{1-\alpha}(y) \|f^+\|_{L^\infty(\Omega \cap B_{r_y})} \right), \quad (3.14)$$

can be obtained for a viscosity solution $u \in \text{Lsc}(\overline{\Omega})$, bounded from below and non-negative on $\partial\Omega$, of the uniformly elliptic differential inequality

$$F(x, u, Du, D^2u) \leq f(x). \quad (3.15)$$

for a continuous function $f^+(x)$.

Proof of Theorem 1. Gathering (3.9) and (3.13), we get at once Theorem 1.

Proof of Theorem 2. Let u be a continuous viscosity solution of the degenerate elliptic equation (2.6) in Ω . For the L^∞ -norm of $\|u\|_{L^\infty(\Omega)}$ we have already obtained a bound (3.7) for all $\alpha \in (0, 1)$.

To estimate the Hölder seminorm $[u]_{\alpha,\Omega}$, let us take a point x of Ω and consider the ball B_r of radius $r > 0$ centred at x . Let

$$M_r = \sup_{\Omega \cap B_r} u, \quad m_r = \inf_{\Omega \cap B_r} u, \quad \omega(r) = M_r - m_r.$$

We also denote by α_1 any exponent α allowed by Theorem 1.

If $r \geq 1$, note that, since $d(y) \leq d_0$ for all $y \in \Omega$, from (1.4) we get

$$\begin{aligned} \omega(r) &\leq 2 \sup_{y \in \Omega \cap B_r} |u(y)| \\ &\leq 2d_0^{\alpha_1} \sup_{y \in \Omega \cap B_r} d^{-\alpha_1}(y) |u(y)| \leq A\delta_{\alpha_1,f}^- r^{\alpha_1}, \end{aligned} \quad (3.16)$$

where $A = 2C_0K_0d_0^{\alpha_1}$.

If $r < 1$, we consider separately the cases: a) $r \leq \frac{1}{2}d(x)$; b) $r > \frac{1}{2}d(x)$.

Case a) Since $r \leq \frac{1}{2}d(x)$, then $B_{r/2} \subset B_r \Subset \Omega$. In this case we have

$$M_r = \sup_{B_r} u, \quad m_r = \inf_{B_r} u, \quad \omega(r) = M_r - m_r,$$

noticing that the non negative functions $u - m_r$ and $M_r - u$ are solutions of the degenerate elliptic differential inequalities (3.1), with $u - m_r$ instead of u , and (2.2) with $v = M_r - u$, respectively.

By structure assumptions (1.1) and (2.3), we can apply to $M_r - u$ and $u - m_r$ the Harnack inequality of C. Imbert, Corollary 3 of [14], which generalizes the uniformly elliptic case (see for instance [8], [16],[1]), to obtain

$$\begin{aligned} M_r - m_{r/2} &\leq C(M_r - M_{r/2} + r\delta_f^-(B_r)) \\ M_{r/2} - m_r &\leq C(m_{r/2} - m_r + r\delta_f^+(B_r)), \end{aligned} \quad (3.17)$$

where

$$\delta_f^\pm(B_r) = \max(\delta, \|f^\pm\|_{L^n(B_r)}), \quad \delta_f(B_r) = \max(\delta, \|f\|_{L^n(B_r)}) \quad (3.18)$$

and C is a non-negative constant depending on $n, \lambda, \Lambda, b_0 K_0 d_0$.

Adding the two inequalities above, we have

$$\omega(r/2) + \omega(r) \leq C(\omega(r) - \omega(r/2) + 2r\delta_f(B_r)) \quad (3.19)$$

from which we have

$$\begin{aligned} \omega(r/2) &\leq \frac{C-1}{C+1}\omega(r) + \frac{2C}{C+1}r\delta_f(B_r) \\ &\leq \gamma\omega(r) + C_1\delta_{\alpha_1, f}r^{\alpha_1}. \end{aligned} \quad (3.20)$$

with $\gamma = \frac{C-1}{C+1} \in (0, 1)$, $C_1 = \frac{2C}{C+1} \in \mathbb{R}_+$ and $\delta_{\alpha_1, f}$ is defined in (1.5) with the exponent $\alpha = \alpha_1$.

Case b) Here we observe that, if $r > \frac{1}{2}d(x)$, then for $y \in B_r$ we have

$$d(y) \leq d(x) + d(x, y) \leq 3r,$$

from which, using (3.9) with $\alpha = \alpha_1$, we deduce

$$\begin{aligned} \omega(r) &\leq 2 \sup_{y \in \Omega \cap B_r} |u(y)| \leq 2 \sup_{y \in \Omega \cap B_r} \left(\frac{3r}{d(y)} \right)^{\alpha_1} |u(y)| \\ &\leq 2 \cdot 3^{\alpha_1} \left(\sup_{y \in \Omega \cap B_r} d^{-\alpha_1}(y) |u(y)| \right) r^{\alpha_1} \leq 2 \cdot 3^{\alpha_1} C_0 K_0 \delta_{\alpha_1, f} r^{\alpha_1}. \end{aligned} \quad (3.21)$$

Letting $B = \max(C_1, 2 \cdot 3^{\alpha_1} C_0 K_0)$, from (3.20) and (3.21) we get

$$\omega(r/2) \leq \gamma \omega(r) + B \delta_{\alpha_1, f} r^{\alpha_1}. \quad (3.22)$$

for all $r \in \mathbb{R}_+$.

Since $r < 1$, by virtue of (3.22) Lemma 8.23 of [13], together with (3.16), yields

$$\omega(r) \leq C_2 (\omega(1)r^\beta + B \delta_{\alpha_1, f} r^{\mu\alpha_1}) \leq C_2 \delta_{\alpha_1, f} (Ar^\beta + Br^{\mu\alpha_1}) \quad (3.23)$$

for any $\mu \in (0, 1)$, where $C_2 = C_2(\gamma)$ and $\beta = \beta(\gamma, \mu)$ are positive constants.

From (3.16) and (3.23) we therefore obtain

$$\omega(r) \leq C_3 \delta_{\alpha_1, f} r^\alpha \quad (3.24)$$

for $\alpha \leq \min(\beta, \mu\alpha_1)$, where C_3 is a positive constant depending on γ , A and B .

Hence, if y is any other point of Ω , letting $|y - x| = r$, we get

$$|u(y) - u(x)| \leq C_3 \delta_{\alpha, f} |x - y|^\alpha$$

and we are done. □

References

- [1] M.E. Amendola, L. Rossi and A. Vitolo, Harnack Inequalities and ABP Estimates for Nonlinear Second Order Elliptic Equations in Unbounded Domains, *Abstr. Appl. Anal.*(2008).

- [2] I. Birindelli, F. Demengel, Eigenvalue, maximum principle and regularity for fully non linear homogeneous operators, *Comm. Pure and Appl. Analysis*, **6** (2007), 335-366.
- [3] I. Birindelli, F. Demengel, Regularity and uniqueness of the first eigenfunction for singular fully non linear operators, *J. Differential Equations*, **249**, (2010), 1089-1110.
- [4] Birindelli, I.; Demengel, F. Eigenvalue and Dirichlet problem for fully-nonlinear operators in non-smooth domains. *J. Math. Anal. Appl.* **352** (2009), no. 2, 822-835.
- [5] X. Cabré, On the Alexandroff–Bakelman–Pucci estimate and the reverse Hölder inequality for solutions of elliptic and parabolic equations, *Comm. Pure Appl. Math.* **48** (1995), 539-570.
- [6] V. Cafagna and A. Vitolo, On the maximum principle for second-order elliptic operators in unbounded domains, *C. R., Math., Acad. Sci. Paris* **334** (2002), n.5, 359–363.
- [7] L.A. Caffarelli, Interior a priori estimates for solutions of fully nonlinear equations, *Ann. Math.* **130** (1989), 189–213.
- [8] L. A. Caffarelli and X. Cabré, “Fully Nonlinear Elliptic Equations”, AMS Colloquium Publications, Providence, Rhode Island, 1995.
- [9] S. Cho and M. Safonov, Hölder regularity of solutions to second-order elliptic equations in nonsmooth domains *Boundary Value Problems* **2007** (2007), Article ID 57928, pp. 24.
- [10] I. Capuzzo Dolcetta and A. Vitolo, A qualitative Phragmén-Lindelöf theorem for fully nonlinear elliptic equations, *J. Differential Equations* **243** (2007), 578-592.
- [11] I. Capuzzo Dolcetta, F. Leoni and A. Vitolo, The Alexandrov–Bakelman–Pucci weak maximum principle for fully nonlinear equations in unbounded domains, *Commun. Partial Differ. Equations* **30** (2005), pp. 1863–1881.
- [12] M.G. Crandall, H. Ishii and P.L. Lions, User’s guide to viscosity solutions of second order partial differential equations, *Bull. Am. Math. Soc., New Ser.* **27** (1992), 1–67.
- [13] D. Gilbarg and N.S. Trudinger, “Elliptic Partial Differential Equations of Second Order”, 2nd ed., Grundlehren Math. Wiss. 224, Springer-Verlag, Berlin-New York, 1983.

- [14] C. Imbert, Alexandroff-Bakelman-Pucci estimate and Harnack inequality for degenerate/singular fully non-linear elliptic equations, *Journal of Differential Equations* **250** (2011), 1553–1574.
- [15] S.Koike, *A Beginners Guide to the Theory of Viscosity Solutions*, MSJ Memoirs 13, Math. Soc. Japan, Tokyo (2004).
- [16] S.Koike and Takahashi, Remarks on regularity of viscosity solutions for fully nonlinear uniformly elliptic PDEs with measurable ingredients, *Adv. Differential Equations* **7** (2002), 493–512.
- [17] B. Sirakov, Solvability of uniformly elliptic fully nonlinear PDE, *Arch. Ration. Mech. Anal.* **195** n.2 (2010), 579–607.
- [18] A. Świech, $W^{1,p}$ -interior estimates for solutions of fully nonlinear uniformly elliptic equations, *Adv. Differential Equations* **2** (1997), 1005–1027.
- [19] N.S. Trudinger, Comparison principles and pointwise estimates for viscosity solutions, *Rev. Mat. Iberoamericana* **4** (1988), 453–468.
- [20] A.Vitolo, On the Maximum Principle for Complete Second-Order Elliptic Operators in General Domains, *J. Differ. Equations* **194** (2003), n.1, pp.166–184.
- [21] A. Vitolo, On the Phragmén-Lindelöf principle for second-order elliptic equations, *J. Math. Anal. Appl.* **300** (2004), 244–259.
- [22] A.Vitolo, A Note on the Maximum Principle for Second-Order Elliptic Equations in General Domains, *Acta Math. Sin., Engl. Ser.* **23** (2007), n.11, pp.1955–1966.