LAPLACE-BELTRAMI OPERATOR FOR THE HEAT CONDUCTION IN POLYMER COATING OF ELECTRONIC DEVICES

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Abstract. In this paper we study a model for the heat conduction in a composite having a microscopic structure arranged in a perodic array. We obtain the macroscopic behaviour of the material via an homogenization procedure, providing the equation satisfied by the effective temperature.

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1. Introduction

The study of thermal, mechanical and electrical properties of composite materials plays an increasingly important role in material sciences due to the fact that these composites have a wide spectrum of applications in industrial processes. An important (though not unique) example of these applications is encapsulation of electronic devices.
It is well known that the increasing miniaturization of such devices poses a big problem in attaining an efficient heat dissipation. As a matter of fact air gaps (e.g. surface mismatch between the electronic components and the heat sinks) decrease dramatically the heat dissipation. To prevent this to happen encapsulation in a polymer coating, e.g. rubber, is employed. An ideal coating should have (quoting from [17]) high thermal conductivity, low coefficient of thermal expansion and low dielectric constant as well. Moreover the material should be soft enough to be easily deformed by applied contact pressure to fill all the gaps between the mating surfaces. The previous considerations justify the use of polymers which satisfy fairly well all the previous requests, with the possible exception of the first one. Unluckily thermal dissipation of silicon rubber and of other composites is not particularly efficient (for example the traditional epoxy resin used in electrical and electronic industry has a poor thermal conductivity and no longer meets the increasingly cooling requirements of electric equipments). For this reason a new kind of materials (elastomeric pads) has been proposed; such materials are made of an elastomeric polymer, such as silicon rubber, reinforced with highly thermally conductive but electrically insulating fillers, such as aluminum nitride, boron nitride, silicon carbide, alumina, silicone nitride, graphene flakes or ceramics. Things are made even more complicated, since

in elastomeric pads (as well as in other reinforced rubbers) nanoparticles fillers have their own film coating separating them from the surrounding polymer (see [20]). This surface enhancement of the nanoparticles is useful, for example, to improve their dispersion (with the purpose of avoiding clusters), and also the electrical properties of the composites.
These materials show an increased thermal conductivity (see [17, 21, 13]) and most of the applied papers in this field focus on the experimental determination of their conductivity coefficients.
For this reason a rigorous mathematical study of these composites seems to us to be justified and also of some interest. The study of the case in which the nanoparticles (without surface coating) are embedded in the polymer is well known and, though interesting from the point of view of applications, it is mathematically standard.
For better understanding the importance of studying also these mathematically simple cases it is sufficient to check [13] in which various experimental formulas are proposed to describe the overall conductivity of the composites. Most of these formulas, though acceptable from the point of view of applications (at least in the isotropic case), are theoretically unjustified, for example in some of these models the weighted average of the conductivities is proposed as a measure of the overall conductivity of the composite.
On the other hand, the study of the more general case, in which we have a polymer filled with nanoparticles whose surface is coated by a very thin film having an active thermal behaviour appears to be a novelty. Motivated by these considerations we are led to investigate the thermal properties of an ideal composite material having a microstructure arranged (for the sake of simplicity) in a periodic array made by two phases separated by a thermally active membrane. We use, as a mathematical description of our model, the differential system of equations given by

$$
\begin{aligned}
\mu^\varepsilon u_{\varepsilon t} - \operatorname{div}(\lambda^\varepsilon \nabla u_\varepsilon) &= 0\,, && \text{in } (\Omega^\varepsilon_{\text{int}} \cup \Omega^\varepsilon_{\text{out}}) \times (0,T);\\
[u_\varepsilon] &= 0\,, && \text{on } \Gamma^\varepsilon \times (0,T);\\
\varepsilon\alpha \frac{\partial u_\varepsilon}{\partial t} - \varepsilon\beta \Delta^B u_\varepsilon &= [\lambda^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon]\,, && \text{on } \Gamma^\varepsilon \times (0,T);\\
u_\varepsilon(x,0) &= \overline{u}_0(x)\,, && \text{in } \Omega^\varepsilon;
\end{aligned}
$$

where $\Omega^\varepsilon_{\text{int}}, \Omega^\varepsilon_{\text{out}}$ denote two disjoint conductive phases, Γ^ε is the separating interface, T is a positive time, $u^{\text{int}}_\varepsilon$, $u^{\text{out}}_\varepsilon$ are the temperatures in the internal and the external conductive phases, respectively, and $[u_\varepsilon] = u^{\text{out}}_\varepsilon - u^{\text{int}}_\varepsilon$. Finally, ν_ε is the normal unit vector to Γ^ε pointing into the external conductive phase. Clearly, the system of equations stated above should be complemented with a boundary condition for u_ε on $\partial\Omega \times (0,T)$, which will be assumed to be a homogeneous Dirichlet boundary condition.
Note that $\alpha, \beta, \mu^\varepsilon, \lambda^\varepsilon$ are positive constants directly linked to the physical properties of the material and, in principle, should not change in the homogenization limit: a kind of stability which is standard in homogenization theory. In particular, we will assume that μ^ε is given by two different constants $\mu_{\text{int}}, \mu_{\text{out}}$ in the two heat conductive phases as well as $\lambda^\varepsilon = \lambda_{\text{int}}, \lambda_{\text{out}}$.

In the model above, the thermal behaviour of the membrane is described by a parabolic equation involving the Laplace-Beltrami operator Δ^B. Such interface equation furnishes the contact temperature of the two diffusive phases in terms of the jump of the heat fluxes at the interface. For this system of equations an existence and uniqueness theorem can be found in [6].
In order to obtain the macroscopic model, we will use the homogenization unfolding method due to Cioranescu-Damlamian-Griso [7, 8] (see also [9, 10] and for the time-depending case [11, 12]) which leads to a two-scale system satisfied by the macroscopic temperature $u(x, t)$ and involving, as usual, also an equation for the heat diffusion at the microscopic level, which is described by a micro-temperature $u_1(x, y, t)$, depending also on the "fast" variable y. We prefer this approach with respect to the two-scale convergence technique introduced by Nguetseng and Allaire [1], [18] because it allows us to identify more easily the differential equation satisfied by the micro-temperature $u_1(x, y, t)$ on the membrane. Indeed, determining the interface differential equation at the microscopic level requires to identify the tangential derivative of the function $u_1(x, y, t)$ on the membrane. Such a result is interesting in itself and is obtained in fact by using the unfolding method (see Proposition 4.17). We note that in order to achieve the same goal using the two-scale technique, it should be unavoidable to introduce the so-called *very weak two-scale convergence* (a refinement of the two-scale convergence) proposed by Holmbon in [16] and then improved in [14, 15] (see also [19, 22]).
In particular, the problem being linear, the system can be decoupled introducing proper cell functions, thus obtaining a parabolic equation for the macroscopic temperature $u(x, t)$ (see (5.20)), where the diffusion coefficients are given in terms of the capacities and the conductivities of the three constitutive materials, i.e. the two thermal phases and the conductive membrane (see (5.21)). This last result is, according to our opinion, of some applied and physical relevance.
An error estimate for our problem, under extra regularity assumptions on the data, can be found in [5] (see the discussion before Proposition 4.17).

The paper is organized as follows. In Section 2 we recall the definition and some properties of the tangential operators (gradient, divergence, Laplace-Beltrami operator), we state our geometrical setting and we present our model. In Section 3, we prove some energy inequalities. In Section 4 we introduce the unfolding method. As already noted, in order to achieve the homogenization result we need to state a new property of the unfolding operator regarding the unfolded tangential derivative of the temperature u_ε (see Proposition 4.17). Finally, in Section 5 we prove the homogenization result.

2. Prelimineries

2.1. **Tangential derivatives.** Let ϕ be a $\mathcal{C}^2$-function, Φ be a $\mathcal{C}^2$-vector function and S a smooth surface with normal unit vector n. We recall that the tangential gradient of ϕ is given by

$$\nabla^B \phi = \nabla \phi - (n \cdot \nabla \phi) n \tag{2.1}$$

and the tangential divergence of Φ is given by

$$\operatorname{div}^B \Phi = \operatorname{div}\left(\Phi - (n \cdot \Phi)n\right) , \tag{2.2}$$

where, taking into account the smoothness of S, the normal vector n can be naturally defined in a small neighborhood of S as a regular field. Moreover, by (2.1) and (2.2), we get that the Laplace-Beltrami operator can be written as

$$\Delta^B \phi = \operatorname{div}^B(\nabla^B \phi) . \tag{2.3}$$

Finally, we recall that on a regular surface S with no boundary (i.e. when $\partial S = \emptyset$) we have

$$\int_S \nabla^B \phi \, \mathrm{d}\sigma = 0 , \qquad \text{and} \qquad \int_S \operatorname{div}^B \Phi \, \mathrm{d}\sigma = 0 . \tag{2.4}$$

2.2. **Geometrical setting.** The typical periodic geometrical setting is displayed in Figure 1. Here we give its detailed formal definition.

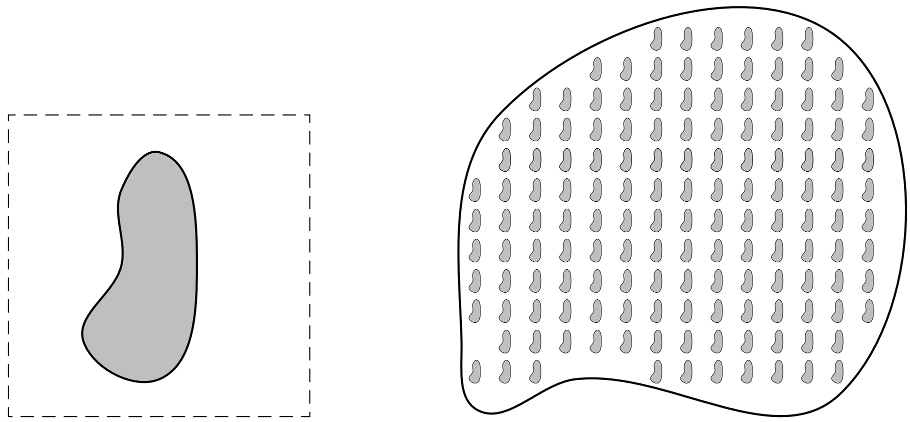

FIGURE 1. *Left*: the periodic cell Y. E_{int} is the shaded region and E_{out} is the white region. *Right*: the region Ω.

Let us introduce a periodic open subset E of $\boldsymbol{R}^N$, so that $E + z = E$ for all $z \in \boldsymbol{Z}^N$. We employ the notation $Y = (0,1)^N$, and $E_{\text{int}} = E \cap Y$, $E_{\text{out}} = Y \setminus \overline{E}$, $\Gamma = \partial E \cap \overline{Y}$. As a simplifying assumption, we stipulate that $\Gamma \cap \partial Y = \emptyset$.
Let Ω be an open connected bounded subset of $\boldsymbol{R}^N$; for all $\varepsilon > 0$ define $\Omega^\varepsilon_{\text{int}} = \Omega \cap \varepsilon E$, $\Omega^\varepsilon_{\text{out}} = \Omega \setminus \overline{\varepsilon E}$, so that $\Omega = \Omega^\varepsilon_{\text{int}} \cup \Omega^\varepsilon_{\text{out}} \cup \Gamma^\varepsilon$, where $\Omega^\varepsilon_{\text{int}}$ and $\Omega^\varepsilon_{\text{out}}$ are two disjoint open subsets of Ω, and $\Gamma^\varepsilon = \partial \Omega^\varepsilon_{\text{int}} \cap \Omega = \partial \Omega^\varepsilon_{\text{out}} \cap \Omega$. The region $\Omega^\varepsilon_{\text{out}}$ [respectively, $\Omega^\varepsilon_{\text{int}}$] corresponds to the outer phase [respectively, the inclusions], while Γ^ε is the interface. We assume that $\operatorname{dist}(\Gamma^\varepsilon, \partial\Omega) \geq \gamma_0 \varepsilon$, for a suitable $\gamma_0 > 0$. We assume also that Ω and E have regular boundary. Finally, let ν denote the normal unit vector to Γ pointing into E_{out}, extended by periodicity to the whole $\boldsymbol{R}^N$, so that $\nu_\varepsilon(x) = \nu(x/\varepsilon)$ denote the normal unit vector to Γ^ε pointing into $\Omega^\varepsilon_{\text{out}}$.

2.3. **Position of the problem.** For later use, we will denote by $H^1_B(\Gamma^\varepsilon)$ the space of Lebesgue measurable functions $u : \Gamma^\varepsilon \to \boldsymbol{R}$ such that $u \in L^2(\Gamma^\varepsilon)$, $\nabla^B u \in L^2(\Gamma^\varepsilon)$. Let us also set

$$\mathcal{X}^\varepsilon_0(\Omega) := H^1_0(\Omega) \cap H^1_B(\Gamma^\varepsilon) \,. \tag{2.5}$$

Let $T > 0$ be a given time, for any spatial domain G, we will denote by $G_T = G \times (0, T)$ the corresponding space–time cylindrical domain over the time interval $(0, T)$.
For every $\varepsilon > 0$ we consider the problem for $u_\varepsilon(x, t)$ stated in the Introduction. We give here a complete formulation for convenience (the operators div and ∇, as well as div^B and ∇^B, act only with respect to the space variable x):

$$\mu^\varepsilon \frac{\partial u_\varepsilon}{\partial t} - \mathrm{div}(\lambda^\varepsilon \nabla u_\varepsilon) = 0 \,, \qquad \text{in } \Omega^\varepsilon_{\mathrm{int}},\ \Omega^\varepsilon_{\mathrm{out}}; \tag{2.6}$$

$$[u_\varepsilon] = 0 \,, \qquad \text{on } \Gamma^\varepsilon; \tag{2.7}$$

$$\varepsilon\alpha \frac{\partial u_\varepsilon}{\partial t} - \varepsilon\beta \Delta^B u_\varepsilon = [\lambda^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon] \,, \qquad \text{on } \Gamma^\varepsilon; \tag{2.8}$$

$$u_\varepsilon(x, t) = 0 \,, \qquad \text{on } \partial\Omega; \tag{2.9}$$

$$u_\varepsilon(x, 0) = \overline{u}_0(x) \,, \qquad \text{in } \Omega. \tag{2.10}$$

Clearly equations (2.6)–(2.9) are in force in the space-time domain, even if for the sake of brevity the interval $(0, T)$ is omitted. This will often be done in the sequel, when no confusion is possible.
We define $\mu^\varepsilon, \lambda^\varepsilon : \Omega \to \boldsymbol{R}$ as

$$\lambda^\varepsilon = \lambda_{\mathrm{int}} \quad \text{in } \Omega^\varepsilon_{\mathrm{int}}, \qquad \lambda^\varepsilon = \lambda_{\mathrm{out}} \quad \text{in } \Omega^\varepsilon_{\mathrm{out}};$$

$$\mu^\varepsilon = \mu_{\mathrm{int}} \quad \text{in } \Omega^\varepsilon_{\mathrm{int}}, \qquad \mu^\varepsilon = \mu_{\mathrm{out}} \quad \text{in } \Omega^\varepsilon_{\mathrm{out}}.$$

Analogously, we define $\mu, \lambda : Y \to \boldsymbol{R}$ as

$$\lambda = \lambda_{\mathrm{int}} \quad \text{in } E_{\mathrm{int}}, \qquad \lambda = \lambda_{\mathrm{out}} \quad \text{in } E_{\mathrm{out}};$$

$$\mu = \mu_{\mathrm{int}} \quad \text{in } E_{\mathrm{int}}, \qquad \mu = \mu_{\mathrm{out}} \quad \text{in } E_{\mathrm{out}}.$$

We also denote

$$[u_\varepsilon] = u^{\mathrm{out}}_\varepsilon - u^{\mathrm{int}}_\varepsilon \,, \tag{2.11}$$

and the same notation will be employed also for other quantities.
We assume that all the constants $\mu_{\mathrm{int}}, \mu_{\mathrm{out}}, \lambda_{\mathrm{int}}, \lambda_{\mathrm{out}}, \alpha, \beta$, involved in equations (2.6) and (2.8) are strictly positive.

Definition 2.1. We say that $u_\varepsilon \in L^2\big(0, T; \mathcal{X}^\varepsilon_0(\Omega)\big)$ is a weak solution of problem (2.6)–(2.10) if

$$- \int_0^t \int_\Omega \mu^\varepsilon u_\varepsilon \frac{\partial \phi}{\partial \tau} \,\mathrm{d}x \,\mathrm{d}\tau + \int_0^t \int_\Omega \lambda^\varepsilon \nabla u_\varepsilon \cdot \nabla \phi \,\mathrm{d}x \,\mathrm{d}\tau - \varepsilon\alpha \int_0^t \int_{\Gamma^\varepsilon} u_\varepsilon \frac{\partial \phi}{\partial \tau} \,\mathrm{d}\sigma \,\mathrm{d}\tau$$

$$+ \varepsilon\beta \int_0^t \int_{\Gamma^\varepsilon} \nabla^B u_\varepsilon \cdot \nabla^B \phi \,\mathrm{d}\sigma \,\mathrm{d}\tau = \int_\Omega \mu^\varepsilon \overline{u}_0 \phi(x, 0) \,\mathrm{d}x + \varepsilon\alpha \int_{\Gamma^\varepsilon} \overline{u}_0 \phi(x, 0) \,\mathrm{d}\sigma \,, \tag{2.12}$$

for every test function $\phi \in \mathcal{C}^\infty(\Omega_T)$ such that ϕ has compact support in Ω for every $t \in (0,T)$ and $\phi(\cdot, T) = 0$ in Ω. □

By [6], for every $\varepsilon > 0$, problem (2.6)–(2.10) admits a unique solution $u_\varepsilon \in L^2\big(0,T;\mathcal{X}_0^\varepsilon(\Omega)\big) \cap \mathcal{C}^0\big([0,T];L^2(\Omega)\cap L^2(\Gamma^\varepsilon)\big)$, if $\overline{u}_0 \in H_0^1(\Omega)$.

3. Energy inequalities

In the following we will assume that the initial data satisfies

$$\overline{u}_0 \in H_0^1(\Omega) \cap H^2(\Omega)\,. \tag{3.1}$$

Firstly we can state the following trace inequality, which can be obtained by rescaling and summing over the ε-cells of Ω the standard trace inequality in Y (see, for instance, [4, formula (7.4) in the proof of Lemma 7.1]).

Proposition 3.1. *Let $w \in H^1(\Omega)$. Then*

$$\int_{\Gamma^\varepsilon} w^2 \,\mathrm{d}\sigma \le \frac{\gamma}{\varepsilon}\left[\int_\Omega w^2\,\mathrm{d}x + \varepsilon^2\int_\Omega |\nabla w|^2\,\mathrm{d}x\right], \tag{3.2}$$

where $\gamma > 0$ is independent of ε.

In particular, if $w \in H_0^1(\Omega)$ then, using the Poincaré's inequality, we simply obtain

$$\int_{\Gamma^\varepsilon} w^2\,\mathrm{d}\sigma \le \frac{\gamma}{\varepsilon}\int_\Omega |\nabla w|^2\,\mathrm{d}x\,. \tag{3.3}$$

By the previous trace inequality we get that $\overline{u}_0$ satisfies

$$\varepsilon\int_{\Gamma^\varepsilon} |\overline{u}_0|^2\,\mathrm{d}\sigma \le \gamma\,, \qquad \varepsilon\int_{\Gamma^\varepsilon} |\nabla^B \overline{u}_0|^2\,\mathrm{d}\sigma \le \gamma\,, \tag{3.4}$$

where $\gamma > 0$ is independent of ε. Notice that, for our purposes, it should be enough to assume that $\overline{u}_0 \in H_0^1(\Omega)$ and satisfies (3.4), but we prefer to assume (3.1) since it is reasonable to choose $\overline{u}_0$ not depending on ε.
We are interested in understanding the limiting behaviour of the heat potential u_ε when $\varepsilon \to 0$; this leads us to look at the homogenization limit of problem (2.6)–(2.10). To this purpose, we first prove some energy estimates for the temperature u_ε. Multiplying (2.6) by u_ε and integrating formally by parts, we obtain

$$\frac{1}{2}\int_0^t\int_\Omega \mu^\varepsilon \frac{\partial u_\varepsilon^2}{\partial \tau}\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_\Omega \lambda^\varepsilon |\nabla u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{\varepsilon\alpha}{2}\int_0^t\int_{\Gamma^\varepsilon} \frac{\partial u_\varepsilon^2}{\partial\tau}\,\mathrm{d}\sigma\,\mathrm{d}\tau + \varepsilon\beta\int_0^t\int_{\Gamma^\varepsilon} |\nabla^B u_\varepsilon|^2(x)\,\mathrm{d}\sigma\,\mathrm{d}\tau = 0\,. \tag{3.5}$$

Then, evaluating the time integral and taking into account the initial condition (2.10), we obtain, for all $0 < t < T$,

$$\frac{1}{2}\int_{\Omega} \mu^{\varepsilon} u_{\varepsilon}^2(t)\,\mathrm{d}x + \int_0^t\!\!\int_{\Omega} \lambda^{\varepsilon}|\nabla u_{\varepsilon}|^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{\varepsilon\alpha}{2}\int_{\Gamma^{\varepsilon}} u_{\varepsilon}^2(t)\,\mathrm{d}\sigma + \varepsilon\beta\int_0^t\!\!\int_{\Gamma^{\varepsilon}} |\nabla^B u_{\varepsilon}|^2(x)\,\mathrm{d}\sigma\,\mathrm{d}\tau = \frac{1}{2}\int_{\Omega} \mu^{\varepsilon}\overline{u}_0^2\,\mathrm{d}x + \frac{\varepsilon\alpha}{2}\int_{\Gamma^{\varepsilon}} \overline{u}_0^2\,\mathrm{d}\sigma\,. \quad (3.6)$$

By (3.4) the right hand side of (3.6) is stable as $\varepsilon \to 0$ and gives the energy estimate

$$\sup_{t\in(0,T)}\int_{\Omega} u_{\varepsilon}^2(t)\,\mathrm{d}x + \int_0^T\!\!\int_{\Omega} |\nabla u_{\varepsilon}|^2\,\mathrm{d}x\,\mathrm{d}\tau + \sup_{t\in(0,T)} \varepsilon\int_{\Gamma^{\varepsilon}} u_{\varepsilon}^2(t)\,\mathrm{d}\sigma + \varepsilon\int_0^T\!\!\int_{\Gamma^{\varepsilon}} |\nabla^B u_{\varepsilon}|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau \le \gamma\,, \quad (3.7)$$

where γ is a constant independent of ε. Multiplying (2.6) by $\frac{\partial u_{\varepsilon}}{\partial t}$ and integrating formally by parts, we obtain

$$\int_0^t\!\!\int_{\Omega} \mu^{\varepsilon}\left(\frac{\partial u_{\varepsilon}}{\partial \tau}\right)^2\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_0^t\!\!\int_{\Omega} \lambda^{\varepsilon}\frac{\partial|\nabla u_{\varepsilon}|^2}{\partial \tau}\,\mathrm{d}x\,\mathrm{d}\tau + \varepsilon\alpha\int_0^t\!\!\int_{\Gamma^{\varepsilon}}\left(\frac{\partial u_{\varepsilon}}{\partial \tau}\right)^2\mathrm{d}\sigma\,\mathrm{d}\tau + \frac{\varepsilon\beta}{2}\int_0^t\!\!\int_{\Gamma^{\varepsilon}} \frac{\partial|\nabla^B u_{\varepsilon}|^2}{\partial \tau}(x)\,\mathrm{d}\sigma\,\mathrm{d}\tau = 0\,. \quad (3.8)$$

Then, evaluating the time integral and taking into account the initial condition (2.10), we obtain, for all $0 < t < T$,

$$\int_0^t\!\!\int_{\Omega} \mu^{\varepsilon}\left(\frac{\partial u_{\varepsilon}}{\partial \tau}\right)^2\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_{\Omega} \lambda^{\varepsilon}|\nabla u_{\varepsilon}|^2(t)\,\mathrm{d}x + \varepsilon\alpha\int_{\Gamma^{\varepsilon}}\left(\frac{\partial u_{\varepsilon}}{\partial \tau}\right)^2\mathrm{d}\sigma\,\mathrm{d}\tau + \frac{\varepsilon\beta}{2}\int_{\Gamma^{\varepsilon}} |\nabla^B u_{\varepsilon}|^2(t)\,\mathrm{d}\sigma = \frac{1}{2}\int_{\Omega} \lambda^{\varepsilon}|\nabla \overline{u}_0|^2\,\mathrm{d}x + \frac{\varepsilon\beta}{2}\int_{\Gamma^{\varepsilon}} |\nabla^B \overline{u}_0|^2\,\mathrm{d}\sigma\,. \quad (3.9)$$

Recalling (3.4), by (3.9) we obtain this further energy estimate

$$\int_0^T\!\!\int_{\Omega}\left(\frac{\partial u_{\varepsilon}}{\partial t}\right)^2\mathrm{d}x\,\mathrm{d}\tau + \sup_{t\in(0,T)}\int_{\Omega} \lambda^{\varepsilon}|\nabla u_{\varepsilon}|^2(t)\,\mathrm{d}x + \varepsilon\int_0^T\!\!\int_{\Gamma^{\varepsilon}}\left(\frac{\partial u_{\varepsilon}}{\partial t}\right)^2\mathrm{d}\sigma\,\mathrm{d}\tau + \sup_{t\in(0,T)} \varepsilon\int_{\Gamma^{\varepsilon}} |\nabla^B u_{\varepsilon}|^2(t)\,\mathrm{d}\sigma \le \gamma\,, \quad (3.10)$$

where γ is a constant independent of ε.

Remark 3.2. Notice that inequalities (3.7) and (3.10) imply that there exists a function u belonging to $L^2\big(0,T;H^1_0(\Omega)\big)\cap H^1\big(\Omega\times(0,T)\big)$ such that, up to a subsequence, $u_\varepsilon \rightharpoonup u$, weakly in $H^1\big(\Omega\times(0,T)\big)$ and $u_\varepsilon \to u$ strongly in $L^2\big(\Omega\times(0,T)\big)$. It will be our purpose to characterize the limit function u. □

4. Definition and main properties of the time-unfolding operator

In this section we define and collect some properties of a space-time version (as in [11, 12]) of the space unfolding operator introduced and developed in [7, 8, 9, 10].
A space-time version of the unfolding operator in a more general framework, in which also a time-microscale is actually present, has been introduced in [2] and [3], to which we also refer for a survey on this topic.
However, in the present case the time variable does not play any special role and can be treated essentially as a parameter, hence most of the properties of this operator can be proven essentially as in the above quoted papers and are therefore omitted. An analogous remark can be done for the other operators which will be introduced in the following. The only real novelty is given by Proposition 4.17, which connects the limit behaviour of the boundary oscillation operator with its tangential derivative. This result is given together with a detailed proof.
Let us set

$$\Xi_\varepsilon = \big\{\xi\in\mathbb{Z}^N\,,\quad \varepsilon(\xi+Y)\subset\Omega\big\}\,,\quad \widehat{\Omega}_\varepsilon = \text{interior}\left\{\bigcup_{\xi\in\Xi_\varepsilon}\varepsilon(\xi+\overline{Y})\right\}\,,$$

$$\Lambda^\varepsilon_T = \widehat{\Omega}_\varepsilon\times(0,T)\,.$$

For $x\in\boldsymbol{R}^N$ we define

$$\left[\frac{x}{\varepsilon}\right]_Y = \left(\left[\frac{x_1}{\varepsilon}\right],\dots,\left[\frac{x_N}{\varepsilon}\right]\right),\qquad \text{so that}\qquad x=\varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y+\left\{\frac{x}{\varepsilon}\right\}_Y\right)\,.$$

Then we introduce the space cell containing x as $Y_\varepsilon(x)=\varepsilon\Big(\left[\frac{x}{\varepsilon}\right]_Y+Y\Big)$.

Definition 4.1. For w Lebesgue-measurable on Ω_T the time-periodic unfolding operator $\mathcal{T}_\varepsilon$ is defined as

$$\mathcal{T}_\varepsilon(w)(x,t,y)=\begin{cases} w\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y+\varepsilon y,t\right)\,, & (x,t,y)\in\Lambda^\varepsilon_T\times Y\,,\\ 0\,, & \text{otherwise.}\end{cases}$$

For w Lebesgue-measurable on Γ^ε_T the boundary unfolding operator $\mathcal{T}^b_\varepsilon$ is defined as

$$\mathcal{T}^b_\varepsilon(w)(x,t,y)=\begin{cases} w\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y+\varepsilon y,t\right)\,, & (x,t,y)\in\Lambda^\varepsilon_T\times \Gamma\,,\\ 0\,, & \text{otherwise.}\end{cases}$$

□

Clearly for w_1, w_2 as in Definition 4.1

$$\mathcal{T}_\varepsilon(w_1w_2)=\mathcal{T}_\varepsilon(w_1)\mathcal{T}_\varepsilon(w_2)\,,\tag{4.1}$$

and the same property holds for the boundary unfolding operator. Note that $\mathcal{T}_\varepsilon^b(w)$ is the trace of the unfolding operator on $\Lambda_T^\varepsilon \times \mathit{\Gamma}$, when both the operators are defined.

We need also an average operator in space defined by

Definition 4.2. Let w be integrable in $\mathit{\Omega}_T$. The space-time average operator is defined by

$$\mathcal{M}^\varepsilon(w)(x,t) = \begin{cases} \dfrac{1}{\varepsilon^N} \displaystyle\int_{Y_\varepsilon(x)} w(\zeta,t)\,\mathrm{d}\zeta\,, & \text{if}\,(x,t)\in\Lambda_T^\varepsilon\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{4.2}$$

□

Remark 4.3. From our definitions it follows

$$\mathcal{M}^\varepsilon(w)(x,t) = \int_Y \mathcal{T}_\varepsilon(w)(x,t,y)\,\mathrm{d}y = \mathcal{M}_Y(\mathcal{T}_\varepsilon(w))(x,t)\,. \tag{4.3}$$

□

Indeed the average operators will be mostly used in connection with the oscillation operators which we define presently.

Definition 4.4. Let w be integrable in $\mathit{\Omega}_T$. The oscillation operator is defined as

$$\mathcal{Z}_\varepsilon(w)(x,t,y) = \left[\mathcal{T}_\varepsilon(w) - \mathcal{M}^\varepsilon(w)\right](x,t,y)\,. \tag{4.4}$$

Analogously, let w be integrable in $\mathit{\Omega}_T$ and also on $\mathit{\Gamma}_T^\varepsilon$. Then the boundary oscillation operator is defined as

$$\mathcal{Z}_\varepsilon{}^b(w)(x,t,y) = \left[\mathcal{T}_\varepsilon^b(w) - \mathcal{M}^\varepsilon(w)\right](x,t,y)\,. \tag{4.5}$$

□

Clearly, the boundary oscillation operator is the trace on $\Lambda_T^\varepsilon \times \mathit{\Gamma}$ of the oscillation operator.

For later use, we will trivially extend our functions, if needed, setting them equal to zero outside $\mathit{\Omega}_T \times Y$.

We collect here some properties of the operators defined above.

Proposition 4.5. *The operator $\mathcal{T}_\varepsilon : L^2(\mathit{\Omega}_T) \to L^2(\mathit{\Omega}_T \times Y)$ is linear and continuous. In addition we have*

$$\|\mathcal{T}_\varepsilon(w)\|_{L^2(\mathit{\Omega}_T\times Y)} \le \|w\|_{L^2(\mathit{\Omega}_T)}\,, \tag{4.6}$$

and

$$\left|\int_{\mathit{\Omega}_T} w\,\mathrm{d}x\,\mathrm{d}\tau - \iint_{\mathit{\Omega}_T\times Y} \mathcal{T}_\varepsilon(w)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau\right| \le \int_{\mathit{\Omega}_T\setminus\Lambda_T^\varepsilon} |w|\,\mathrm{d}x\,\mathrm{d}\tau\,. \tag{4.7}$$

Proposition 4.6. *Let $\{w_\varepsilon\}$ be a sequence of functions in $L^2(\Omega_T)$. If $w_\varepsilon \to w$ strongly in $L^2(\Omega_T)$ as $\varepsilon \to 0$, then*

$$\mathcal{T}_\varepsilon(w_\varepsilon) \to w\,, \quad \textit{strongly in}\, L^2(\Omega_T \times Y)\,. \tag{4.8}$$

If w_ε is a bounded sequence of functions in $L^2(\Omega_T)$, then up to a subsequence

$$\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup \widehat{w}\,, \quad \textit{weakly in}\, L^2\left(\Omega_T \times Y\right)\,, \tag{4.9}$$

and

$$w_\varepsilon \rightharpoonup \mathcal{M}_Y(\widehat{w})\,, \quad \textit{weakly in}\, L^2(\Omega_T)\,. \tag{4.10}$$

Remark 4.7. In particular, if $w \in L^2(\Omega_T)$, we get that $\mathcal{T}_\varepsilon(w) \to w$, for $\varepsilon \to 0$, strongly in $L^2(\Omega_T \times Y)$. □

Remark 4.8. We note that the only cases in which (4.8) holds without assuming the strong convergence of the sequence $\{w_\varepsilon\}$ is when $w_\varepsilon(x,t) = \phi(x,t,\varepsilon^{-1}x)$ where ϕ corresponds to one of the following cases (or sum of them): $\phi(x,t,y) = f_1(x,t)f_2(y)$, with $f_1 f_2 \in L^1(\Omega_T \times Y)$, $\phi \in L^1(Y;\mathcal{C}(\Omega_T))$, $\phi \in L^1(\Omega_T;\mathcal{C}(Y))$. In all such cases we have $\mathcal{T}_\varepsilon(w_\varepsilon) \to \phi$ strongly in $L^2(\Omega_T \times Y)$ (see, for instance, [1, 7, 8] and [3, Remark 2.9]). □

Proposition 4.9. *The operator $\mathcal{T}^b_\varepsilon : L^2(\Gamma^\varepsilon_T) \to L^2(\Omega_T \times \Gamma)$ is linear and continuous. In addition, we have*

$$\|\mathcal{T}^b_\varepsilon(w)\|_{L^2(\Omega_T\times\Gamma)} \le \sqrt{\varepsilon}\|w\|_{L^2(\Gamma^\varepsilon_T)}\,, \tag{4.11}$$

and

$$\int\limits_{\Gamma^\varepsilon_T} w \,\mathrm{d}\sigma\,\mathrm{d}\tau = \frac{1}{\varepsilon}\int\limits_{\Omega_T\times\Gamma} \mathcal{T}^b_\varepsilon(w)\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau\,. \tag{4.12}$$

Note that (4.12) holds since we can choose γ_0 in Subsection 2.2 in such a way that $\Gamma^\varepsilon_T \setminus \Lambda^\varepsilon_T = \emptyset$.

Proposition 4.10. *Assume that $w_\varepsilon \rightharpoonup w$ weakly in $L^2(0,T;H^1_0(\Omega))$. Then $\mathcal{T}^b_\varepsilon(w_\varepsilon) \rightharpoonup w$ weakly in $L^2(\Omega_T \times \Gamma)$.*

Proof. From (4.11) and the trace inequality (3.3), it follows

$$\|\mathcal{T}^b_\varepsilon(w_\varepsilon)\|_{L^2(\Omega_T\times\Gamma)} \le \sqrt{\varepsilon}\|w_\varepsilon\|_{L^2(\Gamma^\varepsilon)} \le \gamma\|\nabla w_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))} \le \gamma\,.$$

Hence there exists $\xi \in L^2(\Omega_T \times \Gamma)$ such that, up to a subsequence, $\mathcal{T}^b_\varepsilon(w_\varepsilon) \rightharpoonup \xi$ weakly in $L^2(\Omega_T \times \Gamma)$. We have only to identify ξ on $\Omega_T \times \Gamma$ with the weak limit w.

To this purpose let us consider vector test functions $\Phi \in \mathcal{C}_c^\infty\big(\Omega_T; \mathcal{C}_\#^\infty(Y)\big)$; then

$$\begin{aligned}
\iint\limits_{\Omega_T\,\Gamma} \xi \Phi(x,t,y)\cdot \nu \,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau &\leftarrow \iint\limits_{\Omega_T\,\Gamma} \mathcal{T}_\varepsilon^b(w_\varepsilon)\Phi(x,t,y)\cdot \nu \,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau \\
&= -\iint\limits_{\Omega_T E_{\mathrm{out}}} \mathcal{T}_\varepsilon(w_\varepsilon)\,\mathrm{div}_y\,\Phi(x,t,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau - \iint\limits_{\Omega_T E_{\mathrm{out}}} \nabla_y \mathcal{T}_\varepsilon(w_\varepsilon)\cdot\Phi(x,t,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau \\
&= -\iint\limits_{\Omega_T E_{\mathrm{out}}} \mathcal{T}_\varepsilon(w_\varepsilon)\,\mathrm{div}_y\,\Phi(x,t,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau - \varepsilon\iint\limits_{\Omega_T E_{\mathrm{out}}} \mathcal{T}_\varepsilon(\nabla_x w_\varepsilon)\cdot\Phi(x,t,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad\to -\iint\limits_{\Omega_T E_{\mathrm{out}}} \widehat{w}(x,y,t)\,\mathrm{div}_y\,\Phi(x,t,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau \qquad (4.13)
\end{aligned}$$

where we used (4.6), (4.9), (4.23) below, and the Gauss-Green formulas. Since $\nabla_y \mathcal{T}_\varepsilon(w_\varepsilon) = \varepsilon \mathcal{T}_\varepsilon(\nabla_x w_\varepsilon) \to 0$ strongly in $L^2(\Omega_T \times Y)$, we obtain that $\widehat{w} \in L^2\big(\Omega_T; H^1(Y)\big)$, and $\nabla_y \widehat{w} = 0$, which implies that $\widehat{w}(x,y,t)$ does not depend on y in $\Omega_T \times E_{\mathrm{out}}$. Moreover by (4.13), it follows that on $\Omega_T \times \Gamma$, $\xi(x,y,t)$ coincides with the trace of $\widehat{w}$ from outside (and hence $\xi(x,y,t) = \xi(x,t)$).
Operating in the same way in E_{int}, we obtain that $\widehat{w}$ does not depend on y even in $\Omega_T \times E_{\mathrm{int}}$ and its trace from inside on $\Omega_T \times \Gamma$ again coincides with $\xi(x,y,t) = \xi(x,t)$. Then $\widehat{w}(x,y,t)$ does not depend on y in the whole of $\Omega_T \times Y$ and therefore, as a consequence of (4.10), we have that $\widehat{w} = w$ in Ω_T, which implies $\xi = w$ on Ω_T.
Thus the whole sequence converges and not only a subsequence and the thesis is achieved. □

Finally we state some results which will be mainly used when we deal with testing functions.

Proposition 4.11. *Let w be a function belonging to $\mathcal{C}(\overline{\Omega}_T)$ then, as $\varepsilon \to 0$,*

$$\mathcal{T}_\varepsilon^b(w) \to w\,, \quad \textit{strongly in}\, L^2\left(\Omega_T \times \Gamma\right)\,. \qquad (4.14)$$

Proof. We have

$$\begin{aligned}
&\iint\limits_{\Omega_T\,\Gamma} |\mathcal{T}_\varepsilon^b(w^2) - w^2|\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad\le \iint\limits_{\Lambda_T^\varepsilon\,\Gamma} |w^2\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y, t\right) - w^2(x,t)|\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau + \|w\|^2_{L^\infty(\Omega_T)}|\Omega_T \setminus \Lambda_T^\varepsilon| \\
&\qquad= \sum_{\xi\in\Xi_\varepsilon} \int_0^T \int\limits_{\xi+\varepsilon Y} \int\limits_{\Gamma} |w^2(\xi+\varepsilon y,t) - w^2(x,t)|\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau + \|w\|^2_{L^\infty(\Omega_T)}|\Omega_T \setminus \Lambda_T^\varepsilon| \\
&\qquad\le T\frac{|\Lambda_T^\varepsilon|}{\varepsilon^N}|\varepsilon Y|\,|\Gamma|\delta + \|w\|^2_{L^\infty(\Omega_T)}|\Omega_T \setminus \Lambda_T^\varepsilon| \le T|\Omega|\,|\Gamma|\delta + \|w\|^2_{L^\infty(\Omega_T)}|\Omega_T \setminus \Lambda_T^\varepsilon|\,,
\end{aligned}$$

where we have that, by the uniform continuity of w on $\overline{\Omega}_T$, for ε sufficiently small,

$$|w^2(\xi+\varepsilon y,t)-w^2(x,t)|<\delta$$

for $y\in\Gamma$, $x\in\xi+\varepsilon Y$ and $t\in[0,T]$. Then, letting first $\varepsilon\to 0$ and then $\delta\to 0$, we get $\big(\mathcal{T}^b_\varepsilon(w)\big)^2=\mathcal{T}^b_\varepsilon(w^2)\to w^2$ strongly in $L^1(\Omega_T\times Y)$ and clearly the same property holds if we replace w^2 with w. Hence, passing to the limit in the equality

$$\iint\limits_{\Omega_T\,\Gamma}|\mathcal{T}^b_\varepsilon(w)-w|^2\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau=\iint\limits_{\Omega_T\,\Gamma}[\big(\mathcal{T}^b_\varepsilon(w)\big)^2+w^2-2\mathcal{T}^b_\varepsilon(w)w]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau\,,$$

the thesis follows. □

As a consequence of Proposition 4.11, taking into account the density of $\mathcal{C}\big([0,T];\mathcal{C}^1(\overline{\Omega})\big)$ in $L^2\big(0,T;H^1(\Omega)\big)$, we can state the following corollary.

Corollary 4.12. *Let w be a function belonging to $L^2\big(0,T;H^1(\Omega)\big)$ then, as $\varepsilon\to 0$,*

$$\mathcal{T}^b_\varepsilon(w)\to w\,,\quad \textit{strongly in}\,L^2\left(\Omega_T\times\Gamma\right)\,. \tag{4.15}$$

Proof. For $w\in L^2\big(0,T;H^1(\Omega)\big)$ and $\{w_k\}\subset\mathcal{C}\big([0,T];\mathcal{C}^1(\overline{\Omega})\big)$ such that, for $k\to+\infty$, $w_k\to w$ strongly in $L^2\big(0,T;H^1(\Omega)\big)$, we obtain (recalling the linearity of the unfolding operator)

$$\begin{aligned}
\iint\limits_{\Omega_T\,\Gamma}|\mathcal{T}^b_\varepsilon(w)-w|^2\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau &\le\gamma\Bigg[\iint\limits_{\Omega_T\,\Gamma}|\mathcal{T}^b_\varepsilon(w)-\mathcal{T}^b_\varepsilon(w_k)|^2\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+\iint\limits_{\Omega_T\,\Gamma}|\mathcal{T}^b_\varepsilon(w_k)-w_k|^2\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau+\iint\limits_{\Omega_T\,\Gamma}|w_k-w|^2\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau\Bigg]\\
&\le\gamma\Bigg[\varepsilon\int\limits_{\Gamma^\varepsilon_T}|w-w_k|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau+\iint\limits_{\Omega_T\,\Gamma}|\mathcal{T}^b_\varepsilon(w_k)-w_k|^2\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau+|\Gamma|\int\limits_{\Omega_T}|w_k-w|^2\,\mathrm{d}x\,\mathrm{d}\tau\Bigg]\\
&\le\gamma\Bigg[\int\limits_{\Omega_T}|w_k-w|^2\,\mathrm{d}x\,\mathrm{d}\tau+\varepsilon^2\int\limits_{\Omega_T}|\nabla w_k-\nabla w|^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+\iint\limits_{\Omega_T\,\Gamma}|\mathcal{T}^b_\varepsilon(w_k)-w_k|^2\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau+\int\limits_{\Omega_T}|w_k-w|^2\,\mathrm{d}x\,\mathrm{d}\tau\Bigg]\,,
\end{aligned}$$

where we used (4.11) and (3.2). Now, taking into account Proposition 4.11 and letting first $\varepsilon\to 0$ and then $k\to+\infty$, the thesis follows. □

Proposition 4.13. *Let $\phi:Y\to\boldsymbol{R}$ be a function extended by Y-periodicity to the whole of $\boldsymbol{R}^N$ and define the sequence*

$$\phi^\varepsilon(x)=\phi\left(\frac{x}{\varepsilon}\right)\,,\qquad x\in\boldsymbol{R}^N\,. \tag{4.16}$$

If ϕ is measurable on Y, then

$$\mathcal{T}_\varepsilon(\phi^\varepsilon)(x,y) = \begin{cases} \phi(y)\,, & (x,y) \in \widehat{\Omega}_\varepsilon \times Y\,, \\ 0\,, & \textit{otherwise.} \end{cases} \tag{4.17}$$

Analogously, if ϕ measurable on Γ, then

$$\mathcal{T}^b_\varepsilon(\phi^\varepsilon)(x,y) = \begin{cases} \phi(y)\,, & (x,y) \in \widehat{\Omega}_\varepsilon \times \Gamma\,, \\ 0\,, & \textit{otherwise.} \end{cases} \tag{4.18}$$

Moreover, if $\phi \in L^2(Y)$ as $\varepsilon \to 0$

$$\mathcal{T}_\varepsilon(\phi^\varepsilon) \to \phi\,, \qquad \textit{strongly in}\, L^2(\Omega \times Y)\,; \tag{4.19}$$

if $\phi \in L^2(\Gamma)$ as $\varepsilon \to 0$

$$\mathcal{T}^b_\varepsilon(\phi^\varepsilon) \to \phi\,, \qquad \textit{strongly in}\, L^2(\Omega \times \Gamma)\,; \tag{4.20}$$

if $\phi \in H^1(Y)$ as $\varepsilon \to 0$

$$\nabla_y(\mathcal{T}_\varepsilon(\phi^\varepsilon)) \to \nabla_y\phi\,, \quad \textit{strongly in}\, L^2(\Omega \times Y)\,. \tag{4.21}$$

Now let us state some properties concerning the behaviour of the unfolding operator with respect to gradients.

Lemma 4.14. *Let $\phi \in L^2\big(0,T;H^1(\Omega \times Y)\big)$, and define*

$$\phi^\varepsilon(x,t) = \phi\left(x,t,\frac{x}{\varepsilon}\right)\,, \qquad (x,t) \in \Omega_T\,, \tag{4.22}$$

where ϕ has been extended by Y-periodicity to $\Omega_T \times \boldsymbol{R}^N$. Then in $\Omega_T \times Y$

$$\nabla_y \mathcal{T}_\varepsilon(\phi^\varepsilon) = \varepsilon \mathcal{T}_\varepsilon\left(\nabla_x \phi\right) + \mathcal{T}_\varepsilon\left(\nabla_y \phi\right)\,. \tag{4.23}$$

Notice that as a consequence of Definitions 4.2 and 4.4 and of Lemma 4.14, if $w \in L^2\big(0,T;H^1_0(\Omega)\big)$

$$\nabla_y \mathcal{Z}_\varepsilon(w) = \nabla_y \mathcal{T}_\varepsilon(w) = \varepsilon \mathcal{T}_\varepsilon(\nabla_x w)\,. \tag{4.24}$$

Similarly, if $w \in L^2\big(0,T;\mathcal{X}^\varepsilon_0(\Omega)\big)$, then

$$\nabla^B_y \mathcal{Z}_\varepsilon{}^b(w) = \nabla^B_y \mathcal{T}^b_\varepsilon(w) = \varepsilon \mathcal{T}^b_\varepsilon(\nabla^B_x w)\,. \tag{4.25}$$

Theorem 4.15. *Let $\{w_\varepsilon\}$ be a sequence converging weakly to w in $L^2\big(0,T;H^1_0(\Omega)\big)$. Then, up to a subsequence, there exists $\widetilde{w} = \widetilde{w}(x,y,t) \in L^2\big(\Omega_T;H^1_\#(Y)\big)$, $\mathcal{M}_Y(\widetilde{w}) = 0$, such that as $\varepsilon \to 0$*

$$\mathcal{T}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widetilde{w}\,, \quad \textit{weakly in}\, L^2(\Omega_T \times Y)\,, \tag{4.26}$$

$$\frac{1}{\varepsilon}\mathcal{Z}_\varepsilon(w_\varepsilon) \rightharpoonup y^c \cdot \nabla w + \widetilde{w}\,, \quad \textit{weakly in}\, L^2(\Omega_T;H^1(Y))\,, \tag{4.27}$$

where

$$y^c = \left(y_1 - \frac{1}{2}\,, y_2 - \frac{1}{2}\,, \cdots\,, y_N - \frac{1}{2}\right)\,.$$

Remark 4.16. Note that by (4.27) and the linearity of the trace operator, it follows

$$\frac{1}{\varepsilon}\mathcal{Z}_\varepsilon{}^b(w_\varepsilon) \rightharpoonup y^c \cdot \nabla w + \widetilde{w}\,, \quad \text{weakly in } L^2(\Omega_T \times \Gamma)\,. \tag{4.28}$$

□

We conclude this subsection with the following result which is, up to our knowledge, new and crucial in order to achieve the rigorous proof of the homogenization theorem. It is worthwhile, in this regard, to stress the fact that, in order to get the homogenized two-scale limit system (5.6)–(5.9), it is fundamental to identify the limit of the Beltrami gradient of u_ε on Γ (i.e. the solution of problem (2.6)–(2.10)) in terms of the Beltrami gradient of the first corrector u_1 (i.e. the function which in Section 5 plays the role of $\widetilde{w}$). In turns this requires to understand what is the sequence related to u_ε converging to u_1. This is not done in the "standard" two-scale approach, since u_1 only appears via its y-gradient. Actually a similar result in the framework of two-scale convergence is obtained in [16, 14, 15], where it is necessary to introduce the concept of "very weak two-scale convergence", which is a refinement of the original one. The identification of the homogenization limit could be obtained using an asymptotic expansion and an error estimate as well (see [5]) but at the price of assuming much more regularity on the data and confining our investigation to the linear case, while on the contrary the approach in this paper can be applied to more general problems having nonlinear source terms.

Proposition 4.17. *Let $\{w_\varepsilon\}$ be a sequence in $L^2\big(0,T;\mathcal{X}_0^\varepsilon(\Omega)\big)$ converging weakly to w in $L^2\big(0,T;H_0^1(\Omega)\big)$, as $\varepsilon \to 0$ and such that*

$$\varepsilon \int_0^T\!\!\int_{\Gamma^\varepsilon} |\nabla_x^B w_\varepsilon|^2 \,\mathrm{d}\sigma\,\mathrm{d}\tau \le \gamma\,, \tag{4.29}$$

where $\gamma > 0$ is a constant independent of ε. Then, for $\widetilde{w}$ the same function as in (4.26)–(4.27), we have that $\nabla_y^B \widetilde{w} \in L^2(\Omega_T \times \Gamma)$ does exist and

$$\nabla_y^B\left(\frac{\mathcal{Z}_\varepsilon{}^b(w_\varepsilon)}{\varepsilon}\right) = \mathcal{T}_\varepsilon^b(\nabla_x^B w_\varepsilon) \rightharpoonup \nabla_x^B w + \nabla_y^B \widetilde{w}\,, \quad \textit{weakly in } L^2(\Omega_T \times \Gamma)\,. \tag{4.30}$$

Proof. By (4.29) and taking into account (4.11), we have

$$\int_{\Omega_T}\!\int_{\Gamma} |\mathcal{T}_\varepsilon^b(\nabla_x^B w_\varepsilon)|^2 \,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau \le \varepsilon \int_0^T\!\!\int_{\Gamma^\varepsilon} |\nabla_x^B w_\varepsilon|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau \le \gamma\,. \tag{4.31}$$

Hence there exists a vector function $\zeta^b \in L^2(\Omega_T \times \Gamma)$ such that, up to a subsequence, $\mathcal{T}_\varepsilon^b(\nabla_x^B w_\varepsilon) \rightharpoonup \zeta^b$ weakly in $L^2(\Omega_T \times \Gamma)$. By (4.25), we obtain

$$\begin{aligned}\int_{\Omega_T}\!\int_{\Gamma} \mathcal{T}_\varepsilon^b(\nabla_x^B w_\varepsilon) \cdot \Psi(y)\phi(x,t)\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau &= \int_{\Omega_T}\!\int_{\Gamma} \frac{1}{\varepsilon}\nabla_y^B \mathcal{Z}_\varepsilon{}^b(w_\varepsilon)\cdot \Psi(y)\phi(x,t)\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau \\ &= -\int_{\Omega_T}\!\int_{\Gamma} \frac{1}{\varepsilon}\mathcal{Z}_\varepsilon{}^b(w_\varepsilon)\,\mathrm{div}_y^B\,\Psi(y)\phi(x,t)\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau\,, \end{aligned}\tag{4.32}$$

for every vector test function $\Phi(x,t,y)$ of the form $\Phi(x,t,y) = \phi(x,t)\Psi(y)$, where $\phi \in \mathcal{C}^\infty_c(\Omega_T)$ and the vector function $\Psi \in \mathcal{C}^\infty_\#(\Gamma)$. Passing to the limit and integrating by parts, it follows

$$\int_{\Omega_T}\int_{\Gamma} \zeta^b \cdot \Psi(y)\phi(x,t)\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau = -\int_{\Omega_T}\int_{\Gamma} (y^c \cdot \nabla_x w + \widetilde{w})\,\mathrm{div}^B_y \Psi(y)\phi(x,t)\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau\,, \tag{4.33}$$

which implies $\xi^b = \nabla^B_y(y^c \cdot \nabla_x w + \widetilde{w}) = \nabla^B_x w + \nabla^B_y \widetilde{w}$, where this last equality follows by an easy calculation taking into account (2.1). This implies that $\nabla^B_y \widetilde{w} = \xi^b - \nabla^B_x w \in L^2(\Omega_T \times \Gamma)$ and, recalling the density of the preceding test functions in $L^2(\Omega_T \times \Gamma)$, (4.30) follows from (4.33). □

5. Main result

Here we prove the main result of the paper; i.e., the homogenization theorem, in which we obtain in a rigorous way that the whole sequence of the solutions u_ε of problem (2.6)–(2.10) converges strongly in $L^2(\Omega_T)$ to the solution of equation (5.20) below.

Theorem 5.1. *Assume that $\overline{u}_0 \in H^1_0(\Omega) \cap H^2(\Omega)$ and that, for every $\varepsilon > 0$, $u_\varepsilon \in L^2\big(0,T;\mathcal{X}^\varepsilon_0(\Omega)\big)$ is the solution of problem (2.6)–(2.10). Then there exist a function $u \in L^2\big(0,T;H^1_0(\Omega)\big)$ and a function $u^1 \in L^2\big(\Omega \times (0,T); H^1_\#(Y)\big)$ such that there exists $\nabla^B_y u^1 \in L^2(\Omega_T \times Y)$ and*

$$u_\varepsilon \to u\,, \qquad \text{strongly in } L^2(\Omega_T); \tag{5.1}$$

$$u_\varepsilon \rightharpoonup u\,, \qquad \text{weakly in } H^1(\Omega_T); \tag{5.2}$$

$$\mathcal{T}_\varepsilon(u_\varepsilon) \to u\,, \qquad \text{strongly in } L^2(\Omega_T \times Y); \tag{5.3}$$

$$\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla_x u + \nabla_y u^1\,, \qquad \text{weakly in } L^2(\Omega_T \times Y); \tag{5.4}$$

$$\mathcal{T}^b_\varepsilon(\nabla^B u_\varepsilon) \rightharpoonup \nabla^B_x u + \nabla^B_y u^1\,, \qquad \text{weakly in } L^2(\Omega_T \times \Gamma); \tag{5.5}$$

and the pair (u,u^1) is the unique weak solution of the two-scale system

$$\widetilde{\mu} u_t - \mathrm{div}\left(\Big(\lambda_0 I - \beta\int_\Gamma \nu\otimes\nu\,\mathrm{d}\sigma\Big)\nabla u\right) - \mathrm{div}\left(\int_Y \lambda\nabla_y u^1\,\mathrm{d}y\right) = 0\,, \quad \text{in } \Omega_T; \tag{5.6}$$

$$-\lambda\,\mathrm{div}_y(\nabla_y u^1 + \nabla_x u) = 0\,, \qquad \text{in } \Omega_T \times (E_{\mathrm{int}} \cup E_{\mathrm{out}}); \tag{5.7}$$

$$\beta\,\mathrm{div}^B_y(\nabla^B_y u^1 + \nabla^B_x u) = -[\lambda(\nabla_y u^1 + \nabla_x u)\cdot\nu]\,, \qquad \text{in } \Omega_T \times \Gamma; \tag{5.8}$$

$$u(x,0) = \overline{u}_0(x)\,, \qquad \text{in } \Omega. \tag{5.9}$$

Proof. Assertions (5.1)–(5.2), up to a subsequence, were proved in Section 3 (see Remark 3.2), while assertion (5.3)–(5.5), still up to a subsequence, follow by Proposition 4.6, Theorem 4.15 and Proposition 4.17. In order to prove that (u,u^1) is the solution of the two-scale system (5.6)–(5.9) we proceed as follows. In (2.12), let us take as test function $\Phi(x,t) = \varphi(x,t) + \varepsilon\phi(x,t)\Psi(\varepsilon^{-1}x)$, with $\varphi,\phi \in \mathcal{C}^\infty\big([0,T];\mathcal{C}^\infty_c(\Omega)\big)$,

$\varphi(\cdot,T)=\phi(\cdot,T)=0$ in Ω, and $\Psi\in\mathcal{C}^\infty_\#(Y)$, so that we can rewrite the weak formulation in the form

$$
\begin{aligned}
&-\int_0^T\!\!\int_\Omega \mu^\varepsilon u_\varepsilon \frac{\partial\varphi}{\partial\tau}\,\mathrm{d}x\,\mathrm{d}\tau-\varepsilon\int_0^T\!\!\int_\Omega \mu^\varepsilon u_\varepsilon \frac{\partial\phi}{\partial\tau}\Psi\,\mathrm{d}x\,\mathrm{d}\tau\\
&+\int_0^T\!\!\int_\Omega \lambda^\varepsilon\nabla u_\varepsilon\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}\tau+\varepsilon\int_0^T\!\!\int_\Omega \lambda^\varepsilon\nabla u_\varepsilon\cdot\nabla_x\phi\Psi\,\mathrm{d}x\,\mathrm{d}\tau+\int_0^T\!\!\int_\Omega \lambda^\varepsilon\nabla u_\varepsilon\cdot\nabla_y\Psi\phi\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad-\varepsilon\alpha\int_0^T\!\!\int_{\Gamma^\varepsilon} u_\varepsilon\frac{\partial\varphi}{\partial\tau}\,\mathrm{d}\sigma\,\mathrm{d}\tau-\varepsilon^2\alpha\int_0^T\!\!\int_{\Gamma^\varepsilon} u_\varepsilon\frac{\partial\phi}{\partial\tau}\Psi\,\mathrm{d}\sigma\,\mathrm{d}\tau\\
&+\varepsilon\beta\int_0^T\!\!\int_{\Gamma^\varepsilon}\nabla^B u_\varepsilon\cdot\nabla^B\varphi\,\mathrm{d}\sigma\,\mathrm{d}\tau+\varepsilon^2\beta\int_0^T\!\!\int_{\Gamma^\varepsilon}\nabla^B u_\varepsilon\cdot\nabla^B_x\phi\Psi\,\mathrm{d}\sigma\,\mathrm{d}\tau+\varepsilon\beta\int_0^T\!\!\int_{\Gamma^\varepsilon}\nabla^B u_\varepsilon\cdot\nabla^B_y\Psi\phi\,\mathrm{d}\sigma\,\mathrm{d}\tau\\
&=\int_\Omega\mu^\varepsilon\overline{u}_0\varphi(x,0)\,\mathrm{d}x+\varepsilon\int_\Omega\mu^\varepsilon\overline{u}_0\phi(x,0)\Psi\,\mathrm{d}x+\varepsilon\alpha\int_{\Gamma^\varepsilon}\overline{u}_0\varphi(x,0)\,\mathrm{d}\sigma+\varepsilon^2\alpha\int_{\Gamma^\varepsilon}\overline{u}_0\phi(x,0)\Psi\,\mathrm{d}\sigma\,.
\end{aligned}
\tag{5.10}
$$

Taking into account (4.1), (4.7) and (4.12) and unfolding, we obtain

$$
\begin{aligned}
&-\int_0^T\!\!\int_\Omega\!\int_Y \mathcal{T}_\varepsilon(\mu^\varepsilon)\mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(\frac{\partial\varphi}{\partial\tau})\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau-\varepsilon\int_0^T\!\!\int_\Omega\!\int_Y \mathcal{T}_\varepsilon(\mu^\varepsilon)\mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(\frac{\partial\phi}{\partial\tau})\mathcal{T}_\varepsilon(\Psi)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau\\
&+\int_0^T\!\!\int_\Omega\!\int_Y \mathcal{T}_\varepsilon(\lambda^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla\varphi)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau+\varepsilon\int_0^T\!\!\int_\Omega\!\int_Y \mathcal{T}_\varepsilon(\lambda^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla_x\phi)\mathcal{T}_\varepsilon(\Psi)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau\\
&+\int_0^T\!\!\int_\Omega\!\int_Y \mathcal{T}_\varepsilon(\lambda^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla_y\Psi)\mathcal{T}_\varepsilon(\phi)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau-\alpha\int_0^T\!\!\int_\Omega\!\int_\Gamma \mathcal{T}^b_\varepsilon(u_\varepsilon)\mathcal{T}^b_\varepsilon(\frac{\partial\varphi}{\partial\tau})\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau\\
&-\varepsilon\alpha\int_0^T\!\!\int_\Omega\!\int_\Gamma \mathcal{T}^b_\varepsilon(u_\varepsilon)\mathcal{T}^b_\varepsilon(\frac{\partial\phi}{\partial\tau})\mathcal{T}^b_\varepsilon(\Psi)\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau+\beta\int_0^T\!\!\int_\Omega\!\int_\Gamma \mathcal{T}^b_\varepsilon(\nabla^B u_\varepsilon)\cdot\mathcal{T}^b_\varepsilon(\nabla^B\varphi)\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau\\
&+\varepsilon\beta\int_0^T\!\!\int_\Omega\!\int_\Gamma \mathcal{T}^b_\varepsilon(\nabla^B u_\varepsilon)\cdot\mathcal{T}^b_\varepsilon(\nabla^B_x\phi)\mathcal{T}^b_\varepsilon(\Psi)\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau+\beta\int_0^T\!\!\int_\Omega\!\int_\Gamma \mathcal{T}^b_\varepsilon(\nabla^B u_\varepsilon)\cdot\mathcal{T}^b_\varepsilon(\nabla^B_y\Psi)\mathcal{T}^b_\varepsilon(\phi)\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau\\
&=\int_\Omega\!\int_Y \mathcal{T}_\varepsilon(\mu^\varepsilon)\mathcal{T}_\varepsilon(\overline{u}_0)\mathcal{T}_\varepsilon(\varphi(x,0))\,\mathrm{d}y\,\mathrm{d}x+\varepsilon\int_\Omega\!\int_Y \mathcal{T}_\varepsilon(\mu^\varepsilon)\mathcal{T}_\varepsilon(\overline{u}_0)\mathcal{T}_\varepsilon(\phi(x,0))\mathcal{T}_\varepsilon(\Psi)\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad+\alpha\int_\Omega\!\int_\Gamma \mathcal{T}^b_\varepsilon(\overline{u}_0\varphi(\cdot,0))\,\mathrm{d}\sigma\,\mathrm{d}x+\varepsilon\alpha\int_\Omega\!\int_\Gamma \mathcal{T}^b_\varepsilon(\overline{u}_0\phi(\cdot,0))\mathcal{T}^b_\varepsilon(\Psi)\,\mathrm{d}\sigma\,\mathrm{d}x+R^\varepsilon\,,
\end{aligned}
\tag{5.11}
$$

where $R^\varepsilon = o(1)$ for $\varepsilon \to 0$.
Then we pass to the limit, taking into account (5.1)–(5.5), Remark 4.7 and Propositions 4.10, 4.11 and 4.13. We get

$$
\begin{aligned}
&- \int_0^T\int_\Omega (\mu_{\mathrm{int}}|E_{\mathrm{int}}| + \mu_{\mathrm{out}}|E_{\mathrm{out}}|)u\frac{\partial\varphi}{\partial\tau}\,\mathrm{d}x\,\mathrm{d}\tau \\
&+ \int_0^T\int_\Omega\int_Y \lambda(\nabla u + \nabla_y u^1)\cdot\nabla\varphi\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^T\int_\Omega\int_Y \lambda(\nabla u + \nabla_y u^1)\cdot\nabla_y\Psi\phi\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad - \alpha|\Gamma|\int_0^T\int_\Omega u\frac{\partial\varphi}{\partial\tau}\,\mathrm{d}x\,\mathrm{d}\tau + \beta\int_0^T\int_\Omega\int_\Gamma (\nabla^B u + \nabla^B_y u^1)\cdot\nabla^B\varphi\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad\qquad + \beta\int_0^T\int_\Omega\int_\Gamma (\nabla^B u + \nabla^B_y u^1)\cdot\nabla^B_y\Psi\phi\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad\qquad = \int_\Omega (\mu_{\mathrm{int}}|E_{\mathrm{int}}| + \mu_{\mathrm{out}}|E_{\mathrm{out}}|)\overline{u}_0\varphi(x,0)\,\mathrm{d}x + \alpha|\Gamma|\int_\Omega \overline{u}_0\varphi(x,0)\,\mathrm{d}x\,.
\end{aligned}
$$

Clearly previous formula is the weak formulation of problem (5.6)–(5.9). In fact, assuming enough regularity for (u, u^1) and taking $\Psi \equiv 0$, integrating by parts with respect to t and with respect to x, we obtain the initial condition $u(x,0) = \overline{u}_0$ (for $x \in \Omega$) and the macroscopic equation

$$
\begin{aligned}
&(\mu_{\mathrm{int}}|E_{\mathrm{int}}| + \mu_{\mathrm{out}}|E_{\mathrm{out}}| + \alpha|\Gamma|)\frac{\partial u}{\partial t} - \operatorname{div}\Big((\lambda_{\mathrm{int}}|E_{\mathrm{int}}| + \lambda_{\mathrm{out}}|E_{\mathrm{out}}|)\nabla u\Big) \\
&\qquad - \operatorname{div}_x\left(\int_Y \lambda\nabla_y u^1\,\mathrm{d}y\right) - \beta\int_\Gamma \operatorname{div}^B_x(\nabla^B u + \nabla^B_y u^1)\,\mathrm{d}y = 0\,, \qquad (5.12)
\end{aligned}
$$

which gives

$$
\begin{aligned}
&(\mu_{\mathrm{int}}|E_{\mathrm{int}}| + \mu_{\mathrm{out}}|E_{\mathrm{out}}| + \alpha|\Gamma|)\frac{\partial u}{\partial t} \\
&\qquad - \operatorname{div}\Big((\lambda_{\mathrm{int}}|E_{\mathrm{int}}| + \lambda_{\mathrm{out}}|E_{\mathrm{out}}| + \beta|\Gamma| - \beta\int_\Gamma \nu\otimes\nu\,\mathrm{d}\sigma)\nabla u\Big) \\
&\qquad\qquad - \operatorname{div}_x\left(\int_Y \lambda\nabla_y u^1\,\mathrm{d}y\right) = 0\,,
\end{aligned}
$$

where we have taken into account that, by (2.2), it follows

$$
-\int_\Gamma \beta\operatorname{div}^B_x\nabla^B_y u^1\,\mathrm{d}\sigma = -\int_\Gamma \beta\operatorname{div}^B_y\nabla^B_x u^1\,\mathrm{d}\sigma = 0\,, \qquad (5.13)
$$

since Γ is a surface with no boundary so that (2.4) holds. Then using (5.12), integrating by parts with respect to y and taking into account the density of product functions in $\mathcal{C}^\infty\big([0,T];\mathcal{C}_c^\infty(\Omega;\mathcal{C}_\#^\infty(Y))\big)$, we obtain

$$
\begin{aligned}
&-\lambda \operatorname{div}_y(\nabla_y u^1 + \nabla u) = 0\,, && (5.14)\\
&\beta \operatorname{div}_y^B(\nabla_y^B u^1 + \nabla^B u) = -[\lambda(\nabla_y u^1 + \nabla u)\cdot\nu]\,. && (5.15)
\end{aligned}
$$

Finally, since the solution (u,u^1) of system (5.6)–(5.9) is unique (see [6] for an investigation in a more general setting), we get that the whole sequence $\{u_\varepsilon\}$ (and not only a subsequence) converges. □

Remark 5.2. Note that in (5.7) and (5.8) the function $u^1(x,y,t)$ can be factorized in terms of ∇u as

$$
u^1(x,y,t) = -\chi(y)\cdot\nabla_x u(x,t) = -\chi_h(y)\frac{\partial u}{\partial x_h}(x,t)\,, \qquad h=1,\dots,N\,, \tag{5.16}
$$

for a vector function $\chi : Y \to \boldsymbol{R}^N$, whose Y-periodic components χ_h have null mean average on Y and satisfy the well-posed system (see [6])

$$
\begin{aligned}
-\lambda \operatorname{div}_y(\nabla_y\chi_h - \boldsymbol{e}_h) &= 0\,, && \text{in } E_{\text{int}},\ E_{\text{out}}; && (5.17)\\
\beta\Delta_y^B(\chi_h - y_h) &= -[\lambda(\nabla_y\chi_h - \boldsymbol{e}_h)\cdot\nu]\,, && \text{on } \Gamma; && (5.18)\\
[\chi_h] &= 0\,, && \text{on } \Gamma. && (5.19)
\end{aligned}
$$

Hence, the two-scale system (5.6)–(5.9) can be decoupled thus obtaining that u satisfies

$$
\widetilde{\mu}u_t - \operatorname{div}\Big((\lambda_0 I + A^{hom})\nabla u\Big) = 0\,, \qquad \text{in } \Omega_T, \tag{5.20}
$$

where

$$
\widetilde{\mu} = \mu_{\text{int}}|E_{\text{int}}| + \mu_{\text{out}}|E_{\text{out}}| + \alpha|\Gamma|\,, \qquad \lambda_0 = \lambda_{\text{int}}|E_{\text{int}}| + \lambda_{\text{out}}|E_{\text{out}}| + \beta|\Gamma|\,,
$$
$$
A^{hom} = \int_\Gamma \Big(-\beta(\nu\otimes\nu) + [\lambda](\nu\otimes\chi)\Big)\,\mathrm{d}\sigma\,. \tag{5.21}
$$

Clearly, equation (5.20) must be complemented with a boundary and an initial condition which are $u=0$ on $\partial\Omega$ and $u(x,0)=\overline{u}_0(x)$ in Ω, respectively, as follows from the microscopic problem (2.6)–(2.10). Notice that, since $\lambda_0 I + A^{hom}$ is a positive definite matrix, equation (5.20) complemented with the previously quoted initial and boundary conditions is a well-posed problem (see [5]).

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