2D Constrained Navier-Stokes Equations

Zdzisław Brzeźniak[a], Gaurav Dhariwal[a,*], Mauro Mariani[b]

[a]Department of Mathematics, University of York, Heslington, York, YO10 5DD, UK
[b]Dipartimento di Matematica Università degli studi di Roma La Sapienza, Piazzale Aldo Moro, 5, Roma, 00185, Italy

Abstract

We study 2D Navier-Stokes equations with a constraint on L^2 energy of the solution. We prove the existence and uniqueness of a global solution for the constrained Navier-Stokes equation on $\mathbb{R}^2$ and $\mathbb{T}^2$, by a fixed point argument. We also show that the solution of constrained Navier-Stokes converges to the solution of Euler equation as viscosity ν vanishes.

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1. Introduction

The motivation for this paper is twofold. Firstly Caglioti *et.al.* in [7] studied the well-posedness and asymptotic behaviour of two dimensional Navier-Stokes equations in the vorticity form with two constraints: constant energy $E(\omega)$ and moment of inertia $I(\omega)$

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \nu \Delta \omega - \nu \operatorname{div}\left[\omega \nabla \left(b\psi + a\frac{|x|^2}{2} \right) \right],$$

which can be rewritten as

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \nu \operatorname{div}\left[\omega \nabla \left(\log \omega - b\psi - a\frac{|x|^2}{2} \right) \right], \tag{1.1}$$

where $\omega = \operatorname{Curl}(u)$, $a = a(\omega)$ and $b = b(\omega)$ are the Lagrange multipliers associated to those constraints and

$$E(\omega) = \int_{\mathbb{R}^2} \psi \omega \, dx, \quad I(\omega) = \int_{\mathbb{R}^2} |x|^2 \omega \, dx, \quad \psi = -\Delta^{-1}\omega.$$

They were able to show the existence of a unique classical global-in-time solution to (1.1) for a family of initial data [7, Theorem 5]. They were also able to prove that the solution to (1.1) converges, as time tends to $+\infty$, to the unique solution of an associated microcanonical variational problem [7, Theorem 8].

Secondly, Rybka [18] and Caffarelli & Lin [6] study the linear heat equation with constraints. Rybka studied heat flow on a manifold $\mathcal{M}$ given by

$$\mathcal{M} = \left\{ u \in L^2(\Omega) \cap C(\Omega) : \int_\Omega u^k(x)\,dx = C_k,\ k = 1,\ldots,N \right\},$$

where Ω denotes a connected bounded region in $\mathbb{R}^2$ with smooth boundary. He proved [18, Theorem 2.5] the existence of the unique global solution for the projected heat equation

$$\begin{cases} \frac{du}{dt} = \Delta u - \sum_{k=1}^{N} \lambda_k u^{k-1} & \text{in } \Omega \subset \mathbb{R}^2, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \qquad u(0,x) = u_0, \end{cases} \tag{1.2}$$

where $\lambda_k = \lambda_k(u)$ are such that u_t is orthogonal to $\mathrm{Span}\left\{u^{k-1}\right\}$. He also showed that the solutions to (1.2) converges to a steady state as time tends to $+\infty$.

On the other hand Caffarelli and Lin initially establish the existence and uniqueness of a global, energy-conserving solution to the heat equation [6, Theorem 1.1]. They were then able to extend these results to more general family of singularly perturbed systems of nonlocal parabolic equations [6, Theorem 3.1]. Their main result was to prove the strong convergence of the solutions to these perturbed systems to some weak-solutions of the limiting constrained nonlocal heat flows of maps into a singular space.

In this paper we consider a problem which links the aforementioned works. We consider Navier-Stokes equations as in [7], but subject to the same energy constraint as in [6, 18]. Contrary to [7] we prove global-in-time existence of the solution but only on a torus, namely in the periodic case. Surprisingly our proof of global existence does not hold for a general bounded domain, although the local existence holds. We also prove our result of global existence of the solution for $\mathbb{R}^2$. We additionally show that, in vanishing viscosity limit, the solution of the constrained equation (1.3) below, converges to the Bardos solution (see [1]) of the Euler equation (formally obtained setting $\nu = 0$).

We are interested in the Cauchy problem

$$\begin{cases} \dfrac{du}{dt} = -\nu Au + \nu |\nabla u|^2\, u - B(u, u), \\ u(0) = u_0, \end{cases} \tag{1.3}$$

where $u \in \mathrm{H}$, and H is a space of divergence free, mean zero vector fields on a torus, see (2.2) below for a precise definition.

The above problem has a local *maximal* solution for each $u_0 \in \mathrm{V} \cap \mathcal{M}$, where V is defined in (2.2) and

$$\mathcal{M} = \{u \in \mathrm{H} : |u| = 1\}.$$

Moreover $u(t) \in \mathcal{M}$ for all times t. This result is true both for NSEs on a bounded domain or with periodic boundary conditions (i.e. on a torus). In a more geometrical fashion, equation (1.3) can be also written as

$$\frac{du}{dt} = -\nabla_{\mathcal{M}}\mathcal{E}(u) - B(u, u),$$

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where $\mathcal{E}(u) = \frac{1}{2}|\nabla u|^2$, $u \in \mathcal{M}$ and $\nabla_{\mathcal{M}}\mathcal{E}(u)$ is the gradient of $\mathcal{E}$ with respect to H-norm projected onto $T_u\mathcal{M}$. The remarkable feature of this is that on a torus $\nabla_{\mathcal{M}}\mathcal{E}(u)$ and $B(u, u)$ are orthogonal in H. This orthogonality holds for the Navier-Stokes without constraint too, i.e. on a torus $\nabla\mathcal{E}(u)$ is orthogonal to $B(u, u)$ in H. The fact that this constraint preserves the orthogonality somehow makes it a natural constraint.

Hence in at least heuristic way

$$\frac{d}{dt}\mathcal{E}(u(t)) = \left\langle \nabla_{\mathcal{M}}\mathcal{E}(u(t)), \frac{du}{dt} \right\rangle_{\mathrm{H}}$$
$$= \langle \nabla_{\mathcal{M}}\mathcal{E}(u(t)), -\nabla_{\mathcal{M}}\mathcal{E}(u(t)) - B(u, u)\rangle_{\mathrm{H}}$$
$$= -|\nabla_{\mathcal{M}}\mathcal{E}(u(t))|^2,$$

so that $\mathcal{E}(u(t))$ is decreasing and thus the $H^{1,2}$ norm of the solution remains bounded.

Next we state the two main results of the paper on a torus.

Theorem 1.1. *Let $u_0 \in V \cap \mathcal{M}$ and $X_T = C([0, T]; V) \cap L^2(0, T; E)$. Then for every $\nu > 0$ there exists a global and locally unique solution $u \in X_T$ of* (1.3).

The space X_T with more details and the precise definition of the solution of (1.3) will be given in the Section 3. Theorem 1.1 will be proved in steps in Sections 3 and 4.

Theorem 1.2. *Let $u_0, u_0^\nu \in V \cap \mathcal{M}$ and u^ν be the solution of* (1.3) *(existence and uniqueness of u^ν follows from Theorem 1.1). Assume that $u_0^\nu \to u_0$ in V as $\nu \downarrow 0$, and that $\mathrm{Curl}(u_0^\nu)$ stays uniformly bounded in $L^\infty(\mathbb{T}^2)$. Then for each $T > 0$, u^ν converges in $C([0, T]; L^2(\mathbb{T}^2))$ to the unique solution u of the limiting equation (namely* (1.3) *with $\nu = 0$).*

We end the introduction with a brief description of the content of the paper. In Section 2, we introduce a constrained Navier-Stokes equation. In Section 3, a precise definition of the solution is given, and local existence and uniqueness are proved, together with some basic properties of the solution. In Section 4, global existence is established. Finally, in Section 5 we prove Theorem 1.2.

2. Constrained Navier-Stokes equations

2.1. General Notations

Let O be either a bounded domain in $\mathbb{R}^2$, $\mathbb{R}^2$ or $\mathbb{T}^2$. For $p \in [1, \infty]$ and $k \in \mathbb{N}$, the Lebesgue and Sobolev spaces of $\mathbb{R}^2$-valued functions will be denoted by $L^p(O, \mathbb{R}^2)$ and $W^{k,p}(O, \mathbb{R}^2)$ respectively, and often L^p and $W^{k,p}$ whenever the context is understood. The usual scalar product on L^2 is denoted by $\langle u, v \rangle$ for $u, v \in L^2$. The associated norm is given by $|u|, u \in L^2$. We also write $W^{k,2}(O, \mathbb{R}^2) := H^k$ and will denote it's norm by $\|\cdot\|_{H^k}$. In particular the scalar product for H^1 is given by

$$\langle u, v \rangle_{H^1} = \langle u, v \rangle + \langle \nabla u, \nabla v \rangle, \quad u, v \in H^1,$$

and thus the norm is

$$\|u\|_{H^1} = \left[|u|^2 + |\nabla u|^2 \right]^{1/2}.$$

In the following two subsections we will introduce some additional spaces. The structure of the spaces will depend on the choice of O.

2.2. Functional setting for $\mathbb{R}^2$

We consider the whole space $\mathbb{R}^2$. We introduce the following spaces:

$$
\begin{aligned}
&\mathrm{H} = \{u \in L^2(\mathbb{R}^2, \mathbb{R}^2) : \nabla \cdot u = 0\}, \\
&\mathrm{V} = H^1 \cap \mathrm{H}.
\end{aligned}
\tag{2.1}
$$

We endow H with the scalar product and norm of L^2 and denote it by $\langle u, \mathrm{v}\rangle_{\mathrm{H}}$, $|u|_{\mathrm{H}}$ respectively for $u, \mathrm{v} \in \mathrm{H}$. We equip the space V with the scalar product and norm of H^1 and will denote it by $\langle \cdot, \cdot \rangle_V$ and $\| \cdot \|_V$ respectively.

Let $\Pi : L^2 \to \mathrm{H}$ be Leray-Helmholtz projection operator which projects the vector fields on the plane of divergence free vector fields. We denote by $A : \mathrm{D}(A) \to \mathrm{H}$, the Stokes operator which is defined by

$$
\begin{aligned}
&\mathrm{D}(A) = \mathrm{H} \cap H^2(\mathbb{R}^2), \\
&Au = -\Pi \Delta u, \quad u \in \mathrm{D}(A).
\end{aligned}
$$

It is well known that A is a self adjoint non-negative operator in H. Note that Δ and Π commute with each other. Moreover

$$
\mathrm{D}((A + I)^{1/2}) = \mathrm{V} \quad \text{and} \quad \langle Au, u \rangle = |\nabla u|^2, \quad u \in \mathrm{D}(A).
$$

From now onwards we will denote $\mathrm{E} := \mathrm{D}(A)$.

2.3. Functional setting for a periodic domain

We denote the bounded periodic domain by $\mathbb{T}^2$ which can be identified to a two dimensional torus. We introduce the following spaces:

$$
\begin{aligned}
&\mathbb{L}_0^2 = \{u \in L^2(\mathbb{T}^2, \mathbb{R}^2) : \int_{\mathbb{T}^2} u(x)\, dx = 0\}, \\
&\mathrm{H} = \{u \in \mathbb{L}_0^2 : \nabla \cdot u = 0\}, \\
&\mathrm{V} = H^1 \cap \mathrm{H}.
\end{aligned}
\tag{2.2}
$$

We endow H with the scalar product and norm of L^2 and denote it by $\langle u, \mathrm{v}\rangle_{\mathrm{H}}$, $|u|_{\mathrm{H}}$ respectively for $u, \mathrm{v} \in \mathrm{H}$. We equip the space V with the scalar product $\langle \nabla u, \nabla \mathrm{v} \rangle_{\mathrm{H}}$ and norm $\|u\|_V, u, \mathrm{v} \in \mathrm{V}$.

One can show that in the case of $\mathbb{T}^2$ V-norm $\| \cdot \|_V$, and H^1-norm $\| \cdot \|_{H^1}$ are equivalent on V.

As before we denote by $A : \mathrm{D}(A) \to \mathrm{H}$, the Stokes operator which is defined by

$$
\begin{aligned}
&\mathrm{D}(A) = \mathrm{H} \cap H^2(\mathbb{T}^2), \\
&Au = -\Delta u, \quad u \in \mathrm{D}(A).
\end{aligned}
$$

It is well known that A is a self adjoint positive operator in H. Moreover

$$
\mathrm{D}(A^{1/2}) = \mathrm{V} \quad \text{and} \quad \langle Au, u \rangle = \|u\|_V^2 = |\nabla u|^2, \quad u \in \mathrm{D}(A).
$$

In the following subsection we will introduce a tri-linear form which is well defined for any general domain O and will state some of it's properties.

2.4. Preliminaries

From now onwards we denote our domain by O which can be either $\mathbb{R}^2$ or $\mathbb{T}^2$. We introduce a continuous tri-linear form $b : L^p \times W^{1,q} \times L^r \to \mathbb{R}$,

$$b(u, \mathrm{v}, w) = \sum_{i,j=1}^{2} \int_O u^i \frac{\partial \mathrm{v}^j}{\partial x^i} w^j \, dx,$$

where $p, q, r \in [1, \infty]$ satisfies

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1.$$

We can define a bilinear map $B : V \times V \to V'$ such that

$$\langle B(u, \mathrm{v}), \phi \rangle = b(u, \mathrm{v}, \phi), \quad \text{for } u, \mathrm{v}, \phi \in V,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between V and V'. If $u \in V, \mathrm{v} \in E$ and $\phi \in H$ then

$$|b(u, \mathrm{v}, \phi)| \leq \sqrt{2} \, |u|_H^{\frac{1}{2}} \|u\|_V^{\frac{1}{2}} \|\mathrm{v}\|_V^{\frac{1}{2}} |\mathrm{v}|_E^{\frac{1}{2}} |\phi|_H.$$

Thus b can be uniquely extended to the tri-linear form (denoted by the same letter)

$$b : V \times E \times H \to \mathbb{R}.$$

We can now also extend the operator B uniquely to a bounded linear operator

$$B : V \times E \to H.$$

The following properties of the tri-linear map b and the bilinear map B are very well established in [19] and Appendix A,

$$b(u, u, u) = 0, \qquad u \in V,$$
$$b(u, w, w) = 0, \qquad u \in V, w \in H^1,$$
$$\langle B(u, u), Au \rangle_H = 0, \quad u \in D(A).$$

The 2D Navier-Stokes equations are given as following:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \nu \Delta u(x,t) + (u(x,t) \cdot \nabla) u(x,t) + \nabla p(x,t) = 0, \\ \nabla \cdot u(x,t) = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{2.3}$$

where $x \in O$ and $t \in [0, T]$ for every $T > 0$; $u \colon O \to \mathbb{R}^2$ and $p \colon O \to \mathbb{R}$ are velocity and pressure of the fluid respectively. ν is the viscosity of the fluid (with no loss of generality, ν will be taken equal to 1 for the rest of the article, except in the Section 5).

With all the notations as defined in the subsections 2.1 and 2.2, the Navier-Stokes equation (2.3) projected on divergence free vector field is given by

$$\begin{cases} \dfrac{du}{dt} + Au + B(u, u) = 0, \\ u(0) = u_0. \end{cases} \tag{2.4}$$

Let us denote the set of divergence free $\mathbb{R}^2$-valued functions with unit L^2 norm, as following

$$\mathcal{M} = \{u \in \mathrm{H} : |u| = 1\}.$$

Then the tangent space at u is defined as,

$$T_u\mathcal{M} = \{\mathrm{v} \in \mathrm{H} : \langle \mathrm{v}, u \rangle_{\mathrm{H}} = 0\}, \quad u \in \mathcal{M}.$$

We define a linear map $\pi_u : \mathrm{H} \to T_u\mathcal{M}$ by

$$\pi_u(\mathrm{v}) = \mathrm{v} - \langle \mathrm{v}, u \rangle_{\mathrm{H}} \, u,$$

then π_u is the orthogonal projection from H into $T_u\mathcal{M}$.

Let $F(u) = Au + B(u, u)$ and $\hat{F}(u)$ be the projection of $F(u)$ on the tangent space $T_u\mathcal{M}$, then

$$\begin{aligned}
\hat{F}(u) &= \pi_u(F(u)) \\
&= F(u) - \langle F(u), u \rangle_{\mathrm{H}} \, u \\
&= Au + B(u, u) - \langle Au + B(u, u), u \rangle_{\mathrm{H}} \, u \\
&= Au - \langle Au, u \rangle_{\mathrm{H}} \, u + B(u, u) - \langle B(u, u), u \rangle_{\mathrm{H}} \, u \\
&= Au - |\nabla u|^2 \, u + B(u, u).
\end{aligned}$$

The last equality follows from the identity that $\langle B(u, u), u \rangle_{\mathrm{H}} = 0$.

Remark 2.1. Since $\langle B(u, u), u \rangle_{\mathrm{H}} = 0$ and $u \in \mathcal{M}$, $B(u, u) \in T_u\mathcal{M}$.

Thus by projecting NSEs (2.4) on the manifold $\mathcal{M}$, we obtain our constrained Navier-Stokes equation which is given by

$$\begin{cases} \dfrac{du}{dt} + Au - |\nabla u|^2 \, u + B(u, u) = 0, \\ u(0) = u_0 \in \mathrm{V} \cap \mathcal{M}. \end{cases} \tag{2.5}$$

3. Local solution : Existence and Uniqueness

In this section we will establish the existence of a local solution of the problem (2.5) by using fixed point method. We obtain certain estimates for non-linear terms of (2.5) using preliminaries from the previous section. After obtaining these estimates we construct a globally Lipschitz map. Some ideas in the Subsection 3.1 are based on [5].

We use the following well established [19] result to obtain the estimates.

Lemma 3.1. *For any open set $\Omega \subset \mathbb{R}^2$ and every $\mathrm{v} \in H^1$, we have*

$$|\mathrm{v}|_{\mathbb{L}^4(\Omega)} \le 2^{1/4} |\mathrm{v}|^{1/2}_{\mathbb{L}^2(\Omega)} |\nabla \mathrm{v}|^{1/2}_{\mathbb{L}^2(\Omega)}, \quad \mathrm{v} \in H^1(\Omega).$$

In what follows we assume that E, V and H are spaces defined before in Section 2.

Lemma 3.2. *Let* $G_1 \colon V \to H$ *be defined by*

$$G_1(u) = |\nabla u|^2 \, u, \quad u \in V.$$

Then there exists $C > 0$ such that for $u_1, u_2 \in V$,

$$|G_1(u_1) - G_1(u_2)|_H \le C\|u_1 - u_2\|_V [\|u_1\|_V + \|u_2\|_V]^2. \tag{3.1}$$

Proof. Let us consider $u_1, u_2 \in V$, then

$$
\begin{aligned}
|G_1(u_1) - G_1(u_2)|_H &= \left| |\nabla u_1|^2 \, u_1 - |\nabla u_2|^2 \, u_2 \right|_H \\
&= \left| |\nabla u_1|^2 \, u_1 - |\nabla u_1|^2 \, u_2 + |\nabla u_1|^2 \, u_2 - |\nabla u_2|^2 \, u_2 \right|_H \\
&= \left| |\nabla u_1|^2 \, (u_1 - u_2) + (|\nabla u_1|^2 - |\nabla u_2|^2) \, u_2 \right|_H \\
&\le |\nabla u_1|^2 \, |u_1 - u_2|_H + [|\nabla u_1| + |\nabla u_2|] \, [|\nabla u_1| - |\nabla u_2|] \, |u_2|_H \\
&\le |\nabla u_1|^2 \, |u_1 - u_2|_H + [|\nabla u_1| + |\nabla u_2|] \, |\nabla (u_1 - u_2)| \, |u_2|_H \\
&\le C \left[|\nabla u_1|^2 \, \|u_1 - u_2\|_V + [|\nabla u_1| + |\nabla u_2|] \, |\nabla (u_1 - u_2)| \, \|u_2\|_V \right] \\
&\le C\|u_1 - u_2\|_V \left[|\nabla u_1|^2 + |\nabla u_2| \, \|u_2\|_V + |\nabla u_1| \, \|u_2\|_V \right],
\end{aligned}
$$

where we have repeatedly used the fact that V is continuously embedded in H. Thus we obtain,

$$|G_1(u_1) - G_1(u_2)|_H \le C\|u_1 - u_2\|_V \, [\|u_1\|_V + \|u_2\|_V]^2.$$

$\square$

Lemma 3.3. *Let* $G_2 \colon E \to H$ *be defined by*

$$G_2(u) = B(u, u), \quad u \in E.$$

Then there exists $\tilde{C} > 0$ such that for $u_1, u_2 \in E$,

$$|G_2(u_1) - G_2(u_2)|_H \le \tilde{C} \left[\|u_1\|_V^{1/2} |u_1|_E^{1/2} \|u_1 - u_2\|_V + \|u_2\|_V \|u_1 - u_2\|_V^{1/2} |u_1 - u_2|_E^{1/2} \right]. \tag{3.2}$$

Proof. Let us take $u_1, u_2 \in E$, then

$$
\begin{aligned}
|G_2(u_1) - G_2(u_2)|_H &= |B(u_1, u_1) - B(u_2, u_2)|_H \\
&= |B(u_1, u_1) - B(u_2, u_1) + B(u_2, u_1) - B(u_2, u_2)|_H \\
&= |B(u_1 - u_2, u_1) + B(u_2, u_1 - u_2)|_H \\
&= |\Pi [(u_1 - u_2) \cdot \nabla u_1] + \Pi [u_2 \cdot \nabla (u_1 - u_2)]|_H \\
&\le |(u_1 - u_2) \cdot \nabla u_1|_H + |u_2 \cdot \nabla (u_1 - u_2)|_H \\
&\le |u_1 - u_2|_{L^4(O)} |\nabla u_1|_{L^4(O)} + |u_2|_{L^4(O)} |\nabla (u_1 - u_2)|_{L^4(O)}.
\end{aligned}
$$

Now using Lemma 3.1 and the embedding of V in H, we obtain,

$$
\begin{aligned}
|G_2(u_1) - G_2(u_2)|_H &\le \sqrt{2} \, |u_1 - u_2|_H^{1/2} |\nabla (u_1 - u_2)|_H^{1/2} |\nabla u_1|_H^{1/2} |\nabla^2 u_1|_H^{1/2} \\
&\quad + \sqrt{2} \, |u_2|_H^{1/2} |\nabla u_2|_H^{1/2} |\nabla (u_1 - u_2)|_H^{1/2} |\nabla^2 (u_1 - u_2)|_H^{1/2} \\
&\le \sqrt{2} C \Big[\|u_1 - u_2\|_V \|u_1\|_V^{1/2} |u_1|_E^{1/2} \\
&\quad + \|u_2\|_V \|u_1 - u_2\|_V^{1/2} |u_1 - u_2|_E^{1/2} \Big].
\end{aligned}
$$

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Thus we obtain the following inequality

$$|G_2(u_1) - G_2(u_2)|_H \leq \tilde{C}\left[\|u_1\|_V^{1/2}|u_1|_E^{1/2}\|u_1 - u_2\|_V + \|u_2\|_V\|u_1 - u_2\|_V^{1/2}|u_1 - u_2|_E^{1/2}\right].$$

$\square$

3.1. Construction of a globally Lipschitz map

Let $\theta : \mathbb{R}_+ \to [0, 1]$ be a C_0^∞ non-increasing function such that

$$\inf_{x \in \mathbb{R}_+} \theta'(x) \geq -1, \quad \theta(x) = 1 \text{ iff } x \in [0, 1] \text{ and } \theta(x) = 0 \text{ iff } x \in [3, \infty)$$

and for $n \geq 1$ set $\theta_n(\cdot) = \theta(\frac{\cdot}{n})$. Observe that if $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function, then for every $x, y \in \mathbb{R}_+$,

$$\theta_n(x)h(x) \leq h(3n), \quad |\theta_n(x) - \theta_n(y)| \leq 3n|x - y|.$$

Set

$$X_T = C([0, T]; V) \cap L^2(0, T; E),$$

with norm

$$|u|_{X_T}^2 = \sup_{t \in [0,T]} \|u(t)\|_V^2 + \int_0^T |u(t)|_E^2 dt.$$

Let us define $G : E \to H$ as

$$G(u) := G_1(u) - G_2(u) = |\nabla u|^2 u - B(u, u). \tag{3.3}$$

Lemma 3.4. *Suppose $G : E \to H$ is a map defined in (3.3). Let $T > 0$, define a map $\Phi_{n,T} : X_T \to L^2(0, T; H)$ by*

$$\Phi_{n,T}(u)(x, t) = \theta_n(|u|_{X_t})G(u)(x, t). \tag{3.4}$$

Then $\Phi_{n,T}$ is globally Lipschitz and moreover, for any $u_1, u_2 \in X_T$,

$$|\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)|_{L^2(0,T;H)} \leq K(n, T)|u_1 - u_2|_{X_T}T^{\frac{1}{4}}, \tag{3.5}$$

where

$$K(n, T) = 3n\left(27n^3T^{1/4} + 9n^2 + 12nT^{1/4} + 2\right),$$

depends on n and T only.

Proof. Assume that $u_1, u_2 \in X_T$. Set

$$\tau_i = \inf\left\{t \in [0, T]; |u_i|_{X_t} \geq 3n\right\}, \quad i = 1, 2.$$

Without loss of generality assume that $\tau_1 \leq \tau_2$. Consider

$$|\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)|_{L^2(0,T;H)} = \left[\int_0^T |\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)|_H^2 \, dt\right]^{\frac{1}{2}}$$

$$= \left[\int_0^T \left|\theta_n(|u_1|_{X_t})G(u_1) - \theta_n(|u_2|_{X_t})G(u_2)\right|_H^2 \, dt\right]^{\frac{1}{2}},$$

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for $i = 1, 2$ $\theta_n(|u_i|_{X_t}) = 0$ for $t \geq \tau_2$, thus we have

$$|\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)|_{L^2(0,T;\mathrm{H})} = \left[\int_0^{\tau_2} \left| \theta_n(|u_1|_{X_t})G(u_1) - \theta_n(|u_2|_{X_t})G(u_2) \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}}$$

$$= \left[\int_0^{\tau_2} \left| \theta_n(|u_1|_{X_t}) \left[G_1(u_1) - G_2(u_1) \right] \right.\right.$$

$$\left.\left. - \theta_n(|u_2|_{X_t}) \left[G_1(u_2) - G_2(u_2) \right] \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}}$$

$$= \left[\int_0^{\tau_2} \left| \theta_n(|u_1|_{X_t})G_1(u_1) - \theta_n(|u_1|_{X_t})G_1(u_2) \right.\right.$$

$$+ \theta_n(|u_1|_{X_t})G_1(u_2) - \theta_n(|u_2|_{X_t})G_1(u_2)$$

$$+ \theta_n(|u_1|_{X_t})G_2(u_2) - \theta_n(|u_1|_{X_t})G_2(u_1)$$

$$\left.\left. + \theta_n(|u_2|_{X_t})G_2(u_2) - \theta_n(|u_1|_{X_t})G_2(u_2) \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}}.$$

Using the Minkowski inequality we get,

$$|\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)|_{L^2(0,T;\mathrm{H})} \leq \left[\int_0^{\tau_2} \left| \theta_n(|u_1|_{X_t}) \left[G_1(u_1) - G_1(u_2) \right] \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}}$$

$$+ \left[\int_0^{\tau_2} \left| \left[\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t}) \right] G_1(u_2) \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}}$$

$$+ \left[\int_0^{\tau_2} \left| \theta_n(|u_1|_{X_t}) \left[G_2(u_2) - G_2(u_1) \right] \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}}$$

$$+ \left[\int_0^{\tau_2} \left| \left[\theta_n(|u_2|_{X_t}) - \theta_n(|u_1|_{X_t}) \right] G_2(u_2) \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}}.$$

Set

$$A_1 = \left[\int_0^{\tau_2} \left| \left[\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t}) \right] G_1(u_2) \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}},$$

$$A_2 = \left[\int_0^{\tau_2} \left| \theta_n(|u_1|_{X_t}) \left[G_1(u_1) - G_1(u_2) \right] \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}},$$

$$A_3 = \left[\int_0^{\tau_2} \left| \left[\theta_n(|u_2|_{X_t}) - \theta_n(|u_1|_{X_t}) \right] G_2(u_2) \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}},$$

$$A_4 = \left[\int_0^{\tau_2} \left| \theta_n(|u_1|_{X_t}) \left[G_2(u_2) - G_2(u_1) \right] \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}}.$$

and hence

$$|\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)|_{L^2(0,T;\mathrm{H})} \leq A_1 + A_2 + A_3 + A_4. \tag{3.6}$$

Since θ_n is a Lipschitz function with Lipschitz constant $3n$ we obtain,

$$A_1^2 = \int_0^{\tau_2} \left| \left[\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t}) \right] G_1(u_2) \right|_H^2 dt$$

$$\leq 9n^2 \int_0^{\tau_2} \left| |u_1|_{X_t} - |u_2|_{X_t} \right|_H^2 |G_1(u_2)|_H^2 dt.$$

Again using the Minkowski inequality we get

$$A_1^2 \leq 9n^2 \int_0^{\tau_2} |u_1 - u_2|_{X_t}^2 |G_1(u_2)|_H^2 dt$$

$$\leq 9n^2 |u_1 - u_2|_{X_T}^2 \int_0^{\tau_2} |G_1(u_2)|_H^2 dt. \tag{3.7}$$

Now consider $\int_0^{\tau_2} |G_1(u_2)|_H^2 dt$; using (3.1) we get

$$\int_0^{\tau_2} |G_1(u_2)|_H^2 dt \leq C \int_0^{\tau_2} \|u_2(t)\|_V^6 dt$$

$$\leq C^2 \left[\sup_{t \in [0,\tau_2]} \|u_2(t)\|_V^6 \right] \int_0^{\tau_2} dt$$

$$\leq C^2 \left[\sup_{t \in [0,\tau_2]} \|u_2(t)\|_V^2 \right]^3 \tau_2.$$

Since

$$|u_2|_{X_{\tau_2}}^2 = \sup_{t \in [0,\tau_2]} \|u_2(t)\|_V^2 + \int_0^{\tau_2} |u_2(t)|_E^2 dt,$$

thus

$$\sup_{t \in [0,\tau_2]} \|u_2(t)\|_V^2 \leq |u_2|_{X_{\tau_2}}^2,$$

and using

$$|u_2|_{X_{\tau_2}} \leq 3n,$$

we get

$$\int_0^{\tau_2} |G_1(u_2)|_H^2 dt \leq C \left[\sup_{t \in [0,\tau_2]} \|u_2(t)\|_V^2 \right]^3 \tau_2$$

$$\leq C |u_2|_{X_{\tau_2}}^6 \tau_2$$

$$\leq C(3n)^6 \tau_2.$$

Hence, the inequality (3.7) takes the form

$$A_1^2 \leq 9n^2 C |u_1 - u_2|_{X_T}^2 (3n)^6 \tau_2,$$

$$A_1 \leq (3n)^4 C |u_1 - u_2|_{X_T} \tau_2^{\frac{1}{2}}. \tag{3.8}$$

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Similarly, since $\theta_n(|u_1|_{X_t}) = 0$ for $t \geq \tau_1$ and $\tau_1 \leq \tau_2$, we have

$$A_2 = \left[\int_0^{\tau_2} \left| \theta_n(|u_1|_{X_t}) \left[G_1(u_1) - G_1(u_2) \right] \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}}$$

$$= \left[\int_0^{\tau_1} \left| \theta_n(|u_1|_{X_t}) \left[G_1(u_1) - G_1(u_2) \right] \right|_{\mathrm{H}}^2 dt \right]^{\frac{1}{2}}.$$

Since $\theta_n(|u_1|_{X_t}) \leq 1$ for $t \in [0, \tau_1)$ and using (3.1), we have

$$A_2^2 \leq \int_0^{\tau_1} |G_1(u_1) - G_1(u_2)|_{\mathrm{H}}^2 dt$$

$$\leq C^2 \int_0^{\tau_1} \|u_1 - u_2\|_{\mathrm{V}}^2 \left[\|u_1\|_{\mathrm{V}} + \|u_2\|_{\mathrm{V}} \right]^4 dt$$

$$\leq C^2 \sup_{t \in [0, \tau_1]} \|u_1 - u_2\|_{\mathrm{V}}^2 \int_0^{\tau_1} \left[\|u_1\|_{\mathrm{V}} + \|u_2\|_{\mathrm{V}} \right]^4 dt$$

$$\leq C^2 |u_1 - u_2|_{X_T}^2 \sup_{t \in [0, \tau_1]} \left[\|u_1\|_{\mathrm{V}} + \|u_2\|_{\mathrm{V}} \right]^4 \int_0^{\tau_1} dt$$

$$\leq C^2 |u_1 - u_2|_{X_T}^2 \left[|u_1|_{X_{\tau_1}} + |u_2|_{X_{\tau_1}} \right]^4 \tau_1.$$

Since $|u_i|_{X_{\tau_i}} \leq 3n$, $i = 1, 2$. We get,

$$A_2^2 \leq C^2 |u_1 - u_2|_{X_T}^2 \left[|u_1|_{X_{\tau_1}} + |u_2|_{X_{\tau_1}} \right]^4 \tau_1$$

$$\leq C^2 |u_1 - u_2|_{X_T}^2 \tau_1 \left[3n + 3n \right]^4$$

$$A_2^2 \leq (6n)^4 C^2 |u_1 - u_2|_{X_T}^2 \tau_1.$$

Thus,

$$A_2 \leq (6n)^2 C |u_1 - u_2|_{X_T} \tau_1^{\frac{1}{2}}. \tag{3.9}$$

Now we consider,

$$A_3^2 = \int_0^{\tau_2} \left| \left[\theta_n(|u_2|_{X_t}) - \theta_n(|u_1|_{X_t}) \right] G_2(u_2) \right|_{\mathrm{H}}^2 dt.$$

Since θ_n is a Lipschitz function with Lipschitz constant $3n$ we obtain,

$$A_3^2 \leq 9n^2 \int_0^{\tau_2} \left| |u_2|_{X_t} - |u_1|_{X_t} \right|_{\mathrm{H}}^2 |G_2(u_2)|_{\mathrm{H}}^2 dt.$$

Using the Minkowski inequality we get

$$A_3^2 \leq 9n^2 \int_0^{\tau_2} \left| u_1 - u_2 \right|_{X_t}^2 |G_2(u_2)|_{\mathrm{H}}^2 dt$$

$$\leq 9n^2 \left| u_1 - u_2 \right|_{X_T}^2 \int_0^{\tau_2} |G_2(u_2)|_{\mathrm{H}}^2 dt. \tag{3.10}$$

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Now consider $\int_0^{\tau_2} \left|G_2(u_2)\right|_{\mathrm{H}}^2 dt$; using (3.2) we get

$$\int_0^{\tau_2} \left|G_2(u_2)\right|_{\mathrm{H}}^2 dt \leq \tilde{C}^2 \int_0^{\tau_2} \|u_2(t)\|_{\mathrm{V}}^3 |u_2|_{\mathrm{E}} dt$$

$$\leq \tilde{C}^2 \left[\sup_{t\in[0,\tau_2]} \|u_2(t)\|_{\mathrm{V}}^2\right]^{\frac{3}{2}} \int_0^{\tau_2} |u_2|_{\mathrm{E}} dt.$$

We apply the Hölder inequality to obtain,

$$\int_0^{\tau_2} \left|G_2(u_2)\right|_{\mathrm{H}}^2 dt \leq \tilde{C}^2 |u_2|_{X_{\tau_2}}^3 \left[\int_0^{\tau_2} |u_2|_{\mathrm{E}}^2 dt\right]^{\frac{1}{2}} \left[\int_0^{\tau_2} dt\right]^{\frac{1}{2}}.$$

Now since $\int_0^{\tau_2} |u_2|_{\mathrm{E}}^2 dt \leq |u_2|_{X_{\tau_2}}^2$ and $|u_2|_{X_{\tau_2}} \leq 3n$,

$$\int_0^{\tau_2} \left|G_2(u_2)\right|_{\mathrm{H}}^2 dt \leq \tilde{C}^2 |u_2|_{X_{\tau_2}}^3 |u_2|_{X_{\tau_2}} \tau_2^{\frac{1}{2}}$$

$$\leq \tilde{C}^2 (3n)^4 \tau_2^{\frac{1}{2}}.$$

Hence, the inequality (3.10) takes form

$$A_3^2 \leq 9n^2 \tilde{C}^2 |u_1 - u_2|_{X_T}^2 (3n)^4 \tau_2^{\frac{1}{2}}$$

$$A_3 \leq (3n)^3 \tilde{C} |u_1 - u_2|_{X_T} \tau_2^{\frac{1}{4}}. \tag{3.11}$$

Since $\theta_n(|u_1|_{X_t}) = 0$ for $t > \tau_1$ and $\tau_1 < \tau_2$ we have,

$$A_4 = \left[\int_0^{\tau_2} \left|\theta_n(|u_1|_{X_t})\left[G_2(u_2) - G_2(u_1)\right]\right|_{\mathrm{H}}^2 dt\right]^{\frac{1}{2}}$$

$$= \left[\int_0^{\tau_1} \left|\theta_n(|u_1|_{X_t})\left[G_2(u_2) - G_2(u_1)\right]\right|_{\mathrm{H}}^2 dt\right]^{\frac{1}{2}}.$$

Since $\theta_n(|u_1|_{X_t}) \leq 1$ for $t \in [0, \tau_1]$ and using (3.2) we have,

$$A_4 \leq \left[\int_0^{\tau_1} \left|G_2(u_2) - G_2(u_1)\right|_{\mathrm{H}}^2 dt\right]^{\frac{1}{2}}$$

$$\leq \tilde{C}\left[\int_0^{\tau_1} \left[\|u_1\|_{\mathrm{V}}^{1/2}|u_1|_{\mathrm{E}}^{1/2}\|u_1 - u_2\|_{\mathrm{V}} + \|u_1 - u_2\|_{\mathrm{V}}^{1/2}|u_1 - u_2|_{\mathrm{E}}^{1/2}\|u_2\|_{\mathrm{V}}\right]^2 dt\right]^{\frac{1}{2}}.$$

Now by the Minkowski inequality,

$$A_4 \leq \tilde{C}\left[\left[\int_0^{\tau_1} |u_1|_{\mathrm{E}}\|u_1 - u_2\|_{\mathrm{V}}^2\|u_1\|_{\mathrm{V}} dt\right]^{\frac{1}{2}} + \left[\int_0^{\tau_1} \|u_2\|_{\mathrm{V}}^2|u_1 - u_2|_{\mathrm{E}}^{1/2}\|u_1 - u_2\|_{\mathrm{V}} dt\right]^{\frac{1}{2}}\right]$$

$$\leq \tilde{C}\left[\sup_{t\in[0,\tau_1]} \|u_1 - u_2\|_{\mathrm{V}}^2\left[\sup_{t\in[0,\tau_1]} \|u_1\|_{\mathrm{V}}^2\right]^{\frac{1}{2}} \int_0^{\tau_1} |u_1|_{\mathrm{E}} dt\right]^{\frac{1}{2}}$$

$$+ \tilde{C}\left[\sup_{t\in[0,\tau_1]} \|u_2\|_{\mathrm{V}}^2\left[\sup_{t\in[0,\tau_1]} \|u_1 - u_2\|_{\mathrm{V}}^2\right]^{\frac{1}{2}} \int_0^{\tau_1} |u_1 - u_2|_{\mathrm{E}} dt\right]^{\frac{1}{2}}.$$

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Since

$$\sup_{t\in[0,\tau_1]} \|u_i\|_V^2 \le |u_i|_{X_{\tau_1}}^2, \qquad \int_0^{\tau_1} |u_1|_E^2 dt \le |u_1|_{X_{\tau_1}}^2, \qquad |u_i|_{X_{\tau_1}} \le 3n, \quad i=1,2,$$

and by using the Hölder inequality we obtain,

$$A_4 \le \tilde{C}\left[|u_1-u_2|_{X_T}^2 |u_1|_{X_{\tau_1}} \left[\int_0^{\tau_1} |u_1|_E^2 dt\right]^{\frac{1}{2}} \left[\int_0^{\tau_1} dt\right]^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

$$+ \tilde{C}\left[|u_1-u_2|_{X_T} |u_2|_{X_{\tau_1}}^2 \left[\int_0^{\tau_1} |u_1-u_2|_E^2 dt\right]^{\frac{1}{2}} \left[\int_0^{\tau_1} dt\right]^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

$$\le \tilde{C}\left[|u_1-u_2|_{X_T}^2 |u_1|_{X_{\tau_1}}^2 \tau_1^{\frac{1}{2}}\right]^{\frac{1}{2}} + \tilde{C}\left[|u_1-u_2|_{X_T}^2 |u_2|_{X_{\tau_1}}^2 \tau_1^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

$$\le \tilde{C}|u_1-u_2|_{X_T}\tau_1^{\frac{1}{4}}\left[3n+3n\right].$$

Thus

$$A_4 \le 6n\tilde{C}|u_1-u_2|_{X_T}\tau_1^{\frac{1}{4}}. \tag{3.12}$$

Now using (3.8), (3.10), (3.11) and (3.12) in (3.6), we obtain

$$|\Phi_{n,T}(u_1)-\Phi_{n,T}(u_2)|_{L^2(0,T;H)} \le (3n)^4 C|u_1-u_2|_{X_T}\tau_2^{\frac{1}{2}} + (6n)^2 C|u_1-u_2|_{X_T}\tau_1^{\frac{1}{2}}$$

$$+ (3n)^3 \tilde{C}|u_1-u_2|_{X_T}\tau_2^{\frac{1}{4}} + 6n\tilde{C}|u_1-u_2|_{X_T}\tau_1^{\frac{1}{4}}$$

$$\le (3n)^4 C|u_1-u_2|_{X_T}T^{\frac{1}{2}} + (6n)^2 C|u_1-u_2|_{X_T}T^{\frac{1}{2}}$$

$$+ (3n)^3 \tilde{C}|u_1-u_2|_{X_T}T^{\frac{1}{4}} + 6n\tilde{C}|u_1-u_2|_{X_T}T^{\frac{1}{4}}$$

$$= K(n,T)|u_1-u_2|_{X_T}T^{\frac{1}{4}},$$

where

$$K(n,T) = 3n\left(27n^3 T^{1/4} + 9n^2 + 12nT^{1/4} + 2\right),$$

is a constant which depends only on n and T. Thus we have proved that $\Phi_{n,T}$ is a Lipschitz function and satisfies (3.2). $\qquad\square$

3.2. Assumptions and definition of a solution

Assume that $E \subset V \subset H$ continuously and $S(t)$ is a family of bounded linear operators on space H such that there exist $C_1, C_2 > 0$ s.t.

A1. For every $T > 0$ and $f \in L^2(0,T;H)$ a function $u = S * f$, defined by

$$u(t) = \int_0^T S(t-r)f(r)dr \quad t \in [0,T],$$

belongs to X_T and

$$|u|_{X_T} \le C_1 |f|_{L^2(0,T;H)}. \tag{3.13}$$

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A2. For every $T > 0$ and $u_0 \in V$ a function $u = S u_0$ defined by

$$u(t) = S(t)u_0,$$

belongs to X_T and

$$|u|_{X_T} \leq C_2 \|u_0\|_V. \tag{3.14}$$

Definition 3.5. • A solution of (2.5) on $[0, T]$, $T \in [0, \infty)$ is a function $u \in X_T$ satisfying

$$u(t) = S(t)u_0 + \int_0^t S(t - r)G(u(r))dr \quad \forall\, t \in [0, T],$$

where $G : E \to H$ is defined by

$$G(u) = |\nabla u|^2 u - B(u, u), \quad u \in E.$$

• Let $\tau \in [0, \infty]$. A function $u \in C([0, \tau), V)$ is a solution to (2.5) on $[0, \tau)$ iff $\forall\, T < \tau$, $u|_{[0,T]} \in X_T$ and satisfies

$$u(t) = S(t)u_0 + \int_0^t S(t - r)G(u(r))dr \quad \forall\, t \in [0, T].$$

Define a function $\Psi_{n,T} : X_T \to X_T$ by

$$\Psi_{n,T}(u) = S(t)u_0 + S * \Phi_{n,T}(u).$$

Lemma 3.6. *u is the unique solution of (2.5) iff u is a fixed point of $\Psi_{n,T}$.*

3.3. Local existence

Lemma 3.7. *Assume that the assumptions (A1)-(A2) hold. Consider a map $\Psi_{n,T} : X_T \to X_T$ defined by*

$$\Psi_{n,T}(u) = S u_0 + S * \Phi_{n,T}(u),$$

where $\Phi_{n,T}$ is as in Lemma 3.4. Then there exists a constant $C_1 > 0$ such that $\Psi_{n,T}$ satisfies following inequality

$$|\Psi_{n,T}(u_1) - \Psi_{n,T}(u_2)|_{X_T} \leq C_1 K(n, T)|u_1 - u_2|_{X_T} T^{\frac{1}{4}}, \quad u_1, u_2 \in X_T, \tag{3.15}$$

where $K(n, T)$ has been introduced in Lemma 3.4. Moreover, $\forall\, \varepsilon \in (0, 1)\, \exists\, T_0 = T_0(n, \varepsilon)$ such that for every $u_0 \in V$, $\Psi_{n,T}$ is an ε-contraction for $T \leq T_0$.

Proof. The map $\Psi_{n,T}$ is evidently well defined. Now for any $u_1, u_2 \in X_T$,

$$|\Psi_{n,T}(u_1) - \Psi_{n,T}(u_2)|_{X_T} = \left| S(t)u_0 + S * \Phi_{n,T}(u_1) - S(t)u_0 + S * \Phi_{n,T}(u_2) \right|_{X_T}$$

$$= \left| S * (\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)) \right|_{X_T},$$

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then by treating $S * (\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2))$ as u and $[\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)] \in L^2(0,T;H)$ as f in inequality (3.13) and using Lemma 3.4 we get

$$|\Psi_{n,T}(u_1) - \Psi_{n,T}(u_2)|_{X_T} \leq C_1 |\Phi_{n,T}(u_1) - \Phi_{n,T}(u_2)|_{L^2(0,T;H)}$$
$$\leq C_1 K(n,T) |u_1 - u_2|_{X_T} T^{\frac{1}{4}},$$

which shows that $\Psi_{n,T}$ is globally Lipschitz and satisfies (3.15).

Let us fix $n \in \mathbb{N}$ and $\varepsilon \in (0,1)$. Since the constant C_1 is independent of T, we can find a $T_0 = T_0(n,\varepsilon)$ such that

$$C_1 K(n,T_0) T_0^{\frac{1}{4}} = \varepsilon,$$

and thus $\Psi_{n,T}$ is an ε-contraction for $T \leq T_0$.

$\square$

Let $\varepsilon \in (0,1)$ then from Lemma 3.7, $\Psi_{n,T}$ is an ε-contraction for $T = T_0(n,\varepsilon)$ and thus by Banach Fixed Point Theorem there exists a unique $u^n \in X_T$ [1] s.t.

$$u^n = \Psi_{n,T}(u^n).$$

This implies that

$$u^n(t) = [\Psi_{n,T}(u^n)](t), \qquad t \in [0,T_0].$$

Let us define

$$\tau_n = \inf\{t \in [0,T_0] : |u^n|_{X_t} \geq n\}.$$

Remark 3.8. If $|u^n|_{X_t} < n$ for each $t \in [0,T_0^n]$ then $\tau_n = T_0^n$.

Theorem 3.9. *Let $R > 0$ be given then $\exists\, T_* = T_*(R)$ such that for every $u_0 \in V$ with $\|u_0\|_V \leq R$ there exists a unique local solution $u : [0,T_*] \to V$ of (2.5).*

Proof. Let $R > 0$ and fix $\varepsilon \in (0,1)$. Let us choose[2] $n = \lfloor \frac{C_2 R}{1-\varepsilon} \rfloor + 1$ where C_2 is as defined in (3.14). Now for these fixed n and ε, $\exists\, T_0(n,\varepsilon)$ such that $\Psi_{n,T}$ is an ε-contraction for all $T \leq T_0$. In particular, it is true for $T = T_0$ and hence by Banach Fixed Point Theorem $\exists!\, u^n \in X_{T_0}$ such that

$$u^n = \Psi_{n,T}(u^n).$$

Note that we have

$$|u^n|_{X_{T_0}} = |\Psi_{n,T}(u^n)|_{X_{T_0}} = |S u_0 + S * \Phi_{n,T}(u^n)|_{X_{T_0}}$$
$$\leq |S u_0|_{X_{T_0}} + |S * \Phi_{n,T}(u^n)|_{X_{T_0}}.$$

---

[1]In fact u^n should have been denoted by $u^{n,T}$ but we have refrained from this.
[2]$\lfloor M \rfloor$ denotes the largest integer less than or equal to M.

Now from (3.14) and Lemma 3.7 we have,

$$|u^n|_{X_{T_0}} \le C_2\|u_0\|_V + \varepsilon|u^n|_{X_{T_0}}.$$

Hence

$$(1 - \varepsilon)|u^n|_{X_{T_0}} \le C_2 R,$$

and so

$$|u^n|_{X_{T_0}} \le \frac{C_2 R}{1 - \varepsilon} \le n.$$

Now since $t \mapsto |\cdot|_{X_t}$ is an increasing function the following holds,

$$|u^n|_{X_t} \le n \quad \forall\, t \in [0, T_0].$$

In particular $|u^n|_{X_{T_0}} \le n$, i.e. $|u^n|_{X_{T_0}}$ is finite and thus $u^n \in X_{T_0}$.
This implies

$$\theta_n(|u^n|_{X_t}) = 1, \quad t \in [0, T_0].$$

Thus for $t \in [0, T_0]$,

$$u^n(t) = S(t)u_0 + \int_0^t S(t - r)G(u^n(r))dr.$$

So u^n on $[0, T_*(R)]$, where $T_* = T_0(n, \varepsilon)$, solves (2.5) and T_* depends only on R.

Thus we have proved the existence of a unique local solution of (2.5) for every initial data $u_0 \in V$, and this unique solution is denoted by u. □

3.4. The solution stays on the manifold $\mathcal{M}$

Lemma 3.10. *If u is the solution of (2.5) on $[0, \tau)$ then $u' \in L^2(0, T; H)$, for every $T < \tau$.*

Proof. Let us fix $T < \tau$. Since u is the solution of (2.5) on $[0, \tau)$ it satisfies

$$\frac{du}{dt} = -Au + |\nabla u|^2 u - B(u, u). \tag{3.16}$$

We will show that RHS of (3.16) belongs to $L^2(0, T; H)$ and hence $u' \in L^2(0, T; H)$.

Since $u \in L^2(0, T; E)$, $Au \in L^2(0, T; H)$. From (3.1) we have

$$\begin{aligned}
\int_0^T \left||\nabla u(t)|^2 u(t)\right|_H^2 dt &\le \int_0^T C^2\|u(t)\|_V^6 dt \\
&\le C^2 \sup_{t \in [0,T]} \|u(t)\|_V^6 \int_0^T dt \\
&\le C^2 T \left[\sup_{t \in [0,T]} \|u(t)\|_V^2\right]^3 \\
&\le C^2 T |u|_{X_T}^6 < \infty,
\end{aligned}$$

thus we have shown that $|\nabla u|^2 u \in L^2(0, T; H)$.

From (3.2) we have,

$$\int_0^T \left| B(u(t), u(t)) \right|_H^2 dt \leq \tilde{C}^2 \int_0^T \|u(t)\|_V^3 |u(t)|_E dt$$

$$\leq \tilde{C}^2 \sup_{t \in [0,T]} \|u(t)\|_V^3 \int_0^T |u(t)|_E dt$$

$$\leq \tilde{C}^2 \left[\sup_{t \in [0,T]} \|u(t)\|_V^2 \right]^{\frac{3}{2}} \left[\int_0^T |u(t)|_E^2 dt \right]^{\frac{1}{2}} \left[\int_0^T dt \right]^{\frac{1}{2}}$$

$$\leq \tilde{C}^2 |u|_{X_T}^3 |u|_{X_T} T^{\frac{1}{2}} < \infty.$$

Thus the non linear term from Navier-Stokes also belongs to $L^2(0, T; H)$ and hence RHS of (3.16) belongs to $L^2(0, T; H)$ which implies $u' \in L^2(0, T; H)$ for all $T < \tau$. $\qquad \square$

The following Lemma is taken from [19]. It proves the existence of an absolute continuous function based on the regularity of the solution and it's time derivative.

Lemma 3.11. *Let* V, H *and* V' *be the Gelfand triple. If a function* $u \in L^2(0, T; V)$ *and its weak derivative* $u' \in L^2(0, T; V')$ *then u is almost everywhere equal to a continuous function* $v : [0, T] \to H$ *such that the function* $[0, T] \ni t \mapsto |v(t)|_H^2 \in \mathbb{R}$ *is absolutely continuous and*

$$\frac{1}{2}|v(t)|_H^2 = \frac{1}{2}|v(0)|^2 + \int_0^t \langle u'(s), u(s) \rangle_H ds, \quad t \in [0, T]. \tag{3.17}$$

Remark 3.12. In the framework of Lemma 3.11, we can identify v with u and so we get

$$\frac{1}{2}|u(t)|_H^2 = \frac{1}{2}|u_0|^2 + \int_0^t \langle u'(s), u(s) \rangle_H ds, \quad t \in [0, \tau). \tag{3.18}$$

Moreover, from Theorem 3.9 and Lemma 3.10

$$\frac{1}{2}\|u(t)\|_V^2 = \frac{1}{2}\|u_0\|_V^2 + \int_0^t \langle u'(s), u(s) \rangle_V ds, \quad t \in [0, \tau), \tag{3.19}$$

where $\langle \cdot, \cdot \rangle_V$ is defined in the Section 2 for $\mathbb{R}^2$ as well as $\mathbb{T}^2$.

Theorem 3.13. *If* $\tau \in [0, \infty]$, $u_0 \in \mathcal{M} \cap V$ *and u is a solution to* (2.5) *on* $[0, \tau)$ *then* $u(t) \in \mathcal{M}$ *for all* $t \in [0, \tau)$.

Proof. Let u be the solution to (2.5) and $u_0 \in \mathcal{M} \cap V$. Let us define $\phi(t) = |u(t)|_H^2 - 1$. Then ϕ is absolutely continuous and by Remark 3.12 and (2.5) we have a.e. on $[0, \tau)$

$$\frac{d}{dt}\phi(t) = \frac{d}{dt}[|u(t)|_H^2 - 1] = 2\langle u'(t), u(t) \rangle_H$$

$$= 2\langle -Au(t) + |\nabla u(t)|^2 u(t) - B(u(t), u(t)), u(t) \rangle_H$$

$$= -2\langle Au(t), u(t) \rangle_H + 2|\nabla u(t)|^2 \langle u(t), u(t) \rangle_H$$

$$= -2|\nabla u(t)|^2 + 2|\nabla u(t)|^2 |u(t)|^2$$

$$= 2|\nabla u(t)|^2 (|u(t)|_H^2 - 1) = |\nabla u(t)|^2 \phi(t).$$

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This on integration gives

$$\phi(t) = \phi(0) \exp\left[\int_0^t |\nabla u(s)|^2 ds\right], \quad t \in [0, \tau).$$

Since $u_0 \in \mathcal{M}$, $\phi(0) = 0$ and also as $u \in X_T$ is the solution of (2.5),

$$\int_0^t |\nabla u(s)|^2 ds \le \int_0^t \|u(s)\|_V^2 \, ds < \infty, \quad t \in [0, \tau).$$

Hence we infer that $|u(t)|_H^2 = 1$ for every $t \in [0, \tau)$. Thus $u(t) \in \mathcal{M}$ for every $t \in [0, \tau)$. $\qquad\square$

Corollary 3.14. *Let the initial data $u_0 \in \mathcal{M}$ and u is the solution to (2.5) on $[0, \tau)$ then $u'(t)$ is orthogonal to $u(t)$ in* H *for every $t \in [0, \tau)$.*

Remark 3.15. We can also prove Theorem 3.9 and Theorem 3.13 for any general bounded domain. Thus we can establish the existence of a local solution to (2.5) for any general bounded domain and $\mathbb{R}^2$.

4. Global solution: Existence and Uniqueness

In this section we will prove the existence of a global solution of (2.5). Lemma A.1 and the Remark 4.1 play crucial role in proving the global existence of the solution. We use stitching argument to extend our solution from $[0, T], T < \infty$ on to the whole real line.

We recall the orthogonality property of the Stokes-operator in the following remark.

Remark 4.1. Note that one can show [20] that on a torus the following identity holds

$$\langle B(u, u), Au \rangle_H = 0, \quad \forall \, u \in V.$$

Let u be the solution of (2.5). We define the energy of our system by

$$\mathcal{E}(u) = \frac{1}{2} |\nabla u|^2.$$

Then

$$\begin{aligned}
\nabla_{\mathcal{M}} \mathcal{E}(u) &= \Pi_u(\nabla \mathcal{E}) \\
&= \Pi_u(Au) \\
&= Au - |\nabla u|^2 \, u.
\end{aligned}$$

Thus, for $u \in \mathcal{M}$

$$|\nabla_{\mathcal{M}} \mathcal{E}(u)|_H^2 = |u|_E^2 - |\nabla u|^4. \tag{4.1}$$

Lemma 4.2. *If u is the local solution of (2.5) on $[0, \tau)$, then*

$$\sup_{s \in [0, \tau)} \|u(s)\|_V \le \|u_0\|_V.$$

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Proof. Let u be the solution of (2.5). Then, from (2.5), Remark 3.12 and Corollary 3.14, for any $t \in [0, \tau)$ we have,

$$
\begin{aligned}
\frac{1}{2}\|u(t)\|_V^2 &= \frac{1}{2}\|u_0\|_V^2 + \int_0^t \langle u'(s), u(s)\rangle_V ds \\
&= \frac{1}{2}\|u_0\|_V^2 + \int_0^t \langle u'(s), u(s)\rangle_H ds + \int_0^t \langle u'(s), Au(s)\rangle_H ds \\
&= \frac{1}{2}\|u_0\|_V^2 + \int_0^t \langle -Au(s) + |\nabla u(s)|^2 u(s) - B(u(s), u(s)), Au(s)\rangle_H ds \\
&= \frac{1}{2}\|u_0\|_V^2 + \int_0^t \left[-\langle Au(s), Au(s)\rangle_H + |\nabla u(s)|^2 \langle u(s), Au(s)\rangle_H \right] ds \\
&\quad - \int_0^t \langle B(u(s), u(s)), Au(s)\rangle_H ds \\
&= \frac{1}{2}\|u_0\|_V^2 + \int_0^t \left[-|u(s)|_E^2 + |\nabla u(s)|^4 \right] ds.
\end{aligned}
$$

Now from Theorem 3.13 we know that $u(t) \in \mathcal{M}$ for every $t \in [0, \tau)$ and hence by using (4.1) we obtain,

$$
\frac{1}{2}\|u(t)\|_V^2 = \frac{1}{2}\|u_0\|_V^2 - \int_0^t \left| [\nabla_{\mathcal{M}} \mathcal{E}(u)](s) \right|_H^2 ds,
$$

and thus

$$
\frac{1}{2}\|u(t)\|_V^2 + \int_0^t \left| [\nabla_{\mathcal{M}} \mathcal{E}(u)](s) \right|_H^2 ds = \frac{1}{2}\|u_0\|_V^2.
$$

Hence we have shown that

$$
\|u(t)\|_V \le \|u_0\|_V, \quad t \in [0, \tau).
$$

$\square$

Lemma 4.3. *Let* $0 \le a < b < c < \infty$ *and* $u \in X_{[a,b]}, v \in X_{[b,c]}$, *such that* $u(b^-) = v(b^+)$. *Then* $z \in X_{[a,c)}$ *where,*

$$
z(t) = \begin{cases} u(t), & t \in [a, b), \\ v(t), & t \in [b, c). \end{cases}
$$

Proof. Let us take $0 \le a < b < c < \infty$ and $u \in X_{[a,b]}, v \in X_{[b,c]}$, such that $u(b^-) = v(b^+)$. Then for any $0 \le t_1 < t_2 < \infty$, using the definition of the norm $|\cdot|_{X_{[t_1,t_2]}}$, we have

$$
\begin{aligned}
|z|_{X_{[a,c]}}^2 &= \sup_{t \in [a,c]} \|z(t)\|_V^2 + \int_a^c |z(t)|_E^2 dt \\
&\le \sup_{t \in [a,b]} \|z(t)\|_V^2 + \sup_{t \in [b,c]} \|z(t)\|_V^2 + \int_a^b |z(t)|_E^2 dt + \int_b^c |z(t)|_E^2 dt
\end{aligned}
$$

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Now by the definition of z we have,

$$|z|^2_{X_{[a,c]}} \leq \sup_{t \in [a,b]} \|u(t)\|^2_V + \sup_{t \in [b,c]} \|v(t)\|^2_V + \int_a^b |u(t)|^2_E dt + \int_b^c |v(t)|^2_E dt$$

$$= \sup_{t \in [a,b]} \|u(t)\|^2_V + \int_a^b |u(t)|^2_E dt + \sup_{t \in [b,c]} \|v(t)\|^2_V + \int_b^c |v(t)|^2_E dt$$

$$= |u|^2_{X_{[a,b]}} + |v|^2_{X_{[b,c]}}.$$

Now since $u \in X_{[a,b]}$ and $v \in X_{[b,c]}$ we have $|z|_{X_{[a,c]}} < \infty$, and thus $z \in X_{[a,c]}$. $\qquad\square$

We will use the following lemma to prove our main result of existence of the global solution.

Lemma 4.4. *Let τ be finite and the initial data $u_0 \in V \cap M$. If $u : [0, \tau] \to V$ is the solution of (2.5) on $[0, \tau]$ and $v : [\tau, 2\tau] \to V$ is the solution of (2.5) on $[\tau, 2\tau]$ such that $u(\tau^-) = v(\tau^+)$, then $z : [0, 2\tau] \to V$ defined as*

$$z(t) = \begin{cases} u(t), & t \in [0, \tau], \\ v(t), & t \in [\tau, 2\tau], \end{cases}$$

is the solution of (2.5) on $[0, 2\tau]$ and $z \in X_{[0,2\tau]}$.

Proof. Since u is the solution of (2.5) on $[0, \tau]$ then $u \in X_{[0,\tau]}$ and similarly $v \in X_{[\tau,2\tau]} := C([\tau, 2\tau]; V) \cap L^2(\tau, 2\tau; E)$. Thus by Lemma 4.3 and the definition of z, $z \in X_{[0,2\tau]}$. Now we are left to show that $z : [0, 2\tau] \to V$ defined as

$$z(t) = \begin{cases} u(t), & t \in [0, \tau], \\ v(t), & t \in [\tau, 2\tau], \end{cases}$$

is the solution of (2.5) on $[0, 2\tau]$. In order to achieve this we will have to show that z satisfies (4.2) for every $t \in [0, 2\tau]$.

$$z(t) = S(t)z(0) + \int_0^t S(t - r)G(z(r))dr. \tag{4.2}$$

For $t \in [0, \tau)$, z satisfies (4.2), since $z(t) = u(t), \forall\, t \in [0, \tau]$ and u is the solution of (2.5) on $[0, \tau]$.
For $t \in [\tau, 2\tau]$, $z(t) = v(t)$ and since v is the solution to (2.5) on $[\tau, 2\tau]$,

$$z(t) = v(t) = S(t - \tau)v(\tau) + \int_\tau^t S(t - r)G(v(r))dr.$$

Now because of continuity of u and v, $v(\tau) = u(\tau)$,

$$z(t) = S(t - \tau)\Big[S(\tau)u_0 + \int_0^\tau S(\tau - r)G(u(r))dr\Big] + \int_\tau^t S(t - r)G(v(r))dr.$$

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Now using the definition of z we obtain,

$$z(t) = S(t)z(0) + \int_0^\tau S(t-r)G(z(r))dr + \int_\tau^t S(t-r)G(z(r))dr$$
$$= S(t)z(0) + \int_0^t S(t-r)G(z(r))dr.$$

Thus z satisfies (4.2) on $[0, 2\tau]$ and hence z is a solution to (2.5) on $[0, 2\tau]$.

$\square$

Proof of Theorem 1.1. Let us take $u_0 \in V$. Put $R = \|u_0\|_V$. By Theorem 3.9 there exists a $T > 0$ such that there exists a unique function $u : [0, T] \to V$ which solves (2.5) on $[0, T]$ and $u \in X_T$. Also by Lemma 4.2 $\|u(T)\|_V \leq R$ thus again by Theorem 3.9 there exists a unique function $v : [T, 2T] \to V$ which solves (2.5) on $[T, 2T]$ and $v \in X_{[T,2T]}$. Now if we define a new function $z : [0, 2T] \to V$ as

$$z(t) = \begin{cases} u(t), & t \in [0, T], \\ v(t), & t \in [T, 2T], \end{cases}$$

then by Lemma 4.4, z is also a solution of (2.5) and $z \in X_{2T}$. Moreover $\|z(2T)\|_V \leq R$. We can keep doing this and extend our solution further and hence obtaining a global solution of (2.5) still denoted by u such that $u \in X_T$ for every $T < \infty$. Each bit of the solution is unique on the respective domain and hence when we glue two unique bits we get a unique extension and thus obtain a unique global solution due to it's construction.

$\square$

5. Convergence to the Euler equation

In this section we are concerned with the convergence of the solution of the constrained Navier-Stokes equation, namely

$$\begin{cases} \dfrac{du}{dt} + \nu Au - \nu|\nabla u|^2 u + B(u, u) = 0, \\ u(0) = u_0^\nu \in V \cap \mathcal{M}, \end{cases} \tag{5.1}$$

as ν vanishes on a torus. The curl of u is defined as $\mathrm{Curl}(u) := D_1 u_2 - D_2 u_1$. We will prove Theorem 1.2 after several preliminary results.

Remark 5.1. Curl is a linear isomorphism between V and $L_0^2(\mathbb{T}^2)$, where

$$L_0^2(\mathbb{T}^2) := \left\{ \omega \in L^2(\mathbb{T}^2) \; : \; \int_{\mathbb{T}^2} \omega(x)dx = 0 \right\}.$$

Moreover for $u \in V$ and some universal constants $C > 0$, $C_p > 0$

$$\|\Delta u\|_{L^2(\mathbb{T}^2)} \leq C\|\nabla\mathrm{Curl}(u)\|_{L^2(\mathbb{T}^2)}, \tag{5.2}$$

$$\|\nabla u\|_{L^p(\mathbb{T}^2)} \leq C_p\|\mathrm{Curl}(u)\|_{L^\infty(\mathbb{T}^2)}. \tag{5.3}$$

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This remark is proved in Appendix B.

Hereafter u^ν is the solution to (5.1), and $\omega^\nu(t, x) := \mathrm{Curl}(u^\nu(t))(x)$. In particular, due to Remark 5.1 and Theorem 3.13, $\omega^\nu \in C([0, T]; L_0^2(\mathbb{T}^2)) \cap L^2(0, T; H^1(\mathbb{T}^2))$. It is then easy to check that ω^ν is a weak solution to

$$
\begin{cases}
\dfrac{d\omega^\nu}{dt} + \nabla \cdot (u^\nu \, \omega^\nu) = \nu \Delta \omega^\nu + \nu \, \|u^\nu\|_V^2 \, \omega^\nu, \\
\omega^\nu(0) = \omega_0^\nu := \mathrm{Curl}(u_0^\nu) \in L_0^2(\mathbb{T}^2).
\end{cases}
\tag{5.4}
$$

Proposition 5.2. *Let us fix $T > 0$, and assume that $\omega_0^\nu \in L^\infty(\mathbb{T}^2)$. Then*

$$
\sup_{t \in [0,T]} |\omega^\nu(t)|_{L^\infty(\mathbb{T}^2)} \leq |\omega_0^\nu|_{L^\infty(\mathbb{T}^2)} \exp\left(\nu \|u_0^\nu\|_V^2 \, T\right),
\tag{5.5}
$$

$$
\nu \int_0^T |\nabla \omega^\nu(t)|_{L^2(\mathbb{T}^2)}^2 dt \leq \tfrac{1}{2} |\omega_0^\nu|_{L^2(\mathbb{T}^2)}^2 + \nu \, T \|u_0^\nu\|_V^2 \, |\omega_0^\nu|_{L^\infty(\mathbb{T}^2)}^2 \exp\left(2\nu \|u_0^\nu\|_V^2 \, T\right).
\tag{5.6}
$$

Proof. Take $h \in C^2(\mathbb{R})$, convex, with bounded second derivative. Then, since $\omega \in C([0, T]; L_0^2(\mathbb{T}^2))$

$$
\begin{aligned}
\langle h(\omega^\nu(t)), \mathbf{1} \rangle &- \langle h(\omega_0^\nu), \mathbf{1} \rangle \\
&= \nu \int_0^t \left[-\langle h''(\omega(s)), |\nabla \omega^\nu|^2(s) \rangle + \|u^\nu(s)\|_V^2 \, \langle h'(\omega^\nu(s)), \omega^\nu(s) \rangle \right] ds \\
&\leq \nu \int_0^t \|u^\nu(s)\|_V^2 \, \langle h'(\omega^\nu(s)), \omega^\nu(s) \rangle \, ds.
\end{aligned}
\tag{5.7}
$$

For $p \geq 2, R > 0$, take

$$
h(w) \equiv h_{p,R}(w) := \begin{cases} |w|^p, & \text{if } |w| \leq R, \\ R^p + p\,R^{p-1}(|w| - R) + \frac{p(p-1)}{2} R^{p-2}(|w| - R)^2, & \text{if } |w| > R. \end{cases}
\tag{5.8}
$$

Then $|h'(w)w| \leq p\,h(w)$ and, since $\|u^\nu(s)\|_V^2 \leq \|u_0^\nu\|_V^2$

$$
\langle h(\omega^\nu(t)), \mathbf{1} \rangle \leq \langle h(\omega_0^\nu), \mathbf{1} \rangle + \nu\,p \int_0^t \|u_0^\nu\|_V^2 \, \langle h(\omega^\nu(s)), \mathbf{1} \rangle \, ds.
\tag{5.9}
$$

By Gronwall inequality

$$
\langle h(\omega^\nu(t)), \mathbf{1} \rangle \leq \langle h(\omega_0^\nu), \mathbf{1} \rangle \exp\left(\nu\,p \|u_0^\nu\|_V^2 \, t\right), \quad t \in [0, T].
\tag{5.10}
$$

Since

$$
|\omega^\nu|_{L^\infty} = \sup_{p,R} \langle h_{p,R}(\omega^\nu), \mathbf{1} \rangle^{1/p},
\tag{5.11}
$$

we get (5.5).

On the other hand, from the first equality in (5.7), taking now $h(w) = w^2/2$

$$
\begin{aligned}
\tfrac{1}{2} |\omega^\nu(T)|_{L^2(\mathbb{T}^2)}^2 + \nu \int_0^T |\nabla \omega^\nu(t)|_{L^2(\mathbb{T}^2)}^2 dt &= \tfrac{1}{2} |\omega_0^\nu|_{L^2(\mathbb{T}^2)}^2 + \nu \int_0^T \|u^\nu(t)\|_V^2 \, |\omega^\nu(t)|_{L^2(\mathbb{T}^2)}^2 dt \\
&\leq \tfrac{1}{2} |\omega_0^\nu|_{L^2(\mathbb{T}^2)}^2 + \nu\,T \|u_0^\nu\|_V^2 \, |\omega_0^\nu|_{L^\infty(\mathbb{T}^2)}^2 e^{2\nu T \|u_0^\nu\|_V^2},
\end{aligned}
$$

where in the last line we used (5.5). Hence (5.6). $\qquad\square$

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Proposition 5.3. *For each $\varphi \in H^2(\mathbb{T}^2)$, and $\nu > 0$*

$$\langle \omega^\nu(t) - \omega^\nu(s), \varphi \rangle \le (t-s) \left(|\omega^\nu|_{L^\infty([0,T] \times \mathbb{T}^2)} + 2\nu \|u_0^\nu\|_V (1 + \|u_0^\nu\|_V^2) \right) |\varphi|_{H^2(\mathbb{T}^2)}. \tag{5.12}$$

Proposition 5.4. *Suppose that, uniformly in ν, u_0^ν is bounded in V and $\mathrm{Curl}(u_0^\nu)$ is bounded in $L^\infty(\mathbb{T}^2)$. Then the sequence u^ν is precompact in $C([0,T]; L^2(\mathbb{T}^2))$.*

Proof. Let us take and fix $\varphi \in H^2(\mathbb{T}^2)$. Also fix $0 \le s < t \le T$. Then from the equation (5.4) and $\|u^\nu(t)\|_V^2 \le \|u_0^\nu\|_V^2$ we get,

$$|\langle u^\nu(t) - u^\nu(s), \varphi \rangle| \le \nu \left| \int_s^t \langle \Delta u^\nu, \varphi \rangle \, dr \right| + \nu \|u_0^\nu\|_V^2 \int_s^t |\langle u^\nu, \varphi \rangle| \, dr + \left| \int_s^t \langle u^\nu \nabla u^\nu, \varphi \rangle \, dr \right|. \tag{5.13}$$

By (5.2), (5.6) and the hypotheses on the initial data, the first term in the r.h.s. is bounded by $C_T |\varphi|_{L^2} (t-s)^{1/2}$ for some constant C_T independent on ν. The second term in the r.h.s. of (5.13) easily enjoys the same bound. As for the third term in the r.h.s., for any $p > 2$, $|u|_{L^\infty} \le C_p(|u|_{L^2} + |\nabla u|_{L^p})$, so that from (5.3) and (5.5), this term is still bounded by $C_T |\varphi|_{L^2}(t-s)^{1/2}$.

Therefore, since u_0^ν is bounded uniformly in $L^2(\mathbb{T}^2)$ by Poincaré inequality, it follows that u^ν is equibounded and equicontinuous in $L^2(\mathbb{T}^2)$ and, by Ascoli-Arzelà theorem, precompact in $C([0,T]; L^2(\mathbb{T}^2))$. $\square$

Proof of Theorem 1.2. Fix $T > 0$. From Proposition 5.3-5.4, from each subsequence we can extract a further subsequence such that $\omega^\nu \to \omega$ in $C([0,T]; H^{-2}(\mathbb{T}^2))$ and weakly in $L^\infty([0,T] \times \mathbb{T}^2)$, $u^\nu \to u$ weakly in $L^\infty([0,T]; V)$ and in $C([0,T]; L^2(\mathbb{T}^2))$. It is immediate to check that $\omega = \mathrm{Curl}(u)$.

Notice that $\omega_0^\nu := \mathrm{Curl}(u_0^\nu)$ converges weakly in $L^\infty(\mathbb{T}^2)$ to $\omega_0 := \mathrm{Curl}(u_0)$. Passing to the limit in the weak formulation of the equation one then has, for each $\varphi \in C^2([0,T] \times \mathbb{T}^2)$

$$\langle \omega(t), \varphi(t) \rangle - \langle \omega_0, \varphi(0) \rangle - \int_0^t \langle \omega(s), \partial_s \varphi(s) \rangle - \int_0^t \langle u \omega, \nabla \varphi \rangle = 0, \tag{5.14}$$

and $\omega(0) = \omega_0$. Recalling that $\omega = \mathrm{Curl}(u)$

$$\langle u(t), \nabla^\perp \varphi(t) \rangle - \langle u_0, \nabla^\perp \varphi(0) \rangle - \int_0^t \langle u(s), \partial_s \nabla^\perp \varphi(s) \rangle - \int_0^t \langle u \cdot \nabla u, \nabla^\perp \varphi \rangle = 0. \tag{5.15}$$

Since $\langle u \omega, \nabla \varphi \rangle = \langle u \cdot \nabla u, \nabla^\perp \varphi \rangle$ holds.

By Bardos uniqueness theorem [1, 9], we conclude that $u^\nu \to u$. $\square$

Appendix A. Orthogonality of bilinear map to the Stokes operator

Lemma A.1. *Let $x \in O$, where $O = \mathbb{R}^2$ or $\mathbb{T}^2$ and $u \in D(A)$, then*

$$\langle B(u, u), Au \rangle_H = 0, \quad \forall\, u \in D(A). \tag{A.1}$$

Proof. The following proof has been modified [20] for $\mathbb{R}^2$.

Let $u \in D(A)$ then, by the definition of $B(u, v)$ and Au,

$$
\begin{aligned}
\langle B(u, u), Au \rangle_H &= \int_O (u(x) \cdot \nabla) u(x) \cdot Au(x)\, dx \\
&= \sum_{i, j, k = 1}^{2} \int_O (u_i D_i u_j)(-\Delta u_j)\, dx \\
&= -\sum_{i, j, k = 1}^{2} \int_O u_i D_i u_j D_k^2 u_j\, dx.
\end{aligned}
$$

Now by integration by parts and the Stokes formula

$$
\begin{aligned}
\langle B(u, u), Au \rangle_H &= -\left(\sum_{i, j, k = 1}^{2} u_i D_i u_j D_k u_j \right)\Big|_{\partial O} + \sum_{i, j, k = 1}^{2} \int_O D_k(u_i D_i u_j) D_k u_j\, dx \\
&= \sum_{i, j, k = 1}^{2} \int_O D_k u_i D_i u_j D_k u_j\, dx + \sum_{i, j, k = 1}^{2} \int_O u_i D_{ki} u_j D_k u_j\, dx.
\end{aligned}
$$

Now we will show that each of the terms in RHS will vanish. We will consider the first term and show that it vanishes.

$$
\begin{aligned}
\sum_{i, j, k = 1}^{2} D_k u_i D_i u_j D_k u_j &= (D_1 u_1)^3 + D_1 u_2 D_2 u_1 D_1 u_1 + D_1 u_1 (D_1 u_2)^2 + (D_1 u_2)^2 D_2 u_2 \\
&\quad + (D_2 u_1)^2 D_1 u_1 + D_2 u_2 (D_2 u_1)^2 + D_2 u_1 D_1 u_2 D_2 u_2 + (D_2 u_2)^3 \\
&= (D_1 u_1 + D_2 u_2)\left[(D_1 u_1)^2 + (D_2 u_2)^2 - D_1 u_1 D_2 u_2 \right] \\
&\quad + D_1 u_2 D_2 u_1 (D_1 u_1 + D_2 u_2) + (D_1 u_2)^2 (D_1 u_1 + D_2 u_2) \\
&\quad + (D_2 u_1)^2 (D_1 u_1 + D_2 u_2).
\end{aligned}
$$

Now since $\nabla \cdot u = D_1 u_1 + D_2 u_2 = 0$, the first term vanishes identically.

The second term vanishes because

$$
\begin{aligned}
2 \sum_{i, j, k = 1}^{2} \int_O u_i D_{ki} u_j D_k u_j\, dx &= \sum_{i, j, k = 1}^{2} \int_O u_i D_i (D_k u_j)^2\, dx \\
&= \left(\sum_{i, j, k = 1}^{2} u_i (D_k u_j)^2 \right)\Big|_{\partial O} - \sum_{i, j, k = 1}^{2} \int_O D_i u_i (D_k u_j)^2\, dx \\
&= -\sum_{j, k = 1}^{2} \int_O (\nabla \cdot u)(D_k u_j)^2\, dx \quad = 0.
\end{aligned}
$$

Thus we have shown that for every $u \in D(A)$, $\langle B(u, u), Au \rangle_H = 0$. $\qquad\square$

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Appendix B. Some results in the support of Section 5

Remark B.1. *If* $\nabla \cdot u = 0$ *and* $\mathrm{Curl}(u) = 0$, *then* u *is constant by Hodge decomposition. In particular if* $u \in V$ *and* $\mathrm{Curl}(u) = 0$, *then* $u = 0$.

Proof of Remark 5.1. We want to show that Curl is a linear isomorphism between V and $L_0^2(\mathbb{T}^2)$. It is clear that the map

$$\mathrm{Curl} : V \ni u \mapsto \omega = \mathrm{Curl(u)} \in L_0^2(\mathbb{T}^2),$$

is linear and continuous. Hence in order to prove the Remark 5.1 it is sufficient to find a continuous linear map

$$\Lambda : L_0^2(\mathbb{T}^2) \to V, \tag{B.1}$$

such that,

$$\mathrm{Curl} \circ \Lambda = \mathrm{id} \text{ on } L_0^2(\mathbb{T}^2), \tag{B.2}$$
$$\Lambda \circ \mathrm{Curl} = \mathrm{id} \text{ on } V. \tag{B.3}$$

Let $\omega \in L_0^2(\mathbb{T}^2)$ then by elliptic regularity [11] (applies also for $p \neq 2$) there exists a unique $\psi \in L_0^2(\mathbb{T}^2) \cap H^2(\mathbb{T}^2)$ such that

$$\Delta \psi = \omega, \tag{B.4}$$

and the map

$$L_0^2 \ni \omega \mapsto \psi \in L_0^2 \cap H^2,$$

is bounded. Let us put $u = \nabla^\perp \psi$, i.e.

$$u = (D_2\psi, -D_1\psi). \tag{B.5}$$

Then $u \in H^1(\mathbb{T}^2)$ and $\nabla \cdot u = 0$ in the weak sense. Thus $u \in V$. Using all of this we define the bounded linear map $\Lambda : L_0^2(\mathbb{T}^2) \ni \omega \mapsto u \in V$. Now we are left to check that (B.2) and (B.3) holds for this Λ.

Let us take $\omega \in L_0^2(\mathbb{T}^2)$ and put $u := \Lambda(\omega) \in V$. Now considering LHS of (B.2),

$$(\mathrm{Curl} \circ \Lambda)(\omega) = \mathrm{Curl(u)} = D_2u_1 - D_1u_2$$
$$= D_2D_2\psi - (-D_1D_1\psi) = \Delta\psi = \omega,$$

where we have used the definitions of ψ and u from (B.4) and (B.5). Hence we have established (B.2).

Now we take $v \in V$ and put $\omega = \mathrm{Curl(v)} \in L_0^2(\mathbb{T}^2)$. Define $\psi \in L_0^2(\mathbb{T}^2) \cap H^2(\mathbb{T}^2)$ by

$$\Delta \psi = \omega. \tag{B.6}$$

Observe that

$$\Delta\varphi = \mathrm{Curl}(D_2\varphi, -D_1\varphi), \quad \varphi \in H^2(\mathbb{T}^2).$$

Thus by (B.6) and the definition of u from (B.5) we obtain

$$\mathrm{Curl(u)} = \mathrm{Curl(v)},$$

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where $u = \nabla^{\perp}\psi \in V$.

Therefore using Remark 1 $u = v$, thus proving that Curl is a linear isomorphism between V and $L_0^2(\mathbb{T}^2)$. It is straightforward to show (5.2). Thus we are left to prove (5.3).

Let us fix $p \in (1, \infty)$ and take $u \in H^{1,p}(\mathbb{T}^2)$. Denote $\omega = \text{Curl}(u) \in L_0^p(\mathbb{T}^2)$. From the first part of the proof there exists a bounded linear map $\Lambda : L_0^p(\mathbb{T}^2) \to H^{1,p}(\mathbb{T}^2)$

$$\Lambda : L_0^p \ni \omega \mapsto u \in H^{1,p},$$

such that

$$\text{Curl} \circ \Lambda = \text{id on } L_0^p(\mathbb{T}^2).$$

In particular, there exists a $C_p' > 0$,

$$|\Lambda\omega|_{H^{1,p}(\mathbb{T}^2)} \le C_p'|\omega|_{L^p(\mathbb{T}^2)}, \quad \omega \in L_0^p(\mathbb{T}^2).$$

Hence

$$|\nabla\Lambda\omega|_{L^p(\mathbb{T}^2)} \le C_p'|\omega|_{L^p(\mathbb{T}^2)}, \quad \omega \in L_0^p(\mathbb{T}^2). \tag{B.7}$$

Taking now $u \in H^{1,p}(\mathbb{T}^2)$. Putting $\omega = \text{Curl}(u)$ so that $\Lambda\omega = u$ from (B.7) we infer (B.8),

$$|\nabla u|_{L^p(\mathbb{T}^2)} \le C_p|\omega|_{L^p(\mathbb{T}^2)}. \tag{B.8}$$

Now since $|\omega|_{L^p(\mathbb{T}^2)} \le |\omega|_{L^\infty(\mathbb{T}^2)}$ for every p, we can establish (5.3). $\qquad\square$

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