The 1-harmonic flow with values into a smooth planar curve

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Abstract

We introduce a notion of solution to the 1-harmonic flow –i. e., the formal gradient flow of the total variation with respect to the L^2-distance– from a domain of $\mathbb{R}^m$ into a connected subset of the image of a smooth Jordan curve. For such notion, we establish existence and uniqueness of solutions to the homogeneous Neumann problem. We also discuss a consistent notion of solution when the target space is a smooth $(n-1)$-dimensional manifold whose geodesics are unique, presenting conjectures and open questions related to it.

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1. Introduction

Let $\Omega \subset \mathbb{R}^m$ be an open bounded domain with Lipschitz continuous boundary and let Σ be an $(n-1)$-dimensional smooth and oriented Riemannian manifold embedded in $\mathbb{R}^n$. Generally speaking, the p-harmonic flow with values into Σ is the formal gradient flow of the p-energy,

$$E_p(u) = \int_\Omega |Du|^p \mathrm{d}x, \quad u : \Omega \to \Sigma,$$

with respect to the L^2-distance, with the constraint that u takes values into Σ. The p-harmonic flow was introduced by Eells and Sampson [10] in the case $p = 2$ for constructing 2-harmonic maps between Riemannian manifolds as long-time limits of solutions to the corresponding 2-harmonic flow. We refer to [24, 26] for referenced discussions of the cases $p = 2$ and $1 < p < \infty$, respectively. Harmonic flows are also prototypes for reaction-diffusion systems arising in various contexts: multi-grain problems [25], theory of liquid crystals [23], ferromagnetism [9], and image processing [29].

Here we are concerned with the case $p = 1$, in which $BV(\Omega)$ is the natural ambient space and $E_1(u)$ corresponds to the total variation of u. The 1-harmonic flow was proposed as a tool to denoise either two-dimensional image gradients and optical flows, in which case $n = 2$ and $\Sigma = \mathbb{S}^1$ [30], or color images by smoothing the chromaticity data while preserving the contrast, in which case $n = 3$ and Σ is an octant of $\mathbb{S}^2$ [31]. When $\Sigma \subseteq \mathbb{S}^{n-1}$, the 1-harmonic

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flow with values into Σ is formally given by

$$u_t - \operatorname{div}\left(\frac{Du}{|Du|}\right) = u|Du|, \quad u \in \Sigma \tag{1.1}$$

(see the Appendix). The most delicate issue in the analysis of (1.1) is to give appropriate interpretations to the bounded matrix $Du/|Du|$ and to the measure $u|Du|$ that appear in (1.1). After results in special cases –piece-wise constant data [16, 19, 20], initial data with "small" energy [18], rotationally symmetric solutions [17, 8, 15]– a general notion of solution has been introduced in [12, 13, 14], yielding existence results for any $n \geq 2$ when $\Sigma = \mathbb{S}^{n-1}_+$ (the first hyper-octant of the sphere), as well as uniqueness ones for $n = 2$. In essence, this notion rewrites (1.1) in the comprehensive form of

$$u_t = \operatorname{div} Z + \mu \tag{1.2}$$

and identifies Z and μ via

$$(Z, Du) = |u^*||Du| \quad \text{and} \quad \mu = \frac{u^*}{|u^*|}|Du|. \tag{1.3}$$

Here (Z, Du) denotes a well-defined pairing which, on smooth functions, coincides with the scalar product of the two matrices, $Z : Du$ (see Section 2).

While the diffuse parts of the measures in (1.3) are very natural, it is not obvious why the jump parts should have the specific form given in (1.3). In order to answer this question, it seems useful to start investigating the more general case in which Σ is not contained in a sphere. In this case, the 1-harmonic flow with values into Σ is formally given by

$$u_t - \operatorname{div}\left(\frac{Du}{|Du|}\right) = \frac{1}{|Du|}\left(DN(u) : Du\right) N(u), \quad u \in \Sigma \tag{1.4}$$

(see the Appendix), where $N : \Sigma \to \mathbb{S}^{n-1}$ denotes one of the two unit normal vector-fields to Σ (note that the right-hand side of (1.4) is invariant under change of orientation). With the exception of Section 5, we concentrate on the simplest possible setting, in which $n = 2$ and $\Sigma \subset \mathbb{R}^2$ is contained in the image of a smooth and regular Jordan curve γ.

Our first goal is to provide an appropriate notion of solution to (1.4). This is done in Section 3, where we argue that, rewriting (1.4) in the comprehensive form (1.2), the natural interpretation to μ is:

$$\mu = \kappa(u)N(u)\,|\tilde{D}u| + \left(T(u_-) - T(u_+)\right)\,\mathcal{H}^{m-1} \llcorner J_u, \tag{1.5}$$

where $T : \Sigma \to \mathbb{S}^1$ is one of the two tangential unit vector-fields to Σ, u_+ follows u_- along the orientation of Σ induced by T, and $\kappa(u) = T(u) \cdot \overline{\nabla} N(u)T(u)$ is the curvature of Σ (here $\overline{\nabla}$ denotes the gradient in the target space, $\mathbb{R}^2$). With this notion, it is easily seen that Z is characterized by

$$(Z, DN(u)) = \kappa(u)\,|\tilde{D}u| + (N(u))^* \cdot \left(T(u_-) - T(u_+)\right)\mathcal{H}^{m-1} \llcorner J_u, \tag{1.6}$$

which is formally equivalent to $Z : Du = |Du|$ when $\kappa \neq 0$ in Σ. We also show that such notion recovers that introduced in [12, 14] when $\Sigma \subsetneq \mathbb{S}^1$.

Our second goal is to prove an existence and uniqueness result based on the above-mentioned notion of solution. This is done in Section 4, where we consider the homogeneous Neumann problem for (1.4):

$$\begin{cases} u_t = \operatorname{div}\left(\dfrac{Du}{|Du|}\right) + \dfrac{1}{|Du|}(DN(u) : Du)N(u), \ u \in \Sigma & \text{in } Q = (0,\infty)\times\Omega \\ \dfrac{Du}{|Du|}\cdot\nu = 0 & \text{on } (0,\infty)\times\partial\Omega \\ u = u_0 & \text{on } \{0\}\times\Omega, \end{cases} \tag{1.7}$$

where ν denotes the outward unit normal to $\partial\Omega$. The arguments extend those in [12]: generally speaking, we argue that u is a solution to (1.7) if and only if $u = \gamma(s)$, where γ is an arc-length parametrization of Σ and s solves the homogeneous Neumann problem for the (unconstrained and scalar) *total variation flow*:

$$s_t = \operatorname{div}\left(\frac{Ds}{|Ds|}\right)$$

([3]; see also [5, 11, 27] for recent discussions on some of its generalizations).

Besides their intrinsic interest, we believe that the two goals above are a first step towards the elaboration of a theory for two distinct generalizations: smooth $(n-1)$-dimensional manifolds on one hand, and *non*-smooth planar curves, such as a Wulff shape, on the other. Concerning the former, in Section 5 we more generally consider an $(n-1)$-dimensional manifold Σ on which each pair of points is connected by a unique geodesic (such as $\mathbb{S}^{n-1}_+$, as considered in [14]). Rewriting (1.4) in the comprehensive form (1.2), we argue in favor of

$$\mu = F_u N(u)\,|\tilde{D}u| + (T(u_-) - T(u_+))\ \mathcal{H}^{m-1}\llcorner J_u \tag{1.8}$$

as the natural interpretation of μ, where:

- F_u is a bounded, scalar, $|Du|$-measurable function which coincides with $\frac{DN(u)}{|Du|} : \frac{Du}{|Du|}$ on smooth functions with $|Du| \neq 0$;
- $T(u_-)$ and $T(u_+)$ are the tangential unit vector-fields along the unique geodesic which goes *from* u_- *to* u_+.

Also, we discuss the analogue of (1.6) and we show that (1.8) coincides both with (1.3) (when $\Sigma = \mathbb{S}^{n-1}_+$) and with (1.5) (when $n = 2$). Finally, we present an open question and a conjecture, which naturally arise from the discussion above, concerning the lower semicontinuity of integral functionals involving the second fundamental form.

2. Preliminaries

2.1. General notation

We denote by $\mathcal{H}^{m-1}$ the $(m-1)$-dimensional Hausdorff measure and by $\mathcal{L}^m$ the m-dimensional Lebesgue measure. We denote by D the gradient with respect to $x \in \Omega$,

$$Df := \left(\frac{\partial f^j}{\partial x_i}\right)_{j=1,\dots,n,\ i=1,\dots,m}, \quad f : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n,$$

while $\overline{\nabla}$ denotes the gradient with respect to $u = (u^1, \dots, u^n) \in \mathbb{R}^n$:

$$\overline{\nabla} h := \left(\frac{\partial h}{\partial u^1}, \dots, \frac{\partial h}{\partial u^n} \right), \quad h : \mathbb{R}^n \to \mathbb{R}.$$

Given an $m \times n$ matrix $Z = (z_i^j)$, we write $z^j = (z_1^j, ..., z_m^j)$, $j \in \{1, \dots, n\}$ and $z_i = (z_i^1, \dots, z_i^n)$, $i \in \{1, ..., m\}$. Given matrices $Z = (z_i^j)$ and $A = (a_i^j)$, we let

$$\operatorname{div} Z := \left(\operatorname{div} z^1, \dots, \operatorname{div} z^n \right), \tag{2.1}$$

where

$$\operatorname{div} z^j = \sum_{i=1}^m \frac{\partial z_i^j}{\partial x_i}, \quad j = 1, \dots, n$$

and

$$Z : A = \sum_{j=1}^n z^j \cdot a^j = \sum_{i=1}^m z_i \cdot a_i = \sum_{j=1}^n \sum_{i=1}^m z_i^j a_i^j. \tag{2.2}$$

Given $w = (w^1, \dots, w^n)$ and $\eta = (\eta_1, ..., \eta_m)$, we have

$$\begin{aligned} w \operatorname{div} \eta &= (w^1 \operatorname{div} \eta, \dots, w^n \operatorname{div} \eta) \\ &= (\operatorname{div} (w^1 \eta) - \eta \cdot D w^1, \dots, \operatorname{div} (w^n \eta) - \eta \cdot D w^n) \\ &\overset{(2.1)}{=} \operatorname{div} (w \otimes \eta) - \eta D w, \end{aligned} \tag{2.3}$$

where $w \otimes \eta$ is the matrix defined by

$$(w \otimes \eta)_i^j = \eta_i w^j, \quad i = 1, \dots, m, \ j = 1, \dots, n.$$

2.2. BV functions and Green's formulas

We use standard notations and properties for measures and functions of bounded variations (see [2]). We sometimes write $u \in BV(\Omega; \Sigma)$, meaning that $u \in BV(\Omega; \mathbb{R}^n)$ and $u(x) \in \Sigma \subset \mathbb{R}^n$ for a.e. $x \in \Omega$.

Let $\mathcal{M}(\Omega; \mathbb{R}^n)$ denote the space of $\mathbb{R}^n$-valued finite Radon measures on Ω (see [2, Def. 1.40]),

$$\begin{aligned} X_p(\Omega; R^n) &:= \{ Z \in L^\infty(\Omega; \mathbb{R}^{mn}) \, : \, \operatorname{div} Z \in L^p(\Omega; \mathbb{R}^n) \}, \\ X_{\mathcal{M}}(\Omega; \mathbb{R}^n) &:= \{ Z \in L^\infty(\Omega; \mathbb{R}^{mn}) \, : \, \operatorname{div} Z \in \mathcal{M}(\Omega; \mathbb{R}^n) \}. \end{aligned}$$

If $Z \in X_{\mathcal{M}}(\Omega; \mathbb{R}^n)$ and $w \in BV(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$, the functional $(Z, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ defined by

$$\langle (Z, Dw), \phi \rangle := - \int_\Omega w^* \phi \, \mathrm{d}(\operatorname{div} Z) - \int_\Omega w \, Z : D\phi \, \mathrm{d}x \tag{2.4}$$

is a Radon measure (note that the first integral on the right hand side of (2.4) is well defined since $|\operatorname{div} Z|(A) = 0$ whenever $\mathcal{H}^{m-1}(A) = 0$, see [7, Proposition 3.1]). In addition, (Z, Dw) is absolutely continuous with respect to $|Dw|$ (see [6, Section 5]): we denote by $\Theta(Z, Dw) \in L^1(\Omega, |Dw|)$ its density, that is,

$$\int_E \mathrm{d}(Z, Dw) = \int_E \Theta(Z, Dw) \, \mathrm{d}|Dw| \quad \text{for any Borel set } E \subseteq \Omega. \tag{2.5}$$

A weak trace on $\partial\Omega$ of the normal component of $z^j \in X_{\mathcal{M}}(\Omega;\mathbb{R})$, denoted by $[z^j,\nu]$, is defined in [4]. Letting

$$[Z,\nu] := ([z^1,\nu],\dots,[z^n,\nu]),$$

the following Green's formula is established in [6, Theorem 5.3]:

$$\int_\Omega w^* \cdot \mathrm{d}(\operatorname{div} Z) + \int_\Omega \mathrm{d}(Z,Dw) = \int_{\partial\Omega} [Z,\nu]\, w \,\mathrm{d}\mathcal{H}^{m-1}. \tag{2.6}$$

Moreover

$$\operatorname{div}(wZ) = w^* \cdot \operatorname{div} Z + (Z,Dw) \quad \text{as measures} \tag{2.7}$$

(see [6], Lemma 5.4 and the discussion below it) and

$$[wZ,\nu] = w[Z,\nu] \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega \tag{2.8}$$

(see [6], Lemma 5.6).

3. The notion of solution: the case $n = 2$

The first point of interest in this paper is to find the appropriate interpretation to equation (1.4). In this section we consider the case $n = 2$, for which an existence and uniqueness result will be proved in Section 4; generalization to $n > 2$ will be discussed in Section 5. We thus make the following assumptions on the target space:

(H) $\tilde\Sigma \subset \mathbb{R}^2$ is the image of a regular Jordan curve $\gamma \in C^2([0,\tilde S])$, $\tilde S > 0$, with $|\gamma'(s)| = 1$ for all $s \in [0,\tilde S]$; $\Sigma \subsetneq \tilde\Sigma$ is a connected subset of $\tilde\Sigma$, i. e. $\Sigma = \operatorname{Im}(\gamma|_{[0,S]})$, $S \in (0,\tilde S)$.

We introduce notation for the tangential, resp. normal, unit vector-fields $T : \tilde\Sigma \to \mathbb{S}^1$, resp. $N : \tilde\Sigma \to \mathbb{S}^1$,

$$T(\gamma(s)) := \gamma'(s), \quad N(\gamma(s)) := (\gamma'(s))^\perp := \left((\gamma^2)', -(\gamma^1)'\right), \tag{3.1}$$

and for the curvature $\kappa : \tilde\Sigma \to \mathbb{R}$, defined through

$$\gamma''(s) =: -\kappa(\gamma(s))N(\gamma(s)) \tag{3.2}$$

(note that $\gamma''(s)\cdot\gamma'(s) \equiv 0$ since $|\gamma'(s)|^2 \equiv 1$, hence κ is well defined). Since $\gamma|_{[0,S]}$ is simple and not closed, its inverse is well defined and continuous:

$$\sigma := (\gamma|_{[0,S]})^{-1} \in C(\Sigma;[0,S]), \quad \sigma(\gamma(s)) = s. \tag{3.3}$$

In order to identify the right hand side of (1.4), we use the following simple observations, which are proved in the Appendix:

Lemma 3.1. *Let Σ as in* (H). *Then*

$$\begin{aligned} \kappa(u) &= T(u)\cdot(\overline{\nabla} N(u) T(u)) &(3.4)\\ &= \left(\operatorname{Tr}(\overline{\nabla} N(u)) - N(u)\cdot(\overline{\nabla} N(u))N(u)\right). &(3.5)\end{aligned}$$

and if $u : \Omega \to \Sigma$ is smooth with $|Du| \neq 0$, then

$$\frac{1}{|Du|}\left(DN(u) : Du\right) N(u) = \kappa(u) N(u) |Du|. \tag{3.6}$$

Note that, because of $n = 2$, $|Du|$ factors out in (3.6): hence the right hand side of (3.6) has a clear meaning for the diffuse part of a BV-function u. In order to identify the natural candidate to represent the left-hand side of (3.6) on J_u, we alternatively resort on the definition of κ. Since $\kappa(u)N(u) = -\gamma''(\sigma(u))$ (cf. (3.2) and (3.3)), at a jump point it is natural to replace $\kappa(u)N(u)$ in (3.6) by minus the difference quotient of the tangential unit vector-field $\gamma'(\sigma(u)) = T(u)$,

$$\frac{T(u_-) - T(u_+)}{|u_+ - u_-|}\,|Du|. \tag{3.7}$$

Here we use the convention that

$$u_+ \text{ follows } u_- \text{ along the orientation of } \Sigma \text{ induced by } \gamma. \tag{3.8}$$

Recalling that $|Du| = |u_+ - u_-|\,\mathcal{H}^{m-1}$ on J_u, (3.6) and (3.7) motivate the following definition:

Definition 3.2. *Let* Σ *as in* (H) *and let* $u \in BV(\Omega; \Sigma)$. *We define the Radon measure:*

$$\mu \;:=\; \kappa(u)N(u)\,|\tilde{D}u| + (T(u_-) - T(u_+))\;\mathcal{H}^{m-1} \llcorner J_u, \tag{3.9}$$

where T, N, *and* κ *are defined in* (3.1)-(3.4) *and* u_-, u_+ *are oriented as in* (3.8).

Remark 3.3. The right-hand side of (3.9) is uniquely determined; in particular, it is *independent of the orientation of* Σ. Indeed: the diffuse part, $\kappa(u)N(u)$, is even with respect to N (cf. (3.4)); concerning the jump part, under a change of orientation, not only u_+ and u_- are switched (cf. (3.8)), but also the tangent vectors $T(u_+)$ and $T(u_-)$ change sign.

Remark 3.4. If Σ was closed, there would be two paths connecting each pair, giving raise to two opposite values of $T(u_-) - T(u_+)$, and in some cases these two paths would have the same length (think of two antipodal points in the circle), making a discrimination between the two impossible: hence, for a closed curve (3.9) should be given as an inclusion rather than as an equality (see. e.g. Definition 2.3 in [12]). Since we wouldn't be able to prove uniqueness of solutions to the resulting problem, we prefer to leave this complication aside.

Remark 3.5. Under the additional assumption that $N(u_1) + N(u_2) \neq 0$ for all $u_1, u_2 \in \Sigma$, we have

$$(T(u_-) - T(u_+))\;\mathcal{H}^{m-1} \llcorner J_u = \pm \frac{(N(u))^*}{|(N(u))^*|}|D^j N(u)|, \tag{3.10}$$

where, however, the sign depends on the orientation of Σ. Indeed, a simple computation shows that $(T(u_-) - T(u_+)) \cdot (N(u_+) + N(u_-)) = 0$; hence

$$(T(u_-) - T(u_+)) = \pm \frac{N(u_+) + N(u_-)}{|N(u_+) + N(u_-)|}|T(u_-) - T(u_+)| = \pm \frac{(N(u))^*}{|(N(u))^*|}|N(u_-) - N(u_+)|$$

and (3.10) follows since $|D^j N(u)| = |N(u_-) - N(u_+)|\mathcal{H}^{m-1} \llcorner J_u$. In particular, it follows from (3.10) and (3.8) that

$$\mu = u|\tilde{D}u| + \frac{u^*}{|u^*|}|Du| \quad \text{when } \Sigma \subsetneq \mathbb{S}^1, \tag{3.11}$$

which coincides with the measure defined in [12, formula (2.1)].

We are now ready to define the concept of solution for problem (1.7).

Definition 3.6. *Let Σ as in* (H)*. A map $u : \Omega \to \Sigma$ is a solution to equation* (1.4) *in Q if*

$$u \in C([0,\infty); L^2(\Omega;\mathbb{R}^2)) \cap L^1_{loc}([0,\infty); BV(\Omega;\mathbb{R}^2)), \quad u_t \in L^2_{loc}((0,\infty); L^2(\Omega;\mathbb{R}^2)) \tag{3.12}$$

and a matrix-valued function $Z \in L^\infty(Q;\mathbb{R}^{2m})$ exists such that

(i) $\|Z\|_\infty \le 1$, $z_i \cdot N(u) = 0$ *a.e. in Q for all $i = 1,\dots,m$,*

and the following holds for a.e. $t > 0$:

(ii) $Z(t) \in X_{\mathcal{M}}(\Omega;\mathbb{R}^2)$*;*
(iii) $u_t(t) - \operatorname{div} Z(t) = \mu(t)$*;*
(iv) $u_t(t) \cdot T(u(t)) = \operatorname{div}(Z(t) \cdot T(u(t)))$ *and* $u_t(t) \cdot N(u(t)) = 0$ *in* $L^2(\Omega)$*,*

with T and N as in (3.1) *and μ as in Definition 3.2.*

The matrix-valued function Z in Definition 3.6 may be characterized as follows:

Proposition 3.7. *Let u and Z as in Definition 3.6. Then*

$$(Z, DN(u)) = \kappa(u)\,|\tilde{D}u| + (N(u))^* \cdot (T(u_-) - T(u_+))\,\mathcal{H}^{m-1} \llcorner J_u. \tag{3.13}$$

In particular,

$$(Z, Du) = |u^*||Du| \quad \text{when } \Sigma \subsetneq \mathbb{S}^1. \tag{3.14}$$

Note that (3.14) coincides with the characterization of Z given in [14, Prop. 3.5].

Proof of Proposition 3.7. Multiplying the equation in (iii) by $(N(u))^*$ we get

$$N(u) \cdot u_t(t) - (N(u))^* \cdot (\operatorname{div} Z(t)) = (N(u))^* \cdot \mu. \tag{3.15}$$

The first term on the left-hand side of (3.15) vanishes by the second identity in (iv). The second one can be rewritten using (2.7) as follows:

$$-(N(u))^* \cdot (\operatorname{div} Z) = -\operatorname{div}(N(u)Z) + (Z, DN(u)) = (Z, DN(u)),$$

where in the last equality we have used (i) in Definition 3.6. Therefore (3.15) is equivalent to

$$(Z, DN(u)) = (N(u))^* \cdot \mu.$$

Using Definition 3.2 we obtain (3.13). If $\Sigma \subsetneq \mathbb{S}^1$, taking e.g. the counter-clockwise orientation, we have $N(u) = u$, $\kappa(u) = 1 = |u|$ (cf. (3.4)), and $(N(u))^* \cdot (T(u_-) - T(u_+)) = |u^*||Du|$ (cf. (3.10) and (3.8)), whence (3.13) translates into $(Z, Du) = |u^*||Du|$. $\square$

The next simple observation, proved in the Appendix, implies that the characterization (3.13) is formally equivalent to $Z : Du = |Du|$ at least when $\tilde{\Sigma}$ is the smooth boundary of a strictly convex domain (in particular, when $\tilde{\Sigma} = \mathbb{S}^1$):

Lemma 3.8. *Let Σ as in* (H) *and assume that $u : \Omega \to \Sigma$ is smooth. If Z is a smooth matrix such that $z_i \cdot N(u) = 0$ for all $i = 1,\dots,m$ and $Z : DN(u) = \kappa(u)\,|Du|$, then $Z(x) : Du(x) = |Du(x)|$ whenever $\kappa(u(x)) \neq 0$.*

Remark 3.9. Besides the jump part of (3.9), Definition 3.6 may be reformulated in terms of the unit normal vector field N alone (as in [12, 14]). Indeed, the first equality in (iv) may be replaced with

$$u_t(t) \wedge N(u(t)) = \operatorname{div}(Z(t) \wedge N(u(t))) \quad \text{in } L^2(\Omega)$$

and, in view of (3.5),

$$\kappa(u) = \operatorname{Tr}(\overline{\nabla} N(u)) - N(u) \cdot \big(\overline{\nabla} N(u)\, N(u)\big)\,.$$

In particular, also in view of (3.11), Definition 3.6 coincides with the one given in [12] when $\Sigma \subsetneqq \mathbb{S}^1$. On the other hand, in general the jump part of (3.9) may be written in terms of N only up to the orientation of Σ, cf. (3.10).

4. Existence and uniqueness of solutions

The second point of interest in this paper is to give an existence and uniqueness result according to the notion of solution illustrated in the previous section. We consider the homogeneous Neumann problem (1.7). Hence, we complement Definition 3.6 with initial and boundary conditions:

Definition 4.1. *Let Σ as in* (H) *and let $u_0 : \Omega \to \Sigma$ be measurable. A map $u : \Omega \to \Sigma$ is a solution to problem* (1.7) *with initial datum u_0 if u is a solution to* (1.4) *in Q according to Definition 3.6 and:*

(v) $[Z(t), \nu] = 0$ *a.e. on* $\partial\Omega$*;*
(vi) $u(0) = u_0$ *in* Ω.

We prove:

Theorem 4.2. *Let Σ as in* (H) *and let $u_0 : \Omega \to \Sigma$ measurable. Then there exists a unique map $u : \Omega \to \Sigma$ which solves problem* (1.7) *with initial datum u_0.*

Our strategy for Theorem 4.2 is inspired by that in [12]. Indeed, we will argue that u is a solution to (1.7) if and only if $s = \sigma(u)$, with σ as in (3.3), is a solution to the following scalar and unconstrained problem:

$$\begin{cases} s_t = \operatorname{div}\left(\dfrac{Ds}{|Ds|}\right) & \text{in } Q \\ \dfrac{Ds}{|Ds|} \cdot \nu = 0 & \text{on } (0,\infty) \times \partial\Omega \\ s = s_0 & \text{on } \{0\} \times \Omega \end{cases} \tag{4.1}$$

Therefore, we begin by providing the necessary details on (4.1).

4.1. The unconstrained scalar problem

We shall be using the following concept of solution to (4.1):

Definition 4.3. *[3, Definition 2.5] A solution to* (4.1) *with initial datum $s_0 \in L^2(\Omega;\mathbb{R})$ is a function*

$$s \in C([0,\infty); L^2(\Omega;\mathbb{R})) \cap W^{1,2}_{loc}((0,\infty); L^2(\Omega;\mathbb{R})) \cap L^1_{loc}([0,\infty); BV(\Omega;\mathbb{R})) \tag{4.2}$$

such that $s(0) = s_0$ *and there exists* $\eta \in L^\infty(Q;\mathbb{R}^m)$ *with* $\|\eta\|_\infty \le 1$ *such that*

$$s_t = \operatorname{div}\eta \quad in\ L^2_{loc}((0,\infty);L^2(\Omega;\mathbb{R})) \tag{4.3}$$

and for a.e. $t > 0$ *and all* $w \in BV(\Omega;\mathbb{R}) \cap L^2(\Omega;\mathbb{R})$ *it holds that*

$$\int_\Omega (s(t) - w)s_t(t)\,\mathrm{d}x = \int_\Omega \mathrm{d}(\eta(t), Dw) - \int_\Omega \mathrm{d}|Ds(t)| \tag{4.4}$$

$$[\eta(t),\nu] = 0 \quad a.e.\ on\ \partial\Omega. \tag{4.5}$$

Note that $\eta(t) \in X_2(\Omega;\mathbb{R})$ for a.e. $t > 0$ in view of (4.3). The pairing and the trace which appear in (4.4), resp. (4.5), are defined exactly as in Section 2, with $w \in BV(\Omega;\mathbb{R}) \cap L^2(\Omega;\mathbb{R})$ and $X_{\mathcal{M}}(\Omega;\mathbb{R})$ replaced by $X_2(\Omega;\mathbb{R})$. In addition, (η, Dw) is absolutely continuous with respect to $|Dw|$ (see [3] Corollary C.7) and (2.6) continues to hold (see [3, Sections C.1 and C.2]).

In the next statement we summarize the results we need on problem (4.1).

Theorem 4.4. *Let* $s_0 \in L^\infty(\Omega;\mathbb{R})$. *Then there exists a unique solution* s *to* (4.1) *in* Q *with initial datum* s_0 *and*

$$(\eta(t), Ds(t)) = |Ds(t)| \quad in\ \mathcal{M}(\Omega)\ for\ a.e.\ t > 0. \tag{4.6}$$

In addition,

(i) *if* $s_0 \in [0, M]$, *then* $s(t) \in [0, M]$ *for a.e.* $t > 0$;
(ii) *in Definition 4.3,* (4.4) *may be replaced by* (4.6).

Proof. The existence and uniqueness part is given in [3, Theorem 2.6]. For the proof of (4.6) and (i) see [14, Theorem 3.3]. Finally, we show that (4.6) implies (4.4):

$$\begin{aligned}\int_\Omega (s(t) - w)s_t(t)\,\mathrm{d}x &\overset{(4.3)}{=} \int_\Omega (s(t) - w)\mathrm{div}\eta \overset{(2.6),(4.5)}{=} -\int_\Omega \mathrm{d}(\eta, D(s(t) - w) \\ &\overset{(4.6)}{=} \int_\Omega \mathrm{d}(\eta(t), Dw) - \int_\Omega \mathrm{d}|Ds(t)|.\end{aligned}$$

□

4.2. Basic relations between s *and* u

For $s \in BV(\Omega;\mathbb{R})$, we assume that the triplet (s_+, s_-, ν_s) is such that

$$s_+(x) > s_-(x) \quad \text{for all } x \in J_s. \tag{4.7}$$

We give a few simple relations, between s and $u = \gamma(s)$, that will be used in both the existence and the uniqueness part of the proof:

Lemma 4.5. *Let* $s \in BV(\Omega;[0,S])$. *Then* $u = \gamma \circ s \in BV(\Omega;\Sigma)$ *with*

$$|Du| \le |Ds|, \quad |\tilde{D}u| = |\tilde{D}s|, \quad and \quad J_u = J_s \tag{4.8}$$

and

$$\mu = -\gamma''(s)|\tilde{D}s| + \big(\gamma'(s_-) - \gamma'(s_+)\big)\,\mathcal{H}^{m-1}\llcorner J_s. \tag{4.9}$$

where μ *is defined by* (3.9).

Proof. Since γ is a Lipschitz C^1 function, the chain rule for BV-functions (see [2, Theorem 3.96]) implies that $u \in BV(\Omega;\mathbb{R}^2)$ (with, obviously, $u \in \Sigma$) and

$$|Du| = |\gamma'(s)||\tilde{D}s| + |\gamma(s_+) - \gamma(s_-)|\mathcal{H}^{m-1} \llcorner J_s,$$

hence (4.8) follows since $|\gamma'| \equiv 1$ and $|\gamma(s_+) - \gamma(s_-)| \leq |s_+ - s_-|$. Finally, recalling the orientation conventions (3.8) and (4.7), (4.9) follows from

$$\begin{aligned}
\mu &\overset{(3.9)}{=} \kappa(u)N(u)|\tilde{D}u| + (T(u_-) - T(u_+))\,\mathcal{H}^{m-1} \llcorner J_u \\
&\overset{(4.8)}{=} \kappa(\gamma(s))N(\gamma(s))|\tilde{D}s| + (T(\gamma(s_-)) - T(\gamma(s_+)))\,\mathcal{H}^{m-1} \llcorner J_s \\
&\overset{(3.1),(3.2)}{=} -\gamma''(s)|\tilde{D}s| + \big(\gamma'(s_-) - \gamma'(s_+)\big)\,\mathcal{H}^{m-1} \llcorner J_s.
\end{aligned}$$

□

4.3. Proof of Theorem 4.2, existence

In the course of the proof of the existence part of Theorem 4.2, we will need to identify pairings of the form $(\eta, D(\gamma^j)'(s))$. We actually prove a slightly more general result which, for a Lipschitz function f, identifies the pairing $(\eta, Df(w))$ in terms of its density with respect to w:

Lemma 4.6. *Let $w \in BV(\Omega;\mathbb{R}) \cap L^2(\Omega;\mathbb{R})$, $\eta \in X_2(\Omega;\mathbb{R})$, and $f \in C^1(\mathbb{R})$ with $\|f'\|_\infty < \infty$. Then, under (4.7),*

$$(\eta, Df(w)) = \Theta(\eta, Dw)\left(f'(w)|\tilde{D}w| + (f(w_+) - f(w_-))\,\mathcal{H}^{m-1} \llcorner J_w\right). \tag{4.10}$$

In addition, if $(\eta, Dw) = |Dw|$ as measures, then $\Theta(\eta, Dw) = 1$ $|Dw|$-a.e. in Ω.

Proof. Let $L > \|f'\|_\infty$, so that the function $w \to Lw + f(w)$ is increasing. Then, it follows from [4, Proposition 2.8] (see also the remark preceding Proposition 2.1 there) that

$$\Theta(\eta, D(Lw + f(w))) = \Theta(\eta, Dw) = \Theta(\eta, LDw) \quad |Dw|\text{-a.e.}. \tag{4.11}$$

Therefore

$$\begin{aligned}
\int_E \mathrm{d}(\eta, Df(w)) &= \int_E (\mathrm{d}(\eta, D(Lw + f(w))) - \mathrm{d}(\eta, LDw)) \\
&\overset{(2.5)}{=} \int_E \Theta(\eta, D(Lw + f(w)))\mathrm{d}|D(Lw + f(s))| - L\int_E \Theta(\eta, LDw)\mathrm{d}|Dw| \\
&\overset{(4.11)}{=} \int_E \Theta(\eta, Dw)\mathrm{d}(|D(Lw + f(w))| - L|Dw|)
\end{aligned}$$

for any Borel set $E \subseteq \Omega$, i.e.,

$$(\eta, Df(w)) = \Theta(\eta, Dw)(|D(Lw + f(w))| - L|Dw|) \quad \text{in } \mathcal{M}(\Omega). \tag{4.12}$$

On the other hand, by the chain rule [2, Theorem 3.96] and since $w \to Lw + f(w)$ is increasing,

$$\begin{aligned}
&|D(Lw + f(w))| - L|Dw| \\
&= f'(w)|\tilde{D}w| + (|Lw_+ + f(w_+) - Lw_- - f(w_-)| - L|w_+ - w_-|)\,\mathcal{H}^{m-1} \llcorner J_w \\
&= f'(w)|\tilde{D}w| + (f(w_+) - f(w_-))\,\mathcal{H}^{m-1} \llcorner J_w.
\end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13) we obtain (4.10). If in addition $(\eta, Dw) = |Dw|$, then

$$\int_E \Theta(\eta, Dw)\mathrm{d}|Dw| = \int_E \mathrm{d}(\eta, Dw) = \int_E \mathrm{d}|Dw| = |Dw|(E),$$

hence $\Theta(\eta, Dw) \equiv 1$ $|Dw|$-a.e. in Ω. □

We are now ready to prove the existence part of Theorem 4.2.

Proof of Theorem 4.2: existence. We define $s_0 : \Omega \to [0, S]$ as $s_0(x) = \sigma(u_0(x))$, where $\sigma = (\gamma_{|_{[0,S]}})^{-1}$. Since u_0 is measurable and σ is continuous (cf. (3.3)), s_0 is also measurable; therefore $s_0 \in L^\infty(\Omega)$. Hence, by Theorem 4.4, there exists a unique solution s to problem 4.1 in Q with initial datum s_0, and $s(t) \in [0, S]$ for a.e. $t > 0$.

Let $u(t) := \gamma(s(t))$. Since $s \in C([0, \infty), L^2(\Omega; \mathbb{R}))$ and γ is Lipschitz continuous, $u \in C([0, \infty), L^2(\Omega; \mathbb{R}^2))$ and $u(0) = \gamma(s(0)) = \gamma(s_0) = u_0$ for a.e. $x \in \Omega$, hence (vi) in Definition 4.1 holds. Moreover, $u_t = \gamma'(s)s_t \in L^2_{loc}((0, \infty); L^2(\Omega; \mathbb{R}^2))$. Since $s(t) \in L^1([0, \infty); BV(\Omega; \mathbb{R}))$, by Lemma 4.5 we have $u \in L^1_{loc}([0, \infty); BV(\Omega; \mathbb{R}^2))$. Hence the regularity properties (3.12) in Definition 3.6 are satisfied.

Let

$$Z := \gamma'(s) \otimes \eta \tag{4.14}$$

with η as in Definition 4.3. Since $\eta \in L^\infty(Q; \mathbb{R}^m)$ with $\|\eta\|_\infty \le 1$, we have $Z \in L^\infty(Q; \mathbb{R}^{2m})$ with $\|Z\|_\infty \le 1$. We now argue for a.e. t and omit dependence on t for notational convenience. Since $\eta \in X_2(\Omega; \mathbb{R})$, it follows immediately from (2.7) (applied with $n = 1$, $Z = \eta$ and $w = (\gamma^j)'(s)$) that

$$\operatorname{div} z^j = ((\gamma^j)'(s))^* \operatorname{div} \eta + (\eta, D(\gamma^j)'(s)) \quad \text{in } \mathcal{D}'(\Omega), \quad j = 1, 2,$$

hence $Z(t) \in X_{\mathcal{M}}(\Omega; \mathbb{R}^2)$ and (ii) in Definition 3.6 holds. Condition (i) in Definition 3.6 is immediate, recalling (4.14) and (3.1). In addition, since $T(u) = T(\gamma(s)) = \gamma'(s)$ and $|\gamma'| = 1$,

$$Z\,T(u) \overset{(4.14)}{=} (\gamma'(s) \otimes \eta)\,\gamma'(s) = (\gamma'(s) \cdot \gamma'(s))\,\eta = \eta. \tag{4.15}$$

Therefore

$$\begin{aligned} u_t \cdot T(u) &= s_t\,\gamma'(s) \cdot T(\gamma(s)) = s_t \overset{(4.1)}{=} \operatorname{div} \eta \overset{(4.15)}{=} \operatorname{div}(Z\,T(u)) && \text{in } L^2(\Omega), \\ u_t \cdot N(u) &= s_t\,\gamma'(s) \cdot N(\gamma(s)) = 0 && \text{in } L^2(\Omega), \end{aligned}$$

hence (iv) in Definition 3.6 holds true. The boundary condition (v) in Definition 4.1 follows directly from (4.5) and (2.8). It remains to show (iii) in Definition 3.6. We have

$$\begin{aligned} u_t &= \gamma'(s)s_t \overset{(4.1)}{=} (\gamma'(s))^* \operatorname{div} \eta \overset{(2.3)}{=} \operatorname{div}(\gamma'(s) \otimes \eta) - \eta D(\gamma'(s)) \\ &\overset{(4.14)}{=} \operatorname{div} Z + \tilde{\mu} \quad \text{in } \mathcal{D}'(\Omega), \end{aligned}$$

where we have set

$$\tilde{\mu} := (-(\eta, D(\gamma^1)'(s)), -(\eta, D(\gamma^2)'(s))). \tag{4.16}$$

Since $(\eta, Ds) = |Ds|$ (cf. Theorem 4.4), Lemma 4.6 (applied with $f = (\gamma^j)'$) yields

$$(\eta, D(\gamma^j)'(s)) = (\gamma^j)''(s)|\tilde{D}s| + \big((\gamma^j)'(s_+) - (\gamma^j)'(s_-)\big)\,\mathcal{H}^{m-1} \llcorner J_s \tag{4.17}$$

for $j = 1, 2$. Therefore

$$\tilde{\mu} \overset{(4.16),(4.17)}{=} -\gamma''(s)|\tilde{D}s| + \big(\gamma'(s_-) - \gamma'(s_+)\big)\,\mathcal{H}^{m-1} \llcorner J_s \overset{(4.9)}{=} \mu$$

and the proof is complete. □

4.4. Proof of Theorem 4.2, uniqueness

In order to prove uniqueness, we will argue that if u is a solution of (1.7), then $s = \sigma(u)$ is a solution to (4.1), and exploit the uniqueness of the latter. For this, we need to guarantee that $s(t, \cdot) \in BV$:

Lemma 4.7. *Let Σ and γ as in* (H). *Then $\sigma := (\gamma|_{[0,S]})^{-1} : \Sigma \to [0, S]$ is Lipschitz continuous. Furthermore, if $u \in BV(\Omega; \Sigma)$ then $s = \sigma \circ u$ is well defined, $s \in BV(\Omega; \mathbb{R})$, and $|Ds| \le C|Du|$, where C is the Lipschitz constant of σ.*

Proof. We recall that σ is well defined in view of (3.3). We claim that σ is Lipschitz continuous in $[0, S]$, i.e., $C \ge 1$ exists such that

$$|\sigma(u_1) - \sigma(u_2)| \le C|u_1 - u_2| \quad \text{for all } u_1, u_2 \in \Sigma, \tag{4.18}$$

which is equivalent to

$$|s_1 - s_2| \le C|\gamma(s_1) - \gamma(s_2)| \quad \text{for all } s_1, s_2 \in [0, S]. \tag{4.19}$$

We assume by contradiction that for any $k \in \mathbb{N}$ there exist $s_{1,k}$ and $s_{2,k}$ such that

$$k|\gamma(s_{1,k}) - \gamma(s_{2,k})| < |s_{1,k} - s_{2,k}|.$$

Since $s_{1,k}, s_{2,k} \in [0, S]$, up to a subsequence (not relabeled) we may assume that $s_{1,k} \to s_1$, $s_{2,k} \to s_2$. Hence we have

$$\frac{|\gamma(s_{1,k}) - \gamma(s_{2,k})|}{|s_{1,k} - s_{2,k}|} < \frac{1}{c_k}, \quad c_k \to \infty \text{ as } k \to \infty. \tag{4.20}$$

We pass to the limit as $k \to \infty$ in (4.20): if $s_1 \ne s_2$, we obtain $\gamma(s_1) = \gamma(s_2)$, in contradiction with the fact that γ is simple and $S < \tilde{S}$; if instead $s_1 = s_2$, we arrive at $|\gamma'(s_1)| = 0$, in contradiction with $|\gamma'(s)| = 1$ for all $s \in [0, \tilde{S}]$. Hence (4.19), i.e. (4.18), holds.

Let now $u \in BV(\Omega; \Sigma)$. Since $u \in \Sigma$, $s = \sigma \circ u$ is well defined. As is well-known (see e.g. [22, Theorem 2.3]), there exists $\hat{\sigma} : \mathbb{R}^2 \to \mathbb{R}$ such that $\hat{\sigma}$ is Lipschitz (with the same constant C of σ) and $\hat{\sigma}|_\Sigma = \sigma$. Using that

$$|D\hat{\sigma}(u(x))| \le C|Du(x)| \quad \text{for all } u \in C^\infty(\Omega; \mathbb{R}^2)$$

and arguing exactly as in the first lines of the proof of Theorem 3.96 in [2], we obtain that $\hat{\sigma} \circ u \in BV(\Omega; \mathbb{R})$. Since $u \in \Sigma$ and $\hat{\sigma}|_\Sigma = \sigma$, we have $s = \hat{\sigma} \circ u$ and the proof is complete. □

We are now ready to prove the uniqueness part of Theorem 4.2.

Proof of Theorem 4.2: uniqueness. Let u be a solution to problem (1.7) in the sense of Definition 4.1. We let $s_0 = \sigma(u_0)$ with σ as in Lemma 4.7. Since σ is Lipschitz, s_0 is measurable and belongs to $L^\infty(\Omega; \mathbb{R})$. In view of Lemma 4.7, for a.e. $t > 0$ the function $s(t) := \sigma(u(t))$ belongs to $BV(\Omega; \mathbb{R})$ and, of course, $u(t, x) = \gamma(s(t, x))$. We will argue that s is a solution to problem 4.1 with initial datum s_0; since such solution s is unique in view of Theorem 4.4 and $u = \gamma(s)$, this implies the desired uniqueness result.

Since $|Ds| \le C|Du|$ (cf. Lemma 4.7), we have $s \in L^1_{loc}([0, \infty); BV(\Omega; \mathbb{R}))$. The other properties in (4.2) follow directly from those of u; in particular, $s(0) = s_0$. We have

$$s_t = (\gamma'(s)s_t \cdot \gamma'(s)) = u_t \cdot T(u) = \operatorname{div}(Z\, T(u)) \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}), \tag{4.21}$$

where we have used (iv) in Definition 3.6. Therefore, letting

$$\eta := Z\,T(u) \in L^\infty(\Omega;\mathbb{R}), \tag{4.22}$$

(4.21) implies that $\eta \in X_2(\Omega;\mathbb{R})$ and that (4.3) holds. In addition, $\|\eta\|_\infty \le \|Z\|_\infty \le 1$ and (4.5) holds in view of (v) in Definition 4.1. It remains to check (4.4). In view of (ii) in Theorem 4.4, it suffices to show that

$$(Z\,T(u), Ds) \overset{(4.22)}{=} (\eta(t), Ds(t)) = |Ds(t)| \quad \text{in } \mathcal{M}(\Omega) \text{ for a.e. } t > 0. \tag{4.23}$$

We argue for a.e. $t > 0$ and we omit the dependence on t for notational convenience. Since $u = \gamma(s)$, we have

$$(\gamma^j(s))' s_t = u^j_t = \operatorname{div} z^j + \mu^j, \quad j = 1, 2, \tag{4.24}$$

where $\mu = (\mu^1, \mu^2)$ is defined in (3.9). Since $z_i = (z_i \cdot T(u))T(u)$ for $i = 1, ..., m$ (see (i) in Definition 3.6), we have

$$Z = (Z\,T(u)) \otimes T(u) \overset{(4.22)}{=} \eta \otimes \gamma'(s),$$

so that (4.24) may be rewritten as

$$(\gamma^j(s))' s_t = \operatorname{div}\big(\eta(\gamma^j)'(s)\big) + \mu^j, \quad j = 1, 2. \tag{4.25}$$

On the other hand,

$$\begin{aligned}(\gamma^j(s))' s_t &\overset{(4.21)}{=} \big((\gamma^j)'(s)\big)^* \operatorname{div}(Z\,T(u)) \overset{(4.22)}{=} \big((\gamma^j)'(s)\big)^* \operatorname{div}\eta \\ &\overset{(2.7)}{=} \operatorname{div}\big((\gamma^j)'(s)\,\eta\big) - (\eta, D(\gamma^j)'(s)), \quad j = 1, 2.\end{aligned} \tag{4.26}$$

Combining (4.25) and (4.26) we obtain

$$-(\eta, D(\gamma^j)'(s)) = \mu^j, \quad j = 1, 2. \tag{4.27}$$

Now, using Lemma 4.6 with $f = (\gamma^j)'$ and $w = s$, we deduce that

$$\begin{aligned}\mu^j &\overset{(4.27)}{=} -(\eta, D(\gamma^j)'(s)) \\ &= \Theta(\eta, Ds)\Big(-(\gamma^j)''(s)|\tilde{D}s| - \big((\gamma^j)'(s_+) - (\gamma^j)'(s_-)\big)\,\mathcal{H}^{m-1}\llcorner J_s\Big)\end{aligned} \tag{4.28}$$

for $j = 1, 2$. On the other hand,

$$\mu \overset{(4.9)}{=} -\gamma''(s)|\tilde{D}s| - \big(\gamma'(s_+) - \gamma'(s_-)\big)\,\mathcal{H}^{m-1}\llcorner J_s. \tag{4.29}$$

From (4.29) and (4.28) we obtain

$$\Theta(\eta, Ds) = 1 \quad |Ds|\text{-a.e.}, \tag{4.30}$$

i.e.,

$$\int_E \mathrm{d}|Ds| \overset{(4.30)}{=} \int_E \Theta(\eta, Ds)\,\mathrm{d}|Ds| = \int_E \mathrm{d}(\eta, Ds)$$

for any Borel set $E \subset \Omega$. Hence (4.23) holds and the proof is complete. □

5. The notion of solution: the case $n \geq 2$

In this section, we return to the first point of interest in this paper: to find an appropriate interpretation to equation (1.4). We look, however, to the general n-dimensional case, assuming that:

(H_n) Σ is an $(n-1)$-dimensional smooth and oriented Riemannian manifold embedded in $\mathbb{R}^n$; furthermore, for any pair $u_-, u_+ \in \Sigma$ there exists a unique (up to the orientation) arc-length parametrized geodesic α in Σ connecting u_- and u_+; we let

$$T(\alpha(s)) := \alpha'(s) \tag{5.1}$$

and we use the convention that

$$u_+ \text{ follows } u_- \text{ along the orientation of } \alpha. \tag{5.2}$$

Using the comprehensive form

$$u_t - \mathrm{div} Z = \underline{\mu},$$

we aim to introduce an interpretation to the right-hand side $\underline{\mu}$ which is consistent with the analysis performed both here and in [12, 13, 14] and which, of course, recovers

$$\underline{\mu} = \frac{1}{|Du|}(DN(u) : Du)N(u) = \left(\overline{\nabla} N(u) Du : \frac{Du}{|Du|}\right) N(u) \tag{5.3}$$

on smooth functions with nonzero gradient. First of all, since N is smooth, in view of (5.3) it is natural to assume that $\underline{\mu}$ is absolutely continuous with respect to $|Du|$, so that we may represent it as

$$\underline{\mu} = \underline{\tilde{\mu}}|\tilde{D}u| + \underline{\mu}^j |D^j u| \tag{5.4}$$

for suitable $|\tilde{D}u|$-, resp. $|D^j u|$-, integrable functions $\underline{\tilde{\mu}}$, resp. $\underline{\mu}^j$. Let us first consider the density of the diffuse part, $\underline{\tilde{\mu}}$. On one hand, since u is approximately continuous on the support of $|\tilde{D}u|$, we expect that the constraint $u \in \Sigma$ is unessential to the representation of $\underline{\tilde{\mu}}$. On the other hand, since the measures Du and $DN(u)$ are both absolutely continuous with respect to $|Du|$, and $\overline{\nabla} N(u)$ is bounded, we have

$$F_u := \frac{DN(u)}{|Du|} : \frac{Du}{|Du|} \in L^1(\Omega; |Du|) \tag{5.5}$$

(here ν/μ denotes the density of ν with respect to μ). In addition, F_u coincides with $\frac{1}{|Du|^2}(DN(u) : Du)$ on smooth functions with nonzero gradient. Therefore, recalling (5.3) and (5.4), we expect that

$$\underline{\tilde{\mu}} = F_u N(u). \tag{5.6}$$

In order to identify the natural candidate to represent $\underline{\mu}^j$, we alternatively resort on two observations. On one hand, for manifold-valued mappings u, geodesics connecting u_+ and u_- are the natural paths on which the jump part is evaluated: for instance, the relaxation of $\int_\Omega |Du| \mathrm{d}x$ coincides, on J_u, with the geodesic distance between u_+ and u_- (see e.g. [1, 21, 28]). On the other hand, along a geodesic α we have (using that α'' is parallel to N)

$$(\overline{\nabla} N(\alpha)\alpha' \cdot \alpha')N(\alpha) \quad = \quad \left((N(\alpha))' \cdot \alpha'\right) N(\alpha) = -(N(\alpha)\cdot \alpha'')N(\alpha) = -\alpha''.$$

Therefore, on J_u it is natural to replace $\left(\frac{DN(u)}{|Du|} : \frac{Du}{|Du|}\right) N(u)$ by minus the difference quotient of the tangential unit vector-field; i.e., recalling (5.1), to let

$$\underline{\mu}^j = \frac{T(u_-) - T(u_+)}{|u_+ - u_-|}. \tag{5.7}$$

Plugging (5.6) and (5.7) into (5.4) and recalling that $|D^j u| = |u_+ - u_-|\mathcal{H}^{m-1} \llcorner J_u$, we obtain what we expect to be the appropriate definition for the right-hand side of (1.4):

$$\underline{\mu} := F_u N(u)|\tilde{D}u| + (T(u_-) - T(u_+))\ \mathcal{H}^{m-1} \llcorner J_u, \tag{5.8}$$

where T is defined in (5.1) and F_u is defined in (5.5). Arguing as in Remark 3.3, it is easily seen that the right-hand side of (5.8) is uniquely determined.

The measure $\underline{\mu}$ in (5.8) coincides both with the measure μ defined in (3.9) (when $n = 2$) and with the characterization given in [14] (when $\Sigma = \mathbb{S}^{n-1}_+$):

Proposition 5.1. *Let $u \in BV(\Omega; \Sigma)$ and let $\underline{\mu}$ be defined by* (5.8)*.*

(i) *If $n = 2$ and Σ is as in (H), then $\underline{\mu} = \mu$, where μ is defined in* (3.9)*;*

(ii) *if $\Sigma = \mathbb{S}^{n-1}_+$, then $\underline{\mu} = u|\tilde{D}u| + \frac{u^*}{|u^*|}|D^j u|$.*

Proof. Concerning (i), the jump parts of $\underline{\mu}$ and μ are identical; hence we only need to prove that

$$F_u = \kappa(u) \quad |\tilde{D}u|\text{-a.e. in } \Omega. \tag{5.9}$$

We obviously have

$$\frac{Du}{|Du|}\left(|\tilde{D}u| + |D^j u|\right) \quad = \quad Du = \tilde{D}u + D^j u = \frac{\tilde{D}u}{|\tilde{D}u|}|\tilde{D}u| + \frac{D^j u}{|D^j u|}|D^j u|.$$

In particular,

$$\frac{Du}{|Du|} = \frac{\tilde{D}u}{|\tilde{D}u|} \quad |\tilde{D}u|\text{-a.e.}. \tag{5.10a}$$

By the same argument,

$$\frac{DN(u)}{|Du|} = \frac{\tilde{D}N(u)}{|\tilde{D}u|} \quad |\tilde{D}u|\text{-a.e.}. \tag{5.10b}$$

In addition, since N is smooth, by the chain rule

$$\tilde{D}N(u) = \overline{\nabla}N(u)\tilde{D}u \quad |\tilde{D}u|\text{-a.e.}. \tag{5.10c}$$

Therefore

$$F_u \quad \overset{(5.5)}{=} \quad \frac{DN(u)}{|Du|} : \frac{Du}{|Du|} \overset{(5.10)}{=} \frac{\overline{\nabla}N(u)\tilde{D}u}{|\tilde{D}u|} : \frac{\tilde{D}u}{|\tilde{D}u|} \quad |\tilde{D}u|\text{-a.e.}. \tag{5.11}$$

By Lemma 4.7, $s = \sigma(u) \in BV(\Omega; [0, S_0])$. Hence $u = \gamma(s)$ and, by the chain rule,

$$\tilde{D}u = \gamma'(s) \otimes \tilde{D}s \quad \text{and} \quad |\tilde{D}u| = |\tilde{D}s|. \tag{5.12}$$

Therefore, using that $\frac{\tilde{D}s}{|\tilde{D}s|} : \frac{\tilde{D}s}{|\tilde{D}s|} = 1$ $|\tilde{D}s|$-a.e. (see e.g. [2, Theorem 1.29]),

$$\begin{aligned} F_u &\overset{(5.11),(5.12)}{=} \frac{\overline{\nabla} N(\gamma(s))\gamma'(s) \otimes \tilde{D}s}{|\tilde{D}s|} : \frac{\gamma'(s) \otimes \tilde{D}s}{|\tilde{D}s|} \\ &= \left(\overline{\nabla} N(\gamma(s))\gamma'(s) \cdot \gamma'(s)\right) \frac{\tilde{D}s}{|\tilde{D}s|} \cdot \frac{\tilde{D}s}{|\tilde{D}s|} \\ &= \overline{\nabla} N(u)T(u) \cdot T(u) \overset{(3.4)}{=} \kappa(u) \quad |\tilde{D}u|\text{-a.e.}, \end{aligned}$$

which proves (5.9).

Concerning (ii), fix e.g. $N(u) = -u$. Again since $\frac{Du}{|Du|} : \frac{Du}{|Du|} = 1$ $|Du|$-a.e., the identification of the diffuse part is immediate. In order to identify the jump part, using the rotational invariance of the equation we may assume without losing generality that

$$u_- = (\sin\theta_-, \cos\theta_-, 0, \dots, 0), \quad u_+ = (\sin\theta_+, \cos\theta_+, 0, \dots, 0), \tag{5.13}$$

so that the geodesic connecting u_- to u_+ belongs to $\Pi := \{(u^1, u^2, 0, \dots, 0) : \ u^1, u^2 \in \mathbb{R}^2\}$. Letting

$$u^\perp := (u^2, -u^1, 0, \dots, 0) \quad \text{for all } u = (u^1, u^2, 0, \dots, 0) \in \Pi$$

and fixing e.g. the clockwise orientation of α (which by (5.13) and (5.2) means that $\theta_- < \theta_+$), we have $T(u) = u^\perp$ and the argument becomes identical to the one in Remark 3.5. $\square$

Remark 5.2. With $\underline{\mu}$ as in (5.8), one can write down a complete definition of solution in the spirit of [14]. In particular, u and Z shall be such that

$$\begin{gathered} u \in C([0,\infty); L^2(\Omega;\mathbb{R}^n)) \cap L^1_{loc}([0,\infty); BV(\Omega;\mathbb{R}^n)), \quad u_t \in L^2_{loc}((0,\infty); L^2(\Omega;\mathbb{R}^n)), \\ Z \in L^\infty(Q;\mathbb{R}^{mn}), \quad \|Z\|_\infty \le 1, \\ z_i \cdot N(u) = u_t \cdot N(u) = 0 \quad \text{a.e. in } Q \text{ for all } i = 1, \dots, m, \\ u_t(t) - \operatorname{div} Z(t) = \underline{\mu}(t) \quad \text{as measures for a.e. } t. \end{gathered}$$

In this case, arguing exactly as in the proof of Proposition 3.7, one finds that Z is such that

$$(Z, DN(u)) \quad = \quad F_u\,|\tilde{D}u| + (N(u))^* \cdot (T(u_-) - T(u_+))\,\mathcal{H}^{m-1} \llcorner J_u. \tag{5.14}$$

In particular $(Z, Du) = |u^*|\,|Du|$ when $\Sigma = \mathbb{S}^{n-1}_+$, which coincides with the characterization given in [14, Proposition 3.5].

Remark 5.3. If both u and Z are smooth, (5.14) coincides with

$$|Du|(Z : DN(u)) = DN(u) : Du \quad \text{in } \Omega. \tag{5.15}$$

In general, it is not obvious that (5.15) yields the expected characterization of Z, $Z : Du = |Du|$. However, this is the case if Σ has positive principal curvatures (up to a change of orientation) and if $Z = fDu$ for some scalar function f. Indeed, the assertion is trivial if $|Du| = 0$. Otherwise, we have

$$(f|Du| - 1)Du : (\overline{\nabla} N(u)Du) = |Du|(Z : DN(u)) - DN(u) : Du \overset{(5.15)}{=} 0,$$

which implies that $f = \frac{1}{|Du|}$ since $|Du| > 0$ and $\overline{\nabla} N$ is positive definite.

It is clear that our analysis leaves the question of existence in the $(n-1)$-dimensional case open. In turn, we believe that an answer to this question will require lower semi-continuity results for functionals involving the second fundamental form. For instance, based on the arguments developed in [14], we expect that the necessity of lower bounds on

$$\mathcal{I}(u) = \inf\left\{\liminf_{n\to\infty}\int_\Omega \frac{1}{|Du_n|}(DN(u_n):Du_n)(N(u))^*\cdot N(u_n):\ u_n\in\Sigma,\ u_n\to u \text{ in } L^1(\Omega)\right\}$$

will naturally emerge from an approximation scheme in order to identify the limits μ and Z (cf. e.g. (5.14)). In this respect, we conjecture that

$$\mathcal{I}(u) \geq F_u|\tilde{D}u| + (N(u))^*(T(u_-)-T(u_+))\mathcal{H}^{n-1}\llcorner J_u \tag{5.16}$$

when Σ is as in (H_n) and, in addition, Σ has positive principal curvatures (up to a change of orientation) and is such that such that $N^*\cdot N$ is positive. When $\Sigma=\mathbb{S}^{N-1}_+$, (5.16) has been shown to be true in [14]. However, in general, it seems that (5.16) does not follow from known lower semi-continuity (in fact, relaxation) theories unless an additional *isotropy condition* is satisfied (see [28]), hence it is itself an open question.

Appendix A.

Appendix A.1. The formal derivation of (1.1) *and* (1.4)

Consider the total variation functional with the constraint that u takes value into Σ:

$$E(u) = \int_\Omega \mathrm{d}|Du|, \quad u:\Omega\to\Sigma.$$

Consider a smooth map u such that $|Du(t,x)|\neq 0$ for all $(t,x)\in Q$. The formal gradient flow of E with respect to the L^2-distance is given by

$$(u_t,v)_{L^2} = -\langle\partial E(u),v\rangle \quad \text{for all } v\in T_u\Sigma \tag{A.1}$$

$$u \in \Sigma. \tag{A.2}$$

Note that, because of (A.2), the class of admissible variations in (A.1) is limited to $v\in T_u\Sigma$, where $T_u\Sigma$ is the tangent plane to Σ at u. Since

$$\langle\partial E(u),v\rangle = \int_\Omega \frac{Du}{|Du|}:Dv\,\mathrm{d}x = -\int_\Omega \operatorname{div}\left(\frac{Du}{|Du|}\right)\cdot v\,\mathrm{d}x \quad \text{for all } v\in T_u\Sigma,$$

(A.1) yields

$$u_t - \operatorname{div}\left(\frac{Du}{|Du|}\right) = \lambda(u)N(u). \tag{A.3}$$

Because of (A.2), we have

$$N(u)\cdot u_t = 0 \quad \text{and} \quad N(u)\cdot D_i u = 0 \quad \text{for all } i=1,\dots,m. \tag{A.4}$$

Multiplying (A.3) by $N(u)$, we obtain

$$\begin{aligned}\lambda(u) &\overset{(A.3)}{=} N(u)\cdot u_t - N(u)\cdot\operatorname{div}\left(\frac{Du}{|Du|}\right)\\ &\overset{(A.4),(2.7)}{=} -\operatorname{div}\left(\frac{N(u)Du}{|Du|}\right) + \left(DN(u):\frac{Du}{|Du|}\right)\\ &\overset{(A.4)}{=} \frac{1}{|Du|}\left(DN(u):Du\right),\end{aligned}$$

whence (1.4). If $\Sigma\subseteq\mathbb{S}^{n-1}$, then (say) $N(u)=-u$, hence $\lambda(u)=-|Du|$ and (1.1) follows.

Appendix A.2. Proof of Lemmas 3.1 and 3.8

Proof of Lemma 3.1. We preliminarily note that if v is such that $|v| = 1$ and Z is a 2×2 matrix, simple computations show that

$$v \cdot Zv + v^\perp \cdot Zv^\perp = \operatorname{Tr} Z, \quad \text{where } v^\perp = (v^2, -v^1). \tag{A.5}$$

We recall that κ is defined through

$$\gamma''(s) = -\kappa(\gamma(s))\, N(\gamma(s)). \tag{A.6}$$

Since $N(\gamma(s)) \cdot \gamma'(s) \equiv 0$, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\big(N(\gamma(s)) \cdot \gamma'(s)\big) = \left(\frac{\mathrm{d}}{\mathrm{d}s} N(\gamma(s))\right) \cdot \gamma'(s) + N(\gamma(s)) \cdot \gamma''(s). \tag{A.7}$$

Therefore

$$\begin{aligned} \kappa(u) \quad &= \quad \kappa(\gamma(s))N(\gamma(s)) \cdot N(\gamma(s)) \overset{\text{(A.6)}}{=} -\gamma''(s) \cdot N(\gamma(s)) \\ &\overset{\text{(A.7)}}{=} \quad \gamma'(s) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}s} N(\gamma(s))\right) = \gamma'(s) \cdot \big(\overline{\nabla} N(\gamma(s))\, \gamma'(s)\big) \\ &= \quad T(u) \cdot \big(\overline{\nabla} N(u)\, T(u)\big), \end{aligned} \tag{A.8}$$

which coincides with (3.4), and using (A.5) we obtain (3.5). On the other hand, in view of (A.4) we have

$$D_i u = |D_i u| T(u) \quad \text{for all } i = 1, \dots, m. \tag{A.9}$$

Then

$$\begin{aligned} DN(u) : Du \quad &\overset{\text{(2.2)}}{=} \quad \sum_{i=1}^{m} D_i u \cdot D_i N(u) = \sum_{i=1}^{m} D_i u \cdot \overline{\nabla} N(u) D_i u \\ &\overset{\text{(A.9)}}{=} \quad \sum_{i=1}^{m} |D_i u|^2 T(u) \cdot \overline{\nabla} N(u) T(u) \\ &= \quad |Du|^2 (T(u) \cdot (\overline{\nabla} N(u) T(u))). \end{aligned} \tag{A.10}$$

Combining (A.8) and (A.10) we obtain (3.6). $\square$

Proof of Lemma 3.8. We recall that, by assumption and using (A.4),

$$z_i = |z_i| T(u) \quad \text{and} \quad z_i \cdot D_i u = |z_i||D_i u| \quad \text{for all } i = 1, \dots, m. \tag{A.11}$$

Then

$$\begin{aligned} Z : DN(u) \quad &\overset{\text{(2.2)}}{=} \quad \sum_{i=1}^{m} z_i \cdot \overline{\nabla} N(u) D_i u \\ &\overset{\text{(A.9),(A.11)}}{=} \quad \left(\sum_{i=1}^{m} |z_i||D_i u|\right) \big(T(u) \cdot \overline{\nabla} N(u) T(u)\big) \\ &\overset{\text{(A.8),(A.11)}}{=} \quad \kappa(u)(Z : Du). \end{aligned}$$

Since by assumption $Z : DN(u) = \kappa(u)|D(u)|$, provided that $\kappa(u) \neq 0$ we conclude that $(Z : Du) = |Du|$. $\square$

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References

[1] R. Alicandro, A. Corbo Esposito, C. Leone, Relaxation in BV of integral functionals defined on Sobolev functions with value in the unit sphere. *J. Convex Anal.* **14** (2007), 69–98.

[2] L. Ambrosio, N. Fusco, D. Pallara, *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Carendon Press, Oxford University Press, New York, 2000.

[3] F. Andreu, V. Caselles, J. M. Mazón, *Parabolic Quasilinear Equations Minimizing Linear Growth Functionals*. Progr. Math., 223, Birkhauser, Boston, 2004.

[4] G. Anzellotti, Pairings between measures and bounded functions and compensated compactness. *Ann, Mat. Pura Appl.* **135** (1983), 293–318.

[5] J. Calvo, Analysis of a class of degenerate parabolic equations with saturation mechanisms. *SIAM J. Math. Anal.* **47** (2015), 2917–2951.

[6] V. Caselles, On the entropy conditions for some flux limited diffusion. *J. Differential Equations* **250** (2011), 3311–3348.

[7] G. Q. Chen, H. Frid, Divergence-measure fields and hyperbolic conservation laws. *Arch. Ration. Mech, Anal.* **147** (1999), 89–118.

[8] R. Dal Passo, L. Giacomelli, S. Moll, Rotationally symmetric 1-harmonic maps from D^2 to S^2. *Calc. Var. Partial Differential Equations* **32** (2008), 533–554.

[9] A. DeSimone, P. Podio-Guidugli, On the continuum theory of deformable ferromagnetic solids. *Arch. Rational Mech. Anal.* **136** (1996), 201–233.

[10] J. Eells, J. H. Sampson, Harmonic mappings of Riemannian manifolds. *Amer. J. Math.* **86** (1964), 109–160.

[11] L. Giacomelli, Finite speed of propagation and waiting-time phenomena for degenerate parabolic equations with linear growth Lagrangian. *SIAM J. Math. Anal.* **47** (2015), 2426–2441.

[12] L. Giacomelli, J. M. Mazón, S. Moll, The 1-harmonic flow with values into $\mathbb{S}^1$. *SIAM J. Math. Anal.* **45** (2013), 1723–1740.

[13] L. Giacomelli, J. M. Mazón, S. Moll, Solutions to the 1-harmonic flow with values into a hyper-octant of the N-sphere. *Appl. Math. Lett.* **26** (2013), 1061–1064.

[14] L. Giacomelli, J. M. Mazón, S. Moll, Solutions to the 1-harmonic flow with values into a hyperoctant of the N-sphere. *Anal. PDE* **7** (2014), 627–671.

[15] L. Giacomelli, S. Moll, Rotationally symmetric 1-harmonic flows from D^2 to S^2: local well-posedness and finite time blowup. *SIAM J. Math. Anal.* **42** (2010), 2791–2817.

[16] Y. Giga, R. Kobayashi, On constrained equations with singular diffusivity. *Methods Appl. Anal.* **10** (2003), 253–277.

[17] Y. Giga, H. Kuroda, On breakdown of solutions of a constrained gradient system of total variation. *Bol. Soc. Parana. Mat.* **22** (2004), 9–20.

[18] e Y. Giga, Y. Kashima, N. Yamazaki, Local solvability of a constrained gradient system of total variation. *Abstr. Appl. Anal.* **8** (2004), 651–682.

[19] Y. Giga, H. Kuroda, N. Yamazaki, An existence result for a discretized constrained gradient system of total variation flow in color image processing. *Interdiscip. Inform. Sci.* **11** (2005), 199–204.

[20] Y. Giga, H. Kuroda, N. Yamazaki, *Global solvability of constrained singular diffusion equation associated with essential variation, Free boundary problems*, 209–218. Internat. Ser. Numer. Math. 154, Birkhäuser, Basel, 2007.

[21] M. Giaquinta, D. Mucci, Relaxation results for a class of functionals with linear growth defined on manifold constrained mappings. *J. Convex Anal.* **15** (2008), 719–751.

[22] e J. Heinonen, *Lectures on Lipschitz analysis*. Report. University of Jyväskylä, Department of Mathematics and Statistics, 100. University of Jyväskylä, Jyväskylä, 2005.

[23] R. van der Hout, Flow alignment in nematic liquid crystals in flows with cylindrical symmetry. *Differential Integral Equations* **14** (2001), 189–211.

[24] N. Hungerbühler, *Heat flow into spheres for a class of energies. Variational problems in Riemannian geometry*, 45–65. Progr. Nonlinear Differential Equations Appl., 59, Birkhäuser, Basel, 2004.

[25] R. Kobayashi, J. A. Warren, W. C. Carter, A continuum model of grain boundaries. *Phys. D* **140** (2000), 141–150.

[26] F. Lin, C. Wang, *The analysis of harmonic maps and their heat flows.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008. xii+267 pp..

[27] P.B. Mucha, M. Muszkieta, P. Rybka, Two cases of squares evolving by anisotropic diffusion. *Adv. Differential Equations* **20** (2015), 773–800.

[28] D. Mucci, Relaxation of isotropic functionals with linear growth defined on manifold constrained Sobolev mappings. *ESAIM Control Optim. Calc. Var.* **15** (2009), 295–321.

[29] G. Sapiro, *Geometric partial differential equations and image analysis.* Cambridge University Press, 2001.

[30] B. Tang, G. Sapiro, V. Caselles, Diffusion of general data on non-flat manifolds via harmonic maps theory: The direction diffusion case. *Int. J. Comput. Vis.* **36** (2000), 149–161.

[31] B. Tang, G. Sapiro, V. Caselles, Color image enhancement via chromaticity diffusion. *IEEE Trans. Image Process.* **10** (2001), 701–707.