WELL-POSEDNESS OF TWO PSEUDO-PARABOLIC PROBLEMS FOR ELECTRICAL CONDUCTION IN HETEROGENEOUS MEDIA

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Abstract. We prove a well-posedness result for two pseudo-parabolic problems, which can be seen as two models for the same electrical conduction phenomenon in heterogeneous media, neglecting the magnetic field. One of the problems is the concentration limit of the other one, when the thickness of the dielectric inclusions goes to zero. The concentrated problem involves a transmission condition through interfaces, which is mediated by a suitable Laplace-Beltrami type equation.

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1. Introduction

Composite materials are experiencing an increasing popularity in different fields of science and applications. In particular, they have a relevant role in the study of electrical conduction. A typical geometry is the one of an electric conductor in which inclusions of a possibly different conductive material are inserted, separated from the surrounding by a dielectric layer, also called membrane. The latter is in principle thick, i.e. N-dimensional, but for reasons of simplicity it is often replaced with a thin, i.e. $(N-1)$-dimensional interface, also in view of the homogenization limit.

It is therefore necessary to investigate the behavior of the composite medium when the thickness η of the insulator vanishes, thus reducing the membrane to a surface. Obviously, a suitable scaling of the relevant physical quantities is required in this process. Some of the authors in [4] performed the limit $\eta \to 0$, assuming essentially that the relative dielectric constant ε_r scales in such a way that it tends to zero as the thickness of the membrane η. In this case, it has been proved in [4] that in the limiting problem the current is continuous across the interfaces and the time derivative of the jump of the potential across the interfaces is proportional to the current. This setting is suitable for electrical conduction in biological tissues of alternating currents at the radiofrequency range, assuming the magnetic field to be negligible (see [2, 3, 5, 6] and the references therein).

It is clear that other scalings are possible. We investigate here a problem arising from the somewhat more traditional scaling of concentration of capacity, where in practice ε_r scales as $1/\eta$ in the limit. This fact produces, in the concentration limit, the appearance of a transmission condition across the resulting $(N-1)$-dimensional membrane ruled by the Laplace-Beltrami operator. Similar problems in the context of thermal diffusion are studied, for instance, in [30], where the authors consider a concentrated capacity model in the framework of abstract evolution equations, and in [8, 9, 10], where the concentration and the homogenization of a parabolic system involving the tangential derivatives on the interface are investigated. For a general survey on tangential operators we refer to [21, 29] and the references therein, as well as for the well-posedness in concentrated capacity problems arising in the framework of the heat conduction we may recall, for instance, [11, 25] and the references therein. Nevertheless, to the best of our knowledge, this problem has not yet been considered in the framework of electric conduction described above. It seems however that it may be physically relevant when dealing with new dielectric materials made available by recent technology (see [1]), where the dielectric constant is very high. The details and motivations of the concentration limit are worked out in a forthcoming paper [7]. However, for the reader's convenience, we provide here a brief formal derivation of the thin membrane problem starting from the thick one, in order to enlighten the connection between the two pseudo-parabolic problems addressed in this paper, on which we focus our main mathematical interest.

More precisely, the thin membrane problem consists of two elliptic equations set in the exterior and interior conductive phases. Here, the dependence on time is merely parametric. However, the two equations are coupled at the interfaces, where the potential is continuous. In addition, on these surfaces the time derivative of the potential solves an equation for the Laplace-Beltrami operator. The jump of the current across the interface acts as a source in this pseudo-parabolic problem. Let us remark that such an equation is due to the dielectric nature of the membranes. This mathematical problem appears to be rather non-standard and new in the literature. Indeed, the elliptic and pseudo-parabolic character of the differential equations imposes as a compatibility condition the fact that the global flux of the potential at each one of the interfaces must be essentially assigned. Problems with a total flux boundary condition can be encountered also in other contexts (see e.g. [17, 23, 24] and the references quoted therein). The novelty in our case is its coupling with an

evolutive problem, and the fact that the solution on the interfaces is not required to be constant. Moreover, the dependence on time and therefore the presence of initial data raise some nontrivial issues.
The necessary presence at the same time of the total flux condition, the evolutive pseudo-parabolic character and the Laplace-Beltrami operator creates technical problems, starting even from the choice of suitable functional spaces, for which we have been unable to find any specific reference in the literature.
Even the mathematical model for thick membranes shares the interesting features of the concentrated scheme. In addition, it exhibits the pseudo-parabolic equation in the bulk of the domain, that is in the dielectric phase, which makes the degeneracy possibly stronger. We recall that pseudo-parabolic problems can appear in several different contexts. For instance, they can model fluid flow in porous media or heat conduction in two-temperature systems and are also used to regularize ill-posed transport problems (see [13, 14, 15, 16, 18, 19, 20, 26, 27, 28, 31] and the references therein). However, also for this problem, to the best of our knowledge, there are no results of well-posedness in the existing literature.
We present here a proof of the well-posedness of both problems introduced above, relying on two different approaches. In both cases, we see that the compatibility condition mentioned above and, independently, the parametric dependence on time in the conductive phases prevent the solution to attain the prescribed initial data other than in the sense of electrical currents, which is actually the correct one from a physical viewpoint (see Remarks 2.4 and 4.6). The problem with a N-dimensional dielectric phase is treated by approximation, introducing a suitable sequence of coercive problems. Our approach to the problem with $(N-1)$-dimensional interfaces is based on a fixed point argument, and must be carefully tuned in such a way that our contractive operator preserves the relevant compatibility conditions. This is made possible by Proposition 2.6 which, though related to results in [24], seems to be new in its present formulation, and allows some regularity estimates (see Remark 4.4).
The paper is organized as follows: in Section 2 we present our main results and the geometrical setting for both problems, together with a formal motivation of the concentration procedure. In Section 3 we give proofs for the problem with thick membranes, and in Section 4 for the one with thin interfaces.

2. Problems and Main Results

2.1. **Tangential derivatives.** Let ϕ be a $\mathcal{C}^2$-function, $\mathbf{\Phi}$ be a $\mathcal{C}^2$-vector function and S a smooth surface in $\mathbb{R}^N$ with normal unit vector n. We recall that the tangential gradient of ϕ is given by

$$\nabla^B \phi = \nabla\phi - (n \cdot \nabla\phi)n \tag{2.1}$$

and the tangential divergence of Φ is given by

$$\begin{aligned}\operatorname{div}^B \mathbf{\Phi} &= \operatorname{div} \mathbf{\Phi} - (n \cdot \nabla \mathbf{\Phi}_i)n_i - (\operatorname{div} n)(n \cdot \mathbf{\Phi}) \\ &= \operatorname{div}^B (\mathbf{\Phi} - (n \cdot \mathbf{\Phi})n) = \operatorname{div} (\mathbf{\Phi} - (n \cdot \mathbf{\Phi})n)\ ,\end{aligned} \tag{2.2}$$

where, taking into account the smoothness of S, the normal vector n can be naturally defined in a small neighborhood of S as $\frac{\nabla d}{|\nabla d|}$, where d is the signed distance from S.

Moreover, we define the Laplace-Beltrami operator as

$$\Delta^B \phi = \operatorname{div}^B(\nabla^B \phi)\,. \tag{2.3}$$

Finally, we recall that on a regular surface S with no boundary (i.e. when $\partial S = \emptyset$) we have

$$\int_S \operatorname{div}^B \mathbf{\Phi} \, \mathrm{d}\sigma = 0\,, \tag{2.4}$$

for any $\mathcal{C}^2$-vector function $\mathbf{\Phi}$.

2.2. **The problem with thick membranes.** We consider first the problem where the insulating membranes have a positive thickness, as displayed in Fig. 1. To this

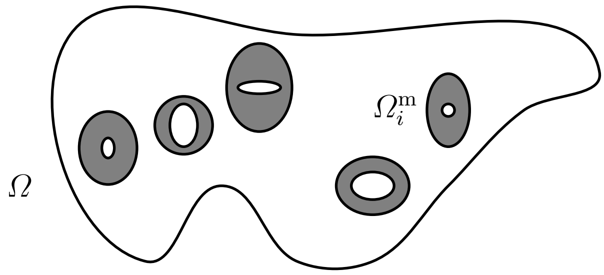

FIGURE 1. The geometrical setting in the case of thick membranes: here Ω^{m} is the gray region, while Ω^{bulk} is the white region. The connected components of Ω^{m} are labelled as Ω_i^{m}.

purpose, let Ω be an open connected bounded subset of $\mathbb{R}^N$ and let us write Ω as $\Omega = \Omega^{\mathrm{bulk}} \cup \Omega^{\mathrm{m}} \cup \partial\Omega^{\mathrm{m}}$, where Ω^{bulk} and Ω^{m} are two disjoint open subsets of Ω. More precisely, Ω^{m} is the union of the isolating membranes with boundary $\partial\Omega^{\mathrm{m}}$, while we assume that $\Omega^{\mathrm{bulk}} = \Omega^{\mathrm{int}} \cup \Omega^{\mathrm{out}}$, where Ω^{out}, Ω^{int} correspond to the two conductive regions separated by the membrane Ω^{m} and we denote $\partial\Omega^{\mathrm{m}} = (\partial\Omega^{\mathrm{int}} \cup \partial\Omega^{\mathrm{out}}) \cap \Omega$. We assume that Ω^{out} is connected and that the boundary $\partial\Omega^{\mathrm{m}}$ is Lipschitz. Finally, given $T > 0$, we denote by $\Omega_T = \Omega \times (0, T)$. More in general, for any spatial domain G, we denote $G_T = G \times (0, T)$.
Let λ_{int}, λ_{out}, α be strictly positive constants and set $A(x) = \lambda_{\mathrm{int}}$ in Ω^{int}, $A(x) = \lambda_{\mathrm{out}}$ in Ω^{out}, $A(x) = 0$ in Ω^{m}, $B(x) = 0$ in $\Omega^{\mathrm{int}} \cup \Omega^{\mathrm{out}}$, $B(x) = \alpha$ in Ω^{m}.
For a given function $\overline{u}_0 \in H^1(\Omega^{\mathrm{m}})$, we will denote by $\widetilde{u}_0$ an extension to the whole of Ω of $\overline{u}_0$, such that $\widetilde{u}_0 \in H^1_0(\Omega)$ and $\|\widetilde{u}_0\|_{H^1_0(\Omega)} \leq \gamma \|\overline{u}_0\|_{H^1(\Omega^{\mathrm{m}})}$. Finally, let $f \in L^2(\Omega_T)$.
We consider the problem for u given by

$$-\operatorname{div}(A\nabla u + B\nabla u_t) = f\,, \qquad \text{in } \Omega_T; \tag{2.5}$$

$$\nabla u(x,0) = \nabla \overline{u}_0(x)\,, \qquad \text{in } \Omega^{\mathrm{m}}. \tag{2.6}$$

Definition 2.1. We say that $u(x,t) \in L^2\big(0,T;H_0^1(\Omega)\big)$ is a weak solution to problem (2.5)–(2.6) if

$$\int_0^T\!\!\int_\Omega A\nabla u\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_\Omega B\nabla u\cdot\nabla\phi_t\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\int_\Omega f\phi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega B\nabla\overline{u}_0\cdot\nabla\phi(0)\,\mathrm{d}x\,, \tag{2.7}$$

for every test function $\phi \in \mathcal{C}^\infty(\overline{\Omega}_T)$ such that ϕ has compact support in Ω for every $t \in (0,T)$ and $\phi(\cdot,T) = 0$ in Ω. □

Here, the operators div and ∇, as well as div^B and ∇^B, act only with respect to the space variable x.

First, we note that, by formally integrating in time and by using Poincaré's inequality and Gronwall's lemma, one can derive from (2.7) the energy inequality

$$\int_0^T\int_{\Omega^{\mathrm{out}}\cup\Omega^{\mathrm{int}}} |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}\tau + \sup_{t\in(0,T)}\int_{\Omega^{\mathrm{m}}}|\nabla u|^2(t)\,\mathrm{d}x \le \gamma(\|f\|^2_{L^2(\Omega)} + \|\nabla\overline{u}_0\|^2_{L^2(\Omega^{\mathrm{m}})})\,, \tag{2.8}$$

where $\gamma = \gamma(\lambda_{\mathrm{int}},\lambda_{\mathrm{out}},\alpha,\Omega^{\mathrm{m}})$. In the next proposition, we make this remark rigorous; this point is not obvious due to the characteristic feature of the problems at hands, that is the fact that they do not necessarily preserve the value of the initial data (which actually may be locally changed by a constant).

Proposition 2.2. *Let $u \in L^2\big(0,T;H_0^1(\Omega)\big)$ be a solution of problem* (2.5)–(2.6)*. Then, u satisfies the energy estimate* (2.8)*.*

Proof. Reasoning as in [22, p.158 ff.], we can prove that $\nabla u \in \mathcal{C}^0\big(0,T;L^2(\Omega^{\mathrm{m}})\big)$. Hence, by Gronwall's lemma and Poincaré's inequality, we obtain that

$$\int_0^T\int_{\Omega^{\mathrm{out}}\cup\Omega^{\mathrm{int}}} |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}\tau + \sup_{t\in(0,T)}\int_{\Omega^{\mathrm{m}}}|\nabla u|^2(t)\,\mathrm{d}x \le \gamma(\|f\|^2_{L^2(\Omega)} + \|\nabla u(0)\|^2_{L^2(\Omega^{\mathrm{m}})})\,. \tag{2.9}$$

Moreover, from the weak formulation (2.7), it follows that

$$\int_\Omega B(\nabla u(0) - \nabla\overline{u}_0)\cdot\nabla\phi\,\mathrm{d}x = 0$$

for all $\phi \in \mathcal{C}^1(\Omega)$. By the asserted continuity of $\nabla u(t)$, we infer that, up to modifying $u(t)$ by a constant $k_i(t)$ in each connected component Ω^{m}_i of Ω^{m}, we may assume that, as $t \to 0$, $u(t) \to v_i$ in $L^2(\Omega^{\mathrm{m}}_i)$. For example we may enforce the condition that $u(t)$ has zero average on each component Ω^{m}_i. Therefore, recalling that $B = 0$ in Ω^{bulk} and setting $g = v_i - \overline{u}_0$ in Ω^{m}_i, it follows that g satisfies the Neumann problem

$$-\mathrm{div}(B\nabla g) = 0\,,\qquad \text{in } \Omega^{\mathrm{m}}_i;$$

$$\frac{\partial g}{\partial\nu} = 0\,,\qquad \text{on } \partial\Omega^{\mathrm{m}}_i;$$

which implies that $v_i = \overline{u}_0 + c_i$, for $c_i \in \mathbb{R}$. Hence $\nabla u(x,0) = \nabla\overline{u}_0(x)$ a.e. in Ω^{m} and the thesis is proven. □

Clearly, estimate (2.8) and the linearity of problem (2.5)–(2.6) ensure that, if a solution does exist in $L^2\big(0,T;H^1_0(\Omega)\big)$, it is unique and depends continuously on $\nabla\overline{u}_0$. Hence, we have just to prove existence. This will be done in Section 3, proving the following general result.

Theorem 2.3. *Let A, B, f and $\overline{u}_0$ be as above. Then, for any given $T>0$, problem (2.5)–(2.6) admits a unique solution $u \in L^2\big(0,T;H^1_0(\Omega)\big)$.*

Remark 2.4. Assume, for the moment, that u has a trace for $t=0$.
Notice that, from (2.5)–(2.7), it follows that the solution u satisfies, for each connected component Ω^{m}_i of Ω^{m},

$$\int\limits_{\partial\Omega^{\mathrm{out}}\cap\partial\Omega^{\mathrm{m}}_i} \lambda_{\mathrm{out}}\frac{\partial u}{\partial\nu}\,\mathrm{d}\sigma = \int\limits_{\Omega^{\mathrm{int}}_i\cup\Omega^{\mathrm{m}}_i} f\,\mathrm{d}x\,, \tag{2.10}$$

in a weak sense (see Remark 4.1). In general, this condition with u replaced by $\overline{u}_0$ is not automatically satisfied for any choice of the initial datum $\overline{u}_0$. However, reasoning as in Proposition 2.6 with $\Gamma_i = \partial\Omega^{\mathrm{out}}\cap\partial\Omega^{\mathrm{m}}_i$, it is possible to prove that $\overline{u}_0$ can be modified by a suitable constant c_i in every Ω^{m}_i in such a way that (2.10) is fulfilled. Clearly, this does not affect the initial condition for ∇u, in accordance with the fact that problem (2.5)–(2.7) requires an initial condition only for the gradient of the solution and not for the solution itself. Therefore, the solution does not assume exactly the initial condition $\overline{u}_0$; roughly speaking, it "rearranges by itself" the prescribed initial condition by adding to $\overline{u}_0$ in each connected component of Ω^{m} the previously quoted constant c_i, in order for (2.10) to hold true.
However, without suitable assumptions on the source f with respect to the time dependence (for instance $f \in H^1\big(0,T;L^2(\Omega)\big)$), it is not possible to guarantee that u has a trace for $t=0$ in the whole of Ω, since in $\Omega^{\mathrm{bulk}}\cup\Omega^{\mathrm{m}}$ the problem (2.5) displays only a parametric dependence on t. □

2.3. **Formal concentration.** We devote this subsection to formally justify the relationship between the problem with thick membranes and the one with thin membranes. The rigorous proof of this result can be found in [7].
Assume that Ω^{m} is, indeed, a tubular neighborhood of a smooth $(N-1)$-dimensional regular surface Γ with thickness $\eta << 1$ and with a finite number of connected components strictly contained in Ω. Redefine $\Omega^{\mathrm{m}} = \Gamma_\eta$, $\Omega^{\mathrm{int}} = \Omega^{\mathrm{int}}_\eta$ and $\Omega^{\mathrm{out}} = \Omega^{\mathrm{out}}_\eta$, so that our domain becomes $\Omega = \Omega^{\mathrm{int}}_\eta \cup \Omega^{\mathrm{out}}_\eta \cup \Gamma_\eta$.
Let λ_{int}, λ_{out}, α be as in Subsection 2.2 and define $A^\eta(x) = \lambda_{\mathrm{int}}$ in $\Omega^{\mathrm{int}}_\eta$, $A^\eta(x) = \lambda_{\mathrm{out}}$ in $\Omega^{\mathrm{out}}_\eta$, $A^\eta(x) = 0$ in Γ_η, $B^\eta(x) = 0$ in $\Omega^{\mathrm{int}}_\eta\cup\Omega^{\mathrm{out}}_\eta$, $B^\eta(x) = \alpha/\eta$ in Γ_η. The choice of the scaling $1/\eta$ is designed to let the specific permittivity of the interface to blow up as $\eta\to 0$. This is essential to allow conduction "along" the concentrated membrane, as required by the fact that in the thin interface model we have a Laplace-Beltrami equation on the membrane Γ.

Denoting by u^η the solution of problem (2.5)-(2.6), we can rewrite its new weak formulation as

$$\int_0^T \int_{\Omega_\eta^{\mathrm{int}} \cap \Omega_\eta^{\mathrm{out}}} A^\eta \nabla u^\eta \cdot \nabla \phi \,\mathrm{d}x\,\mathrm{d}t - \frac{\alpha}{\eta} \int_0^T \int_{\Gamma_\eta} \nabla u^\eta \cdot \nabla \phi_t \,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_0^T \int_\Omega f\phi \,\mathrm{d}x\,\mathrm{d}t + \frac{\alpha}{\eta} \int_{\Gamma_\eta} \nabla \overline{u}_0 \cdot \nabla \phi(0) \,\mathrm{d}x \,, \quad (2.11)$$

for every test function $\phi \in \mathcal{C}^\infty(\overline{\Omega}_T)$ such that ϕ has compact support in Ω for every $t \in (0, T)$ and $\phi(\cdot, T) = 0$ in Ω.
In order to pass to the limit for $\eta \to 0$ in the previous equation, we consider smooth test functions ϕ^η as before such that $\nabla \phi^\eta \sim \nabla^B \phi^\eta$ in Γ^η and $\nabla \phi^\eta$ is stable in $\Omega_\eta^{\mathrm{int}} \cup \Omega_\eta^{\mathrm{out}}$. Such testing functions can be constructed by a suitable process of interpolation (see [7, 8]). Inserting ϕ^η in (2.11), it formally follows

$$\int_0^T \int_{\Omega_\eta^{\mathrm{int}} \cap \Omega_\eta^{\mathrm{out}}} A^\eta \nabla u^\eta \cdot \nabla \phi^\eta \,\mathrm{d}x\,\mathrm{d}t - \frac{\alpha}{\eta}\eta \int_0^T \int_{\Gamma} \nabla u^\eta \cdot \nabla^B \phi_t^\eta \,\mathrm{d}\sigma\,\mathrm{d}t$$

$$\sim \int_0^T \int_\Omega f\phi^\eta \,\mathrm{d}x\,\mathrm{d}t + \frac{\alpha}{\eta}\eta \int_{\Gamma} \nabla \overline{u}_0 \cdot \nabla^B \phi^\eta(0) \,\mathrm{d}\sigma \,. \quad (2.12)$$

Thus, taking into account that $\nabla u^\eta \cdot \nabla^B \phi_t^\eta = \nabla^B u^\eta \cdot \nabla^B \phi_t^\eta$ and $\nabla \overline{u}_0 \cdot \nabla^B \phi^\eta(0) = \nabla^B \overline{u}_0 \cdot \nabla^B \phi^\eta(0)$, and passing to the limit for $\eta \to 0$, we obtain

$$\int_0^T \int_\Omega \lambda \nabla u \cdot \nabla \phi \,\mathrm{d}x\,\mathrm{d}\tau - \alpha \int_0^T \int_\Gamma \nabla^B u \cdot \nabla^B \phi_t \,\mathrm{d}\sigma\,\mathrm{d}\tau$$

$$= \int_0^T \int_\Omega f\phi \,\mathrm{d}x\,\mathrm{d}\tau + \alpha \int_\Gamma \nabla^B \overline{u}_0 \cdot \nabla^B \phi(x, 0) \,\mathrm{d}\sigma \,,$$

which is exactly the weak formulation of the problem with thin membranes (see (2.13)–(2.17)).

2.4. **The problem with thin membranes.** The typical geometrical setting is displayed in Figure 2. Here we give, for the sake of clarity, its detailed formal definition. Let Ω be an open connected bounded subset of $\mathbb{R}^N$ such that $\Omega = \Omega_{\mathrm{int}} \cup \Omega_{\mathrm{out}} \cup \Gamma$, where Ω_{int} and Ω_{out} are two disjoint open subsets of Ω, and $\Gamma = \partial\Omega_{\mathrm{int}} \cap \Omega = \partial\Omega_{\mathrm{out}} \cap \Omega$. The region Ω_{out} [respectively, Ω_{int}] corresponds to the outer phase [respectively, the inclusions], while Γ is the interface. We assume that Ω_{out} is connected (while Ω_{int} could be connected or not), Γ is the union of a finite number (say $m \geq 1$) of connected components Γ_i, and $\mathrm{dist}(\Gamma, \partial\Omega) > 0$. We assume also that $\Omega_{\mathrm{out}}, \Omega_{\mathrm{int}}$ have regular

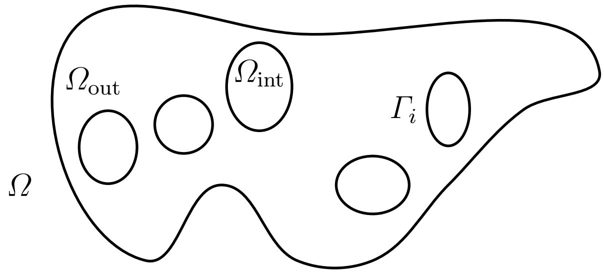

FIGURE 2. The geometrical setting in the case of thin membranes: The connected components of Γ are labelled as Γ_i.

boundary (that is $\partial\Omega$ and Γ are smooth). Finally, let ν denote the normal unit vector to Γ pointing into Ω_{out}.

Let us consider the problem

$$-\operatorname{div}(\lambda\nabla u) = f\,, \qquad \text{in } (\Omega^{\text{int}} \cup \Omega^{\text{out}}) \times (0,T); \tag{2.13}$$

$$[u] = 0\,, \qquad \text{on } \Gamma_T; \tag{2.14}$$

$$-\alpha\Delta^B u_t = [\lambda\nabla u \cdot \nu]\,, \qquad \text{on } \Gamma_T; \tag{2.15}$$

$$u(x,t) = 0\,, \qquad \text{on } \partial\Omega \times (0,T); \tag{2.16}$$

$$\nabla^B u(x,0) = \nabla^B \overline{u}_0(x)\,, \qquad \text{on } \Gamma, \tag{2.17}$$

where we denote

$$[u] = u^{\text{out}} - u^{\text{int}}\,; \tag{2.18}$$

the same notation being employed also for other quantities. Here, $\lambda = \lambda_{\text{int}}$ in Ω_{int}, $\lambda = \lambda_{\text{out}}$ in Ω_{out} and $\lambda_{\text{int}}, \lambda_{\text{out}}, \alpha, f$ are as in the previous section, while $\overline{u}_0 \in H^1(\Gamma)$. Since problem (2.13)–(2.17) is not standard, in order to define a proper notion of weak solution, let us set

$$\mathcal{X}_0(\Omega) := \{u \in H^1_0(\Omega) : tr|_\Gamma(u) \in H^1(\Gamma)\}\,. \tag{2.19}$$

Definition 2.5. We say that $u \in L^2\big(0,T;\mathcal{X}_0(\Omega)\big)$ is a weak solution of problem (2.13)–(2.17) if

$$\int_0^T\!\!\int_\Omega \lambda\nabla u \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}\tau - \alpha\int_0^T\!\!\int_\Gamma \nabla^B u \cdot \nabla^B \phi_t \,\mathrm{d}\sigma\,\mathrm{d}\tau = \int_0^T\!\!\int_\Omega f\phi\,\mathrm{d}x\,\mathrm{d}\tau + \alpha\int_\Gamma \nabla^B\overline{u}_0 \cdot \nabla^B\phi(x,0)\,\mathrm{d}\sigma\,, \tag{2.20}$$

for every test function $\phi \in \mathcal{C}^\infty(\overline{\Omega}_T)$ such that ϕ has compact support in Ω for every $t \in (0,T)$ and $\phi(\cdot,T) = 0$ in Ω. $\square$

We also state the following energy inequality, which can be obtained (via a regularization process in the spirit of Proposition 2.2) by taking $\phi = u$ in (2.20) and integrating by parts with respect to t:

$$\int_0^T \int_\Omega |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}\tau + \sup_{t \in (0,T)} \int_\Gamma |\nabla^B u|^2 \, \mathrm{d}\sigma \leq \gamma \left(\|f\|_{L^2(\Omega)} + \|\overline{u}_0\|_{H^1(\Gamma)} \right) , \tag{2.21}$$

where γ depends on $\lambda_{\mathrm{int}}, \lambda_{\mathrm{out}}, \alpha$ and the Poincaré constant for Ω. Clearly, uniqueness of solutions follows trivially by the previous energy inequality, hence it remains to prove existence. To this purpose, we first need the following technical result (for similar problems, see [17, 23, 24]).

Proposition 2.6. *Let $\Gamma = \cup_i \Gamma_i$, $i = 1, \ldots, m$, and assume that $h_i \in H^1(\Gamma_i)$. Then, there exist $c_i \in \mathbb{R}$, $i = 1, \ldots, m$, such that the solution to problem*

$$\Delta w = 0 , \qquad \text{in } \Omega_{\mathrm{out}}; \tag{2.22}$$

$$w = 0 , \qquad \text{on } \partial\Omega; \tag{2.23}$$

$$w = h_i + c_i , \qquad \text{on } \Gamma_i,\ i = 1, \ldots, m, \tag{2.24}$$

satisfies the further condition

$$\int_{\Gamma_i} \frac{\partial w}{\partial \nu} \, \mathrm{d}\sigma = \ell_i , \qquad i = 1, \ldots, m , \tag{2.25}$$

where ℓ_i are given numbers. Moreover, the constants c_i are unique.

Remark 2.7. In Proposition 2.6, the assumption $h_i \in H^1(\Gamma_i)$ may be relaxed to $h_i \in H^{1/2}(\Gamma_i)$, but in our following application (see Theorem 2.9), h shall belong to $H^1(\Gamma)$, since we will need to consider $\nabla^B h$. □

Remark 2.8. In order to state a periodic version of Proposition 2.6, in which Ω is replaced by $Y = (0,1)^N$ and periodic boundary conditions are assigned instead of (2.23), we need to impose the natural compatibility condition given by $\sum \ell_i = 0$. Notice that, in this case, uniqueness is ensured up to a global additive constant. □

The regular dependence of the constants c_i on the data will play a role in our proof of the following existence result, see Remark 4.4.

Theorem 2.9. *Let $T > 0$, $f \in L^2(\Omega_T)$ and $\overline{u}_0 \in H^1(\Gamma)$. Then, problem* (2.13)–(2.17) *admits a unique solution $u \in L^2\big(0, T; \mathcal{X}_0(\Omega)\big)$. Moreover, $\nabla^B u \in H^1\big(0, T; L^2(\Gamma)\big)$.*

Also in this case only the initial gradient is attained by the solution; see also Remark 4.6.

Remark 2.10. Notice that $\Delta^B u_t$ in (2.15) should be rewritten in the form $\mathrm{div}^B \left((\nabla^B u)_t \right)$, as it is done in (4.39) (a similar remark applies to (2.5)), since u_t is not defined as a Sobolev derivative. However, the weak formulations (2.20) and (2.7) still are correct. □

3. Proof of existence for the problem with thick membranes

We remark that this problem is non-standard since the principal part of the equation is not coercive. For this reason, we are led to introduce a coercive perturbation of it and to prove the well-posedness of this δ-perturbed problem. Then, in order to obtain existence for the original problem, we pass to the limit for $\delta \to 0$, once we have obtained suitable estimates independent of δ.

Theorem 3.1. *Given $\delta > 0$ and $f \in L^2(\Omega_T)$, set $B^\delta(x) = \alpha$ in Ω^m and $B^\delta(x) = \delta$ in $\Omega^{int} \cup \Omega^{out}$ and let A and $\widetilde{u}_0$ be defined as above. Then, for any fixed $T > 0$, the problem*

$$-\operatorname{div}\left(B^\delta \nabla u_t^\delta + A\nabla u^\delta\right) = f\,, \qquad \text{in } \Omega_T; \tag{3.1}$$

$$\nabla u^\delta(x,0) = \nabla \widetilde{u}_0\,, \qquad \text{in } \Omega; \tag{3.2}$$

$$u^\delta(x,t) = 0\,, \qquad \text{on } \partial\Omega \times (0,T), \tag{3.3}$$

admits a unique solution $u^\delta \in L^2\big(0,T; H_0^1(\Omega)\big) \cap H^1(\Omega_T)$.

Proof. First, we note that the weak formulation of problem (3.1)–(3.3) reads as

$$-\int_0^T\!\!\int_\Omega B^\delta \nabla u^\delta \cdot \nabla \phi_t \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega A\nabla u^\delta \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\int_\Omega f\phi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega B^\delta \nabla\widetilde{u}_0 \cdot \nabla\phi(0)\,\mathrm{d}x\,, \tag{3.4}$$

for every test function $\phi \in \mathcal{C}^\infty(\overline{\Omega}_T)$ such that ϕ has compact support in Ω for every $t \in (0,T)$ and $\phi(\cdot,T) = 0$ in Ω. Moreover, using Poincaré's inequality and Gronwall's lemma, we have also the energy estimate

$$\begin{aligned}\sup_{t\in(0,T)} \int_{\Omega^{\mathrm{m}}} |\nabla u^\delta|^2 \,\mathrm{d}x + \delta \sup_{t\in(0,T)} \int_{\Omega^{\mathrm{int}}\cup\Omega^{\mathrm{out}}} |\nabla u^\delta|^2\,\mathrm{d}x + \int_0^T \int_{\Omega^{\mathrm{int}}\cup\Omega^{\mathrm{out}}} |\nabla u^\delta|^2 \,\mathrm{d}x\,\mathrm{d}t \\ \le \gamma \left[\int_0^T\!\!\int_\Omega f^2\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega^{\mathrm{m}}} |\nabla \overline{u}_0|^2\,\mathrm{d}x + \delta \int_{\Omega^{\mathrm{int}}\cup\Omega^{\mathrm{out}}} |\nabla\widetilde{u}_0|^2\,\mathrm{d}x \right] \le \gamma\,, \end{aligned} \tag{3.5}$$

where γ depends on $\lambda_{\mathrm{int}}, \lambda_{\mathrm{out}}, \alpha, \|\nabla\overline{u}_0\|_{L^2(\Omega^{\mathrm{m}})}, \|f\|_{L^2(\Omega)}$, the Poincaré constant for Ω, but not on δ. As a consequence of this energy estimate, we obtain that, if a solution does exist, then it is unique.

In order to prove existence for problem (3.1)–(3.3), for any given function $h \in L^2\big(0,T; H_0^1(\Omega)\big)$, we consider the auxiliary problem

$$-\operatorname{div}\left(B^\delta \nabla v^\delta\right) = \operatorname{div}(A\nabla h) + f\,, \qquad \text{in } \Omega; \tag{3.6}$$

$$v^\delta = 0\,, \qquad \text{on } \partial\Omega. \tag{3.7}$$

Since B^δ is a strictly positive $L^\infty(\Omega)$-function, problem (3.6)–(3.7) is a standard Dirichlet problem, so that, for a.e. $t \in (0,T)$, it admits a unique solution $v^\delta \in$

$L^2\big(0,T;H_0^1(\Omega)\big)$, satisfying for a.e. $t\in(0,T)$ the following energy estimate:

$$\int\limits_{\Omega^{\mathrm{m}}} |\nabla v^\delta|^2\,\mathrm{d}x+\delta\int\limits_{\Omega^{\mathrm{int}}\cup\Omega^{\mathrm{out}}} |\nabla v^\delta|^2\,\mathrm{d}x\le\gamma(\|h\|_{H_0^1(\Omega)}\|\nabla v^\delta\|_{L^2(\Omega)}+\|f\|_{L^2(\Omega)}\|v^\delta\|_{L^2(\Omega)})\,,$$

which gives

$$\|v^\delta\|_{H_0^1(\Omega)}\le\gamma_\delta(\|h\|_{H_0^1(\Omega)}+\|f\|_{L^2(\Omega)})\,,$$

where γ_δ depends on $\alpha,\lambda_{\mathrm{int}},\lambda_{\mathrm{out}},\delta$ and the Poincaré constant for Ω. Integrating with respect to time the previous inequality, we obtain

$$\|v^\delta\|_{L^2(0,T;H_0^1(\Omega))}\le\gamma_\delta(\|h\|_{L^2(0,T;H_0^1(\Omega))}+\|f\|_{L^2(\Omega_T)})\,. \tag{3.8}$$

Hence, we can define the function

$$u^\delta(x,t)=\widetilde{u}_0(x)+\int\limits_0^t v^\delta(x,\tau)\,\mathrm{d}\tau\,, \tag{3.9}$$

which actually belongs to $L^2\big(0,T;H_0^1(\Omega)\big)\cap H^1(\Omega_T)$. Now, let us choose $\overline{T}$ in such a way that $\gamma_\delta\overline{T}<1$ and consider the linear operator $L:L^2\big(0,\overline{T};H_0^1(\Omega)\big)\to L^2\big(0,\overline{T};H_0^1(\Omega)\big)\cap H^1(\Omega_T)\subset L^2\big(0,\overline{T};H_0^1(\Omega)\big)$ defined by $L(h)=u^\delta$, where u^δ is given by (3.9). Clearly, L is a contraction since

$$\|L(h_1)-L(h_2)\|^2_{L^2(0,\overline{T};H_0^1(\Omega))}=\|u_1^\delta-u_2^\delta\|^2_{L^2(0,\overline{T};H_0^1(\Omega))}=\|\int\limits_0^t(v_1^\delta-v_2^\delta)\,\mathrm{d}\tau\|^2_{L^2(0,\overline{T};H_0^1(\Omega))}$$

$$\le\int\limits_0^{\overline{T}}\left(t\int\limits_0^t\|v_1^\delta-v_2^\delta\|^2_{H_0^1(\Omega)}\,\mathrm{d}\tau\right)\,\mathrm{d}t\le\frac{\overline{T}^2}{2}\int\limits_0^{\overline{T}}\|v_1^\delta-v_2^\delta\|^2_{H_0^1(\Omega)}\,\mathrm{d}\tau$$

$$\le\frac{\overline{T}^2}{2}\|v_1^\delta-v_2^\delta\|^2_{L^2(0,\overline{T};H_0^1(\Omega))}\le\gamma_\delta^2\frac{\overline{T}^2}{2}\|h_1-h_2\|^2_{L^2(0,\overline{T};H_0^1(\Omega))}<\frac{1}{2}\|h_1-h_2\|^2_{L^2(0,\overline{T};H_0^1(\Omega))}.$$

Therefore, there exists a unique fixed point $u^\delta\in L^2(0,\overline{T};H_0^1(\Omega))$, given by (3.9), where v^δ satisfies

$$-\operatorname{div}\left(B^\delta\nabla v^\delta\right)=\operatorname{div}(A\nabla u^\delta)+f\,,\qquad\text{in }\Omega\times(0,\overline{T}),$$

which is nothing else than the equation (3.1), since $v^\delta=u_t^\delta$. Finally, since $\overline{T}$ depends on $\alpha,\lambda_{\mathrm{int}},\lambda_{\mathrm{out}},\delta$, but not on the initial condition $\overline{u}_0$, we can repeat the previous fixed point argument in the intervals $(\overline{T},2\overline{T})$, $(2\overline{T},3\overline{T})$ and so on, so that we can recover the whole interval $(0,T)$ by iteration. Here, we employ the regularity in time of u^δ to define the trace of $u^\delta(t)$ at all time levels. Then, the thesis is achieved, once we take into account that conditions (3.2), (3.3) are clearly satisfied by the function u^δ thus constructed. □

Proof of Theorem 2.3. For any $\delta>0$, let $u^\delta\in L^2\big(0,T;H_0^1(\Omega)\big)\cap H^1(\Omega_T)$ be the solution of problem (3.1)–(3.3). By the Poincaré inequality and integrating the inequality

(3.5) with respect to the time t in the interval $(0,T)$, we obtain

$$\|u^\delta\|_{L^2(0,T;H^1_0(\Omega))} \leq \gamma\,,$$

where γ is independent of δ. Then, passing to a subsequence if needed, it follows that there exists $u \in L^2\big(0,T;H^1_0(\Omega)\big)$ such that $u^\delta \rightharpoonup u$ for $\delta \to 0$ in $L^2\big(0,T;H^1_0(\Omega)\big)$. Then, passing to the limit in the weak formulation (3.4) and taking into account that $B^\delta \to B$ strongly in $L^\infty(\Omega)$, we obtain that the limit u satisfies (2.7); i.e., u is a solution of problem (2.5)–(2.6). By uniqueness, it follows that the whole sequence (u^δ) coverges to u and u is the unique solution of problem (2.5)–(2.6). □

4. Proof of existence for the problem with thin interfaces

The main goal of this section is to prove that problem (2.13)–(2.17) admits a solution $u \in L^2\big(0,T;\mathcal{X}_0(\Omega)\big)$.

Remark 4.1. Since the solution of problem (2.22)–(2.24), in general, belongs only to $H^1(\Omega_{\text{out}})$, we have to precise the meaning of condition (2.25). It is quite a standard result (see, for instance, [12]), but we prefer to recall it here for the reader's convenience.

To this purpose, assume that, given $h \in H^1(\Gamma)$, $(h_n) \in \mathcal{C}^\infty(\Gamma)$ is a sequence of smooth functions such that $h_n \to h$ strongly in $H^1(\Gamma)$. Let us set $H^1_\Gamma(\Omega_{\text{out}}) := \{u \in H^1(\Omega_{\text{out}}) \,:\, \text{s.t. } u = 0 \text{ on } \partial\Omega\}$ and, for $n \in \mathbb{N}$, define the linear functional $A_n : H^1_\Gamma(\Omega_{\text{out}}) \to \mathbb{R}$ as

$$A_n(\phi) = \int\limits_\Gamma \frac{\partial u_n}{\partial \nu}\phi \,\mathrm{d}\sigma = -\int\limits_{\Omega_{\text{out}}} \nabla u_n \cdot \nabla \phi \,\mathrm{d}x\,, \tag{4.1}$$

where $u_n \in \mathcal{C}^\infty(\Omega_{\text{out}})$ is the smooth solution of problem (2.22)–(2.24) corresponding to the boundary datum $h_n \in \mathcal{C}^\infty(\Gamma)$ and c_i, $i = 1,\dots,m$, as above. By standard energy estimate, and taking into account the linearity of the problem, we obtain

$$\|u_n - u_m\|_{H^1(\Omega_{\text{out}})} \leq \gamma \|h_n - h_m\|_{H^1(\Gamma)}\,, \qquad \forall n,m \in \mathbb{N}\,, \tag{4.2}$$

so that $u_n \to u \in H^1_\Gamma(\Omega_{\text{out}})$ strongly in $H^1(\Omega_{\text{out}})$, where u is still the solution of problem (2.22)–(2.24) corresponding to h.

Then, by (4.1), we obtain that there exists a limit functional denoted by $A : H^1_\Gamma(\Omega_{\text{out}}) \to \mathbb{R}$ and defined by

$$A(\phi) = -\int\limits_{\Omega_{\text{out}}} \nabla u \cdot \nabla \phi \,\mathrm{d}x\,. \tag{4.3}$$

In particular, by fixing $i \in \{1,\dots,m\}$ and by taking $\widehat{\phi}_i \in H^1_\Gamma(\Omega_{\text{out}})$ such that $\widehat{\phi}_i \equiv 1$ on Γ_i and $\widehat{\phi}_i = 0$ on Γ_j, for $j \neq i$, we have that

$$\int\limits_{\Gamma_i} \frac{\partial u_n}{\partial \nu}\,\mathrm{d}\sigma = -\int\limits_{\Omega_{\text{out}}} \nabla u_n \cdot \nabla \widehat{\phi}_i \,\mathrm{d}x = A_n(\widehat{\phi}_i) \to A(\widehat{\phi}_i)\,, \tag{4.4}$$

so that we can state that, for a given H^1 solution of problem (2.22)–(2.24), condition (2.25) is understood in the sense that $A(\widehat{\phi}_i) = 0$, for $i = 1,\dots,m$, where A is here the operator associated to w. □

Proof of Proposition 2.6. For any $h \in H^1(\Gamma_j)$, denote by $u_j[h] \in H^1_\Gamma(\Omega_{\text{out}})$ the solution of the standard elliptic problem

$$\Delta v = 0\,, \qquad \text{in } \Omega_{\text{out}}; \tag{4.5}$$

$$v = 0\,, \qquad \text{on } \partial\Omega; \tag{4.6}$$

$$v = 0\,, \qquad \text{on } \Gamma_i,\ i = 1, \dots, m,\ i \neq j; \tag{4.7}$$

$$v = h\,, \qquad \text{on } \Gamma_j. \tag{4.8}$$

Then, taking into account the linearity of the problem, the solution to (2.22)–(2.24) can be written in the form

$$w = \sum_{j=1}^{m} u_j[h_j] + u_j[c_j]\,. \tag{4.9}$$

Then, the conditions (2.25) become

$$\sum_{j=1}^{m} \left\{ \int_{\Gamma_i} \frac{\partial u_j[h_j]}{\partial \nu}\, \mathrm{d}\sigma + c_j \int_{\Gamma_i} \frac{\partial u_j[1]}{\partial \nu}\, \mathrm{d}\sigma \right\} = \ell_i\,, \quad i = 1, \dots, m\,. \tag{4.10}$$

Upon defining

$$a_{ij} = \int_{\Gamma_i} \frac{\partial u_j[1]}{\partial \nu}\, \mathrm{d}\sigma\,, \qquad \text{and} \qquad G_i = -\sum_{j=1}^{m} \int_{\Gamma_i} \frac{\partial u_j[h_j]}{\partial \nu}\, \mathrm{d}\sigma + \ell_i\,, \tag{4.11}$$

we can rewrite (4.10) as

$$\sum_{j=1}^{m} a_{ij} c_j = G_i\,, \qquad i = 1, \dots, m\,. \tag{4.12}$$

We claim that the previous linear system has a unique solution $(c_1, \dots, c_m)$. Indeed, assume, by contradiction, that the corresponding homogeneous system

$$\sum_{j=1}^{m} a_{ij} d_j = 0\,, \qquad i = 1, \dots, m\,, \tag{4.13}$$

admits a nonzero solution. It is easily seen that the function $w = \sum_j u_j[d_j]$ solves (2.22)–(2.24) with $h_j = 0$ and $c_j = d_j$, for every $j = 1, \dots, m$, and it satisfies also the conditions (2.25) with $\ell_i = 0$, $i = 1, \dots, m$, since

$$\int_{\Gamma_i} \frac{\partial w}{\partial \nu}\, \mathrm{d}\sigma = \sum_{j=1}^{m} d_j \int_{\Gamma_i} \frac{\partial u_j[1]}{\partial \nu}\, \mathrm{d}\sigma = \sum_{j=1}^{m} a_{ij} d_j = 0\,, \qquad i = 1, \dots, m\,. \tag{4.14}$$

Now, let $k \in \{1, \dots, m\}$ be an index such that

$$d_k = \max_{j=1,\dots,m} d_j$$

Then, if $d_k \geq 0$, by Hopf's Lemma it follows that $\frac{\partial w}{\partial \nu} < 0$ on Γ_k, which contradicts (4.14) for $i = k$. A similar argument holds when $d_k < 0$. Hence, the linear system

(4.13) admits only the null solution, which implies that the matrix $\mathcal{A} := [a_{ij}]$ has a trivial kernel, that is (4.12) has a unique solution. $\square$

Remark 4.2. Let $h \in H^1(\Gamma)$ be a given function and consider the problem

$$-\operatorname{div}(\lambda\nabla u) = 0\,, \qquad \text{in } \Omega_{\text{int}} \cup \Omega_{\text{out}}; \tag{4.15}$$

$$u = 0\,, \qquad \text{on } \partial\Omega; \tag{4.16}$$

$$u = h\,, \qquad \text{on } \Gamma = \bigcup_{i=1}^{m} \Gamma_i. \tag{4.17}$$

As a consequence of the previous lemma, where we take $\ell_i = 0$, for $i = 1, \dots, m$, it follows that the set

$$\mathcal{H}_0(\Gamma) = \{h \in H^1(\Gamma) \text{ s.t. the solution } u \in H^1_0(\Omega) \text{ to problem (4.15) –(4.17)} \\ \text{satisfies} \int\limits_{\Gamma_i} \lambda_{\text{out}} \frac{\partial u^{\text{out}}}{\partial \nu}\, \mathrm{d}\sigma = 0,\ \forall i = 1, \dots, m\} \tag{4.18}$$

is a non-empty linear space. Moreover, it is not difficult to see that $\mathcal{H}_0(\Gamma)$ is also a closed subspace of $H^1(\Gamma)$. Indeed, if (h_n) is a sequence in $\mathcal{H}_0(\Gamma)$ strongly converging to h in $H^1(\Gamma)$, for $n \to +\infty$, it follows that the corresponding sequence (u_n) of solutions to problem (4.15)–(4.17), with h replaced by h_n, strongly converges in $H^1_0(\Omega)$ to the solution $u \in H^1_0(\Omega)$ of problem (4.15)–(4.17) corresponding to the limit function h. Moreover, passing to the limit, for $n \to +\infty$, in the weak formulation of (4.15) for Ω_{out}, we obtain

$$0 = -\int\limits_{\Gamma_i} \lambda_{\text{out}} \frac{\partial u_n}{\partial \nu}\, \mathrm{d}\sigma = \int\limits_{\Omega_{\text{out}}} \lambda_{\text{out}} \nabla u_n \cdot \nabla\varphi\, \mathrm{d}x \to \int\limits_{\Omega_{\text{out}}} \lambda_{\text{out}} \nabla u \cdot \nabla\varphi\, \mathrm{d}x\,,$$

for every $\varphi \in H^1_0(\Omega)$ supported in a neighbourhood of Γ_i and $\varphi = 1$ on Γ_i. This implies that

$$0 = \int\limits_{\Omega_{\text{out}}} \lambda_{\text{out}} \nabla u \cdot \nabla\varphi\, \mathrm{d}x = -\int\limits_{\Gamma_i} \lambda_{\text{out}} \frac{\partial u}{\partial \nu}\, \mathrm{d}\sigma\,,$$

and this can be repeated for every $i = 1, \dots, m$, so that $h \in \mathcal{H}_0(\Gamma)$. Hence, $\mathcal{H}_0(\Gamma)$ is closed, which implies that it is a Banach space. $\square$

Remark 4.3. Let $h_j \in H^1(\Gamma_j)$. For $i = 1, \dots, m$, if we fix $\widehat{\phi}_i \in H^1_\Gamma(\Omega_{\text{out}})$ such that $\widehat{\phi}_i = 1$ on Γ_i, $\widehat{\phi}_i = 0$ on Γ_j for $j \neq i$, by (4.3) and (4.4) we have

$$\left|\sum_{j=1}^{m} \int\limits_{\Gamma_i} \lambda_{\text{out}} \frac{\partial u_j[h_j]}{\partial \nu}\, \mathrm{d}\sigma\right| = \left|\sum_{j=1}^{m} \int\limits_{\Omega_{\text{out}}} \lambda_{\text{out}} \nabla u_j[h_j] \cdot \nabla\widehat{\phi}_i\, \mathrm{d}x\right| \\ \le \gamma \sum_{j=1}^{m} \|u_j[h_j]\|_{H^1(\Omega_{\text{out}})}\, \|\widehat{\phi}_i\|_{H^1(\Omega_{\text{out}})} \le \gamma \|h\|_{H^1(\Gamma)}\,, \tag{4.19}$$

where we set $u[h] = \sum_j u_j[h_j]$ and the last inequality is due to the standard energy inequality for problem (2.22)–(2.24) and the fact that the test function $\widehat{\phi}_i$ is fixed.

Given $h, g \in H^1(\Gamma)$ with $h\big|_{\Gamma_j} = h_j$, $g\big|_{\Gamma_j} = g_j$ and $h_j, g_j \in H^1(\Gamma_j)$, set G_i^h, G_i^g the corresponding numbers defined in (4.11). Then, by the linearity of problem (4.5)–(4.8) and by (4.19), it follows that

$$|G_i^h - G_i^g| \le \gamma \|h - g\|_{H^1(\Gamma)} . \tag{4.20}$$

Hence, by the first equality in (4.11) and by (4.12), we have

$$|c_j^h - c_j^g| \le \gamma \|h - g\|_{H^1(\Gamma)} , \tag{4.21}$$

where we have taken into account that the constant matrix $\mathcal{A} := [a_{ij}]$ by Proposition 2.6 is invertible and depends only on the geometry. Here, we have employed also for c_j^h and c_j^g the same notation used for G_i^h and G_i^g. □

Remark 4.4. We note that, if in (4.8) we take $h_j = h_j(t)$ with $h_j \in L^2\big(0, T; H^1(\Gamma)\big)$, by standard regularity results for elliptic equations, it follows that the solution $u_j[h_j]$ of problem (4.5)–(4.8) belongs to the space $L^2\big(0, T; H^1_\Gamma(\Omega_{\text{out}})\big)$. Then, if also $\ell_i \in L^2(0, T)$, by (4.11) it follows that $G_i \in L^2(0, T)$ and, by (4.12), the same holds also for c_j. Analogously, if $h_j \in H^1\big(0, T; H^1(\Gamma)\big)$ and $\ell_i \in H^1(0, T)$, then $u_j[h_j] \in H^1\big(0, T; H^1_\Gamma(\Omega_{\text{out}})\big)$ and again $G_i \in H^1(0, T)$ and the same holds also for c_j. □

Proof of Theorem 2.9. For a.e. $t \in (0, T)$, let $\overline{u}(t) \in H^1_0(\Omega)$ be the unique solution of the standard Dirichlet problem

$$-\operatorname{div}(\lambda \nabla \overline{u}(t)) = f(t) , \quad \text{in } \Omega_{\text{int}} \cup \Omega_{\text{out}}; \tag{4.22}$$

$$\overline{u}(t) = 0 , \qquad \text{on } \Gamma. \tag{4.23}$$

Clearly, $\overline{u}$ satisfies

$$\|\overline{u}(t)\|_{H^1_0(\Omega)} \le \gamma \|f(t)\|_{L^2(\Omega)} . \tag{4.24}$$

Moreover, set

$$\ell_j(t) = -\int_{\partial\Omega} \lambda_{\text{out}} \frac{\partial \overline{u}(t)}{\partial n} \,\mathrm{d}\sigma - \int_{\Omega^j_{\text{int}} \cup \Omega_{\text{out}}} f(t) \,\mathrm{d}x + \sum_{i \ne j} \int_{\Gamma_i} \lambda_{\text{out}} \frac{\partial \overline{u}(t)}{\partial \nu} \,\mathrm{d}\sigma , \tag{4.25}$$

where n denotes the outer normal vector to $\partial\Omega$.
For a.e. $t \in (0, T)$, let us define

$$\mathcal{H}_\ell(\Gamma) = \{h \in H^1(\Gamma) \text{ s.t. the solution } u \in H^1_0(\Omega) \text{ to problem (4.15) –(4.17)}$$
$$\text{satisfies} \int_{\Gamma_i} \lambda_{\text{out}} \frac{\partial u^{\text{out}}}{\partial \nu} \,\mathrm{d}\sigma = \ell_i(t),\ \forall i = 1, \dots, m\} \tag{4.26}$$

where $\ell_i(t)$, $i = 1, \dots, m$, are defined in (4.25). Following a similar argument as in Remark 4.2, one can easily prove that $\mathcal{H}_\ell(\Gamma)$ endowed with the distance defined by

$$d_{\mathcal{H}_\ell}(h_1, h_2) = \|h_1 - h_2\|_{H^1(\Gamma)} , \qquad \forall h_1, h_2 \in \mathcal{H}_\ell(\Gamma) ,$$

is a complete metric space. Let $h \in L^2(0, T; \mathcal{H}_\ell(\Gamma))$ and, for a.e. $t \in (0, T)$, let $\widetilde{u}(t)$ be the solution of problem (4.15)–(4.17) with h replaced by $h(t)$. Clearly, the unique solution $\widetilde{u}(t) \in H^1_0(\Omega)$ satisfies

$$\|\widetilde{u}(t)\|_{H^1_0(\Omega)} \le \gamma \|h(t)\|_{H^1(\Gamma)} , \tag{4.27}$$

where $\|h(t)\|_{H^1(\Gamma)}$ depends obviously on $\ell_1, \dots, \ell_m$, and hence on $\|f(t)\|_{L^2(\Omega)}$. Set $u(t) = \overline{u}(t) + \widetilde{u}(t)$, which satisfies

$$-\operatorname{div}(\lambda \nabla u) = f(t)\,, \qquad \text{in } \Omega_{\text{int}} \cup \Omega_{\text{out}}; \tag{4.28}$$

$$u = 0\,, \qquad \text{on } \partial\Omega; \tag{4.29}$$

$$u = h(t)\,, \qquad \text{on } \Gamma. \tag{4.30}$$

Starting from $u(t)$, solve the problem

$$-\alpha \Delta^B v(t) = \left[\lambda \frac{\partial u(t)}{\partial \nu}\right], \qquad \text{in } \Gamma; \tag{4.31}$$

$$\int_{\Gamma_i} v(t)\,\mathrm{d}\sigma = 0\,. \qquad i = 1, \dots, m. \tag{4.32}$$

First, we note that $\left[\lambda \frac{\partial u(t)}{\partial \nu}\right] \in H^{-1}(\Gamma)$; indeed, taking into account that $u(t) \in H^1_0(\Omega)$ solves (4.28) and setting, for $w \in H^1(\Gamma)$,

$$\langle \left[\lambda \frac{\partial u(t)}{\partial \nu}\right], w\rangle := -\int_{\Omega_{\text{out}}} \lambda_{\text{out}} \nabla u(t) \cdot \nabla w\,\mathrm{d}x - \int_{\Omega_{\text{int}}} \lambda_{\text{int}} \nabla u(t) \cdot \nabla w\,\mathrm{d}x\,,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Gamma)$ and $H^1(\Gamma)$ and w is assumed to be extended from $H^1(\Gamma)$ to an $H^1_0(\Omega)$-function such that $\|w\|_{H^1_0(\Omega)} \le \gamma \|w\|_{H^1(\Gamma)}$, it follows that $\left[\lambda \frac{\partial u(t)}{\partial \nu}\right]$ is a linear and continuous functional on $H^1(\Gamma)$, since

$$|\langle \left[\lambda \frac{\partial u(t)}{\partial \nu}\right], w\rangle| \le \gamma \|u(t)\|_{H^1_0(\Omega)} \|w\|_{H^1_0(\Omega)} \le \gamma (\|h(t)\|_{H^1(\Gamma)} + \|f(t)\|_{L^2(\Omega)}) \|w\|_{H^1(\Gamma)}\,. \tag{4.33}$$

Moreover, since $u(t)$ satisfies (4.28) inside Ω_{int}, it follows that

$$\int_{\Gamma_i} \lambda_{\text{int}} \frac{\partial u^{\text{int}}(t)}{\partial \nu}\,\mathrm{d}\sigma = -\int_{\Omega^i_{\text{int}}} f(t)\,\mathrm{d}x\,,$$

while

$$\int_{\Gamma_i} \lambda_{\text{out}} \frac{\partial u^{\text{out}}(t)}{\partial \nu}\,\mathrm{d}\sigma = \int_{\Omega_{\text{out}}} f(t)\,\mathrm{d}x - \sum_{j \ne i} \int_{\Gamma_j} \lambda_{\text{out}} \frac{\partial \overline{u}(t)}{\partial \nu}\,\mathrm{d}\sigma + \int_{\partial\Omega} \lambda_{\text{out}} \frac{\partial \overline{u}(t)}{\partial n}\,\mathrm{d}\sigma + \ell_i(t) =$$

$$\int_{\Omega_{\text{out}}} f(t)\,\mathrm{d}x - \sum_{j \ne i} \int_{\Gamma_j} \lambda_{\text{out}} \frac{\partial \overline{u}(t)}{\partial \nu}\,\mathrm{d}\sigma + \int_{\partial\Omega} \lambda_{\text{out}} \frac{\partial \overline{u}(t)}{\partial n}\,\mathrm{d}\sigma$$

$$-\int_{\partial\Omega} \lambda_{\text{out}} \frac{\partial \overline{u}(t)}{\partial n}\,\mathrm{d}\sigma - \int_{\Omega^i_{\text{int}} \cup \Omega_{\text{out}}} f(t)\,\mathrm{d}x + \sum_{j \ne i} \int_{\Gamma_j} \lambda_{\text{out}} \frac{\partial \overline{u}(t)}{\partial \nu}\,\mathrm{d}\sigma = -\int_{\Omega^i_{\text{int}}} f(t)\,\mathrm{d}x$$

by the choice of $h \in \mathcal{H}_\ell(\varGamma)$. Therefore, the compatibility condition

$$\int_{\varGamma_i} \left[\lambda \frac{\partial u(t)}{\partial \nu}\right] \mathrm{d}\sigma = 0$$

is satisfied, so that, from standard results, for a.e. $t \in (0,T)$, there exists a unique solution $v(t) \in H^1(\varGamma)$ of (4.31) such that

$$\|v(t)\|_{H^1(\varGamma)} \leq \gamma \| \left[\lambda \frac{\partial u(t)}{\partial \nu}\right] \|_{H^{-1}(\varGamma)} \leq \gamma(\|h(t)\|_{H^1(\varGamma)} + \|f(t)\|_{L^2(\varOmega)}), \tag{4.34}$$

where the last inequality is due to (4.33).
Define

$$w(x,t) = \overline{u}_0(x) + \int_0^t v(x,\tau)\,\mathrm{d}\tau \in H^1\big((0,T); H^1(\varGamma)\big), \tag{4.35}$$

and

$$\widetilde{w}(x,t) = w(x,t) + \sum_{i=1}^m \widetilde{c}_i(t)\chi_{\varGamma_i}(x), \tag{4.36}$$

where, for $i = 1, \dots, m$, $\widetilde{c}_i(t)$ are the constants given in Proposition 2.6 with $h_i(x,t) = w(x,t)$ and ℓ_i given by (4.25). Now, set $\overline{T} = (2\sqrt{\gamma})^{-1}$, where γ is the constant given in the last line of (4.37), and consider the operator $L : L^2\big(0,\overline{T}; \mathcal{H}_\ell(\varGamma)\big) \to L^2\big(0,\overline{T}; \mathcal{H}_\ell(\varGamma)\big)$, defined by $L(h) = \widetilde{w}$, with $\widetilde{w}$ given by (4.36). Clearly, L is a contraction since

$$\int_0^{\overline{T}} \left[d_{\mathcal{H}_\ell}\big(L(h_1), L(h_2)\big)\right]^2 \,\mathrm{d}t = \|\widetilde{w}_1 - \widetilde{w}_2\|^2_{L^2(0,\overline{T};H^1(\Gamma))}$$

$$= \|\int_0^t (v_1 - v_2)\,\mathrm{d}\tau + \sum_{i=1}^m \big(\widetilde{c}_i^1(t) - \widetilde{c}_i^2(t)\big)\chi_{\Gamma_i}\|^2_{L^2(0,\overline{T};H^1(\Gamma))}$$

$$\leq \gamma \left\{ \int_0^{\overline{T}} \left(t \int_0^t \|v_1(\tau) - v_2(\tau)\|^2_{H^1(\Gamma)} \,\mathrm{d}\tau \right) \mathrm{d}t + \sum_{i=1}^m |\Gamma_i| \int_0^{\overline{T}} \big(\widetilde{c}_i^1(t) - \widetilde{c}_i^2(t)\big)^2 \,\mathrm{d}t \right\}$$

$$\leq \gamma \left\{ \int_0^{\overline{T}} \left(t \int_0^t \|v_1(\tau) - v_2(\tau)\|^2_{H^1(\Gamma)} \,\mathrm{d}\tau \right) \mathrm{d}t + \sum_{i=1}^m |\Gamma_i| \int_0^{\overline{T}} \|w_1(t) - w_2(t)\|^2_{H^1(\Gamma_i)} \,\mathrm{d}t \right\}$$

$$\leq \gamma \left\{ \int_0^{\overline{T}} \left(t \int_0^t \|v_1(\tau) - v_2(\tau)\|^2_{H^1(\Gamma)} \,\mathrm{d}\tau \right) \mathrm{d}t + \int_0^{\overline{T}} \left(t \int_0^t \|v_1(\tau) - v_2(\tau)\|^2_{H^1(\Gamma)} \,\mathrm{d}\tau \right) \mathrm{d}t \right\}$$

$$\leq \gamma \overline{T}^2 \int_0^{\overline{T}} \|v_1(\tau) - v_2(\tau)\|^2_{H^1(\Gamma)} \,\mathrm{d}t \leq \gamma \overline{T}^2 \|h_1 - h_2\|^2_{L^2(0,\overline{T};H^1(\Gamma))}$$

$$= \frac{1}{2}\|h_1 - h_2\|^2_{L^2(0,\overline{T};H^1(\Gamma))}, \tag{4.37}$$

where we reason as in (4.21) (with h and g replaced by w_1 and w_2 respectively). We also use an obvious version of (4.34) written for $f = 0$, which readily follows from the definition of v. Therefore, there exists a unique fixed point $\widetilde{w} \in L^2\big(0, \overline{T}; \mathcal{H}_\ell(\Gamma)\big)$. Consider the function u defined as in (4.28)–(4.30) with $h = \widetilde{w}$ being the fixed point. Then, according to our definition,

$$\nabla^B u(x,t) = \nabla^B \widetilde{w}(x,t) = \nabla^B w(x,t) = \nabla^B \overline{u}_0(x) + \int_0^t \nabla^B v(x,\tau)\,\mathrm{d}\tau\,, \tag{4.38}$$

where v is the solution of (4.31)–(4.32) for the just defined u. Then, clearly,

$$-\alpha \operatorname{div}^B(\nabla^B u)_t = [\lambda \nabla u \cdot \nu]\,. \tag{4.39}$$

Finally, $\nabla^B u(x,0) = \nabla^B \overline{u}_0(x)$; then, u solves our problem (2.13)–(2.17) in $(0, \overline{T})$. Note that actually $\nabla^B u$ is continuous in time (in the L^2-norm) either owing to an analogue of Proposition 2.2 or simply by (4.38) above, so that we may choose $\nabla^B u(x, \overline{T})$ as a new initial data. In this fashion we cover the interval $(0, T)$ with a finite number of steps of width $\overline{T}$, which does not depend on the initial data. □

Corollary 4.5. *Let $T > 0$ and $\overline{u}_0 \in H^1(\Gamma)$. If $f \in H^1\big(0, T; L^2(\Omega)\big)$, then the solution u of problem* (2.13)*–*(2.17) *belongs to $H^1\big(0, T; \mathcal{X}_0(\Omega)\big)$.*

Proof. It is a direct consequence of the construction in (4.36), when we take into account that $\ell_j \in H^1(0,T)$, $j = 1,\dots,m$, as it follows from (4.25), and we recall Remark 4.4. $\square$

Remark 4.6. Clearly, in the case $f \in H^1\big(0,T;L^2(\Omega)\big)$, the previous corollary implies that $u(x,0)$ is defined a.e. in Ω; however, in general, it does not coincide with $\overline{u}_0$, but it is "rearranged" by adding in each connected component of Γ the suitable constant provided by Proposition 2.6, as explained in Remark 2.4. $\square$

Remark 4.7. We note that, if $\Omega = Y = (0,1)^N$ and we replace condition (2.16) with the requirement that the solution is Y-periodic with respect to the spatial variable and has null mean average on Y for a.e. $t \in (0,T)$, then the corresponding problem (2.13)–(2.15), (2.17) still admits a unique solution $u \in L^2\big(0,T;H^1_\#(Y) \cap H^1(\Gamma)\big)$, when also the initial datum $\overline{u}_0$ and the source f are assumed to be Y-periodic and f has null mean average on Y. $\square$

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