HOMOGENIZATION OF SINGULAR ELLIPTIC SYSTEMS WITH NONLINEAR CONDITIONS ON THE INTERFACES

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ABSTRACT. We prove existence and homogenization results for elliptic problems involving a singular lower order term and defined on a domain with interfaces having a nonlinear response. The considered system of equations can model heat or electrical conduction in composite media.

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1. Introduction

In many models it is useful to describe the macroscopic behavior of a body taking into account its microscopic structure. Often, these problems deal with composite media, where there are two finely mixed regions, occupied by materials with different physical properties (an analogous situation occurs for porous media, where one of these regions is empty). The geometry of the medium is modeled intersecting the domain $\Omega \subset \mathbb{R}^N$ occupied by the body with a lattice of period ε. The typical configuration is obtained considering an ε-scaling of the unit cube $Y \subset \mathbb{R}^N$, identified with the flat torus in $\mathbb{R}^N$ (however, other richer geometries can be considered; for example, quasi periodic structures [15, 16, 17]). Therefore, the mathematical description is given by means of equations which typically are the Euler-Lagrange equation or the gradient flow of energy functionals depending on a micro-variable $y = \frac{x}{\varepsilon}$.
This occurs, for instance, in modelling problems of electrostatic, magnetostatic or heat diffusion (for details on some of these physical models see, for instance, [11, 24, 31, 32] and the references quoted therein).

The aim of the homogenization theory is to study the limit, as $\varepsilon \to 0$, of the solutions of these problems at the ε-scale, trying to find suitable limit equations or limit functionals, describing the effective behavior of the macroscopic phenomena.
In order to handle with this kind of problems, several techniques have been developed as, for example, Γ-convergence (in the variational framework) [15]; G-convergence and H-convergence [18, 21, 29, 30], two-scale convergence [1, 2, 3, 28], unfolding technique [19, 20].
In this paper, we consider a model for heat conduction in a composite medium with two finely mixed phases having a periodic active interface and a singular source (for the introduction of a singular source in the model, see [23]). We assume that the heat flow across the interface is related to the jump of the temperature (on the interface) by means of a nonlinear relation.
More precisely, if $\Omega \subseteq \mathbb{R}^N$ is the region occupied by the material, we denote by Ω_1^ε and Ω_2^ε the two different phases separated by the interface Γ^ε; namely, $\Omega_1^\varepsilon = \Omega \cap \varepsilon E$, where E is a periodic open subset of $\mathbb{R}^N$, and $\Omega_2^\varepsilon = \Omega \setminus \overline{\Omega_1^\varepsilon}$. For $\theta \in (0,1)$ and $k \in \{0,1\}$, we consider the problem

$$\begin{aligned}
-\operatorname{div}(\lambda_1 \nabla u_\varepsilon) &= f/u_\varepsilon^\theta, && \text{in } \Omega_1^\varepsilon;\\
-\operatorname{div}(\lambda_2 \nabla u_\varepsilon) &= f/u_\varepsilon^\theta, && \text{in } \Omega_2^\varepsilon;\\
\lambda_1 \nabla u_\varepsilon \cdot \nu &= \lambda_2 \nabla u_\varepsilon \cdot \nu, && \text{on } \Gamma^\varepsilon;\\
\tfrac{1}{\varepsilon^{1-k}} g\left(\tfrac{[u_\varepsilon]}{\varepsilon^k}\right) &= \lambda_2 \nabla u_\varepsilon \cdot \nu_\varepsilon, && \text{on } \Gamma^\varepsilon;\\
u_\varepsilon &> 0, && \text{in } \Omega;\\
u_\varepsilon &= 0, && \text{on } \partial\Omega\,,
\end{aligned}$$

where λ_1, λ_2 are strictly positive constant, $[u_\varepsilon]$ denotes the jump of u_ε across Γ^ε and ν_ε is the normal unit vector to Γ^ε pointing into Ω_2^ε.
In the previous system, in addition to the nonlinear singular source term f/u_ε^θ (with f a strictly positive source), there is also a suitable nonlinearity g in the interface condition. In particular, we consider two different nonlinear responses of the interface, called "weak nonlinearity" (when $k = 0$) and "strong nonlinearity" (when $k = 1$).
From the homogenization point of view, these two cases display different behaviors. If $k = 0$, despite the presence of the nonlinearity in the microscopic problem, the two-scale homogenized system provides on the interface a linear condition, linking the flux (of both the macro and the micro states) with the jump of the micro state (see (5.39)). Therefore, as in the linear case, the limit system can be decoupled, leading to a single-scale equation for the effective temperature. This motivates the term "weak nonlinearity" for this case. On the contrary, if $k = 1$, the homogenized problem preserves a nonlinear relation in the interface (see (5.9)), so that the system cannot be decoupled. For this reason, this is called "strong nonlinearity".
We recall that the previous problem, in the case of linear response of the interface, has been considered in [12, 22], in different geometrical settings, while the parabolic version of the above system, without singular source, was studied in [6] in the weak nonlinear case and in [9] in the strong one.

Our main results are an existence and uniqueness theorem and a homogenization theorem for the previous system of equations, whose proof is obtained via the two-scale convergence technique. We point out that, even though in the limit problem there is no interaction between the nonlinear singular source term and the nonlinear interface condition, they are deeply linked in the proof of both the existence and the homogenization results. Therefore, we have to carefully readapt the techniques already developed in [12] and previously introduced in [22].
Similar models, in the framework of elasticity and electrical or thermal conduction in composite materials, have been treated in [25, 26, 27] and, more recently, in [4, 5, 6, 7, 8, 9, 10, 13, 14, 22].
The paper is organized as follows. In Section 2, we fix some notations, we define the suitable functional spaces and we recall the definitions and some properties of the two-scale convergence. We state our problem in Section 3. The main results of the paper are proven in Sections 4 and 5. More precisely, in Section 4 we prove the existence and uniqueness for the problem at the ε-scale, and in Section 5, we deduce the homogenized system.

2. Notation and preliminary results

For $N \geq 3$, let $\Omega \subset \mathbb{R}^N$ be an open, connected and bounded set, while E denotes a periodic open subset of $\mathbb{R}^N$ (i.e. $E + z = E \quad \forall z \in \mathbb{Z}^N$). We assume that Ω and E have Lipschitz continuous boundary. For all $\varepsilon > 0$, we define the two open sets $\Omega_1^\varepsilon = \Omega \cap \varepsilon E$ and $\Omega_2^\varepsilon = \Omega \setminus \overline{\varepsilon E}$. We set $\Gamma^\varepsilon = \partial\Omega_1^\varepsilon \cap \Omega = \partial\Omega_2^\varepsilon \cap \Omega$, so that $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$. Moreover, we assume that $\operatorname{dist}(\Gamma^\varepsilon, \Omega) \geq \gamma_0\varepsilon$, for a suitable $\gamma_0 > 0$. We set $Y = (0,1)^N$ and $E_1 = E \cap Y$, $E_2 = Y \setminus \overline{E}$, $\Gamma = \partial E \cap Y$ and we assume that $E_1 \subset\subset Y$, so that E_2 is connected.
We denote by ν_ε the normal unit vector to Γ^ε pointing into Ω_2^ε and by ν the normal unit vector to Γ pointing into E_2.
For $u : \Omega \to \mathbb{R}$, $u^{(1)}$ and $u^{(2)}$ denote the restriction of u to Ω_1^ε and Ω_2^ε, respectively. On Γ^ε, we define $[u] := u^{(2)} - u^{(1)}$, where, with abuse of notation, here $u^{(2)}$ and $u^{(1)}$ denotes the trace of u on Γ^ε from Ω_2^ε and Ω_1^ε, respectively. We use the same notation for functions defined in the unit cell Y, where $u^{(2)}$ and $u^{(1)}$ stands for the restriction of u to E_2 and E_1, respectively. In the following, the symbol $[\cdot]$ will be used also to denote the jump across Γ^ε for other quantities.
In the sequel, x and y will denote the macro and micro-variable, respectively, so that, for $u : \Omega \times Y \to \mathbb{R}$, $\nabla_x u$, $\nabla_y u$ and $\operatorname{div}_x u$, $\operatorname{div}_y u$ denote the gradient and the divergence of u computed with respect to the variables x and y, respectively. When no confusion is possible, we write ∇u for $\nabla_x u$ and $\operatorname{div} u$ for $\operatorname{div}_x u$.
For $\xi, \eta \in \mathbb{R}^N$, $\xi \otimes \eta$ denotes the matrix whose entries are $(\xi \otimes \eta)_{ij} = \xi_i \eta_j$ and $(\mathbf{e}_1, \ldots, \mathbf{e}_N)$ is the standard euclidian basis of $\mathbb{R}^N$. In the sequel, C will denote a positive constant, which may vary from line to line.

We set

$$V_0^\varepsilon(\Omega) = \{u : \Omega \to \mathbb{R} \,|\, u^{(1)} \in H^1(\Omega_1^\varepsilon),\, u^{(2)} \in H^1(\Omega_2^\varepsilon),\, u = 0 \text{ on } \partial\Omega\}\,,$$

and

$$\mathfrak{L}_0^\varepsilon(\Omega) = \{u : \overline{\Omega} \to \mathbb{R} \,|\, u^{(1)} \in \mathrm{Lip}(\overline{\Omega_1^\varepsilon}), u^{(2)} \in \mathrm{Lip}(\overline{\Omega_2^\varepsilon}),\, u = 0 \text{ on } \partial\Omega\}.$$

Analogously, we set

$$V_\#(Y) = \{v : Y \to \mathbb{R} \,|\, v \text{ is } Y\text{-periodic, } v^{(1)} \in H^1_\#(E_1),\, v^{(2)} \in H^1_\#(E_2)\},$$

and

$$\mathfrak{L}_\#(Y) = \{v : \overline{Y} \to \mathbb{R} \,|\, v \text{ is } Y\text{-periodic, } v^{(1)} \in \mathrm{Lip}(\overline{E_1}), v^{(2)} \in \mathrm{Lip}(\overline{E_2})\}.$$

Here Y is identified with the flat torus in $\mathbb{R}^N$, so that, for every subset E of the flat torus Y, $H^1_\#(E)$ corresponds to the space of the H^1-functions $v : E \to \mathbb{R}$, such that v and ∇v coincide on opposite sides of $\partial E \cap \partial Y$.
Notice that, if $u \in V_0^\varepsilon(\Omega)$, then $[u] \in L^2(\Gamma^\varepsilon)$ and, analogously, if $v \in V_\#(Y)$, then $[v] \in L^2(\Gamma)$.

We recall the following Poincaré's inequality (see [6, Lemma 7.1])

Theorem 2.1. *There exists $C > 0$, independent of ε, such that*

$$\int_\Omega v^2 \,\mathrm{d}x \le C \left\{ \int_\Omega |\nabla v|^2 \,\mathrm{d}x + \frac{1}{\varepsilon} \int_{\Gamma^\varepsilon} [v]^2 \,\mathrm{d}\sigma \right\} \qquad \forall v \in V_0^\varepsilon(\Omega). \tag{2.1}$$

We recall some basic definitions and properties of the *two-scale convergence* technique. For more details see, for instance, [1, 2, 3, 9, 25] and the references therein.

Definition 2.2. *A function $\varphi \in L^2(\Omega \times Y)$ is said an admissible test function if φ is Y-periodic with respect to the second variable and satisfies:*

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi^2\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y \varphi^2(x,y) \,\mathrm{d}x \,\mathrm{d}y.$$

We denote by $\mathcal{A}(\Omega)$ the space of the admissible test functions on Ω.

Remark 2.3. *The space $\mathcal{C}^0(\overline{\Omega}; \mathcal{C}^0_\#(\overline{Y}))$ or, more in general, the spaces $L^2(\Omega; \mathcal{C}^0_\#(\overline{Y}))$ and $L^2_\#(Y; \mathcal{C}^0(\overline{\Omega}))$ are contained in $\mathcal{A}(\Omega)$. Moreover, if $\varphi(x,y) = \varphi_1(x)\varphi_2(y)$ with $\varphi_1 \in L^p(\Omega)$ and $\varphi_2 \in L^q_\#(Y)$, $p^{-1} + q^{-1} = 2^{-1}$, then $\varphi \in \mathcal{A}(\Omega)$.*

Definition 2.4 (Two-scale convergence). *Let $\{u_\varepsilon\} \subset L^2(\Omega)$ and $u_0 \in L^2(\Omega \times Y)$. We say that $\{u_\varepsilon\}$ two-scale converges to u_0 in $L^2(\Omega \times Y)$ as $\varepsilon \to 0$ (and we write $u_\varepsilon \xrightarrow{2-sc} u_0$) if*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y u_0(x,y)\varphi(x,y) \,\mathrm{d}x \,\mathrm{d}y\,,$$

for every $\varphi \in \mathcal{A}(\Omega)$.

Definition 2.5 (Two-scale convergence on surfaces)**.** *Let $\{w_\varepsilon\} \subset L^2(\Gamma^\varepsilon)$ and $w_0 \in L^2(\Omega \times \Gamma)$. We say that $\{w_\varepsilon\}$ two-scale converges to w_0 in $L^2(\Omega\times\Gamma)$ as $\varepsilon \to 0$ (and, as above, we use the notation $w_\varepsilon \overset{2-sc}{\longrightarrow} w_0$) if*

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\Gamma^\varepsilon} w_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}\sigma = \int_\Omega\int_\Gamma w_0(x,y)\varphi(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma(y)\,,$$

for every $\varphi \in \mathcal{C}^0(\overline{\Omega}; \mathcal{C}^0_\#(\overline{Y}))$.

Theorem 2.6. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$. Then there exist a subsequence of $\{u_\varepsilon\}$ (still denoted by $\{u_\varepsilon\}$) and a function $u_0 \in L^2(\Omega \times Y)$ such that $u_\varepsilon \overset{2-sc}{\longrightarrow} u_0$ in $L^2(\Omega \times Y)$.*

Proposition 2.7. *Let $\{u_\varepsilon\}$ be a sequence of functions in $L^2(\Omega)$, which two-scale converges to a limit $u_0(x,y) \in L^2(\Omega \times Y)$. Then, u_ε converges weakly to $u(x) = \int_Y u_0(x,y)dy$ in $L^2(\Omega)$. Furthermore, we have*

$$\liminf_{\varepsilon\to 0} ||u_\varepsilon||_{L^2(\Omega)} \ge ||u_0||_{L^2(\Omega\times Y)} \ge ||u||_{L^2(\Omega)}.$$

Theorem 2.8. *Let $\{w_\varepsilon\} \subset L^2(\Gamma^\varepsilon)$. Assume that there exists $C > 0$, independent of ε, such that*

$$\varepsilon \int_{\Gamma^\varepsilon} |w_\varepsilon|^2 \,\mathrm{d}\sigma \le C\,, \qquad \forall \varepsilon > 0.$$

Then, there exist a subsequence of $\{w_\varepsilon\}$ (still denoted by $\{w_\varepsilon\}$) and a function $w_0 \in L^2(\Omega \times \Gamma)$ such that $w_\varepsilon \overset{2-sc}{\longrightarrow} w_0$ in $L^2(\Omega \times \Gamma)$.

3. Statement of the problems

Let λ_1, λ_2 be positive constants. Let $\lambda_\varepsilon : \Omega \to \mathbb{R}$ and $\lambda : Y \to \mathbb{R}$ be defined as

$$\lambda_\varepsilon(x) = \begin{cases} \lambda_1, & \text{if } x \in \Omega_1^\varepsilon\,; \\ \lambda_2, & \text{if } x \in \Omega_2^\varepsilon\,; \end{cases} \qquad \text{and} \qquad \lambda(y) = \begin{cases} \lambda_1, & \text{if } y \in E_1\,; \\ \lambda_2, & \text{if } y \in E_2\,, \end{cases}$$

and set $\lambda_0 = \lambda_1|E_1| + \lambda_2|E_2|$.

In the following, we will assume that $f \in L^{\frac{2}{1+\theta}}(\Omega)$, with $\theta \in (0,1)$, is a strictly positive function a.e. in Ω, even if some of the results presented below can be proved also for a nonnegative source f (this fact will be remarked, when it will be the case). For $k \in \{0,1\}$, we consider the problem

$$\begin{aligned} -\operatorname{div}(\lambda_\varepsilon \nabla u_\varepsilon) &= \frac{f}{u_\varepsilon^\theta}, && \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon; \\ [\lambda_\varepsilon \nabla u_\varepsilon \cdot \nu] &= 0, && \text{on } \Gamma^\varepsilon; \\ \frac{1}{\varepsilon^{1-k}} g\left(\frac{[u_\varepsilon]}{\varepsilon^k}\right) &= \lambda_2 \nabla u_\varepsilon^{(2)} \cdot \nu_\varepsilon, && \text{on } \Gamma^\varepsilon; \\ u_\varepsilon &> 0, && \text{in } \Omega; \\ u_\varepsilon &= 0, && \text{on } \partial\Omega\,, \end{aligned} \tag{3.1}$$

where $g \in C^0(\mathbb{R})$, with $g(0) = 0$, and satisfies

$$\exists\, \beta > 0 : \quad \big(g(s_1) - g(s_2)\big)(s_1 - s_2) \ge \beta (s_1 - s_2)^2 \quad \forall\, s_1, s_2 \in \mathbb{R}\,. \tag{3.2}$$

Moreover, if $k = 1$, we assume that

$$\exists\, \alpha > 0 : \quad |g(s_1) - g(s_2)| \le \alpha |s_1 - s_2| \quad \forall\, s_1, s_2 \in \mathbb{R}\,, \tag{3.3}$$

while, if $k = 0$, we assume $g \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ (which implies (3.3)).

Definition 3.1. *We say that $u_\varepsilon \in V_0^\varepsilon(\Omega)$ is a weak solution of (3.1) if $u_\varepsilon > 0$ a.e. in Ω and it satisfies*

$$\left| \int_\Omega \frac{f}{u_\varepsilon^\theta} \psi \, \mathrm{d}x \right| < +\infty\,, \tag{3.4}$$

$$\int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla \psi \, \mathrm{d}x + \int_{\Gamma^\varepsilon} \frac{1}{\varepsilon^{1-k}} g\left(\frac{[u_\varepsilon]}{\varepsilon^k} \right) [\psi] \, \mathrm{d}\sigma = \int_\Omega \frac{f}{u_\varepsilon^\theta} \psi \, \mathrm{d}x \tag{3.5}$$

for every $\psi \in V_0^\varepsilon(\Omega)$.

We remark that the finiteness of the left-hand side of (3.5) implies (3.4); however we prefer to require it explicitly, since it is crucial in the proof of existence and homogenization results. Moreover, taking into account that u_ε and f are positive and recalling that $\psi = \psi^+ - \psi^-$, condition (3.4) can be equivalently rewritten for $\psi > 0$ and without the absolute value, or even in the apparently stronger form $\int_\Omega \frac{f}{u_\varepsilon^\theta} |\psi| \, \mathrm{d}x < +\infty$.

4. Existence and uniqueness

We first prove a result of existence and uniqueness for system (3.1). The proof of this result follows some arguments in the Appendix of [12], where the linear case is considered. However, we will sketch the proof in order to highlight the differences due to the presence of the nonlinearity g.

We notice that, as in [12], the following result holds also in the more general case, where the source f is assumed only to be nonnegative, but not identically equal to zero both in Ω_1 and in Ω_2.

Theorem 4.1. *For every $\varepsilon > 0$ fixed, the problem (3.1) admits a unique solution $u_\varepsilon \in V_0^\varepsilon(\Omega)$ strictly positive in Ω.*

Proof. For the sake of simplicity and without loss of generality, we fix $\varepsilon = 1$ and we will omit it so that, similarly as in Section 2, we write $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ and we set

$$V_0(\Omega) = \{u : \Omega \to \mathbb{R} \mid u^{(1)} \in H^1(\Omega_1),\, u^{(2)} \in H^1(\Omega_2),\, u = 0 \text{ on } \partial\Omega\},$$

endowed with the norm defined by

$$\|u\|_{V_0(\Omega)} := \|\nabla u\|_{L^2(\Omega_1 \cup \Omega_2)} + \|[u]\|_{L^2(\Gamma)}\,.$$

We set $\lambda(x) = \lambda_1$ a.e. in Ω_1 and $\lambda(x) = \lambda_2$ a.e. in Ω_2.

First, for $F \in L^2(\Omega)$, we consider the problem

$$\begin{aligned} -\operatorname{div}(\lambda \nabla u) &= F, && \text{in } \Omega_1 \cup \Omega_2;\\ [\lambda \nabla u \cdot \nu] &= 0, && \text{on } \Gamma;\\ g([u]) &= \lambda \nabla u^{(2)} \cdot \nu, && \text{on } \Gamma;\\ u &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

whose existence can be obtained by the Direct Methods of Calculus of Variations. Indeed, let $J : V_0(\Omega) \to V_0(\Omega)$ be the functional defined as

$$J(u) = \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Gamma G([u]) \,\mathrm{d}\sigma - \int_\Omega F\, u \,\mathrm{d}x,$$

where $G(s) = \int_0^s g(t)\,\mathrm{d}t$. The Euler-Lagrange equation of J is

$$\int_\Omega \lambda \nabla u \cdot \nabla \psi \,\mathrm{d}x + \int_\Gamma g([u])[\psi] \,\mathrm{d}\sigma = \int_\Omega F\, \psi \,\mathrm{d}x\,, \qquad \forall \psi \in V_0(\Omega), \tag{4.2}$$

which is the weak form of (4.1). By (3.2) and the fact that $G(0) = 0$, we get that $G(s) \geq \frac{\beta}{2} s^2$ and hence J is coercive in $V_0(\Omega)$. Therefore, if $\{u_h\}$ is a minimizing sequence for J, there exists $u \in V_0$ such that, up to a subsequence, there holds

$$\begin{aligned} &u_h \to u\,, && \text{strongly in } L^2(\Omega),\\ &\nabla u_h \rightharpoonup \nabla u\,, && \text{weakly in } L^2(\Omega),\\ &[u_h] \rightharpoonup [u]\,, && \text{strongly in } L^2(\Gamma). \end{aligned} \tag{4.3}$$

Taking into account that the first term of J is lower semi-continuous with respect to the weak L^2-convergence and that the last two terms are continuous with respect to the strong L^2-convergence, we obtain that u is a minimizer of J in $V_0(\Omega)$. Moreover u is unique because J is strictly convex.

Then, the main idea is to approximate our problem by means of the sequence of systems

$$\begin{aligned} -\operatorname{div}(\lambda \nabla u_n) &= \frac{f_n}{\left(u_n + \frac{1}{n}\right)^\theta}, && \text{in } \Omega_1 \cup \Omega_2;\\ [\lambda \nabla u_n \cdot \nu] &= 0, && \text{on } \Gamma;\\ g([u_n]) &= \lambda \nabla u_n^{(2)} \cdot \nu, && \text{on } \Gamma;\\ u_n &\geq 0, && \text{in } \Omega;\\ u_n &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{4.4}$$

where $f_n = \min\{f(x), n\}$ and whose weak formulation is

$$\int_\Omega \lambda \nabla u_n \cdot \nabla \psi \,\mathrm{d}x + \int_\Gamma g([u_n])[\psi] \,\mathrm{d}\sigma = \int_\Omega \frac{f_n}{\left(u_n + \frac{1}{n}\right)^\theta}\, \psi \,\mathrm{d}x\,, \qquad \forall \psi \in V_0(\Omega). \tag{4.5}$$

Repeating the strategy of the linear case (see [12], proof of Theorem A.2) and exploiting property (3.2) of g, existence for (4.4) can be obtained applying the Schauder's

Theorem to the non singular problem

$$\begin{cases} -\operatorname{div}(\lambda \nabla u_n) &= \frac{f_n}{\left(|w|+\frac{1}{n}\right)^\theta}\,, & \text{in } \Omega_1 \cup \Omega_2\,; \\ [\lambda \nabla u_n \cdot \nu] &= 0\,, & \text{on } \Gamma\,; \\ g([u_n]) &= \lambda_2 \nabla u_n^{(2)} \cdot \nu\,, & \text{on } \Gamma\,; \\ u_n &= 0\,, & \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

with $w \in L^2(\Omega)$, which is analogous to the problem (4.1) and whose existence is proved above. This provides the existence of a solution for (4.5) with $f_n/(u_n + 1/n)^\theta$ replaced by $f_n/(|u_n| + 1/n)^\theta$. In order to prove that $u_n \geq 0$, we follow a similar idea as in [12, Proof of Theorem A.2]. Indeed, taking $\psi = -u_n^-$ as testing function in (4.5) modified, as before, with $f_n/(u_n + 1/n)^\theta$ replaced by $f_n/(|u_n| + 1/n)^\theta$, we get

$$\int_\Omega \lambda |\nabla u_n^-|^2 \,\mathrm{d}x \leq -\int_\Omega \lambda \nabla u_n \cdot \nabla u_n^- \,\mathrm{d}x + \int_\Gamma g([u_n])[-u_n^-]\,\mathrm{d}\sigma$$

$$= -\int_\Omega \frac{f_n}{\left(|u_n| + \frac{1}{n}\right)^\theta} u_n^- \,\mathrm{d}x \leq 0\,, \tag{4.7}$$

where, in the last inequality we have taken into account that f is nonnegative a.e. in Ω and in the first inequality we used the fact that $g([u_n])[-u_n^-] \geq 0$. Indeed, recalling (3.2), we obtain

- if $0 \geq u_n^{(2)} \geq u_n^{(1)}$, then $g([u_n]) \geq 0$ and $-(u_n^{(2)})^- + (u_n^{(1)})^- \geq 0$;
- if $u_n^{(2)} \geq 0 \geq u_n^{(1)}$, then $g([u_n]) \geq 0$ and $-(u_n^{(2)})^- + (u_n^{(1)})^- = (u_n^{(1)})^- \geq 0$;
- if $u_n^{(2)} \geq u_n^{(1)} \geq 0$ or $u_n^{(1)} \geq u_n^{(2)} \geq 0$, then $(u_n^{(2)})^- = 0 = (u_n^{(1)})^-$;
- if $u_n^{(1)} \geq 0 \geq u_n^{(2)}$, then $g([u_n]) \leq 0$ and $-(u_n^{(2)})^- + (u_n^{(1)})^- = -(u_n^{(2)})^- \leq 0$;
- if $0 \geq u_n^{(1)} \geq u_n^{(2)}$, then $g([u_n]) \leq 0$ and $-(u_n^{(2)})^- + (u_n^{(1)})^- \leq 0$.

Therefore, by (4.7), it follows that $\nabla u_n^- = 0$ a.e. in Ω, so that $u_n^- = 0$ a.e. in Ω_2 (i.e. $u_n \geq 0$ a.e. in Ω_2), because of the homogenous boundary condition, while u_n^- equals a non-negative constant (let us write $u_n^- = \gamma^2$) a.e. in Ω_1. In order to assure that $\gamma^2 = 0$, assume, by contradiction, that $\gamma^2 > 0$. Since $u_n \in H^1(\Omega)$, this implies that $u_n = -\gamma^2$ a.e. in Ω_1 and hence $[u_n] = u_n^{(2)} - u_n^{(1)} = u_n^{(2)} + \gamma^2 > 0$ a.e. on Γ, which implies $g([u_n]) > 0$ a.e. on Γ, once we recall (3.2). Moreover, by (4.7), we get

$$\gamma^2 \int_\Gamma g([u_n])\,\mathrm{d}\sigma = \int_\Gamma g([u_n])(u_n^{(1)})^- \,\mathrm{d}\sigma = \int_\Gamma g([u_n])[-u_n^-]\,\mathrm{d}\sigma \leq 0\,,$$

which is a contradiction, if we take into account that $\int_\Gamma g([u_n])\,\mathrm{d}\sigma > 0$. Therefore, $\gamma^2 = 0$, so that $u_n^- = 0$ a.e. in Ω_1 and we have proved that $u_n \geq 0$ a.e. in Ω. Thus u_n is a solution of problem (4.4).

Finally, similarly as in [22, Proof of Theorem 4.5], uniqueness is a consequence of the positivity of f and of the decreasing monotonicity of the function $s \in (0, +\infty) \mapsto \frac{1}{\left(s+\frac{1}{n}\right)^\theta}$. Indeed, assume that there exist two different solutions u_n and $\overline{u}_n$ for problem

(4.5) then, setting $U_n = u_n - \overline{u}_n$ and $\lambda_{\min} = \min(\lambda_1, \lambda_2)$, taking $\psi = U_n$ in (4.5) written for u_n and $\overline{u}_n$, respectively, and finally subtracting the two equality, it follows

$$\begin{aligned}\lambda_{\min} \int_{\Omega} |\nabla U_n|^2 \,\mathrm{d}x + \beta \int_{\Gamma} [U_n]^2 \,\mathrm{d}\sigma &= \lambda_{\min} \int_{\Omega} |\nabla U_n|^2 \,\mathrm{d}x + \beta \int_{\Gamma} ([u_n] - [\overline{u}_n])^2 \,\mathrm{d}\sigma \\ &\le \int_{\Omega} \lambda \nabla U_n \cdot \nabla U_n \,\mathrm{d}x + \int_{\Gamma} (g([u_n]) - g([\overline{u}_n]))[U_n] \,\mathrm{d}\sigma \\ &= \int_{\Omega} \left(\frac{f_n}{\left(u_n + \frac{1}{n}\right)^{\theta}} - \frac{f_n}{\left(\overline{u}_n + \frac{1}{n}\right)^{\theta}} \right) (u_n - \overline{u}_n) \,\mathrm{d}x \le 0 \,,\end{aligned}$$

where, in the first inequality we use (3.2). Therefore, $u_n - \overline{u}_n = U_n = 0$ and the uniqueness of the solution of (4.5) is proved.
Moreover, by a standard procedure we obtain the following energy estimate for the solution u_n of (4.4):

$$\int_{\Omega} |\nabla u_n|^2 \,\mathrm{d}x + \int_{\Gamma} [u_n]^2 \,\mathrm{d}\sigma \le C, \tag{4.8}$$

where $C > 0$ does not depend on n. Hence, up to a subsequence, we get for u_n the same convergence stated in (4.3) for u_h and the limit u is nonnegative. Now, we let $n \to +\infty$ in (4.5). We note that the limit of the left-hand side is standard, because g is continuous and the sequence $\{[u_n]\}$ is strongly convergent. In order to pass to the limit in the right-hand side, we write

$$\int_{\Omega} \frac{f_n}{\left(u_n + \frac{1}{n}\right)^{\theta}} \psi \,\mathrm{d}x = \int_{\Omega \cap \{0 \le u_n \le \delta\}} \frac{f_n}{\left(u_n + \frac{1}{n}\right)^{\theta}} \psi \,\mathrm{d}x + \int_{\Omega \cap \{u_n > \delta\}} \frac{f_n}{\left(u_n + \frac{1}{n}\right)^{\theta}} \psi \,\mathrm{d}x := I^1_{n,\delta} + I^2_{n,\delta}$$

and, as in [12, proof of Theorem A.1], we obtain

$$\lim_{\delta \to 0} \lim_{n \to +\infty} I^2_{n,\delta} = \int_{\Omega \cap \{u > 0\}} \frac{f}{u^{\theta}} \psi \,\mathrm{d}x \,.$$

The crucial point is to prove that

$$\lim_{\delta \to 0} \lim_{n \to +\infty} I^1_{n,\delta} = 0 \,.$$

To this aim, following again the same approach as in [12, proof of Theorem A.1], we consider the function $Z_\delta : \mathbb{R} \to [0, +\infty)$ defined by

$$Z_\delta(s) = \begin{cases} 1\,, & \text{if } 0 \le s \le \delta\,; \\ -\frac{s}{\delta} + 2\,, & \text{if } \delta \le s \le 2\delta\,; \\ 0\,, & \text{if } s \ge 2\delta\,, \end{cases} \tag{4.9}$$

and we use as test function in (4.5) the function $Z_\delta(u_n)\psi$ (with ψ as above and $\psi \geq 0$), getting

$$I^1_{n,\delta} \leq \int_\Omega \lambda \nabla u_n \cdot \nabla \psi Z_\delta(u_n)\,\mathrm{d}x + \int_\Gamma g([u_n])\left(Z_\delta(u_n^{(2)})\psi^{(2)} - Z_\delta(u_n^{(1)})\psi^{(1)}\right)\,\mathrm{d}\sigma. \quad (4.10)$$

However, in the present case we have to proceed very carefully with the estimates, due to the presence of the nonlinearity g. Indeed, using (3.3) and recalling that $g(t) > 0$ if $t > 0$ and $g(t) < 0$ if $t < 0$ and the definition of Z_δ, we have

$$\begin{aligned}
&\int_\Gamma g([u_n])[Z_\delta(u_n)\psi]\,\mathrm{d}\sigma = \int_\Gamma g([u_n])Z_\delta(u_n^{(2)})\psi^{(2)}\,\mathrm{d}\sigma - \int_\Gamma g([u_n])Z_\delta(u_n^{(1)})\psi^{(1)}\,\mathrm{d}\sigma \\
\leq &\int_{\Gamma\cap\{u_n^{(1)}\leq u_n^{(2)}\}} g([u_n])Z_\delta(u_n^{(2)})\psi^{(2)}\,\mathrm{d}\sigma - \int_{\Gamma\cap\{u_n^{(2)}\leq u_n^{(1)}\}} g([u_n])Z_\delta(u_n^{(1)})\psi^{(1)}\,\mathrm{d}\sigma \\
\leq &\int_{\Gamma\cap\{u_n^{(1)}\leq u_n^{(2)}\leq 2\delta\}} g([u_n])\psi^{(2)}\,\mathrm{d}\sigma - \int_{\Gamma\cap\{u_n^{(2)}\leq u_n^{(1)}\leq 2\delta\}} g([u_n])\psi^{(1)}\,\mathrm{d}\sigma \\
\leq &\alpha \int_{\Gamma\cap\{u_n^{(1)}\leq u_n^{(2)}\leq 2\delta\}} (|u_n^{(2)}| + |u_n^{(1)}|)\psi^{(2)}\,\mathrm{d}\sigma + \alpha \int_{\Gamma\cap\{u_n^{(2)}\leq u_n^{(1)}\leq 2\delta\}} (|u_n^{(2)}| + |u_n^{(1)}|)\psi^{(1)}\,\mathrm{d}\sigma
\end{aligned} \quad (4.11)$$

and hence by (4.10) and (4.11) we infer

$$I^1_{n,\delta} \leq \int_\Omega \lambda \nabla u_n \cdot \nabla \psi Z_\delta(u_n)\,\mathrm{d}x + 4\alpha\delta ||\psi^{(2)} + \psi^{(1)}||_{L^1(\Gamma)}. \quad (4.12)$$

Taking into account that $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$, $u_n \to u$ strongly in $L^2(\Omega)$ and that Z_δ is continuous, we get

$$\lim_{n\to+\infty} I^1_{n,\delta} \leq \int_\Omega \lambda \nabla u \cdot \nabla \psi Z_\delta(u)\,\mathrm{d}x + 4\alpha\delta ||\psi^{(2)} + \psi^{(1)}||_{L^1(\Gamma)}.$$

Then, passing to the limit as $\delta \to 0$, we get

$$\lim_{\delta\to 0}\lim_{n\to+\infty} I^1_{n,\delta} \leq \int_{\Omega\cap\{u=0\}} \lambda \nabla u \cdot \nabla \psi\,\mathrm{d}x = 0\,, \quad (4.13)$$

where we used that $\nabla u = 0$ a.e. on the level set $\{u = 0\}$. Clearly, we have paid attention to choose $\delta \notin \mathcal{C} = \{\delta > 0 : |\{u(x) = \delta\}| > 0\}$, which is at most countable. Finally, the uniqueness and the strict positivity of the solution u can be proved as in [12, proof of Theorem A.1]. □

5. Homogenization

Once stated the existence and the uniqueness of the solution to problem (3.1) for every $\varepsilon > 0$, we focus our attention in passing to the limit for $\varepsilon \to 0$. We find a two-scale

homogenized system (see (5.6)–(5.11)), involving a pair of functions (u, u_1), where u depends only on the macro-variable x, while u_1 depends also on the microvariable y (thus keeping memory of the behavior at the micro-scale).
We stress again the fact that, when $k = 1$ (i.e. in the "strong" nonlinear case), differently from the linear case, u_1 cannot be factorized in terms of the gradient of u (see (5.9)), so that the homogenized system remains coupled. This calls for our assumption on the strict positivity of the source f, in order to assure that the homogenization limit u is strictly positive a.e. in Ω. Indeed, the strong maximum principle, which is the standard tool, generally used to this aim, cannot be applied to the resulting homogenized two-scale system (5.6)–(5.9).

5.1. **The strongly nonlinear case:** $k = 1$.

Theorem 5.1. *For $\varepsilon > 0$, let $u_\varepsilon \in V_0^\varepsilon(\Omega)$ be the weak solution of the problem* (3.1)*. Then, there exist $u \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega; V_\#(Y))$ with $\int_Y u_1(x,y)\,\mathrm{d}y = 0$ a.e. in Ω, such that, as $\varepsilon \to 0$, we have*

$$u_\varepsilon \to u\,, \qquad \textit{strongly in } L^2(\Omega)\,; \tag{5.1}$$

$$u_\varepsilon \xrightarrow{2-sc} u\,, \qquad \textit{in } L^2(\Omega \times Y)\,; \tag{5.2}$$

$$\chi_{\Omega\setminus\Gamma^\varepsilon} \nabla u_\varepsilon \xrightarrow{2-sc} \nabla u + \nabla_y u_1\,, \qquad \textit{in } L^2(\Omega \times Y)\ : \tag{5.3}$$

$$\frac{1}{\varepsilon}[u_\varepsilon] \xrightarrow{2-sc} [u_1]\,, \qquad \textit{in } L^2(\Omega; L^2(\Gamma))\,. \tag{5.4}$$

Moreover,

$$\left| \int_\Omega \frac{f}{u^\theta} \varphi \,\mathrm{d}x \right| < +\infty\,, \qquad \forall \varphi \in H_0^1(\Omega)\,, \tag{5.5}$$

and the pair (u, u_1) solves

$$-\operatorname{div}\left(\lambda_0 \nabla u + \int_Y \lambda \nabla_y u_1 \,\mathrm{d}y \right) = \frac{f}{u^\theta}\,, \qquad \textit{in } \Omega\,; \tag{5.6}$$

$$-\operatorname{div}_y \left(\lambda(\nabla u + \nabla_y u_1)\right) = 0\,, \qquad \textit{in } \Omega \times (E_1 \cup E_2)\,; \tag{5.7}$$

$$[\lambda(\nabla u + \nabla_y u_1) \cdot \nu] = 0\,, \qquad \textit{on } \Omega \times \Gamma\,; \tag{5.8}$$

$$g([u_1]) = \lambda_2(\nabla u + \nabla_y u_1) \cdot \nu\,, \qquad \textit{on } \Omega \times \Gamma\,; \tag{5.9}$$

$$u > 0\,, \qquad \textit{in } \Omega\,; \tag{5.10}$$

$$u = 0\,, \qquad \textit{on } \partial\Omega\,, \tag{5.11}$$

where λ_0 and λ are defined at the beginning of Subsection 3.

Remark 5.2. *Notice that the problem* (5.6)*–*(5.11) *admits at most one pair of solutions (u, u_1). Indeed, assume by contradiction that (u^i, u_1^i), for $i = 1, 2$ are two pairs of solutions and denote by $U = u^1 - u^2$ and $U_1 = u_1^1 - u_1^2$. Using U as test function* *in* (5.6) *written for* u^1 *and* U_1 *as test function in* (5.7) *written for* u_1^1*, adding the two equations, integrating by parts and using* (5.8)–(5.9)*, we get*

$$\int_\Omega\int_Y \lambda(\nabla u^1 + \nabla_y u_1^1)\cdot\nabla U\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega\int_Y \lambda(\nabla u^1 + \nabla_y u_1^1)\cdot\nabla_y U_1\,\mathrm{d}x\,\mathrm{d}y \\ + \int_\Omega\int_\Gamma g([u_1^1])[U_1]\,\mathrm{d}x\,\mathrm{d}\sigma(y) = \int_\Omega \frac{f}{(u^1)^\theta} U\,\mathrm{d}x\,.$$

Repeating the same procedure for (u^2, u_1^2) *and subtracting the equation for* (u^2, u_1^2) *from the equation for* (u^1, u_1^1)*, it follows*

$$\int_\Omega\int_Y \lambda|\nabla U + \nabla_y U_1|^2\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega\int_\Gamma \left(g([u_1^1]) - g([u_1^2])\right)[U_1]\,\mathrm{d}x\,\mathrm{d}\sigma(y) \\ = \int_\Omega f\left(\frac{1}{(u^1)^\theta} - \frac{1}{(u^2)^\theta}\right)(u^1 - u^2)\,\mathrm{d}x\,. \quad (5.12)$$

Recalling the property (3.2)*, we have*

$$\left(g([u_1^1]) - g([u_1^2])\right)\left([u_1^1] - [u_1^2]\right) \geq \beta\left([u_1^1] - [u_1^2]\right)^2 = \beta[U_1]^2\,. \quad (5.13)$$

By (5.12) *and* (5.13) *we get*

$$\begin{aligned} \beta\int_\Omega\int_\Gamma [U_1]^2\,\mathrm{d}x\,\mathrm{d}\sigma(y) &\leq \int_\Omega f\left(\frac{1}{(u^1)^\theta} - \frac{1}{(u^2)^\theta}\right)(u^1 - u^2)\,\mathrm{d}x\,, \\ \int_\Omega\int_Y \lambda|\nabla U + \nabla_y U_1|^2\,\mathrm{d}x\,\mathrm{d}y &\leq \int_\Omega f\left(\frac{1}{(u^1)^\theta} - \frac{1}{(u^2)^\theta}\right)(u^1 - u^2)\,\mathrm{d}x\,. \end{aligned} \quad (5.14)$$

Since the function $s \in (0, +\infty) \mapsto \frac{1}{s^\theta}$ *is decreasing, the right-hand side in the previous equalities is non positive, which implies* $[U_1] = 0$ *and* $|\nabla U + \nabla_y U_1| = 0$*. Moreover,*

$$\int_\Omega |\nabla U|^2\,\mathrm{d}x + \int_\Omega\int_Y |\nabla_y U_1|^2\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega |\nabla U|^2\,\mathrm{d}x + \int_\Omega\int_Y |\nabla_y U_1|^2\,\mathrm{d}x\,\mathrm{d}y \\ + 2\int_\Omega \nabla u\cdot\left(\int_Y \nabla_y U_1\,\mathrm{d}y\right)\mathrm{d}x = \int_\Omega\int_Y |\nabla U + \nabla_y U_1|^2\,\mathrm{d}x\,\mathrm{d}y = 0\,,$$

where we have taken into account that $\int_Y \nabla_y U_1\,\mathrm{d}y = 0$*, because of the* Y*-periodicity of* U_1 *and the fact that* $[U_1] = 0$*. Thus,* $\nabla U = \nabla_y U_1 = 0$*, which implies* $U = 0$ *in* Ω*, since it satisfies the homogeneous boundary condition, and* $U_1 = 0$*, since it has null mean average on* Y*.*

Proof of Theorem 5.3.

Step 1. We first state some a priori estimates. Let u_ε be the weak solution of problem (3.1). Taking $\psi = u_\varepsilon$ in (3.5) and recalling (3.2), we get

$$\int_\Omega |\nabla u_\varepsilon|^2 \,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}\sigma \le \int_\Omega \lambda_\varepsilon |\nabla u_\varepsilon|^2 \,\mathrm{d}x + \int_{\Gamma^\varepsilon} g\left(\frac{[u_\varepsilon]}{\varepsilon}\right)[u_\varepsilon]\,\mathrm{d}\sigma$$

$$= \int_\Omega f u_\varepsilon^{1-\theta} \,\mathrm{d}x \le ||f||_{L^{\frac{2}{1+\theta}}(\Omega)} ||u_\varepsilon||^{1-\theta}_{L^2(\Omega)}. \tag{5.15}$$

By Theorem 2.1, it follows

$$||u_\varepsilon||^{1-\theta}_{L^2(\Omega)} \le C\left[\int_\Omega |\nabla u_\varepsilon|^2 \,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}\sigma\right]^{\frac{1-\theta}{2}}. \tag{5.16}$$

Hence, (5.15) and (5.16) imply that there exists $C > 0$ such that

$$\int_\Omega |\nabla u_\varepsilon|^2 \,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}\sigma \le C\|f\|^{\frac{2}{1+\theta}}_{L^{\frac{2}{1+\theta}}(\Omega)} \qquad \forall\, \varepsilon > 0. \tag{5.17}$$

Moreover, by (5.16) and (5.17) we also get that there exists $C > 0$ such that

$$\int_\Omega u_\varepsilon^2 \,\mathrm{d}x \le C||f||^{\frac{2}{1+\theta}}_{L^{\frac{2}{1+\theta}}(\Omega)} \qquad \forall \varepsilon > 0. \tag{5.18}$$

Finally, arguing as in [12, Proposition 3.2], there holds

$$\left|\int_\Omega \frac{f}{u_\varepsilon^\theta}\psi(x)dx\right| \le \max(\lambda_1, \lambda_2)||\nabla\psi||_{L^2(\Omega)}||\nabla u_\varepsilon||_{L^2(\Omega)} \quad \forall\, \psi \in H^1_0(\Omega). \tag{5.19}$$

Moreover, by (3.3) and (5.17), it follows also that

$$\varepsilon\int_{\Gamma^\varepsilon} g^2\left(\frac{[u_\varepsilon]}{\varepsilon}\right) \mathrm{d}\sigma \le C\|f\|^{\frac{2}{1+\theta}}_{L^{\frac{2}{1+\theta}}(\Omega)}, \tag{5.20}$$

for a suitable constant $C > 0$.
Step 2. We now use the above estimates to achieve suitable compactness properties for the two-scale convergence, which allows to obtain the homogenized limit problem. By (5.17),(5.18) and [25, Proposition 5.5] we get that (5.1)–(5.4) hold. Hence, taking into account (5.17) and (5.1) and passing to the limit in (5.19), when $\varepsilon \to 0$, by Fatou's Lemma we get (5.5). Moreover, by Theorem 2.8, there exists $\mu \in L^2(\Omega \times \Gamma)$ such that, up to a subsequence,

$$g([u_\varepsilon]/\varepsilon) \xrightarrow{2-sc} \mu\,, \qquad \text{for } \varepsilon \to 0. \tag{5.21}$$

We recall that u is nonnegative, being the limit of the sequence of positive solutions u_ε. Hence, taking into account that f is strictly positive, by (5.5) we infer that u is strictly positive a.e. in Ω.

In order to pass to the two-scale limit in (3.5), we choose as test function

$$\psi(x) = \varphi(x) + \varepsilon\Phi\left(x, \frac{x}{\varepsilon}\right) \tag{5.22}$$

with $\varphi \in \mathcal{C}_c^1(\Omega)$ and $\Phi \in \mathcal{C}_c^1(\Omega; \mathfrak{L}_\#(Y))$. Then, we get

$$\int\limits_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x + \varepsilon \int\limits_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla_x \Phi \,\mathrm{d}x + \int\limits_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla_y \Phi \,\mathrm{d}x \\ + \varepsilon \int\limits_{\Gamma^\varepsilon} g\left(\frac{[u_\varepsilon]}{\varepsilon}\right)[\Phi]\,\mathrm{d}\sigma = \int\limits_\Omega \frac{f}{u_\varepsilon^\theta}\varphi\,\mathrm{d}x + \varepsilon \int\limits_\Omega \frac{f}{u_\varepsilon^\theta}\Phi\,\mathrm{d}x. \tag{5.23}$$

By (5.3),(5.4) and (5.21), as $\varepsilon \to 0$, the left-hand side of (5.23) converges to

$$\int\limits_\Omega\int\limits_Y \lambda(\nabla u + \nabla_y u_1)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y + \int\limits_\Omega\int\limits_Y \lambda(\nabla u + \nabla_y u_1)\cdot\nabla_y\Phi\,\mathrm{d}x\,\mathrm{d}y \\ + \int\limits_\Omega\int\limits_\Gamma \mu[\Phi]\,\mathrm{d}x\,\mathrm{d}\sigma(y). \tag{5.24}$$

We now focus our attention on the right-hand side of (5.23) and we set

$$I_\varepsilon := \int\limits_\Omega \frac{f}{u_\varepsilon^\theta}\varphi\,\mathrm{d}x\,, \qquad J_\varepsilon := \varepsilon\int\limits_\Omega \frac{f}{u_\varepsilon^\theta}\Phi\,\mathrm{d}x\,. \tag{5.25}$$

As proved in [12],

$$J_\varepsilon \to 0\,, \qquad \text{as } \varepsilon \to 0. \tag{5.26}$$

In order to study the limit of I_ε, having in mind the decomposition $\varphi = \varphi^+ - \varphi^-$ (notice again that the Lipschitz continuity of φ is enough for our purposes), we may assume $\varphi \geq 0$. Moreover, similarly as in the proof of Theorem 4.1, we have to split the singular term into the part near to and far away from the singularity. To this purpose, we write

$$I_\varepsilon = \int\limits_{\Omega\cap\{0<u_\varepsilon\leq\delta\}} \frac{f}{u_\varepsilon^\theta}\varphi\,\mathrm{d}x + \int\limits_{\Omega\cap\{u_\varepsilon>\delta\}} \frac{f}{u_\varepsilon^\theta}\varphi\,\mathrm{d}x := I^1_{\varepsilon,\delta} + I^2_{\varepsilon,\delta}\,. \tag{5.27}$$

where, by the Lebesgue Dominated Convergence Theorem and taking into account that $0 \leq \frac{f}{u_\varepsilon^\theta}\varphi \leq \frac{f}{\delta^\theta}\varphi \in L^1(\Omega)$ in the set $\{u_\varepsilon > \delta\}$ (here it is crucial that φ is bounded), we get

$$\lim_{\delta\to0}\lim_{\varepsilon\to0} I^2_{\varepsilon,\delta} = \int\limits_{\Omega\cap\{u>0\}} \frac{f}{u^\theta}\varphi\,\mathrm{d}x = \int\limits_\Omega \frac{f}{u^\theta}\varphi\,\mathrm{d}x\,, \tag{5.28}$$

once we have taken $\delta \notin \mathcal{C} = \{\delta > 0 : |\{u(x) = \delta\}| > 0\}$, which is at most countable (exactly as in [12, Proof of Theorem 4.1]) and we recall that $u > 0$ a.e. in Ω.

Moreover, using as test function in (3.5) the function $Z_\delta(u_\varepsilon)\varphi$, with Z_δ defined in (4.9) and φ as above, and recalling that $s \in [0, +\infty) \mapsto Z_\delta(s)$ is decreasing and that $g(t) > 0$ if $t > 0$ and $g(t) < 0$ if $t < 0$, we arrive at

$$
\begin{aligned}
I^1_{\varepsilon,\delta} &\leq \int\limits_{\Omega} \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi \, Z_\delta(u_\varepsilon) \, \mathrm{d}x \\
&= \int\limits_{\Omega} \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi [Z_\delta(u_\varepsilon) - Z_\delta(u)] \, \mathrm{d}x + \int\limits_{\Omega} \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi \, Z_\delta(u) \, \mathrm{d}x
\end{aligned} \tag{5.29}
$$

since

$$
\int\limits_{\Gamma^\varepsilon} g\left(\frac{[u_\varepsilon]}{\varepsilon}\right) \left(Z_\delta(u_\varepsilon^{(2)}) - Z_\delta(u_\varepsilon^{(1)})\right) \varphi \, \mathrm{d}x \leq 0
$$

and

$$
\int\limits_{\Omega\cap\{\delta\leq u_\varepsilon\leq 2\delta\}} \frac{f}{u_\varepsilon^\theta} Z_\delta(u_\varepsilon)\varphi \, \mathrm{d}x \geq 0.
$$

In order to pass to the two-scale limit in (5.29), we have to take into account that $\lambda_\varepsilon \nabla u_\varepsilon$ is bounded in $L^2(\Omega)$ and $Z_\delta(u_\varepsilon) - Z_\delta(u) \to 0$ strongly in $L^2(\Omega)$ (since $s \mapsto Z_\delta(s)$ is continuous and (5.1) holds), so that the first integral in the second line of (5.29) vanishes, while in the second integral, thanks to Remark 2.3, we can take $\lambda_\varepsilon \nabla\varphi Z_\delta(u)$ as admissible test function for the two-scale convergence. Therefore, recalling that u is strictly positive a.e. in Ω (as proved at the beginning of Step 2), it follows that $|\{u = 0\}| = 0$; then, we get

$$
\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I^1_{\varepsilon,\delta} \leq \int\limits_{\Omega\cap\{u=0\}} \int\limits_{Y} |\lambda(\nabla u + \nabla_y u_1)| \, |\nabla\varphi| \, \mathrm{d}x \, \mathrm{d}y = 0. \tag{5.30}
$$

Then, passing to the limit for $\varepsilon \to 0$ in (5.23), by (5.24), (5.28), (5.30) and taking into account the density of our test functions in $H^1_0(\Omega) \times L^2(\Omega; V_\#(Y))$, we obtain

$$
\begin{aligned}
&\int\limits_{\Omega\times Y} \lambda(\nabla u + \nabla_y u_1) \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}y + \int\limits_{\Omega\times Y} \lambda(\nabla u + \nabla_y u_1) \cdot \nabla_y \Phi \, \mathrm{d}x \, \mathrm{d}y \\
&\qquad + \int\limits_{\Omega}\int\limits_{\Gamma} \mu[\Phi] \, \mathrm{d}x \, \mathrm{d}\sigma(y) = \int\limits_{\Omega} \frac{f}{u^\theta}\varphi \, \mathrm{d}x \, .
\end{aligned} \tag{5.31}
$$

It remains to identify μ. To this purpose, we follow the Minty monotone operators method. Let us consider a sequence of test function $\psi_k(x) = \phi_0^k(x) + \varepsilon\phi_1^k\left(x, \frac{x}{\varepsilon}\right) + t\varepsilon\phi_2\left(x, \frac{x}{\varepsilon}\right)$, with $\phi_0^k \in \mathcal{C}^1_c(\Omega)$, $\phi_0^k \to u$ strongly in $H^1_0(\Omega)$, $\phi_1^k \in \mathcal{C}^1\big(\overline{\Omega}; \mathfrak{L}_\#(Y)\big)$, with ϕ_1^k vanishing on $\partial\Omega$, $\phi_1^k \to u_1$ strongly in $L^2\big(\Omega; V_\#(Y)\big)$, and $\phi_2 \in \mathcal{C}^1_c\big(\Omega; \mathfrak{L}_\#(Y)\big)$.

Taking into account the monotonicity assumption (3.2) on g we obtain

$$\int\limits_{\Omega} \lambda_\varepsilon (\nabla u_\varepsilon - \nabla \psi_k) \cdot (\nabla u_\varepsilon - \nabla \psi_k)\,\mathrm{d}x + \varepsilon \int\limits_{\Gamma^\varepsilon} \left(g\left(\frac{[u_\varepsilon]}{\varepsilon}\right) - g\left(\frac{[\psi_k]}{\varepsilon}\right)\right) \left(\frac{[u_\varepsilon]}{\varepsilon} - \frac{[\psi_k]}{\varepsilon}\right) \mathrm{d}\sigma \geq 0\,. \quad (5.32)$$

Notice that the function $u_\varepsilon - \psi_k$ can be taken as a test function in the weak formulation (3.5); hence, inequality (5.32) can be rewritten as

$$\int\limits_{\Omega} \frac{f}{u_\varepsilon^\theta}(u_\varepsilon - \psi_k)\,\mathrm{d}x - \int\limits_{\Omega} \lambda_\varepsilon \nabla \psi_k \cdot (\nabla u_\varepsilon - \nabla \psi_k)\,\mathrm{d}x - \varepsilon \int\limits_{\Gamma^\varepsilon} g\left(\frac{[\psi_k]}{\varepsilon}\right) \left(\frac{[u_\varepsilon]}{\varepsilon} - \frac{[\psi_k]}{\varepsilon}\right) \mathrm{d}\sigma \geq 0\,. \quad (5.33)$$

Hence, passing to the two-scale limit as $\varepsilon \to 0$, it follows

$$\int\limits_{\Omega} \frac{f}{u^\theta}(u - \phi_0^k)\,\mathrm{d}x - \iint\limits_{\Omega\, Y} \lambda\big(\nabla \phi_0^k + \nabla_y(\phi_1^k + t\phi_2)\big) \cdot \big(\nabla u + \nabla_y u_1 - \nabla \phi_0^k - \nabla_y(\phi_1^k + t\phi_2)\big)\,\mathrm{d}x\,\mathrm{d}y - \iint\limits_{\Omega\, \Gamma} g\left([\phi_1^k + t\phi_2]\right)\left([u_1] - [\phi_1^k + t\phi_2]\right)\,\mathrm{d}x\,\mathrm{d}\sigma(y) \geq 0\,, \quad (5.34)$$

where we have taken into account that

$$\int\limits_{\Omega} \frac{f}{u_\varepsilon^\theta} u_\varepsilon\,\mathrm{d}x = \int\limits_{\Omega} f u_\varepsilon^{1-\theta}\,\mathrm{d}x \to \int\limits_{\Omega} \frac{f}{u^\theta} u\,,$$

as follows by (5.1) and the Lebesgue Dominated Convergence Theorem, and

$$\int\limits_{\Omega} \frac{f}{u_\varepsilon^\theta} \psi_k\,\mathrm{d}x \to \int\limits_{\Omega} \frac{f}{u^\theta} \phi_0^k\,,$$

as follows by (5.26), (5.28) and (5.30).
Now, letting $k \to +\infty$ and taking into account that $\phi_0^k \to u$ strongly in $H^1(\Omega)$ and $\phi_1^k \to u_1$ strongly in $L^2(\Omega; V_\#(Y))$, we obtain

$$\int\limits_{\Omega} \int\limits_{Y} \lambda(\nabla u + \nabla_y u_1 + t\nabla_y \phi_2) \cdot t\nabla_y \phi_2\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\Omega} \int\limits_{\Gamma} g\left([u_1 + t\phi_2]\right) t[\phi_2]\,\mathrm{d}x\,\mathrm{d}\sigma(y) \geq 0\,, \quad (5.35)$$

since, by (5.31) with $\varphi = u - \phi_0^k$ and $\Phi \equiv 0$, it follows

$$\left| \int\limits_{\Omega} \frac{f}{u^\theta}(u - \phi_0^k)\,\mathrm{d}x \right| = \left| \int\limits_{\Omega}\int\limits_{Y} \lambda(\nabla u + \nabla_y u_1) \cdot (\nabla u - \nabla \phi_0^k)\,\mathrm{d}x \right|$$
$$\leq C \|\nabla u + \nabla_y u_1\|_{L^2(\Omega\times Y)} \|\nabla u - \nabla \phi_0^k\|_{L^2(\Omega)} \to 0\,.$$

Moreover, again by (5.31), with $\varphi = 0$ and $\Phi = \phi_2$, we get that (5.35) can be written as

$$t^2 \int\limits_{\Omega}\int\limits_{Y} \lambda \nabla_y \phi_2 \cdot \nabla_y \phi_2\,\mathrm{d}x\,\mathrm{d}y$$
$$- t \int\limits_{\Omega}\int\limits_{\Gamma} \mu[\phi_2]\,\mathrm{d}x\,\mathrm{d}\sigma(y) + t \int\limits_{\Omega}\int\limits_{\Gamma} g\left([u_1 + t\phi_2]\right)[\phi_2]\,\mathrm{d}x\,\mathrm{d}\sigma(y) \geq 0\,. \quad (5.36)$$

Assuming first $t > 0$ and then $t < 0$, dividing by t the previous equation and then letting $t \to 0$, we obtain

$$\int\limits_{\Omega}\int\limits_{\Gamma} \mu[\phi_2]\,\mathrm{d}x\,\mathrm{d}\sigma(y) = \int\limits_{\Omega}\int\limits_{\Gamma} g\left([u_1]\right)[\phi_2]\,\mathrm{d}x\,\mathrm{d}\sigma(y)\,,$$

which gives

$$\mu = g\left([u_1]\right)\,. \quad (5.37)$$

Therefore, by (5.31) and (5.37), it follows that the pair (u, u_1) satisfies

$$\int\limits_{\Omega}\int\limits_{Y} \lambda(\nabla u + \nabla_y u_1) \cdot \nabla\varphi\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\Omega}\int\limits_{Y} \lambda(\nabla u + \nabla_y u_1) \cdot \nabla_y \Phi\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int\limits_{\Omega}\int\limits_{\Gamma} g([u_1])[\Phi]\,\mathrm{d}x\,\mathrm{d}\sigma(y) = \int\limits_{\Omega} \frac{f}{u^\theta}\varphi\,\mathrm{d}x. \quad (5.38)$$

Finally, taking first $\varphi = 0$ and then $\Phi = 0$ in (5.38), and recalling that $u > 0$ a.e. in Ω, we obtain that u is a weak solution of the problem (5.6)–(5.11). Moreover, by Remark 5.2, it follows that the whole sequence $\{u_\varepsilon\}$ converges and the thesis is accomplished.

□

5.2. **The weakly nonlinear case:** $k = 0$**.**

Theorem 5.3. *For $\varepsilon > 0$, let $u_\varepsilon \in V_0^\varepsilon(\Omega)$ be the weak solution of the problem* (3.1)*. Then, there exist $u \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega; V_\#(Y))$ with $\int_Y u_1(x,y)\,\mathrm{d}y = 0$ a.e. in Ω, satisfying* (5.1)-(5.5)*, as $\varepsilon \to 0$. Moreover, the pair (u, u_1) solves* (5.6)–(5.8),(5.10),(5.11) *and*

$$g'(0)[u_1] = \lambda_2(\nabla u + \nabla_y u_1) \cdot \nu\,, \qquad \text{on } \Omega \times \Gamma \quad (5.39)$$

instead of (5.9)*.*

Proof. It is easy to check that, as in Step 1 of the proof of Theorem 5.3, we get that (5.1)-(5.5) and (5.10) still hold and, in particular, from (5.17), we have also

$$\frac{1}{\varepsilon}\int_{\Gamma^\varepsilon}[u_\varepsilon]^2\,\mathrm{d}\sigma \le C, \tag{5.40}$$

with C independent on ε. In order to pass to the two-scale limit in (3.5), we choose the same test function ψ defined in (5.22) and we obtain

$$\int_\Omega \lambda_\varepsilon \nabla u_\varepsilon\cdot\nabla\varphi\,\mathrm{d}x + \varepsilon\int_\Omega \lambda_\varepsilon\nabla u_\varepsilon\cdot\nabla_x\Phi\,\mathrm{d}x + \int_\Omega \lambda_\varepsilon\nabla u_\varepsilon\cdot\nabla_y\Phi\,\mathrm{d}x$$

$$+\int_{\Gamma^\varepsilon} g\left([u_\varepsilon]\right)[\Phi]\,\mathrm{d}\sigma = \int_\Omega \frac{f}{u_\varepsilon^\theta}\varphi\,\mathrm{d}x + \varepsilon\int_\Omega \frac{f}{u_\varepsilon^\theta}\Phi\,\mathrm{d}x. \tag{5.41}$$

The right hand-side and the first three terms in the left hand-side of (5.41) can be treated as in the case $k=1$. The main difference occurs in the limit of the fourth integral. In particular, recalling that $g(0)=0$, we have

$$\varepsilon\int_{\Gamma^\varepsilon}\frac{1}{\varepsilon}g([u_\varepsilon])[\Phi]\,\mathrm{d}\sigma = g'(0)\varepsilon\int_{\Gamma^\varepsilon}\frac{[u_\varepsilon]}{\varepsilon}[\Phi]\,\mathrm{d}\sigma + \frac{\varepsilon}{2}\int_{\Gamma^\varepsilon}\frac{[u_\varepsilon]^2}{\varepsilon}g''(\xi_\varepsilon)[\Phi]\,\mathrm{d}\sigma, \tag{5.42}$$

for a suitable ξ_ε, with $|\xi_\varepsilon|\le|[u_\varepsilon]|$. Since $g\in C^2(\mathbb{R})\cap W^{2,\infty}(\mathbb{R})$ and $\Phi\in\mathcal{C}^1_c(\Omega;\mathfrak{L}_\#(Y))$, using (5.40), we obtain $\int_{\Gamma^\varepsilon}\frac{[u_\varepsilon]^2}{\varepsilon}g''(\xi_\varepsilon)[\Phi]\,\mathrm{d}\sigma$ is uniformly bounded with respect to ε. Hence, using (5.4), as $\varepsilon\to 0$, it follows

$$\int_{\Gamma^\varepsilon} g([u_\varepsilon])[\Phi]\,\mathrm{d}\sigma \to g'(0)\int_{\Omega\times\Gamma}[u_1][\Phi]\,\mathrm{d}\sigma\,.$$

□

Remark 5.4. *Arguing as in Remark 5.2, also in this case the homogenized system has a unique solution* $(u,u_1)\in H^1_0(\Omega)\times L^2(\Omega;V_\#(Y))$ *with* $\int_Y u_1(x,y)\,\mathrm{d}y=0$ *a.e. in* Ω. *Moreover, as pointed out in the Introduction, due to the linear structure of the left hand-side in* (5.39), *we can factorize* u_1 *in terms of* ∇u, *decoupling the homogenized system and characterizing* u *by means of a single elliptic equation. More precisely, we set*

$$u_1(x,y)=\chi(y)\cdot\nabla u(x)\,,$$

where $\chi=(\chi_1,\dots,\chi_N)$ *and* $\chi_j\in V_\#(Y)$ *with* $\int_Y\chi_j\,\mathrm{d}y=0$, *for each* $j=1,\dots,N$, *satisfying*

$$\begin{aligned}
&-\operatorname{div}_y(\lambda(\nabla_y\chi_j+\mathbf{e}_j))=0\,, &&\text{in } E_1\cup E_2\,;\\
&[\lambda(\nabla_y\chi_j+\mathbf{e}_j)\cdot\nu]=0\,, &&\text{on } \Gamma\,;\\
&g'(0)\,[\chi_j]=\lambda_2(\nabla_y\chi_j+\mathbf{e}_j)\cdot\nu\,, &&\text{on } \Gamma\,.
\end{aligned}$$

Closely following [12, Remark 4.3]*, we obtain the following single scale homogenized problem*

$$\begin{aligned} & -\operatorname{div}(A_{hom}\nabla u) = \frac{f}{u^\theta}\,, && \textit{in } \Omega\,; \\ & u > 0\,, && \textit{in } \Omega\,; \\ & u = 0\,, && \textit{on } \partial\Omega\,, \end{aligned}$$

where the symmetric and positive definite matrix A_{hom} is defined as

$$A^{ij}_{hom} = \lambda_0\delta_{ij} + \int\limits_Y \lambda\partial_i\chi_j \,\mathrm{d}y = \int\limits_Y \lambda\nabla(\chi_i + y_i)\cdot\nabla(\chi_j + y_j)\,\mathrm{d}y + g'(0)\int\limits_\Gamma [\chi_i][\chi_j]\,\mathrm{d}\sigma\,,$$

(see, also, [26, end of Section 3.1]*).*

Remark 5.5. *We point out that Theorem 5.3 holds also in the more general assumption $f \geq 0$, as a consequence of the decoupling of the two-scale homogenized system. Indeed, in this case, it is possible to apply the strong maximum principle to the resulting single-scale equation (as done in* [12, Theorem 4.1]*), thus obtaining the strict positivity of the homogenization limit u.*

6. Declarations

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