A NEW NONAUTONOMOUS CHAIN RULE IN BV

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Abstract. The aim of this note is to present a new nonautonomous chain rule formula for the distributional derivative of the composite function $v(x) = B(x, u(x))$, where $u : \mathbb{R}^N \to \mathbb{R}$ is a scalar function of bounded variation and B admits a special integral form in terms of a locally bounded function $b(x,t)$, with $b(\cdot,t)$ of bounded variation. It is an useful tool especially in view to applications to semicontinuity results for integral functional (see [1, 8, 9, 10]) and to conservation laws (see [5, 6]).

1. Introduction

In this note we present a new nonautonomous chain rule formula in the scalar case for the distributional derivative of the composite function $v(x) = B(x, u(x))$, with $u : \mathbb{R}^N \to \mathbb{R}$ a scalar function of bounded variation and $B(x,t) = \int_0^t b(x,s)ds$, where $b(x,t)$ is locally bounded (which implies that $B(x,\cdot)$ is Lipschitz continuous) and $b(\cdot,t)$ has bounded variation.

In 1967, A.I. Vol'pert in [13] considers a general B in the autonomous case and by requiring the Lipschitz continuity of B, proved that the following identity holds in the sense of measures:

$$Dv = \nabla B(u)\nabla u\, \mathcal{L}^N + \nabla B(\widetilde{u})D^c u + [B(u^+) - B(u^-)]\, \nu_u\, \mathcal{H}^{N-1} \llcorner J_u\,, \tag{1}$$

where

$$Du = \nabla u\, \mathcal{L}^N + D^c u + \nu_u\, \mathcal{H}^{N-1} \llcorner J_u \tag{2}$$

is the usual decomposition of Du in its absolutely continuous part ∇u with respect to the Lebesgue measure $\mathcal{L}^N$, its Cantor part $D^c u$ and its jumping part, which is represented by the restriction of the $(N-1)$-dimensional Hausdorff measure to the jump set J_u. Moreover, ν_u denotes the measure

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theoretical unit normal to J_u, $\widetilde{u}$ is the approximate limit and u^+, u^- are the approximate limits from both sides of J_u.

The validity of (1) is stated also in the vectorial case (see Theorem 3.96 in [4] for $B \in C^1$). The situation is significantly more complicated if B is only a Lipschitz continuous function. In this case, the general chain rule is false, while a weaker form of the formula was proved by Ambrosio and Dal Maso in [3] (see also [12]).

On the other hand, in some recent papers a remarkable effort is devoted to establish chain rule formulas with an explicit dependence on the space variable x (see [1, 5, 8, 9, 10]). Notice that the new term of derivation with respect to x needs a particular attention. The proofs are achieved by regularizing $B(\cdot, t)$ with fixed t, by applying the Ambrosio-Dal Maso formula to the regularized functions and finally by passing to the limit in each term.

More recently, a very general nonautonomous formula is proven in [2] for vector functions $u \in BV$. Here, the first assumption is a C^1 dependence of $B(x, \cdot)$ with an uniform bound on $\partial_t B(x,t)$. Concerning the x-derivative, it is required the existence of a Radon measure σ bounding from above all measures $|D_x B(\cdot, t)|$, uniformly with respect to $t \in \mathbb{R}$.

The aim of this note is to consider the special case of

$$B(x,t) = \int_0^t b(x,s)ds.$$

In the spirit of Theorem 3.1 below proved in [9] we find a chain rule in this situation. We assume that b is BV in x and it is locally bounded (then $B(\cdot, t)$ is BV and $B(x, \cdot)$ is Lipschitz continuous) and we find an explicit form for the term involving the x-derivation, which is described in [9] by a Fubini's type inversion of integration order.

In the spirit of [2] we require the existence of a Radon measure $\overline{\sigma}$ bounding from above all measures $|D_x b(\cdot, t)|$, uniformly with respect to $t \in \mathbb{R}$. We prove that for any $u \in BV_{\text{loc}}$ the composite function $v(x) = B(x, u(x))$ belongs to BV_{loc} and it is shown the existence of a countably $\mathcal{H}^{N-1}$-rectifiable set $\overline{\mathcal{N}}$, independent of u and containing the jump set of $B(\cdot, t)$ for every $t \in \mathbb{R}$, such that the jump set of v is contained in $\overline{\mathcal{N}} \cup J_u$. A chain rule is obtained (see Theorem 4.2) by requiring further uniformity conditions, but without assuming any continuity assumptions. The result here presented will be proven in a forthcoming paper.

2. Definitions and preliminaries

In this section we recall some preliminary results and basic definitions (see [4] and [11]).
Let E be a measurable subset of $\mathbb{R}^N$. The *density* $D(E;x)$ of E at a point $x \in \mathbb{R}^N$ is defined by

$$D(E;x) = \lim_{\varrho \to 0} \frac{\mathcal{L}^N(E \cap B_\rho(x))}{\omega_N \rho^N},$$

whenever this limit exists, where ω_N is the measure of the unit ball and $B_\rho(x)$ denotes the ball centered at x with radius ρ.

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u : \Omega \to \mathbb{R}$ be a measurable function. The *upper and lower approximate limits* of u at a point $x \in \Omega$ are defined as

$$\begin{aligned} u^+(x) &= \inf\{t \in \mathbb{R} : D(\{u > t\}; x) = 0\}, \\ u^-(x) &= \sup\{t \in \mathbb{R} : D(\{u < t\}; x) = 0\}, \end{aligned} \tag{3}$$

respectively. The quantities $u^+(x)$, $u^-(x)$ are well defined (possibly equal to $\pm\infty$) at every $x \in \Omega$, and $u^-(x) \le u^+(x)$. The functions u^+, $u^- : \Omega \to [-\infty, \infty]$ are Borel measurable.
We say that u is *approximately continuous* at a point $x \in \Omega$ if $u^+(x) = u^-(x) \in \mathbb{R}$. In this case, we set $\widetilde{u}(x) = u^+(x) = u^-(x)$ and call $\widetilde{u}(x)$ the *approximate limit* of u at x. The set of all points in Ω where u is approximately continuous is a Borel set which will be denoted by C_u and called the set of *approximate continuity* of u. The set $S_u = \Omega \setminus C_u$ will be referred to as the set of *approximate discontinuity* of u.
Finally, by u^* we denote the *precise representative* of u which is defined by

$$u^*(x) = \frac{u^+(x) + u^-(x)}{2}$$

if $u^+(x), u^-(x) \in \mathbb{R}$, $u^*(x) = 0$ otherwise.

A locally integrable function u is said to be *approximately differentiable* at a point $x \in C_u$ if there exists $\nabla u(x) \in \mathbb{R}^N$ such that

$$\lim_{\rho \to 0} \frac{1}{\rho^{N+1}} \int_{B_\rho(x)} |u(y) - \widetilde{u}(x) - \langle \nabla u(x), y - x \rangle| \, dy = 0 \,. \tag{4}$$

Here, $\langle \cdot, \cdot \rangle$ stands for scalar product in $\mathbb{R}^N$. The vector $\nabla u(x)$ is called the *approximate differential* of u at x.

A function $u \in L^1(\Omega)$ is said to be of *bounded variation* if its distributional gradient Du is an $\mathbb{R}^N$-valued Radon measure in Ω and the total variation $|Du|$ of Du is finite in Ω. The space of all functions of bounded variation in Ω is denoted by $BV(\Omega)$, while the notation $BV_{\rm loc}(\Omega)$ will be reserved for the space of those functions $u \in L^1_{\rm loc}(\Omega)$ such that $u \in BV(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.

Let $u \in BV(\Omega)$. Then it can be proved that

$$\lim_{\rho\to 0} \fint_{B_\rho(x)} |u(y) - \widetilde{u}(x)|\, dy = 0 \qquad \text{for } \mathcal{H}^{N-1}\text{–a.e. } x \in C_u$$

and that u is approximately differentiable for $\mathcal{L}^N$-a.e. x. Moreover, the functions u^- and u^+ are finite $\mathcal{H}^{N-1}$-a.e. and for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ there exists a unit vector $\nu_u(x)$ such that

$$\lim_{\rho\to 0} \fint_{B^\pm_\rho(x;\nu_u(x))} |u(y) - u^\pm(x)|\, dy = 0\,, \tag{5}$$

where $B^+_\rho(x;\nu_u(x)) = \{y \in B_\rho(x) : \langle y - x, \nu_u(x)\rangle > 0\}$, and $B^-_\varrho(x;\nu_u(x))$ is defined analogously. The set of all points in S_u where the equalities in (5) are satisfied is called the *jump set* of u and is denoted by J_u.
If u is a BV function, we denote by $D^a u$ the absolutely continuous part of Du with respect to Lebesgue measure. The singular part, denoted by $D^s u$, is split into two more parts, the *jump part* $D^j u$ and the *Cantor part* $D^c u$, defined by

$$D^j u = D^s u \llcorner J_u, \qquad\qquad D^c u = D^s u - D^j u\,.$$

Finally, we denote by $\widetilde{D}u$ the *diffuse part* of Du, defined by

$$\widetilde{D}u = D^a u + D^c u\,.$$

3. The chain rule in $BV(\mathbb{R}^N)$ proven in [9]

In the paper [9] the authors deal with a general chain rule formula in $BV(\mathbb{R}^N)$ for functions whose dependence in x is BV. More precisely, the following theorem is proved for particular functions of the type $B(x,t) = \int_0^t b(x,s)\, ds$.

Theorem 3.1. *Let $b : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Borel function. Assume that*

(α) *the function $b(x,t)$ is locally bounded;*
(β) *for every $t \in \mathbb{R}$ the function $b(\cdot,t) \in BV(\mathbb{R}^N)$;*

(γ) *for any compact set* $H \subset \mathbb{R}$,

$$\int_H |D_x b(\cdot, t)|(\mathbb{R}^N)\, dt < +\infty\,,$$

where $D_x b(\cdot, t)$ *is the distributional gradient of the map* $x \mapsto b(x, t)$.

Then for every $u \in BV(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$, *the function* $v : \mathbb{R}^N \to \mathbb{R}$, *defined by*

$$v(x) := \int_0^{u(x)} b(x, t)\, dt\,,$$

belongs to $BV_{\text{loc}}(\mathbb{R}^N)$ *and for any* $\phi \in C^1_0(\mathbb{R}^N)$ *we have*

$$(6)\quad \int_{\mathbb{R}^N} \nabla\phi(x) v(x)\, dx$$

$$= -\int_{-\infty}^{+\infty} dt \int_{\mathbb{R}^N} \operatorname{sgn}(t) \chi^*_{\Omega_{u,t}}(x)\phi(x)\, dD_x b(x, t) - \int_{\mathbb{R}^N} \phi(x) b(x, u(x))\nabla u(x)\, dx$$

$$- \int_{\mathbb{R}^N} \phi(x)\widetilde{b}(x, \widetilde{u}(x))\, dD^c u - \int_{J_u} \phi(x) \left[\int_{u^-(x)}^{u^+(x)} b^*(x, t)\, dt\right] \nu_u(x)\, d\mathcal{H}^{N-1},$$

where J_u *is the jump set of* u, $\Omega_{u,t} = \{x \in \mathbb{R}^N : t$ *belongs to the segment of endpoints* 0 *and* $u(x)\}$ *and* $\chi^*_{\Omega_{u,t}}$ *and* $b^*(\cdot, t)$ *are, respectively, the precise representatives of* $\chi_{\Omega_{u,t}}$ *and* $b(\cdot, t)$.

Remark 3.2. *Notice that* $b^*(x, t) = (b^+(x, t) + b^-(x, t))/2$, *where* $b^+(x, t)$ *and* $b^-(x, t)$ *are the upper and lower approximate limits of* $b(\cdot, t)$ *at a point* x. *The function* $b(\cdot, t)$ *is approximately continuous at a point* x *if* $b^+(\cdot, t) = b^-(\cdot, t) \in \mathbb{R}$. *In this case, we set* $\widetilde{b}(\cdot, t) = b^+(\cdot, t) = b^-(\cdot, t)$. *By Lemma 3.1 in* [9] *the functions* $\widetilde{b}(x, t)$, $b^+(x, t)$, $b^-(x, t)$ *and* $b^*(x, t)$ *are locally bounded Borel functions. Moreover, if* $b(x, t) \equiv b(t)$, *then (6) reduces to the well known chain rule formula for the composition of BV functions with a Lipschitz function, while, in the special case that* $b(x, t) \equiv b(x)$, *(6) gives the formula for the derivative of the product of two BV functions.*

4. An explicit chain rule

In this section we will present the result and we will write more explicitly the first term appearing in the right hand side of formula (6).

Let $b : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Borel function. Assume that

(i) the function $b(x, t)$ is locally bounded;

(ii) for every $t \in \mathbb{R}$ the function $b(\cdot, t) \in BV(\mathbb{R}^N)$;

(iii) the measure

$$\overline{\sigma} := \bigvee_{t\in\mathbb{R}} |D_x b(\cdot,t)|$$

is a Radon measure, where $\bigvee$ denotes the least upper bound in the space of nonnegative Borel measures.

Remark 4.1. *As in Remark 3.5 in* [2]*, since we will consider* $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$*, condition* (iii) *can be replaced by the following local version*

(iii)$_{\mathrm{loc}}$ *for every compact set* $H \subset \mathbb{R}$ *the measure*

$$\overline{\sigma}_H := \bigvee_{t\in H} |D_x b(\cdot,t)|$$

is a Radon measure.

For simplicity we will omit the explicit dependence of $\overline{\sigma}$ *on* H*. By* (iii)*, we have that* $\overline{\sigma} \ll \mathcal{H}^{N-1}$ *and, if we define*

$$\overline{\mathcal{N}} = \Big\{x \in \mathbb{R}^N : \liminf_{r\to 0} \frac{\overline{\sigma}(B_r(x))}{r^{N-1}} > 0\Big\},$$

then $\overline{\mathcal{N}}$ *is a* $\mathcal{H}^{N-1}$*-rectifiable set. We omit the dependence of* $\overline{\mathcal{N}}$ *of* H *in the local version (see Remark 3.5 in* [2]*).*

Moreover we consider the following assumptions:

(iv) there exists a Borel set $\mathcal{N}_0 \subset \mathbb{R}^N$ with $\mathcal{L}^N(\mathcal{N}_0) = 0$ such that the approximate differential $\nabla_x b(x,t)$ of the function $y \mapsto b(y,t)$ at x exists for every $x \in \mathbb{R}^N \setminus \mathcal{N}_0$ and for every $t \in \mathbb{R}$ and

$$\frac{dD_x b(\cdot,t)}{d\mathcal{L}^N}(x) = \nabla_x b(x,t)$$

for every $x \in \mathbb{R}^N \setminus \mathcal{N}_0$ and for every $t \in \mathbb{R}$;

(v) there exists a Borel set $\mathcal{N}_1 \subseteq \mathbb{R}^N$ with $\overline{\sigma}(\mathcal{N}_1) = 0$ such that the following limit

$$\lim_{r\downarrow 0} \frac{D^c_x b(\cdot,t)(B_r(x))}{\overline{\sigma}(B_r(x))} = \frac{dD^c_x b(\cdot,t)}{d\overline{\sigma}}(x)$$

exists for every $x \in \mathbb{R}^N \setminus \mathcal{N}_1$ and for every $t \in \mathbb{R}$ and this equality holds, where $\frac{dD^c_x b(\cdot,t)}{d\overline{\sigma}}(x)$ is Radon-Nikodým derivative at x of the Cantor part of the measure $D_x b(\cdot,t)$ w.r.t. $\overline{\sigma}$;

(vi) there exists a Borel set $\mathcal{N}_2 \subset \mathbb{R}^N$ with $\mathcal{H}^{N-1}(\mathcal{N}_2) = 0$ such that the one-sided limits $b^+(x,t)$ and $b^-(x,t)$ defined by

$$\lim_{r\downarrow 0} \fint_{B_r^\pm(x)} |b(y,t) - b^\pm(x,t)| dy = 0$$

exist for every $x \in \mathbb{R}^N \setminus \mathcal{N}_2$ and for every $t \in \mathbb{R}$, where $B_r^\pm(x)$ are the two half balls determined by the normal $\nu_{\overline{\mathcal{N}}}$, and

$$\frac{dD_x^j b(\cdot,t)}{d\mathcal{H}^{N-1}}(x) = [b^+(x,t) - b^-(x,t)]\nu_{\overline{\mathcal{N}}}(x)$$

for every $x \in \mathbb{R}^N \setminus \mathcal{N}_2$ and for every $t \in \mathbb{R}$.

By (vi) the functions $b^\pm : (\mathbb{R}^N \setminus \mathcal{N}_2) \times \mathbb{R} \to \mathbb{R}$ are locally bounded Borel functions.

Moreover for all $x \in \mathbb{R}^N \setminus (\overline{\mathcal{N}} \cup \mathcal{N}_2)$ and $t \in \mathbb{R}$ there exists the limit

$$\widetilde{b}(x,t) = \lim_{r\to 0} \fint_{B_r(x)} b(y,t)\, dy.$$

For all $x \in \mathbb{R}^N \setminus (\overline{\mathcal{N}} \cup \mathcal{N}_2)$ the function $t \mapsto \widetilde{b}(x,t)$ is a locally bounded Borel functions. If assumptions (i)–(vi) hold, then for every $t \in \mathbb{R}$ the following decomposition formula holds

$$\begin{aligned} (D_x b)(\cdot,t) =& (\nabla_x b)(x,t)\, \mathcal{L}^N + \frac{dD_x^c b(\cdot,t)}{d\overline{\sigma}}(x)\overline{\sigma} \\ &+ \left[b^+(x,t) - b^-(x,t)\right] \nu_{\overline{\mathcal{N}}}(x)\, \mathcal{H}^{N-1} \llcorner \overline{\mathcal{N}}\,, \end{aligned} \tag{7}$$

in the sense of measures.

Theorem 4.2. *Let $b : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Borel function satisfying* (i)–(vi)*. Then, for every $u \in BV(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$, the function $v : \mathbb{R}^N \to \mathbb{R}$, defined by*

$$v(x) := \int_0^{u(x)} b(x,t)\, dt\,,$$

belongs to $BV_{\rm loc}(\mathbb{R}^N)$ *and for any* $\phi \in C_0^1(\mathbb{R}^N)$ *we have*

$$
\begin{aligned}
(8)\qquad & \int_{\mathbb{R}^N} \nabla\phi(x) v(x)\,dx \\
= \; - & \int_{\mathbb{R}^N} \phi(x) \left[\int_0^{u(x)} \nabla_x b(x,t)\,dt\right] dx - \int_{\mathbb{R}^N} \phi(x) b(x,u(x)) \nabla u(x)\,dx \\
- & \int_{\mathbb{R}^N} \phi(x) \left[\int_0^{\widetilde{u}(x)} \frac{dD_x^c b}{d\overline{\sigma}}(x,t)\,dt\right] d\overline{\sigma} - \int_{\mathbb{R}^N} \phi(x) \widetilde{b}(x,\widetilde{u}(x))\,dD^c u \\
- & \int_{\overline{\mathcal{N}} \cup J_u} \phi(x) \left[\int_0^{u^+(x)} b^+(x,t)\,dt - \int_0^{u^-(x)} b^-(x,t)\,dt\right] \nu_{\overline{\mathcal{N}} \cup J_u}(x)\,d\mathcal{H}^{N-1},
\end{aligned}
$$

where it is understood that for $\mathcal{H}^{N-1}$*-a.e.* $x \in \overline{\mathcal{N}} \cap J_u$ *the normal* $\nu_{\overline{\mathcal{N}} \cup J_u}$ *is choosen equal to* $\nu_{\overline{\mathcal{N}}}$ *.*

Corollary 4.3. *Let* $b : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *be a Borel function satisfying*

(i) *the function* $b(x,t)$ *is locally bounded;*

(ii) *for every* $t \in \mathbb{R}$ *the function* $b(\cdot,t) \in W^{1,1}(\mathbb{R}^N)$ *and there exists a Borel set* $\mathcal{N}_1 \subseteq \mathbb{R}^N$ *such that* $\mathcal{H}^{N-1}(\mathcal{N}_1) = 0$ *such that*

$$b(x,t) = \widetilde{b}(x,t)$$

for every $x \in \mathbb{R}^N \setminus \mathcal{N}_1$ *and every* $t \in \mathbb{R}$*;*

(iii) *for every compact set* $H \subseteq \mathbb{R}$ *the function*

$$g_H(x) := \sup_{t \in H} |\nabla_x b(x,t)|$$

belongs to $L^1_{\rm loc}(\mathbb{R}^N)$*;*

(iv) *there exists a Borel set* $\mathcal{N}_2 \subseteq \mathbb{R}^N$ *such that* $\mathcal{L}^N(\mathcal{N}_2) = 0$ *such that the approximate gradient* $\nabla_x b(x,t)$ *of the function* $y \mapsto b(y,t)$ *at* x *exists for every* $x \in \mathbb{R}^N \setminus \mathcal{N}_2$ *and every* $t \in \mathbb{R}$*.*

Then, for every $u \in BV(\mathbb{R}^N) \cap L^\infty_{\rm loc}(\mathbb{R}^N)$*, the function* $v : \mathbb{R}^N \to \mathbb{R}$*, defined by*

$$v(x) := \int_0^{u(x)} b(x,t)\,dt\,,$$

belongs to $BV_{\rm loc}(\mathbb{R}^N)$ *and for any* $\phi \in C_0^1(\mathbb{R}^N)$ *we have*

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla\phi(x)v(x)dx &= \int_{\mathbb{R}^N} \phi(x)\left[\int_0^{u(x)} \nabla_x b(x,t)\,dt\right]dx \\ &- \int_{\mathbb{R}^N} \phi(x) b(x,u(x))\nabla u(x)\,dx - \int_{\mathbb{R}^N}\phi(x)\widetilde{b}(x,\widetilde{u}(x))\,dD^c u \\ &- \int_{J_u}\phi(x)\left[\int_{u^-(x)}^{u^+(x)} \widetilde{b}(x,t)\,dt\right]\nu_u(x)\,d\mathcal{H}^{N-1}.\end{aligned}$$

Remark 4.4. *This corollary improves Proposition 1.2 in* [8] *where* $\mathcal{N}_2 = \emptyset$ *and* $b(x,\cdot)$ *is continuous for a.e.* $x \in \mathbb{R}^N$.

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