THE POISSON LIE ALGEBRA, RUMIN'S COMPLEX AND BASE CHANGE

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ABSTRACT. Results from the forthcoming papers [BDK4, D3] are announced. We introduce a *singular current construction*, or *base change*, for pseudoalgebras which may be used to obtain a primitive Lie pseudoalgebra of type H from a suitable one of type K. When applied to representations, it derives the pseudo de Rham complex of type H from that of type K — which is related to Rumin's construction from [Ru] — both with standard coefficients and with nontrivial Galois coefficients. In the latter case, the construction yields exact complexes of modules for the Poisson linearly compact Lie algebra P_N exhibiting a nontrivial central action.

1. INTRODUCTION

The notion of (Lie) pseudoalgebra over a (cocommutative) Hopf algebra was introduced in [BDK1] as a generalization of Lie conformal algebras, which have proved useful in dealing with locality of formal distributions and the description of both vertex algebras [K, D1, D2, DM2] and Poisson vertex algebras [DeK]. However, one of their most natural applications is the study of discrete representations over linearly compact Lie algebras, as the annihilation algebra functor may be used to associate with (commutative, associative, Lie) pseudoalgebras the corresponding linearly compact algebras and representations of the latter can often by lifted to the pseudoalgebraic language [BDK2, BDK3].

A special role among primitive (i.e., those that cannot be obtained by means of a nontrivial current construction as in [BDK1, Section 4.2]) Lie pseudoalgebras is played by those of type H, that correspond to the Hamiltonian family in Cartan's description of simple infinite-dimensional linearly compact Lie algebras [Ca]. Indeed, Lie pseudoalgebras $H(\mathfrak{d}, \chi, \omega)$ are the only finite primitive ones over $\mathcal{U}(\mathfrak{d})$, whose annihilation algebra has a non trivial center; this, however, acts trivially on every irreducible pseudoalgebra representation.

Here, we announce results from the forthcoming paper [D3] which generalize and clarify [DM1]: we show how to use the concept of representations with coefficients [DM1] so as construct *projective representations* of $L = H(\mathfrak{d}, \chi, \omega)$ that correspond to irreducible discrete representations of the annihilation algebra with a nontrivial central action. We show how to set up the machinery and explain how to construct such modules starting from irreducible reps of a suitably chosen Lie pseudoalgebra $K(\mathfrak{d}, \theta)$ by means of a singular current construction, or *base change*.

The distinction between regular and singular tensor modules already observed in [BDK2, BDK3, BDK4] generalizes to the present setting, as can be shown by explicit computation of Tor-spaces. As a byproduct of our techniques, we obtain an alternate more conceptual proof of the non-exactness of the pseudocomplex of de Rham type of the Lie pseudoalgebras $H(\mathfrak{d}, 0, \omega)$.

The present constructions also sheds light on the similarity of type H and type K structures, especially when reformulated with a pseudoalgebraic language.

2. PRIMITIVE LIE PSEUDOALGEBRAS OF TYPE H AND K

In this paper, definitions and notation concerning pseudoalgebras and Hopf algebras follow those found in [BDK1].

Henceforth, $\mathfrak{d} \neq 0$ will be a finite dimensional Lie algebra and $H = U(\mathfrak{d})$ its universal enveloping algebra. A simple Lie H-pseudoalgebra L is said to be *primitive* if its annihilation

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Lie algebra $\mathcal{L} := H^* \otimes_H L$ is (a central extension of) one of the simple linearly compact Lie algebras from Cartan's classification [Ca].

Here, we are interested in primitive Lie pseudoalgebras of type H and K, i.e., those such that $\mathcal{L}$ is isomorphic to either the Poisson Lie algebra P_N, which centrally extends H_N, or the contact Lie algebra K_N. It is showed in [BDK1] that N must then equal $\dim \mathfrak{d}$, so that primitive Lie pseudoalgebras of type H (resp. K) only exist when $\mathfrak{d}$ is even (resp. odd) dimensional. There are further constraints that prevent, for instance, $\mathfrak{d}$ from being abelian in the K-type case.

Proposition 2.1 ([BDK1, Section 8.5])**.** *Let L be a Lie H-pseudoalgebra of type H (resp. K). Then $L = He$ is a free H-module of rank 1 and*

$$[e * e] = (r + s \otimes 1 - 1 \otimes s) \otimes_H e,$$

where $0 \neq r \in \bigwedge^2 \mathfrak{d}$, $s \in \mathfrak{d}$ satisfy

$$[r, \Delta(s)] = 0, \qquad ([r_{12}, r_{13}] + r_{12}s_3) + \text{ cyclic permutations} = 0,$$

and

- $\operatorname{supp} r = \mathfrak{d}$ *if L is of type H;*
- $s \notin \operatorname{supp} r$ *and* $\operatorname{supp} r + \mathbf{k}s = \mathfrak{d}$ *in type K,*

where $\operatorname{supp} r \subset \mathfrak{d}$ *denotes the subspace supporting r.*

We shall then call $(r, s, \mathfrak{d})$ a *datum of type H (resp. K)*.

Example 2.1. If $(r, s, \mathfrak{d})$ is a datum of type H, then r is non-degenerate and may be used to identify $\mathfrak{d}$ with its dual. Thus r and s translate to a 2-form $\omega \in \bigwedge^2 \mathfrak{d}^*$ and a trace form $\chi \in \mathfrak{d}^*$ respectively, satisfying

$$\mathrm{d}\chi = 0, \qquad \mathrm{d}\omega + \chi \wedge \omega = 0.$$

This means that ω is a 2-cocycle of $\mathfrak{d}$ with values in the one-dimensional $\mathfrak{d}$-module $\mathbf{k}_\chi$ defined by χ, yielding an abelian extension

$$0 \to \mathbf{k}_\chi \to \mathfrak{d}' \to \mathfrak{d} \to 0.$$

More explicitly, if $\mathfrak{d}' = \mathfrak{d} \oplus \mathbf{k}c$ as vector spaces, then

$$[g, h]' = [g, h] + \omega(g, h)c, \qquad [g, c]' = \chi(g)c$$

extend to a Lie bracket on $\mathfrak{d}'$. If $\chi = 0$, then ω is a 2-cocycle of $\mathfrak{d}$ and $\mathfrak{d}'$ is the corresponding central extension.

By [BDK1, Remark 8.6], if we identify $\mathfrak{d}$ as a subspace of $\mathfrak{d}'$, then $(r, s + c, \mathfrak{d}')$ is a datum of type K. Notice that not all data of type K are obtained by this construction.

3. Currents and base change

3.1. **Rings of coefficients.**

Let H be a Hopf algebra over $\mathbf{k}$ with comultiplication Δ. A *left H-comodule* is a vector space C over $\mathbf{k}$ endowed with a $\mathbf{k}$-linear map

$$\begin{array}{rccc} \Lambda : & C & \longrightarrow & H \otimes C \\ & c & \longmapsto & c_{(1)} \otimes c_{(2)} \end{array}$$

such that

$$(\Delta \otimes \mathrm{id}_C) \circ \Lambda = (\mathrm{id}_H \otimes \Lambda) \circ \Lambda, \tag{3.1}$$

$$(\epsilon \otimes \mathrm{id}_C) \circ \Lambda = \mathrm{id}_C. \tag{3.2}$$

Λ is called the *comodule structure map* of C, or simply *coaction*. Repeated application of Λ defines the *n-fold coaction maps* $\Lambda^n : C \to H^{n-1} \otimes C$, $n > 1$. We employ the notation

$$\Lambda^2(c) = (\Delta \otimes \mathrm{id}_C) \circ \Lambda(c) = (\mathrm{id}_H \otimes \Lambda) \circ \Lambda(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}, \tag{3.3}$$

and similarly $\Lambda^{n-1}(c) = c_{(1)} \otimes \ldots \otimes c_{(n)}$. Notice that this can be misleading, as only the last tensor factor $c_{(n)}$ lies in C, whereas all others are elements of H. The (left) counit axiom (3.2) rewrites as

$$(\epsilon \otimes \mathrm{id}_C) \circ \Lambda(c) = \epsilon(c_{(1)})c_{(2)} = c. \tag{3.4}$$

Once again, the map ϵ can only be applied on elements of H, as no counit is defined on C, hence no right counit axiom may be required to hold.

An H-*comodule algebra* is a unital $\mathbf{k}$-algebra D endowed with an algebra homomorphism

$$\begin{array}{rccc} \Lambda: & D & \longrightarrow & H \otimes D \\ & d & \longmapsto & d_{(1)} \otimes d_{(2)}, \end{array}$$

making D into an H-comodule. From now on we will call an H-comodule algebra D a *ring of coefficients*, or simply *roc*, over H. When D is a roc over H, we will also say that (H, D) is a *roc*.

Lemma 3.1. *Let $\phi : H_1 \longrightarrow H_2$ be a Hopf algebra homomorphism and D be a roc over H_1 with comodule map $\Lambda_1 : D \to H_1 \otimes D$. Then $\Lambda_2 = (\phi \otimes 1)\Lambda_1$ makes (H_2, D) into a roc.*

Example 3.1. Every Hopf algebra H is a roc over itself, with comodule structure map given by Δ.

Example 3.2. Let H' be a Hopf subalgebra of a Hopf algebra H. Since H' is a roc over itself and the inclusion from H' to H is a Hopf homomorphism from H' to H, then H' is a roc over H. In particular $H' = \mathbf{k} \subset H$ is a roc over H.

Definition 3.1. Let (D_1, Λ_1) and (D_2, Λ_2) be rings of coefficients over H_1 and H_2 respectively. Let $\phi : H_1 \to H_2$ be a Hopf algebra homomorphism and $\psi : D_1 \to D_2$ be an algebra homomorphism. The pair $(\phi, \psi) : (H_1, D_1) \longrightarrow (H_2, D_2)$ is a *roc homomorphism* if

$$\Lambda_2(\psi(d)) = (\phi \otimes \psi)\Lambda_1(d), \tag{3.5}$$

for every $d \in D_1$.

Example 3.3. If Λ_1, Λ_2 are as in Lemma 3.1, then $(\phi, \mathrm{id}_D) : (H_1, D) \mapsto (H_2, D)$ is a roc homomorphism.

Notice that if $(\phi, \psi) : (H, D) \to (H', D')$ is a roc homomorphism, then $I = \ker \phi$ is a Hopf ideal of H_1, and $J = \ker \psi$ is an algebra ideal of D satisfying $\Lambda(J) \subset I \otimes D + H \otimes J$. Every pair $(I, J) \subset (H, D)$ satisfying the above requirement is an *ideal* of the roc (H, D). Clearly, whenever (I, J) is an ideal of (H, D), there exists a unique roc structure on $(H/I, D/J)$ making the natural projection (π_I, π_J) into a roc homomorphism.

3.2. **Lie pseudoalgebra representations with coefficients.** Let (H, D) be a roc, L a Lie H-pseudoalgebra, M a left D-module. A (left) *pseudoaction of* L on M with coefficients in D is an $(H \otimes D)$-linear map

$$\begin{array}{rccc} * : & L \otimes M & \longrightarrow & (H \otimes D) \otimes_D M \\ & a \otimes m & \mapsto & a * m. \end{array}$$

This pseudoaction defines a Lie pseudoalgebra representation (with coefficients) if

$$[a * b] * m = a * (b * m) - (b * (a * m))^{\sigma_{12}}, \tag{3.6}$$

for any $a, b \in L$, $m \in M$, where we have extended $*$ to $(H^{\otimes(i+j-1)} \otimes D)$-linear maps

$$* : \quad (H^{\otimes i} \otimes_H L) \otimes ((H^{\otimes(j-1)} \otimes D) \otimes_D M) \quad \longrightarrow \quad (H^{\otimes(i+j-1)} \otimes D) \otimes_D M,$$

by

$$(F \otimes_H a) * (G \otimes_D m) = (F \otimes G)(\Delta^{i-1} \otimes \Lambda^{j-1})(a * m), \tag{3.7}$$

where $F \in H^{\otimes i}, G \in H^{\otimes(j-1)} \otimes D, a \in L, m \in M, i, j \geq 1$.

Notice that when $D = H$ the above notion of representation coincides with usual Lie pseudoalgebra representations, as considered in [BDK1]. A representation M is *finite* if it is finitely generated as a D-module. It is *irreducible* if the only D-submodules $N \subset M$ satisfying $L * N \subset (H \otimes D) \otimes_D N$ are the trivial ones.

3.3. **Base change on pseudoalgebras.** Let H, H' be Hopf algebras, L a Lie H-pseudoalgebra. Every Hopf algebra homomorphism $\phi : H \longrightarrow H'$ endows H' with a right H-module structure so that we may consider the tensor product $L' = \phi_* L := H' \otimes_H L$, which is a left H'-module. It is not difficult to show that the Lie H-pseudoalgebra structure on L induces a corresponding Lie H'-pseudoalgebra structure on L', whose pseudobracket satisfies

$$[(h' \otimes_H a) * (k' \otimes_H b)] = \sum_i (h'\phi(h^i) \otimes k'\phi(k^i)) \otimes_{H'} (1 \otimes_H e_i), \tag{3.8}$$

if $[a * b] = \sum_i (h^i \otimes k^i) \otimes_H e_i$, $a, b \in L$.

If $\phi : H \to H'$, and $L' = \phi_* L$, we will say that L' is obtained from L by *extension of scalars* or *base change*. Let us see a few examples.

3.3.1. *Current construction.* Let $H \subset H'$ be cocommutative Hopf algebras, $\iota : H \to H'$ the inclusion homomorphism. Then ι_* coincides with the current construction *Cur* $_H^{H'}$ described in [BDK1, bla]. Clearly, if $H = H'$, then $\iota = \mathrm{id}_H$ and ι_* is canonically isomorphic to the identity functor.

3.3.2. *Algebra of* 0*-modes.* Let A be a Lie conformal algebra, viewed as a Lie pseudoalgebra over $H = \mathbf{k}[\partial]$. Then the counit $\varepsilon : H \to \mathbf{k}$ is a Hopf algebra homomorphism, and $\varepsilon_* A$ coincides with the algebra of Fourier 0-modes of A, see [K].

3.3.3. $K(\mathfrak{d}, \theta)$ *and* $H(\mathfrak{d}, \chi, \omega)$. Let $L = He$ be a primitive Lie pseudoalgebra of type H corresponding to the datum $(r, s, \mathfrak{d})$ and χ be the corresponding 1-form. If we set $\overline{\partial} := \partial + \chi(\partial)$, then the Lie pseudobracket on L may be rewritten as

$$[e * e] = \sum_i (\overline{\partial_i} \otimes \overline{\partial^i}) \otimes_H e.$$

Consider the datum $(r, s, \mathfrak{d}' = \mathfrak{d} \oplus \mathbf{k}c)$ of type K constructed in Example 2.1. The Lie pseudobracket of the corresponding Lie pseudoalgebra of type K then rewrites as

$$[e' * e'] = (\sum_i \bar{\partial}_i \otimes \bar{\partial}^i + c \otimes 1 - 1 \otimes c) \otimes_{H'} e'.$$

The canonical projection $\pi : \mathfrak{d}' \twoheadrightarrow \mathfrak{d}'/\mathbf{k} \simeq \mathfrak{d}$ extends to a Hopf algebra homomorphism $\pi : H' \to H$ mapping c to 0. It is then easy to see that $\pi_* L' = L$. Notice that π is not injective, so that π_* cannot, and should not, be understood in terms of the abovementioned standard current construction.

3.4. **Base change on representations.** It is possible to change scalars on both a Lie pseudoalgebra and its representation (with coefficients), once we make sure to employ a roc homomorphism.

Proposition 3.1. *Let $(H, D), (H', D')$ be rocs, $\Phi = (\phi, \psi) : (H, D) \to (H', D')$ a roc homomorphism, L a Lie pseudoalgebra over H, M a representation of L with coefficients in D. Then there exists a natural pseudoalgebra action of $L' = \phi_* L$ on $M' = \psi_* M := D' \otimes_D M$ with coefficients in D' satisfying*

$$(h' \otimes_H a) * (d' \otimes_D m) = \sum_i (h'\phi(h^i) \otimes d'\phi(d^i)) \otimes_{D'} (1 \otimes_D e_i), \tag{3.9}$$

if $a * m = \sum_i (h^i \otimes d^i) \otimes_H e_i$, $a \in L$, $m \in M$.

4. Galois objects and projective representations

The possibility of straightening a pseudoalgebra action on the right makes it possible to generalize many results from [BDK1]. This naturally occurs when the representation takes its coefficients in a *Galois roc*. Representations of a Lie H-pseudoalgebra with coefficients in a Galois roc are called *projective*, in analogy with [Dc].

4.1. **Straightening on the right.**

Definition 4.1. A roc (H, D) is *Galois* if D is the Galois map

$$\begin{array}{rccc} \beta: & D\otimes D & \longrightarrow & H\otimes D \\ & d\otimes d' & \mapsto & (1\otimes d)\Delta(d') \end{array} \tag{4.1}$$

is a linear isomorphism.

The map β factors via $D\otimes_{D^{\text{co-}H}} D$, where $D^{\text{co-}H} = \{d\in D | \Delta(d) = 1\otimes d\}$. Therefore, in order for (H, D) to be Galois, one needs $D^{\text{co-}H} = \mathbf{k}$. This implies that $\mathbf{k}\subset D$ is a Hopf-Galois extension, hence D is a Galois object, thus justifying the terminology. Properties of the inverse map β^{-1} are well understood. We use a Sweedler-like notation for β by setting $\beta^{-1}(h\otimes 1) = h^{[1]}\otimes h^{[2]}$.

Proposition 4.1. *Let $g, h\in H$, $d\in D$. Then:*

$$h^{[2]}{}_{(1)}\otimes h^{[1]}h^{[2]}{}_{(2)} = h\otimes 1 \tag{4.2}$$

$$h^{[1]}h^{[2]} = \epsilon(h)1_D \tag{4.3}$$

$$(gh)^{[1]}\otimes(gh)^{[2]} = h^{[1]}g^{[1]}\otimes g^{[2]}h^{[2]} \tag{4.4}$$

$$d_{(2)}(d_{(1)})^{[1]}\otimes(d_{(1)})^{[2]} = 1\otimes d \tag{4.5}$$

Proof. Compute β on $\beta^{-1}(h\otimes 1) = h^{[1]}\otimes h^{[2]}$ in order to obtain (4.2). Then, applying $\epsilon\otimes\mathrm{id}_D$ gives (4.3). Equations (4.4) and (4.5) are proved by applying β, which is invertible, on both sides. □

Lemma 4.1. *Let (H, D) be a Galois roc, let M be a right D-module and N be a left D-module. Then the map*

$$\begin{array}{rccc} \tau^R: & (H\otimes M)\otimes_D N & \longrightarrow & M\otimes N \\ & (h\otimes m)\otimes_D n & \longmapsto & mh^{[1]}\otimes h^{[2]}n, \end{array} \tag{4.6}$$

is a well defined linear isomorphism.

Proof. Using (4.2), the linear map extending $m\otimes n\mapsto(1\otimes m)\otimes_D n$ is easily checked to be an explicit inverse to τ^R. Thus, we only need to worry about well definedness of τ^R. However,

$$(h\otimes m)\otimes_D dn = (h\otimes m)\Delta(d)\otimes_D n = (hd_{(1)}\otimes md_{(2)})\otimes_D n$$

gets mapped to

$$md_{(2)}(hd_{(1)})^{[1]}\otimes(hd_{(1)})^{[2]}n = md_{(2)}(d_{(1)})^{[1]}h^{[1]}\otimes h^{[2]}(d_{(1)})^{[2]}n,$$

due to (4.4), and this equals $mh^{[1]}\otimes h^{[2]}dn$ thanks to (4.5). □

Corollary 4.1. *Let (H, D) be a Galois roc, M be a right D-module, N a left D-module. Then every element $\alpha\in(H\otimes M)\otimes_D N$ can be expressed as a finite sum*

$$\alpha = \sum_i (1\otimes m^i)\otimes_D n_i, \tag{4.7}$$

where both $m^i\in M$ and $n_i\in N$ are linearly independent over $\mathbf{k}$, and $\sum_i m^i\otimes n_i\in M\otimes N$ is uniquely determined by α.

We will refer to (4.7) as to a *right-straightened* expression in $(H\otimes M)\otimes_D N$.

4.2. **Galois rocs and twists: the Weyl roc.**
Let H be a cocommutative Hopf algebra, $\sigma : H \otimes H \to \mathbf{k}$ a Hopf 2-cocycle. Then [Sw] the twisted product $H_\sigma = \mathbf{k}\#_\sigma H$ is a comodule algebra over H, and a Hopf-Galois extension of $\mathbf{k}$. All Galois rocs (H, D), where D satisfies the normal basis condition, are obtained in this way; for instance, when H is pointed, e.g., when it is cocommutative, all Galois objects satisfy the normal basis condition.

We are going to give an alternate construction of this fact in a special case that is of interest to us: let $(r, s, \mathfrak{d})$ be a datum of type H and $(r, s + c, \mathfrak{d}')$ the corresponding datum of type K as from Example 2.1. Set $H = U(\mathfrak{d}), H' = U(\mathfrak{d}')$. Then $\mathbf{k}c$ is an abelian ideal of $\mathfrak{d}'$, which is central when $s = 0$ — which corresponds to $\chi = 0$. Choose $\lambda \in \mathbf{k}$ and set $I_\lambda = H' \cdot (c - \lambda)$; notice that c, hence $c - \lambda$, is central if and only if $\chi = 0$.

Lemma 4.2.
- *I_0 is a Hopf ideal of H;*
- *If $\chi = 0$, then (I_0, I_λ) is a roc ideal of (H, H) for all $\lambda \in \mathbf{k}$.*

Proof. By construction, $[\mathfrak{d}', c] \subset \mathbf{k}c$, hence $cH' \subset H'c$, thus showing that $I_0 = H'c = cH'$ is a two-sided ideal of H'. However, when $\lambda\chi = 0$, $c - \lambda$ is central in H', and all I_λ are also two-sided ideals. As

$$\Delta(c - \lambda) = c \otimes 1 + 1 \otimes (c - \lambda), \qquad S(c) = -c,$$

then

$$\Delta(I_\lambda) \subset I_0 \otimes H' + H' \otimes I_\lambda, \qquad S(I_0) \subset I_0,$$

whence both claims follow immediately. □

For every choice of $\lambda \in \mathbf{k}$, let $\psi^\lambda : H' \longrightarrow H'/I_\lambda := D_\lambda$ be the natural projection, and denote by

$$\Lambda^\lambda : D_\lambda = H'/I_\lambda \to (H' \otimes H')/(I_0 \otimes H' + H' \otimes I_\lambda) = D_0 \otimes D_\lambda$$

the map induced by Δ. Notice that D_0 identifies with H as a Hopf algebra.

Proposition 4.2.
- *The Hopf algebra homomorphism $\phi : H' \to H$ induced by the surjection $\mathfrak{d}' \to \mathfrak{d}$ coincides with $\psi^0 : H' \to D_0 \simeq H$.*
- *The maps Λ^λ make each D_λ into a roc over H, and all pairs $(\phi, \psi^\lambda), \lambda \in \mathbf{k}$ are roc homomorphism.*

Remark 4.1. Since $\Delta(I_\lambda) \subset I_0 \otimes H' + H' \otimes I_\lambda$, the map $\Lambda^\lambda : H'/I_\lambda \to H'/I_0 \otimes H'/I_\lambda$ induced by Δ always defines a comodule structure. However, H'/I_λ carries a compatible algebra structure, hence is a roc, only when I_λ is two-sided. When $\lambda\chi \neq 0$, the 2-sided ideal of H' generated by I_λ coincides with the whole H'.

It is not difficult to show that the rocs (H, D_λ) are all Galois. We have already mentioned that D_0 is isomorphic to H. The algebras $D_\lambda, \lambda \neq 0$ are all isomorphic as algebras, and also as H-comodules, but not as rocs over H. If $\mathfrak{d}$ is abelian of dimension $N = 2n$, $\chi = 0$ and ω is symplectic, then $D_\lambda, \lambda \neq 0$ is isomorphic to the Weyl algebra A_n. In all cases, D_λ is a noetherian domain.

5. An exact sequence of projective tensor modules of $H(\mathfrak{d}, 0, \omega)$

Finite irreducible representations (with standard coefficients) of primitive Lie pseudoalgebras of type W, S, K have been considered in [BDK2, BDK3]. In all cases, one is able to locate a class of representations called *tensor module* and to prove that every finite irreducible representation arises as a quotient of a suitably chosen tensor module. Tensor modules are generically irreducible, but there are exceptions, called *singular* that can be put together in exact complexes of representations; then the image (or equivalently, the kernel) of each morphism in the complex provides a maximal and irreducible submodule of the relevant tensor module. Tensor modules are always free as H-modules.

These exceptional complexes possess a geometrical interpretation: in types W and S they arise as twists [BDK2, Section 4.2] the *pseudification* of the de Rham complex, whereas in type K they are related to the complex [Ru] introduced by Rumin in the context of contact manifolds.

Let us review the type K case more closely. Say $(r, s, \mathfrak{d}')$ is a datum of type K corresponding to the Lie pseudoalgebra $K(\mathfrak{d}', \theta')$; here $\mathfrak{d}' = \mathfrak{d} + \mathbf{k}s$ and $\mathfrak{d}$ is the support of r. Tensor modules for $K(\mathfrak{d}', \theta')$ are parametrized by finite dimensional representations of the Lie algebra $\mathfrak{d}' \oplus \mathfrak{csp}\,\mathfrak{d}$ as in [BDK3, Section 5.2]. For the sake of simplicity, we will ignore here the action of $\mathfrak{d}'$, which can be recovered by applying a twist; then singular $K(\mathfrak{d}', \theta')$-tensor modules occur when the $\mathfrak{csp}\,\mathfrak{d}$-action restricts to one of the fundamental representations of $\mathfrak{sp}\,\mathfrak{d} \subset \mathfrak{csp}\,\mathfrak{d}$, for a suitable choice of the scalar action.

The corresponding exact complex of singular tensor modules [BDK3, Theorem 6.1] is then

$$0 \to \Omega^0(\mathfrak{d}')/I^0(\mathfrak{d}') \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^N(\mathfrak{d}')/I^N(\mathfrak{d}') \xrightarrow{\mathrm{d}^R} J^{N+1}(\mathfrak{d}') \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} J^{2N+1}(\mathfrak{d}'), \tag{5.1}$$

where $\Omega^i(\mathfrak{d}')$ are modules of *pseudoforms* and $I^i(\mathfrak{d}'), J^i(\mathfrak{d}')$ are suitable submodules related to Rumin's construction. The last d morphism is not surjective, as the complex provides a resolution of $\mathbf{k}$ viewed as a trivial H-module via the counit.

Let now $(r, 0, \mathfrak{d})$ be a datum of type H, and consider the primitive Lie pseudoalgebra $L' = K(\mathfrak{d}', \theta') = H'e'$ of type K associated with the datum of type K constructed in Example 2.1. We have seen in Section 3.3.3 that $L = H(\mathfrak{d}, 0, \omega) = He$ coincides with the base change $\phi_* L'$, where $\phi : H' \to H$ is the Hopf algebra homomorphism induced by mapping the central element $c \in \mathfrak{d}' \subset H'$ to 0.

As c is central in H', it may be specialized to any scalar value; let $\psi^\lambda : H' \to H'/H'(c-\lambda) \simeq D_\lambda$ denote the natural projection, so that $\phi = \psi_0$. We have already seen in Proposition 4.2 that each pair $(\phi, \psi_\lambda) : (H', H') \to (H, D_\lambda)$ is a roc homomorphism, which we may use to extend scalars on (5.1) obtaining the following sequence of $H(\mathfrak{d}, 0, \omega) = \phi_* K(\mathfrak{d}', \theta')$-modules

$$\begin{aligned} 0 \to \psi^\lambda_* \left(\Omega^0(\mathfrak{d}')/I^0(\mathfrak{d}')\right) &\xrightarrow{\psi^\lambda_* \mathrm{d}} \cdots \xrightarrow{\psi^\lambda_* \mathrm{d}} \psi^\lambda_* \left(\Omega^N(\mathfrak{d}')/I^N(\mathfrak{d}')\right) \\ &\xrightarrow{\psi^\lambda_* \mathrm{d}^R} \psi^\lambda_* J^{N+1}(\mathfrak{d}') \xrightarrow{\psi^\lambda_* \mathrm{d}} \cdots \xrightarrow{\psi^\lambda_* \mathrm{d}} \psi^\lambda_* J^{2N+1}(\mathfrak{d}'), \end{aligned} \tag{5.2}$$

which is certainly a complex, by functoriality of ψ^λ_*.

Recall that (5.1) is a projective resolution of the trivial left H'-module $\mathbf{k}$, so that the homology of (5.2) computes $\mathrm{Tor}^{H'}_\bullet(D_\lambda, \mathbf{k})$, which can also be computed by choosing a projective resolution of the right H'-module D_λ and tensoring it by $\mathbf{k}$. This is easily done, by noticing that $c - \lambda$ is a nonzero divisor in H' so that

$$0 \to H' \xrightarrow{\cdot(c-\lambda)} H' \xrightarrow{\psi^\lambda} D_\lambda \to 0$$

is exact. When $\lambda \neq 0$, applying $\otimes_{H'} \mathbf{k}$ to the above resolution yields

$$0 \to \mathbf{k} \xrightarrow{\cdot(-\lambda)} \mathbf{k} \to 0,$$

which is manifestly exact. This shows that $\mathrm{Tor}^{H'}_i(D_\lambda, \mathbf{k}) = 0$ for all $i > 0$, so that (5.2) is an exact complex of projective $H(\mathfrak{d}, 0, \omega)$-modules when $\lambda \neq 0$; furthermore, the last connecting homomorphism is surjective as $D_\lambda \otimes_{H'} \mathbf{k} = 0$. By using right-straightening as from Section 4.1, one may show that projective $H(\mathfrak{d}, 0, \omega)$-modules in (5.2) and their twists are the only singular ones, so that we have a complete analogy with results from [BDK2, BDK3].

The exact sequence (5.2) may be employed to exhibit submodules of reducible projective tensor modules. A classification of their maximal submodules, hence of their irreducible quotients, will be made explicit in [D3].

One may proceed similarly when $L = H(\mathfrak{d}, \chi, \omega)$ and $\lambda = 0$, which gives $D_0 \simeq H$ and yields modules with standard coefficients. In this case, one obtains exactness of the sequence (5.2) everywhere but at the second to last module. This sequence coincides with the pseudo de

Rham complex from [BDK4] and keeps being a resolution of the trivial module $H \otimes_{H'} \mathbf{k} \simeq \mathbf{k}$. This allows one to control the one-dimensional non-exactness of the complex of de Rham type for primitive Lie pseudoalgebras of type H, thus providing a more conceptual homological explanation.

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