Skew Brownian diffusions across Koch interfaces

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Abstract

We consider planar skew Brownian motion (BM) across pre-fractal Koch interfaces $\partial\Omega^n$ and moving on $\overline{\Omega^n} \cup \Sigma^n = \Omega^n_\varepsilon$. We study the asymptotic behaviour of the corresponding multiplicative additive functionals when thickness of Σ^n and skewness coefficients vanish with different rates.

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1 Introduction

Diffusions on irregular domains have been investigated by many authors as well as the construction of reflecting Brownian motions on non smooth domains ([9, 22, 29, 30]). However, if the domain D is Lipschitz, then we can construct the usual reflecting BM as in [9]. Let $D \subset \mathbb{R}^d$, $d \geq 2$, a bounded Lipschitz domain. Existence and uniqueness of the solution to $dX_t = dB_t + \mathbf{n}(X_t)dL^{\partial D}_t$ have been investigated in [6, 7] when $\mathbf{n}(z)$ is the inward normal vector at $z \in \partial D$ and $L^{\partial D}_t$ is the local time of X on the boundary of D. In particular, $L^{\partial D}_t$ is a non-decreasing process such that $\int_0^\infty \mathbf{1}_D(X_s)dL^{\partial D}_s = 0$ that is, the process does not increase inside D. The local time can be associated with the surface measure ([8, 9]) in the sense of the Revuz correspondence. Moreover, convergence of reflecting BM in varying domain has been also investigated (see for example [14] and the references therein). In [8] the authors studied the Robin problem on fractal domains in the framework of the so called *trap domains* (see [15]). In this paper we consider Robin problems on snowflake domains by using the homogenization results obtained in [19, 20] with the approach of insulating layers (see, for example, [1, 13] in smooth layers). More precisely, the fractal layer is approximated by a two-dimensional insulating thin layer with vanishing thickness and decreasing conductivity. Therefore, the emerging operators have discontinuous coefficients on the pre-fractal interfaces and so we consider skew Brownian motions. The skew BM has been introduced in [33, 34, 52] and constructed to model permeable barrier in [44, 45]. An interesting surveys can be found in [39]. It has been also investigated by many researchers as a tool in applied sciences. Applications to a single interface have been developed in [4, 33, 34, 40, 43, 46, 52]. Recent results on multidimensional skew BM can be found in [5]. In [38, 53] the authors approach homogenization problems.

As well described in [34, pag. 272], it is possible to construct a reflecting BM B^+ on $\Omega \subset D$ by considering a BM B on D and the occupation time $\mathfrak{f}$ of B on Ω. That is, $B(\mathfrak{f}^{-1})$ is identical in law to B^+. It is also shown in [34] that by killing $B(\mathfrak{f}^{-1})$ at a random time T with conditional law $\mathbb{P}.(T > t|B(\mathfrak{f}^{-1})) = \exp - \int \ell(\mathfrak{f}^{-1}(t), x)\kappa(dx)$, one obtains the connection with the motion driven by the Feynman-Kac generator (ℓ is a local time and κ is a killing rate). An interesting connection has been also given by verifying a conjecture of Feller. Indeed, an elastic BM on $[0, \infty)$ with elastic condition $\gamma u(0) = (1 - \gamma)u'(0)$, $\gamma \in (0, 1)$ is identical in law to B^+ killed according

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with the conditional law $\mathbb{P}.(T > t|B^+) = \exp -\frac{\gamma}{1-\gamma}\ell^+(t,0)$. We notice that the special cases $\gamma = 1$ or $\gamma = 0$ correspond to Dirichlet or Neumann conditions.

In our analysis, we mainly focus on occupation measures and stopping times with conditional law $\mathbb{P}.(T > t|X) = \exp -c_n \int_{\partial\Omega^n} \ell^{n+}(t,dy)$. In particular, we consider a sequence of exponential random variables ζ^n with parameter $c_n \in [0,\infty]$ from which we construct a sequence of stopping times $\widehat{\zeta^{\Omega^n}}$ depending on the time the process spends on (or cross) the pre-fractal interfaces. The process we are dealing with is a skew planar BM (on a bounded domain $\Omega^n_\varepsilon \supset \Omega^n$ with pre-fractal interface $\partial\Omega^n$). We say that the BM in Ω^n_ε is skew meaning that it has different probability to stay in either Ω^n or $\Omega^n_\varepsilon \setminus \Omega^n$. We have a transmission condition on the boundary $\partial\Omega^n$. A key role is played by the fact that the pre-fractal and fractal Koch domains are non trap. Thus, the fact that the semi-permeable barrier is given by the pre-fractal curve $\partial\Omega^n$ does not affect our discussion in terms of occupation measures. We denote by $B^{\nu,*}_t$ the skew (modified) planar BM on Ω^n_ε and we focus on multiplicative additive functional $M^n_t = \mathbf{1}_{(t<\zeta^{\Omega^n_\varepsilon})}$ of $B^{\nu,*}_t$. Our aim is to investigate the asymptotic behaviour of M^n_t when thickness (of Σ^n) and skewness coefficients vanish with different rates according with c_n (see Theorem 6.1). Our approach is based on the study of the asymptotic behaviour of $\widehat{M^n_t} = \mathbf{1}_{(t<\widehat{\zeta^{\Omega^n}})}$.

Concerning the Dirichlet problem on $D \subset \mathbb{R}^d$, the connection between variational and probabilistic approach to diffusion equations with killing has been investigated for example in [10]. Boundary value problems with varying domains has been also investigated in [16, 50] where a key role is played by the capacity induced by a regular Dirichlet form.

The plan of the paper is the following. Section 2 introduces notation and definitions of the pre-fractal and fractal Koch curves. Moreover, we recall the homogenization results obtained in [20]. Section 3 gives some basic aspects about positive continuous additive functionals and random times. In Section 4 we consider skew BM across a regular layer. The skew BM across irregular boundaries is introduced in Section 5. Our main results are collected and discussed in Section 6.

2 Notation and preliminary results

In this section we introduce the notation and some preliminary results. We recall the definition of the Koch curve with endpoints $A = (0,0)$, and $B = (1,0)$. We consider the family $\Psi^\alpha = \{\psi^\alpha_1, \dots, \psi^\alpha_4\}$ of contractive similitudes $\psi^\alpha_i : \mathbb{C} \to \mathbb{C}$, $i = 1,\dots,4$, with contraction factor α^{-1}, $2 < \alpha < 4$,

$$\psi^\alpha_1(z) = \frac{z}{\alpha}, \qquad \psi^\alpha_2(z) = \frac{z}{\alpha}e^{i\theta(\alpha)} + \frac{1}{\alpha},$$
$$\psi^\alpha_3(z) = \frac{z}{\alpha}e^{-i\theta(\alpha)} + \frac{1}{2} + i\sqrt{\frac{1}{\alpha} - \frac{1}{4}}, \qquad \psi^\alpha_4(z) = \frac{z-1}{\alpha} + 1,$$

where $\theta(\alpha) = \arcsin\left(\frac{\sqrt{\alpha(4-\alpha)}}{2}\right)$.

By the general theory of self-similar fractals (see [27]), there exists a unique closed bounded set K_α which is *invariant* with respect to Ψ^α, that is,

$$K_\alpha = \cup_{i=1}^4 \psi^\alpha_i(K_\alpha). \tag{2.1}$$

We recall that K_α supports a unique self-similar Borel measure μ_α which is equivalent to the d_f-dimensional Hausdorff measure where $d_f = \frac{\log 4}{\log \alpha}$. Let K^0 be the line segment of unit length that has as endpoints $A = (0,0)$ and $B = (1,0)$. We set, for each n in $\mathbb{N}$,

$$K^1_\alpha = \bigcup_{i=1}^4 \psi^\alpha_i(K^0), \qquad K^2_\alpha = \bigcup_{i=1}^4 \psi^\alpha_i(K^1_\alpha), \qquad \dots, K^{n+1}_\alpha = \bigcup_{i=1}^4 \psi^\alpha_i(K^n_\alpha); \tag{2.2}$$

K^n_α is the so-called n-th pre-fractal curve. Moreover, the iterates K^n_α converge to the self-similar set K_α in the Hausdorff metric, when n tends to infinity. Let Ω^0 be the triangle with vertices $A = (0,0), B = (1,0)$, and $C = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$. We construct on the side with endpoints A and B the pre-fractal Koch curve defined before, which will be denoted by $K^n_{1,\alpha}$ and the Koch curve

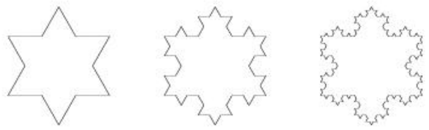

Figure 2.1: The pre-fractal domains.

defined before, which will be denoted by $K_{1,\alpha}$. In a similar way, we construct on the other sides the analogous pre-fractal Koch curves (the Koch curves) denoting by $K^n_{2,\alpha}$ and $K^n_{3,\alpha}$ (by $K_{2,\alpha}$ and $K_{3,\alpha}$) the curves with endpoints B and C, and C and A, respectively. We denote by Ω^n_α the pre-fractal domain that is the set bounded by the pre-fractal Koch curves $K^n_{j,\alpha}$, $j = 1, 2, 3$. Moreover, we denote by Ω_α the snowflake that is the set bounded by the Koch curves $K_{j,\alpha}$, $j = 1, 2, 3$. (see Figure 2.1). We denote by Σ^0_1 the *open set condition* triangle of vertices $A = (0, 0)$, $B = (1, 0)$ and $C = (1/2, b/2)$ where $b = \tan(\frac{\theta}{2})$.

Following the construction in [18], for every n and ε, we define the fiber $\Sigma^n_{1,\alpha}$, *ε-neighborhood* of $K^n_{1,\alpha}$ to be the (open) set

$$\Sigma^n_{1,\alpha} = \bigcup_{i|n} \Sigma^{i|n}_{1,\alpha},$$

where

$$\Sigma^{i|n}_{1,\alpha} = \psi_{i|n}(\Sigma^0_1)$$

(see Figure 2.2). We proceed in a similar way in order to construct the fiber $\Sigma^n_{j,\alpha}$, *ε-neighborhood* of $K^n_{j,\alpha}$ $(j = 2, 3)$ and, we define the fiber Σ^n_α, *ε-neighborhood* of $\partial\Omega^n$,

$$\Sigma^n_\alpha = \bigcup_{j=1}^{3} \Sigma^n_{j,\alpha}$$

and

$$\Omega^n_{\varepsilon,\alpha} = \overline{\Omega^n_\alpha} \bigcup \Sigma^n_\alpha.$$

From now on, we omit α when it does not give rise to misunderstanding, by writing simply Ω instead of Ω_α and similar expressions. We note that

$$\Omega^n \subset \Omega^{n+1} \subset \Omega \subset \Omega^{n+1}_\varepsilon \subset \Omega^n_\varepsilon.$$

We define a *weight* w^n as follows. Let P – for some $i|n$ – belong to the boundary $\partial(\Sigma^{i|n}_1)$ of $\Sigma^{i|n}_{1,\varepsilon}$ and let $P^\perp$ be the orthogonal projection of P on $K^{i|n}_1$. If (x, y) belongs to the segment with end-points P and $P^\perp$, we set, in our current notation,

$$w^n_1(x, y) = \frac{3|P - P^\perp|}{3 + b^2},$$

where $|P - P^\perp|$ is the (Euclidean) distance between P and $P^\perp$ in $\mathbb{R}^2$. We proceed in a similar way in order to construct the weights w^n_j on Σ^n_j $(j = 2, 3)$ and we define w^n on Ω^n_ε

$$w^n(x, y) = \begin{cases} w^n_j(x, y) & \text{if} \quad (x, y) \in \Sigma^n_j \\ 1 & \text{if} \quad (x, y) \in \overline{\Omega}^n. \end{cases} \tag{2.3}$$

Associated with the weight w^n, we consider the Sobolev spaces $H^1(\Omega^n_\varepsilon; w^n)$ and $H^1_0(\Omega^n_\varepsilon; w^n)$, defined as the completion of $C^\infty(\overline{\Omega^n_\varepsilon})$ and $C^\infty_0(\Omega^n_\varepsilon)$, respectively, in the norm

$$\|u\|_{H^1(\Omega^n_\varepsilon; w^n)} = \left(\int_{\Omega^n_\varepsilon} u^2 dxdy + \int_{\Omega^n_\varepsilon} |\nabla u|^2 w^n dxdy \right)^{\frac{1}{2}}. \tag{2.4}$$

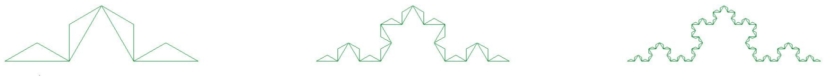

Figure 2.2: The fibers.

We define the coefficients

$$a^n_\varepsilon(x,y) = \begin{cases} c_n\sigma_n\, w^n(x,y) & \text{if} \quad (x,y)\in\Sigma^n \\ 1 & \text{if} \quad (x,y)\in\overline{\Omega}^n, \end{cases} \tag{2.5}$$

where

$$c_n > 0 \tag{2.6}$$

and

$$\sigma_n = \frac{\alpha^n}{4^n}. \tag{2.7}$$

From now on, when it does not give rise to misunderstanding, we denote by C positive, possibly different constants that do not depend on n and on ε. The following theorem states the existence and the uniqueness of the variational solution of the reinforcement problem. We consider the bilinear form associated with the reinforcement problem

$$a_n(u,v) := \int_{\Omega^n_\varepsilon} a^n_\varepsilon \nabla u\, \nabla v\, dxdy + \delta_n \int_{\Omega^n_\varepsilon} u\, v\, dxdy \tag{2.8}$$

where a^n_ε is defined in (2.5), (2.7), (2.6), and $\delta_n > 0$.

Theorem 2.1. *Let σ_n be as in (2.7) and $d_n \in \mathbb{R}$. Then, for any $f_n \in L^2(\Omega^n_\varepsilon)$, there exists one and only one solution u_n of the following problem*

$$\begin{cases} \textit{find } \; u_n \in H^1_0(\Omega^n_\varepsilon; w^n) \quad \textit{such that} \\ a_n(u_n, v) = \int_{\Omega^n_\varepsilon} f_n\, v\, dxdy + \sigma_n d_n \int_{\partial\Omega^n} v\; ds \quad \forall\, v \in H^1_0(\Omega^n_\varepsilon; w^n), \end{cases} \tag{2.9}$$

where $a_n(\cdot,\cdot)$ is defined in (2.8). Moreover, u_n is the only function that realizes the minimum of the energy functional

$$\min_{v\in H^1_0(\Omega^n_\varepsilon; w^n)} \Big\{ a_n(v,v) - 2\int_{\Omega^n_\varepsilon} f_n\, v\, dxdy - 2\sigma_n d_n \int_{\partial\Omega^n} v\; ds \Big\}. \tag{2.10}$$

In the following theorems, we state the existence and uniqueness of the variational solution of the Robin, Neumann, and Dirichlet problems on the domain Ω. We consider the bilinear form associated with the Robin problem

$$a_{c_0}(u,v) := \int_\Omega \nabla u\, \nabla v\, dxdy + \delta_0 \int_\Omega u\, v\, dxdy + c_0 \int_{\partial\Omega} \gamma_0 u\, \gamma_0 v\, d\mu \tag{2.11}$$

where μ is the measure on $\partial\Omega$ that coincides, on each K_j $j = 1,2,3$, with the Hausdorff measure defined before and $\gamma_0 u$ denotes the trace of the function u on the boundary of Ω, that is for v in $L^1_{loc}(D)$, where D is an arbitrary open set of $\mathbb{R}^2$, the trace operator γ_0 is defined as

$$\gamma_0 v(P) := \lim_{r\to 0} \frac{1}{m(B(P,r)\cap D)} \int_{B(P,r)\cap D} v(x,y)\, dx\, dy \tag{2.12}$$

at every point $P \in \overline{D}$ where the limit exists (see, for example, page 15 in [35]). From now on, we suppress γ_0 in the notation, when it does not give rise to misunderstanding, by writing simply v instead of $\gamma_0 v$ and similar expressions. We assume that

$$c_0 \geq 0,\ \delta_0 \geq 0, \text{ and } \max(c_0, \delta_0) > 0. \tag{2.13}$$

Theorem 2.2. *Let us assume* (2.13) *and* $d \in \mathbb{R}$. *Then, for any* $f \in L^2(\Omega)$, *there exists one and only one solution* u *of the following problem*

$$\begin{cases} \text{find } u \in H^1(\Omega) \quad \text{such that} \\ a_{c_0}(u,v) = \int_\Omega f\, v\, dxdy + d \int_{\partial\Omega} v\, d\mu \qquad \forall\, v \in H^1(\Omega) \end{cases} \tag{2.14}$$

where $a_{c_0}(\cdot,\cdot)$ *is defined in* (2.11). *Moreover,* u *is the only function that realizes the minimum of the energy functional*

$$\min_{v \in H^1(\Omega)} \Big\{ a_{c_0}(v,v) - 2\int_\Omega f\, v\, dxdy - 2d \int_{\partial\Omega} v\, d\mu \Big\}. \tag{2.15}$$

In a similar way, we prove the following result. We consider the bilinear form associated with the Dirichlet problem and

$$a_\infty(u,v) := \int_\Omega \nabla u\, \nabla v\, dxdy + \delta_0 \int_\Omega u\, v\, dxdy. \tag{2.16}$$

We assume that

$$\delta_0 \geq 0. \tag{2.17}$$

Theorem 2.3. *Let us assume* (2.17). *Then, for any* $f \in L^2(\Omega)$, *there exists one and only one solution* u *of the following problem*

$$\begin{cases} \text{find } u \in H^1_0(\Omega) \quad \text{such that} \\ a_\infty(u,v) = \int_\Omega f\, v\, dxdy \qquad \forall\, v \in H^1_0(\Omega) \end{cases} \tag{2.18}$$

where $a_\infty(\cdot,\cdot)$ *is defined in* (2.16). *Moreover,* u *is the only function that realizes the minimum of the energy functional*

$$\min_{v \in H^1_0(\Omega)} \Big\{ a_\infty(v,v) - 2\int_\Omega f\, v\, dxdy \Big\}. \tag{2.19}$$

We recall the notion of $M-$*convergence* of functionals, introduced in [41], (see also [42]).

Definition 2.1. *A sequence of functionals* $F^n : H \to (-\infty, +\infty]$ *is said to* $M-$*converge to a functional* $F : H \to (-\infty, +\infty]$ *in a Hilbert space* H, *if*

(a) For every $u \in H$ *there exists* u_n *converging strongly to* u *in* H *such that*

$$\limsup F^n[u_n] \leq F[u], \quad as \quad n \to +\infty. \tag{2.20}$$

(b) For every v_n *converging weakly to* u *in* H

$$\liminf F^n[v_n] \geq F[u], \quad as \quad n \to +\infty. \tag{2.21}$$

Let Ω^* be an open regular domain such that $\Omega^* \supset \overline{\Omega^n_\varepsilon}$, for all n : in order to fix notation we choice as Ω^* the ball with the center in the point $P_0 = (\frac{1}{2}, -\frac{1}{2})$ and radius 1. We consider the sequence of weighted energy functionals in $L^2(\Omega^*)$

$$F^n[u] = \begin{cases} \int_{\Omega^n_\varepsilon} a^n_\varepsilon(x,y) |\nabla u|^2 dxdy + \delta_n \int_{\Omega^n_\varepsilon} u^2 dxdy & \text{if } u|_{\Omega^n_\varepsilon} \in H^1_0(\Omega^n_\varepsilon; w^n) \\ +\infty & \text{otherwise in } L^2(\Omega^*) \end{cases} \tag{2.22}$$

(the coefficients a_ε^n are defined in (2.5), (2.7), (2.6), $\delta_n > 0$) and

$$F_{c_0}[u] = \begin{cases} \int_\Omega |\nabla u|^2 dxdy + \delta_0 \int_\Omega u^2 dxdy + c_0 \int_{\partial\Omega} u^2 d\mu & \text{if } u|_\Omega \in H^1(\Omega) \\ +\infty & \text{otherwise in } L^2(\Omega^*). \end{cases} \tag{2.23}$$

Moreover, we consider the case where the layer is *weakly insulating* (see (2.30) below) and we introduce the following functional (2.24) in $L^2(\Omega^*)$

$$F_\infty[u] = \begin{cases} \int_\Omega |\nabla u|^2 dxdy + \delta_0 \int_\Omega u^2 dxdy & \text{if} \quad u|_\Omega \in H_0^1(\Omega) \\ +\infty & \text{otherwise in } L^2(\Omega^*). \end{cases} \tag{2.24}$$

In order to study the asymptotic behaviour of the functions u_n, we fix the further assumptions

$$f_n,\, f \in L^2(\Omega^*), \text{ and } f_n \to f \ \in L^2(\Omega^*), \text{as } n \to +\infty, \tag{2.25}$$

$$\delta_n > 0 \text{ and } \delta_n \to \delta_0 \text{ as } n \to +\infty, \tag{2.26}$$

$$c_n > 0 \text{ and } c_n \to c_0 \text{ as } n \to +\infty, \tag{2.27}$$

$$d_n, d \in \mathbb{R}, \text{ and } d_n \to d \text{ as } n \to +\infty. \tag{2.28}$$

We also introduce the following results which have been proved in [20] and turn out to be useful further on.

Proposition 2.1. *Let σ_n be as in* (2.7)*. Then, for every sequence $g_n \in H^1(\Omega^{(\xi)})$ weakly converging towards g^* in $H^1(\Omega^{(\xi)})$, we have*

$$\sigma_n \int_{\partial\Omega^n} g_n ds \to \int_{\partial\Omega} g^* \, d\mu \ \ , \textit{as } \ n \to +\infty. \tag{2.29}$$

Theorem 2.4. *Let us assume* (2.27) *and* (2.26)*. Then, the sequence of functionals $F^n_{\varepsilon(n)}$, defined in (2.22), $M-$converges in $L^2(\Omega^*)$ to the functional F_{c_0} defined in* (2.23) *as $n \to +\infty$.*

Now we consider the case when the conductivity of the thin fibers vanishes slower than the thickness of the fiber: more precisely, we suppose

$$c_n w^n \to 0, \ c_n \to +\infty. \tag{2.30}$$

Theorem 2.5. *Let us assume* (2.30) *and* (2.26)*. Then the sequence of functionals $F^n_{\varepsilon(n)}$, defined in (2.22), $M-$converges in $L^2(\Omega^*)$ as $n \to +\infty$ to the energy functional $F_\infty[u]$ defined in (2.24).*

3 Positive continuous additive functionals and random times

We recall some basic aspects and introduce some notations. Let E be a locally compact separable metric space and m be a positive Radon measure on X such that supp$[m] = E$. A Dirichlet form $\mathcal{E}$ with domain $D(\mathcal{E})$ is a Markovian closed symmetric form on $L^2(E, m)$ (see [30, Chapter 1]).

We say that A_t, $t \geq 0$ is a positive continuous additive functional (PCAF) and write $A_t \in \mathbf{A}_c^+$ denoting by $\mathbf{A}_c^+$ the totality of PCAFs of an m-symmetric process X (see [24, A.3.1] for details). More precisely, we say that $A_t \in \mathbf{A}_c^+$ if

A.1) A_t, $t \geq 0$ is $\mathcal{F}_t$-measurable ($\{\mathcal{F}_t\}$ is the minimum completed admissible filtration),

A.2) there exists a set $\Lambda \in \mathcal{F}_\infty$ and an exceptional set $N \subset E$ with Cap$(N) = 0$ such that $\mathbb{P}_x(\Lambda) = 1$ for all $x \in E \setminus N$, $\theta_t \Lambda \subset \Lambda$ for all $t > 0$; for every $\omega \in \Lambda$, $A_t(\omega) : t \mapsto A_t(\omega)$ is continuous, $A_0(\omega) = 0$; for all $s, t \geq 0$ $A_{s+t}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$ where θ_t, $t \geq 0$ is the (time) translation semigroup, A.3) for all $\omega \in \Lambda$, $A_t(\omega) : t \mapsto A_t(\omega)$ is non-decreasing.

Hereafter, we write $\langle v, u\rangle_\mu = \int_E v(x)u(x)\mu(dx)$ and, in some case, we simply write $\langle v, \mu\rangle$ with obvious meaning of the notation. Moreover, $\mu(dx) = dx$ is the Lebesgue measure in which case we also write $\mu = m$.

A positive Radon measure μ for which ([30, pag. 74])

$$\int |v| d\mu \leq C\sqrt{\mathcal{E}_1(v, v)}, \quad \forall\, v \in D(\mathcal{E}) \cap C_0(E) \tag{3.1}$$

where

$$\mathcal{E}_\lambda(u, v) = \mathcal{E}(u, v) + \lambda\langle u, v\rangle \tag{3.2}$$

is said of finite energy integral and formula (3.1) holds if and only if there exists, for each $\lambda > 0$, a unique function $U_\lambda\mu \in D(\mathcal{E})$ (where $U_\lambda\mu$ is a λ-potential) such that

$$\mathcal{E}_\lambda(U_\lambda\mu, v) = \int v(x)\mu(dx). \tag{3.3}$$

We recall that ([30, pag. 64]), for an open set $B \subset E$ and $\mathcal{L}_B = \{v \in D(\mathcal{E}) : v \geq 1\, m\text{-}a.e.\ on\ B\}$, the capacity is defined as $\mathrm{Cap}(B) = \inf_{u\in\mathcal{L}_B} \mathcal{E}_1(u, u)$ if $\mathcal{L}_B \neq \emptyset$ and $\mathrm{Cap}(B) = \infty$ if $\mathcal{L}_B = \emptyset$. We say that a Borel measure μ on E is a smooth measure and write $\mu \in S = S(E)$ if [30, pag. 80]

μ.1) μ charges no set of zero capacity;

μ.2) there exists an increasing sequence $\{F_n\}$ of closed sets such that $\mu(F_n) < \infty$ and $\mathrm{Cap}(K \setminus F_n) \to 0$ for all compact sets K.

The class of smooth measures S is therefore large and it contains all positive Radon measures charging no set of zero capacity. By [30, Lemma 2.2.3], all measures of finite energy are smooth. We use the notation introduced in [30] and denote by $S_0 \subset S$ the set of positive Radon measure of finite energy integrals, by $S_{00} \subset S_0$ the set of finite measures with $\|U_1\mu\|_\infty < \infty$.

Let us consider $\mu_A \in S$ and $A_t \in \mathbf{A}_c^+$ associated with the m-symmetric process $X = (\{X_t\}_{t\geq 0}; \mathfrak{F}; \mathbb{P}_x, x \in E)$ with $\mathbb{P}_m(\Lambda) = \int_E \mathbb{P}_x(\Lambda)\, m(dx)$ and $\mathbb{P}_x(\Lambda) = \mathbb{P}_x(X_t \in \Lambda)$ for $\Lambda \in \mathfrak{F}$. Then, the measure μ_A and the PCAF A_t are in the Revuz correspondence if, for any $f \in \mathcal{B}_+(E)$ (the set of non-negative and measurable functions on E), we have that

$$\langle f, \mu_A\rangle = \lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}_m\left[\int_0^t f(X_s)dA_s\right] = \lim_{\lambda\to\infty} \lambda\mathbb{E}_m\left[\int_0^\infty e^{-\lambda t} f(X_t)dA_t\right]. \tag{3.4}$$

We say that μ_A is the Revuz measure of $A \in \mathbf{A}_c^+$ and if $\mu_A \in S$, then there exists a unique (up to equivalence) PCAF $\{A_t\}_{t\geq 0}$ with Revuz measure μ_A ([30, Theorem 5.1.4 and Theorem 5.1.3]). Throughout, we write μ instead of μ_A if no confusion arises. Moreover, we introduce

$$R_\lambda f(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\lambda t} f(X_t)dt\right] \quad \text{and} \quad U_A^\lambda f(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\lambda t} f(X_t)dA_t\right] \tag{3.5}$$

(see [47] for a complete discussion).

We introduce some further notation and basic aspects. In the following sections we consider the killed process

$$X_t = \begin{cases} \widetilde{X}_t, & t < \tau \\ \partial, & t \geq \tau \end{cases} \tag{3.6}$$

where ($\widetilde{X}_t \in E$ and ∂ is the "coffin state" not in E) τ will be a suitable random time and $\mathbf{P}_t f(x) = \mathbb{E}_x[f(X_t)] = \mathbb{E}_x[f(\widetilde{X}_t); t < \tau]$, $x \in E$ is the associated semigroup. In particular, we consider the following cases: i) $\tau = \zeta^E$ is a random time such that $(\zeta^E < t) \equiv (L_t^{\partial E} > \zeta)$ and ζ is an exponential random variable, with parameter $c_0 \in (0, \infty)$, independent from X; ii) $\tau = \infty$ under suitable conditions; iii) $\tau = \tau_E$ is the exit time of X from E.

Thus X_t, $t \in [0, \infty]$, is a Markov process with state space $E_\partial := E \cup \{\partial\}$. The transition function is not conservative according with the cemetery point $\{\partial\}$, that is $\mathbb{P}_x(X_t = \partial) \geq 0$,

$\forall\, x \in E_\partial$, $t \geq 0$. In particular, X is conservative if $\mathbb{P}_x(\zeta^E < \infty) = 0$ for every $x \in E$ where we denote by ζ^E also the lifetime of the process on E. Since X_t is a Markov process, $\mathbb{P}_x(X_0 = x) = 1$ for all $x \in E_\partial$ and $\mathbb{P}_\partial(X_t = \partial) = 1$ for all t. Our discussion is mainly concerned with trap domains. A point $x \in E_\partial$ is called a *trap* of X if $\mathbb{P}_x(X_t = x) = 1$ for every $t \geq 0$. We give the definition of trap domain further on in the text. In i) we have introduced the local time process $L^{\partial E}_t = L^{\partial E}_t(X)$ which is the PCAF increasing when X hits the boundary ∂E. It is well known that, the lifetime of the process follows the law $\mathbb{P}_x(\zeta^E > t | X_t) = e^{-c_0 L^{\partial E}_t}$ for every $x \in E$ and $t > 0$. Thus, $L^{\partial E}_0 = 0$ and $\mathbb{P}_x(\zeta^E > 0) = 1$. $L^{\partial E}$ is the occupation time of X on ∂E. For $\Lambda \subseteq E$, we denote by $\Gamma^\Lambda_t(X) = meas\{s \in [0,t] \,:\, X_s \in \Lambda\}$ the occupation time process of X on Λ. The semigroup $\mathbf{P}_t$ is strongly continuous and we use the fact that $\lambda R_\lambda f \to f$ and $\lambda \langle U^\lambda_A f, m\rangle \to \langle f, \mu\rangle$ as $\lambda \to \infty$ where μ is the Revuz measure associated with the additive functional A and therefore, to the random time $\tau \in [\tau_E, \infty]$. In particular, if $\mathbb{P}_x(\tau = \infty) = 1$, then for the planar BM B, $\forall\, \Lambda \subseteq \mathbb{R}^2$, $\mathbb{P}_x$-almost surely, $\overline{\Gamma^\Lambda_t}(B) = \int_0^\infty e^{-\delta s} \mathbf{1}_\Lambda(B_s) ds = \infty$ if $\delta = 0$.

We say that X^n converges in law to X and write $X^n \stackrel{law}{\to} X$ if $\mathbb{E} f(X^n) \to \mathbb{E} f(X)$ as $n \to \infty$ for every continuous and bounded function f.

4 Transmission condition on regular interfaces

In this section we consider the probabilistic approach of thin layer when Ω is a disc. Actually, we provide a sketch of proof for the problem with collapsing annulus by following two approaches. The purpose is to underline the main differences with the fractal case investigated in the next sections. Notice also that speed measure and scale function characterize uniquely one-dimensional diffusions.

First approach. Let us consider a BM X on $\mathbb{R}^d$ (with $d \geq 2$) started away from zero. For $\theta \in [0, 2\pi)$, $r \geq 0$ we can write, $\mathbb{P}_x(X_t \in dy) = \mathbb{P}_{(\theta', r')}(\Theta_t \in d\theta, R_t \in dr)$ where $R = |X|$ is a Bessel process. In particular, R and Θ are the radial and the angular part of X. It is also well-known that a skew-product representation is given in term of (R, Θ) where $R = |X|$ is a Bessel process and $\Theta = X/|X| = B(\int R^{-2} ds)$ with B an independent BM on the sphere $\mathbb{S}^{d-1}$ [34, pag. 269]. Here Θ is a time-changed BM on $\mathbb{S}^{d-1}$.

Let B^ν be a (skew) planar BM on the disc C_2 with $C_1 \subset C_2$ (centred at the same point $(0,0)$, with radius $r_1 < r_2 = r_1 + \varepsilon$, $\varepsilon > 0$) and transmission condition on ∂C_1, Dirichlet condition on ∂C_2. Let $\mathcal{L}_n$ be the governing operator of B^ν. We examine in this section the classical Chan, Terence Occupation times of compact sets by planar Brownian motion. Ann. Inst. H. Poincar Probab. Statist. 30 (1994), no. 2, 317329.case corresponding to the (formal) problem

$$\begin{aligned} \mathcal{L}_n u_n &= -f_n \quad \text{on } C_2 \\ (1-\nu)\, \partial_{\mathbf{n}} u_n\big|_{\partial C_1 -} &= \nu\, \partial_{\mathbf{n}} u_n\big|_{\partial C_1 +} \\ u_n\big|_{\partial C_1 -} &= u_n\big|_{\partial C_1 +} \\ u_n\big|_{\partial C_2} &= 0 \end{aligned}$$

where $\partial_{\mathbf{n}}$ is the normal derivative and we denote by ∂C_1- and ∂C_1+ the boundary from the interior and from the exterior of C_1. Let us consider the sequences $\nu = \nu(n)$, $\varepsilon = \varepsilon(n)$, $c = c(n)$, $n \in \mathbb{N}$. Our aim is to study the asymptotic behaviour of the solution as $n \to \infty$ and $\nu, \varepsilon \to 0$ with different rate given by the elastic coefficient $c \geq 0$. The problem above can be associated with B^ν started away from the origin. The process is partially (normally) reflected on ∂C_1 and totally absorbed in ∂C_2.

A reflecting BM on a disc can be constructed (in law) by considering suitable time change and rotation ([34, pag. 272]). The time change in this case is a stochastic clock given by an additive functional of the radial motion as indicated before. Denote by $C_{1,2}$ the annulus $C_2 \setminus \overline{C_1}$. Thus, for $0 < r, r' < r_2$ and $0 < \theta, \theta' \leq 2\pi$, $\mathbb{P}_x(B^\nu_t \in dy) = \mathbb{P}_{(\theta', r')}(\Theta_t \in d\theta, R^\nu_t \in dr)$ where R^ν is a skew Bessel process on $(0, r_2)$ such that, $\mathbb{P}_{r_1}(R^\nu_t \in (r_1, r_2)) = \nu$. We have normal reflection on ∂C_1 and

$$\forall\, x \in \partial C_1, \qquad \mathbb{P}_x(B^\nu_t \in dy) = \frac{d\theta}{2\pi} \mathbb{P}_{r_1}(R^\nu_t \in dr) \tag{4.1}$$

which means that, the BM can move from ∂C_1 according with an uniformly distributed angle Θ for the choice of the starting point. Therefore, we have that

$$\mathbb{P}_{\partial C_1}(R^\nu_t \in C_{1,2}) = \nu \quad \text{and} \quad \mathbb{P}_{\partial C_1}(R^\nu_t \in C_1) = 1-\nu.$$

Let R be the part of the Bessel process $\widetilde{R}$ on (r_1, r_2) with $\widetilde{R} \in (0,\infty)$. We cut the excursions of $\widetilde{R}$ on $(0, r_1)$ by considering a time change given by the inverse of $\Gamma^{(r_1,r_2)}_t(\widetilde{R})$. We do the same with $\Gamma^{(0,r_1)}_t(\widetilde{R})$. As in [34, pag. 115] we can obtain a skew motion by considering the ν portion of $\Gamma^{(r_1,r_2)}_t(\widetilde{R})$ and the $1-\nu$ portion of $\Gamma^{(0,r_1)}_t(\widetilde{R})$. Thus, it is possible to consider a suitable time change, say $\mathfrak{f}^{-1}$, in order to obtain partial (normal) reflection on r_1 and, $R^\nu = R_{\mathfrak{f}^{-1}}$ is a Bessel process on (r_0, r_2) with transmission condition on r_1. The skew BM constructed in this way has the skew-product representation involving the time-changed Bessel process $R_{\mathfrak{f}^{-1}}$ where the BM on the circle is identical in law to the original process (if $\Theta^\nu_t = B(T^\nu_t)$ where B is independent from T^ν_t defined above and obtained by R^ν, then $\Theta^\nu_t \stackrel{law}{=} \Theta_t$ where $\Theta_t = B(T_t)$ and T_t is obtained by R). Thus, the only process we consider is the radial part R_t time-changed by $\mathfrak{f}^{-1}$, that is R^ν. The Bessel process can start from zero and then it is instantaneously reflected. It never hits the origin at some $t > 0$. The mean exit time

$$v_\varepsilon(r) = \mathbb{E}[\tau_{(r_0,r_2)}(R^\nu) | R^\nu_0 = r \in (r_0, r_2)] = \mathbb{E}_r \tau_{C_2}$$

can be explicitly written by following standard techniques for one-dimensional diffusions (see for example [37]) and, as $\varepsilon \to 0$, $\nu \to 0$ according with $\nu/\epsilon \to c$, we find that it solves

$$\begin{array}{ll} v_0'' = -1 & \\ v_0(r_0) = 0 & \\ v_0(r_1) = 0 & \text{if } c = \infty \\ v_0'(r_1) = -c\, v_0(r_1) & \text{if } c \in [0,\infty). \end{array}$$

This corresponds to the study of u_n with $f_n = \mathbf{1}$. Due to isotropy and the discussion about the angular part of the planar BM, we arrive at the solution u_∞ of the problem above. Therefore, the boundary conditions on r_1 depend on the limit of the ratio between the skewness coefficient ν and the thickness coefficient ε.

Second approach. Alternatively, we can approach the problem as follows. Our result in fractal domains can be reformulated here (in regular domains) by considering the stopping time $T_{c(n)}$ and the fact that $(T_{c(n)} > t | B^\nu) \equiv (T_{c(n)} > t | R^\nu)$ in view of the previous discussion. In particular we consider the lifetime ζ^{C_2} of R^ν and the sequence of random times $\widehat{T_{c(n)}} = \inf\{s \in (0, \zeta^{C_2}] : L^{r_1}_s > \zeta^n\}$ with conditional law $\mathbb{P}_x(\widehat{T_{c(n)}} > t | R^\nu) = \exp - c(n) L^{r_1}_t$ where $L^{r_1}_t$ is the symmetric local time of R^ν at r_1. That is, we consider ζ^n as exponential random variable with parameter $c(n)$ and independent from R^ν. Thus, under the assumption that $\lim_{n\to\infty} c(n) = \lim_{n\to\infty} \nu(n)/\varepsilon(n)$, we study the asymptotic behaviour of

$$u_n(r) = \mathbb{E}_r\left[\int_0^{\tau_{(r_0, r_1+\varepsilon)}} f_n(\widetilde{R^\nu_s}) ds\right] = \mathbb{E}_r\left[\int_0^\infty f_n(\widetilde{R^\nu_t}) M^n_t dt\right]$$

where $M^n_t = \mathbf{1}_{(t < T_{c(n)})}$ by means of the asymptotic behaviour of

$$\widehat{u_n}(r) = \mathbb{E}_r\left[\int_0^\infty f_n(\widetilde{R^\nu_t}) \widehat{M^n_t} dt\right]$$

where $\widehat{M^n_t} = \mathbf{1}_{(t < \widehat{T_{c(n)}})}$ and (assume here $x \in C_1$ for the reader's convenience)

$$\mathbb{P}_x(\widehat{T_{c(n)}} > t | R^\nu) \stackrel{law}{\to} \left\{ \begin{array}{ll} \mathbf{1}, & c(n) \to 0, \\ \exp - c_0 L^{r_1}_t, & c(n) \to c_0, \\ \mathbf{1}_{(t < \tau_{C_1})}, & c(n) \to \infty, \end{array} \right. \qquad \text{as } n \to \infty.$$

We use the same notation for the local times meaning that $L_t^{r_1}(R^{\nu(n)}) \to L_t^{r_1}(R^+)$ in law where R^+ is a reflecting Bessel process on $(0, r_1)$. Thus, we estimate the true stopping time T by $\widehat{T}$ and exploit the fact that $\widehat{T} \leq T$ with probability one. This immediately follows by observing that $T = \zeta^{C_2}$ and by the definition of $\widehat{T}$. The convergence of $R^{\nu(n)}$ can be obtained by considering that $\mathbb{P}_r(R_t^{\nu(n)} > M) \leq M^{-1}\mathbb{E}R_t^{\nu(n)}$ and the moment is bounded.

Remark 4.1. *For a compact subset $K \in \mathbb{R}^d$ ([48, Theorem 22.7])*

$$\mathbb{P}_x(B_t \in K \text{ for some } t > 0) = \int G(x,y)\mu_K(dy) = G\mu_K(x)$$

is a potential of a unique measure μ_K concentrated on ∂K. The capacity $Cap(K) = \inf\{\mathcal{E}(\mu) : G\mu \geq 1 \text{ on } K\}$ where $\mathcal{E}(\mu) = \int G(x,y)\mu(dx)\mu(dy)$ can be defined from $\mu_K(K)$.

Define $\sigma_K = \sup\{s > 0\,;\, B_s \in K\}$ with $\sup \emptyset = 0$, then for $x \in \mathbb{R}^d$, $y \in K$, $t > 0$, we have that ([25, 31])

$$\mathbb{P}_x(B_{\sigma_K} \in dy, \sigma_K \in dt) = p(t,x,y)\mu_K(dy)dt \tag{4.2}$$

and we recover an interesting connection between elastic coefficient and capacity. Consider $K = \overline{C_1}$, the last exit time can be therefore rewritten as $\sigma_K = \inf\{s > 0 : L_s^{r_1} > \zeta^n\}$ where now, ζ^n is the time the process spends on (or cross) r_1 before absorption in r_2.

Remark 4.2. *Notice that we used isotropy and skew product representation which are not suitable tools for approaching our fractal problem. In particular, if we consider the Koch domain Ω, the normal vector does not exist at almost all boundary points. However it is possible to define the Robin boundary condition in the sense of the dual of certain Besov spaces (see [17, Theorem 4.2]).*

5 Transmission conditions on irregular interfaces

In this section we introduce the modified skew BM $B_t^{\nu,*}$, $t \geq 0$ on Ω_ε^n. The parameter $\nu \in [0,1]$ is the so called skewness parameter. Skew BM is a process with associated Dirichlet form in $L^2(\Omega^*, \mathfrak{m}_\nu)$ given by

$$\mathcal{E}(u,v) = \frac{1}{2}\int \nabla u \, \nabla v \, d\mathfrak{m}_\nu \tag{5.1}$$

where $\mathfrak{m}_\nu(x) = 2(1-\nu)\mathbf{1}_{\Omega^n}(x) + 2\nu\mathbf{1}_{\Sigma^n}(x)$ and it can be associated with discontinuous diffusion coefficients. We focus on the sequence of elliptic operators

$$L_n \, u = -\operatorname{div}\,(a_\varepsilon^n(x,y)\,\nabla u) \tag{5.2}$$

in divergence form with coefficients given in (2.5). The discontinuous coefficients a_ε^n in (5.2) introduce the transmission condition in the $L^2(\partial\Omega^n)$

$$\nabla u \cdot \mathbf{n}\big|_{y-} = c_n \sigma_n \, \nabla u \cdot \mathbf{n}\big|_{y+} \quad \forall\, y \in \partial\Omega^n \tag{5.3}$$

(where $\mathbf{n}$ is the outer normal to Ω^n, $y^- = y \in \overline{\Omega^n} \cap \partial\Omega^n$ and $y^+ = y \in \overline{\Sigma^n} \cap \partial\Omega^n$, we recall that $w^n|_{\partial\Omega^n} = 1$) and therefore, the corresponding diffusion behaves like a modified BM. For a given n, the operator (5.2) can be regarded as the governing operator of the planar skew BM $\widetilde{B^\nu} = (\{\widetilde{B_t^\nu}\}_{t\geq 0}; \mathfrak{F}^\nu; \mathbb{P}_x^n, x \in \Omega_\varepsilon^n)$ from which we define the killed process B^ν. Let $\mathcal{L}_n$ be the governing operator of B^ν on $L^2(\Omega_\varepsilon^n, dx)$ with

$$\mathcal{D}(\mathcal{L}_n) = \Big\{ u \in L^2(\Omega_\varepsilon^n, dx), \quad u|_{\Omega^n} \in H^2(\Omega^n), \quad u|_{\Sigma^n} \in H^2(\Sigma^n),$$
$$: \, u\big|_{\partial\Omega_\varepsilon^n} = 0, \quad u \text{ is continuous on } \partial\Omega^n \text{ and satisfies (5.3)} \Big\}.$$

Then, the transition function $\mathbf{P}^n_t f(x) = \mathbb{E}^n_x[f(B^\nu_t)] = \mathbb{E}^n_x[f(\widetilde{B^\nu_t});\, t < \tau_{\Omega^n_\varepsilon}]$ with transition kernel p^ν where ν depends on the coefficients a^n_ε and therefore, on (5.3), is governed by

$$\frac{\partial u}{\partial t} = \mathcal{L}_n\, u \quad \text{on} \quad \Omega^n_\varepsilon \tag{5.4}$$

and $\mathcal{L}_n f := \frac{1}{2} L_n f,\ f \in \mathcal{D}(\mathcal{L}_n)$. The parabolic equation (5.4) can be rewritten by considering the infinitesimal generator $\widetilde{\mathcal{L}}_n := \frac{1}{2}\Delta$ on $L^2(\widetilde{\mathfrak{m}}_\nu)$ with $\mathcal{D}(\widetilde{\mathcal{L}}_n) = \mathcal{D}(\mathcal{L}_n)$ (see [24, pag 356] for details) where

$$\widetilde{\mathfrak{m}}_\nu(x) = \mathbf{1}_{\Omega^n}(x) + c_n \sigma_n w^n\, \mathbf{1}_{\Sigma^n}(x). \tag{5.5}$$

From the transition kernel p^ν we can write

$$\mathbb{P}^n_x(\widetilde{B^\nu_t} \in \Lambda, t < \tau_{\Omega^n_\varepsilon}) = \int_\Lambda p^\nu(t,x,y)\, dy \quad x \in \Omega^n_\varepsilon \tag{5.6}$$

for some Borel set $\Lambda \in \mathfrak{F}^\nu$ with the (first) exit time

$$\tau_{\Omega^n_\varepsilon} = \inf\{s > 0 \,:\, \widetilde{B^\nu_s} \notin \Omega^n_\varepsilon\}. \tag{5.7}$$

We refer to B^ν as a modified skew BM in the sense that it depends on both the skewness coefficient ν and the weight w^n. The process B^ν represents a Brownian diffusion of a particle with transmission condition (5.3) on the pre-fractal $\partial\Omega^n$. Thus, according with (4.1), for $x \in \partial\Omega^n$ we have that $\mathbb{P}^n_x(B^\nu_t \in \Omega^n) = (1-\nu)\sigma_n$, $\mathbb{P}^n_x(B^\nu_t \in \Sigma^n) = \nu\sigma_n$ and this explains the reason for which we refer to ν as a skewness coefficient. The BM is partially reflected when it hits $\partial\Omega^n$. This means that, when it hits the pre-fractal curve, it moves with different probability toward Ω^n or Σ^n. Due to the isotropic nature of the motion, $\forall\, x \in \partial\Omega^n$ the process starting from x moves with the same probability $1-\nu$ or ν. In particular, $\mathbb{P}_{\partial\Omega^n}(B^\nu_t \in \Sigma^n) = \int_{\partial\Omega^n} \mathbb{P}^n_x(B^\nu_t \in \Sigma^n) ds = \nu$ and $\mathbb{P}_{\partial\Omega^n}(B^\nu_t \in \Omega^n) = 1-\nu$. Let $\nu = \nu(n)$ be a sequence such that $\nu(n) \to 0$ as $n \to \infty$. Since condition (5.3) holds $\forall\, y \in \partial\Omega^n$, from the construction we present here, it must be that

$$\frac{a^{n+}_\varepsilon}{a^{n-}_\varepsilon + a^{n+}_\varepsilon} \frac{1}{\nu(n)\sigma_n} \to 1 \qquad \text{uniformly on } \partial\Omega^n \text{ as } n \to \infty$$

that is, $\nu(n)/c_n w^n \to 1$ as $n \to \infty$. In particular, the transmission condition (5.3) says also that

$$\frac{\nu(n)}{1-\nu(n)} \frac{1}{c_n w^n} \to 1 \qquad \text{uniformly on } \Sigma^n \text{ as } n \to \infty \tag{5.8}$$

In view of (5.8), we also refer to ν as transmission parameter. However, due to the fact that $w|_{\partial\Omega^n} = 1$, we must pay particular attention on the pre-fractal boundary.

We also introduce the following representation

$$\widetilde{B^\nu_t} = \begin{cases} B^w_t & \text{on } \mathbb{R}^2 \setminus \overline{\Omega^n} \quad \text{with probability } \nu \\ B_t & \text{on } \Omega^n \qquad\quad \text{with probability } 1-\nu \end{cases} \tag{5.9}$$

where B^w and $B = B^1$ are independent Brownian motions with total mass respectively on $\mathbb{R}^2 \setminus \overline{\Omega^n}$ and Ω^n and depending on w^n with $w^n \neq 1$ only outside $\overline{\Omega^n}$. Thus, B^ν_t equals B^w_t with probability ν and B_t with probability $1-\nu$; in this case, notice also that, $B = B^+$ on Ω^n and B^w is a BM reflected on $\partial\Omega^n$ and absorbed on $\partial\Omega^n_\varepsilon$.

Let us consider the perturbed Dirichlet form on $L^2(E,m)$ written as

$$\mathcal{E}^\mu_\lambda(u,v) = \mathcal{E}_\lambda(u,v) + \langle u,v\rangle_\mu, \quad u,v \in D(\mathcal{E}) \cap L^2(E,\mu) \tag{5.10}$$

where $\mathcal{E}_\lambda$ has been introduced in (3.2). Let $\mu \in S$ and $A_t \in \mathbf{A}^+_c$, then the transition function

$$\mathbf{P}^\mu_t f(x) = \mathbb{E}_x[e^{-A_t} f(\widetilde{X_t})] \tag{5.11}$$

is associated with the regular form $(\mathcal{E}_0^\mu, D(\mathcal{E}_0^\mu))$ and μ is the Revuz measure of A_t (see [30, Theorem 6.1.1 and Theorem 6.1.2]). We simply write $\mathbf{P}_t$ instead of $\mathbf{P}_t^\mu$.

We follow the characterization of trap domain given in [8, 15]. Consider the reflected BM B^+ on $\overline{D}$ and an open connected set $D \subset \mathbb{R}^d$, $d \geq 2$ with finite volume. Let $\mathcal{B} \subset D$ be an open ball with non-zero radius and denote by $\tau_{\partial\mathcal{B}} = \inf\{s \geq 0 \,:\, B_s^+ \in \partial\mathcal{B}\}$ the hitting time of the reflecting BM $B^+ \in D \setminus \mathcal{B}$.

Definition 5.1. *The set D is a trap domain if*

$$\sup_{x \in D\setminus\mathcal{B}} \mathbb{E}_x\, \tau_{\partial\mathcal{B}} = \infty. \tag{5.12}$$

Otherwise, D is a non-trap domain.

Notice that the definition above does not depend on the choice of $\mathcal{B}$ ([15, Lemma 3.3]). In both Lipschitz domains Ω^n and Σ^n the process B^ν behaves like a BM B^+ reflecting on $\partial\Omega^n$. As shown in [8, 15], the pre-fractal and fractal Koch domains are *non-trap*. Then, $\forall\, n$, Ω^n and Σ^n are non trap for B^ν. Condition (5.12) can be rewritten in analytic way as follows

$$\sup_{x\in D\setminus\mathcal{B}} \int_{D\setminus\mathcal{B}} G^+(x,y)dy = \infty$$

where G^+ is the Green function of B^+ on D and $D = \Omega^n$ or $D = \Sigma^n$.

The process on Ω^n_ε is a transient BM for which $\mathbf{P}^n_t \mathbf{1}_{\Omega^n_\varepsilon}(x) = \mathbb{P}_x(\tau_{\Omega^n_\varepsilon} > t)$ and $\mathbb{E}_x \tau_{\Omega^n_\varepsilon} = \int \mathbf{P}^n_t \mathbf{1}_{\Omega^n_\varepsilon}(x)dt < \infty$. Nevertheless, we are looking for asymptotic results concerning also a non transient limit process. Thus, for the skew BM B^ν_t, $t < \tau_{\Omega^n_\varepsilon}$, we introduce the Green function $G^\nu_n(x,y) = \int_0^\infty e^{-\delta_n t} p^\nu(t,x,y)dt$ for which we write

$$G^\nu_n f(x) = \int G^\nu_n(x,y)\, f(y)\, dy = \mathbb{E}^n_x\left[\int_0^{\tau_{\Omega^n_\varepsilon}} e^{-\delta_n t} f(\widetilde{B^\nu_t})dt\right] \tag{5.13}$$

where $\mathbb{E}^n_x$ is the expectation under (5.6). Furthermore, we write $G^\nu_n f(x) = \int G_n(x,y)\, f(y)\, \mathfrak{m}_\nu(y)dy$ where $G_n(\cdot,\cdot) = \int_0^\infty p(t,\cdot,\cdot)dt$ is the Green function of a BM B on Ω^n_ε. Throughout, we consider the PCAF (in the strict sense, that is, in A.2) Λ is the defining set and N is an empty set) $A^n_t = \int_0^t f(X^n_s)ds$ and a stopping time T_n together with ([23, Lemma 2.1])

$$\lim_{t\downarrow 0} \mathbb{E}^n_x A^n_t = \lim_{t\downarrow 0} \left[\mathbb{E}^n_x[A^n_t\,;\, t < T_n] + \mathbb{E}^n_x[A^n_t\,;\, t \geq T_n]\right] = \lim_{t\downarrow 0} \mathbb{E}^n_x[A^n_t\,;\, t < T_n]. \tag{5.14}$$

Let us introduce the following measures on Ω^n_ε

$$m^n_\varepsilon(dx) = \mathbf{1}_{\Omega^n\cup\Sigma^n}(x)\, dx + \mathbf{1}_{\partial\Omega^n}(x)\, ds \tag{5.15}$$

and

$$\mathfrak{m}^n_\varepsilon(dx) = \mathfrak{m}_{\nu(n)}(x)\, dx + 2\nu(n)\sigma_n \mathbf{1}_{\partial\Omega^n}(x)\, ds \tag{5.16}$$

where the measure on the pre-fractal curve is taken according with Proposition 2.1. Notice that $\mathfrak{m}_\nu$ is related to $\widetilde{\mathfrak{m}}_\nu$ by means of (5.8).

We also introduce the process $B^{\nu,*}_t$ which is the m^n_ε-symmetric extension of B^ν to $\Omega^n \cup \partial\Omega^n \cup \Sigma^n$ (see [24, Definition 7.5.8 and Definition 7.7.1]). We can also consider the $\mathfrak{m}^n_\varepsilon$-symmetric extension of $\widetilde{B}$ according with the generator $\widetilde{\mathcal{L}}_n$ (and considering (5.5), (5.8)). Thus, we write (5.6) as $\mathbb{P}^n_x(B^\nu_t \in \Lambda)$ and

$$\mathbb{P}^n_{m^n_\varepsilon}(B^{\nu,*}_t \in \Lambda) = \int_{\Omega^*} \mathbb{P}^n_x(B^{\nu,*}_t \in \Lambda)\, m^n_\varepsilon(dx). \tag{5.17}$$

To be precise, we say that $B^{\nu,*}_t$ is the m^n_ε-symmetric extension of B^ν meaning that

$$m^n_\varepsilon\big|_{\Omega^n\cup\Sigma^n} = m$$

where m is a Lebesgue measure such that $m(\{e^*\}) = 0$ for the extra point e^* identifying every point $x \in \partial\Omega^n$. Thus, $E = \Omega^n \cup \Sigma^n$ is the state space of B^ν and $E^* = E \cup \{e^*\}$ is a representation of the state space of $B^{\nu,*}$. For the sake of clarity we notice that, by following [24], we should say "m-symmetric extension". The process $B^{\nu,*}$ is the part process of $\widetilde{B^{\nu,*}_t}$ on Ω^n_ε where $\widetilde{B^{\nu,*}_t}$ equals B^* on Ω^n and $B^{w,*}$ on $\mathbb{R}^2 \setminus \Omega^n$ according with representation (5.9).

5.1 Local time and Occupation measure

The skew BM is a Markov process with continuous paths (and discontinuous local time). The boundary local time is a PCAF defined as an occupation time process on the boundary (see [21, 26] for example). Moreover, we deal with a modified skew BM depending on the weights w^n. For a given n, we introduce the occupation density $\ell^n_t(x)$, $x \in \Omega^*$, $t \geq 0$ such that, for $\Lambda \in \Omega^*$, the following occupation formula holds true

$$\int_0^{t\wedge\tau_{\Omega^n_\varepsilon}} \mathbf{1}_\Lambda(\widetilde{B^{\nu,*}_s})ds = \int_0^t \mathbf{1}_\Lambda(B^{\nu,*}_s)ds = \int_\Lambda \ell^n_t(y; B^{\nu,*})\, m^n_\varepsilon(dy) = \int_\Lambda \ell^n_{t\wedge\tau_{\Omega^n_\varepsilon}}(y; \widetilde{B^{\nu,*}})\, m^n_\varepsilon(dy) \quad (5.18)$$

where m^n_ε is the measure (5.15) and $B^{\nu,*}$ is the m^n_ε-symmetric extension of B^ν. With some abuse of notation we do not distinguish here between absolutely continuity of the occupation density on $\Lambda \subset \Omega^n_\varepsilon$ or $\Lambda \subset \partial\Omega^n$. For the sake of simplicity we use the same symbol ℓ^n_t for a density w.r.t. m^n_ε. In particular, with (3.4) in mind, as $t \to 0$ we have that

$$\begin{aligned} &\frac{1}{t}\mathbb{E}^n_{m^n_\varepsilon}\left[\int_0^t f(B^{\nu,*}_s)d\Gamma^\Lambda_s\right] \to \int_{\Omega^n_\varepsilon} f(x)\mathbf{1}_\Lambda(x)dx,\\ &\frac{1}{t}\mathbb{E}^n_{m^n_\varepsilon}\left[\int_0^t f(B^{\nu,*}_s)dL^\Lambda_s\right] \to \sigma_n \int_{\partial\Omega^n} f(x)\mathbf{1}_\Lambda(x)ds. \end{aligned} \quad (5.19)$$

The occupation density $\ell^n_t(y; B^{\nu,*})$ must be discontinuous on $\partial\Omega^n$ (and continuous on $\Omega^n \cup \Sigma^n$ with different "speed" measures depending on w^n). In particular, $\ell^n_t(y; B^{\nu,*})m^n_\varepsilon(dy) = \ell^n_t(y; X)\mathfrak{m}^n_\varepsilon(dy)$ whereas, the modified motion X (equals B^* or $B^{w,*}$ according with (5.9) and) behaves like a BM on Ω^n_ε. The occupation density can be therefore written by considering the "right" (from the exterior, Σ^n) and "left" (from the interior, Ω^n) densities $\ell^{n+}_t(y; B^{\nu,*}) = 2\nu(n)\sigma_n\ell^n_t(y; B^{\nu,*})$ and $\ell^{n-}_t(y; B^{\nu,*}) = 2(1-\nu(n))\sigma_n\ell^n_t(y; B^{\nu,*})$. Thus,

$$\int_0^t \mathbf{1}_{\partial\Omega^n}(B^{\nu,*}_s)ds = L^{\partial\Omega^n}_t(B^{\nu,*}) = (L^{\partial\Omega^n-}_t + L^{\partial\Omega^n+}_t)/2 = \sigma_n \int_{\partial\Omega^n} \ell^n_t(y; B^{\nu,*})\, m^n_\varepsilon(dy) \quad (5.20)$$

is written in terms of $L^{\partial\Omega^n-}_t = L^{\partial\Omega^n-}_t(B^{\nu,*})$ and $L^{\partial\Omega^n+}_t = L^{\partial\Omega^n+}_t(B^{\nu,*})$, say "left" and "right" local time. Recall that we are dealing with the BM $B^{\nu,*} = B^*$ on Ω^n and $B^{\nu,*} = B^{w,*}$ on $\overline{\Sigma}^n$ with probability respectively given by $1-\nu$ and ν as in (5.9). We have that $L^{\partial\Omega^n-}_t(B^{\nu,*}) = L^{\partial\Omega^n}_t(B^*)$ and $L^{\partial\Omega^n+}_t(B^{\nu,*}) = L^{\partial\Omega^n}_t(B^{w,*})$ according with (5.9), that is

$$L^{\partial\Omega^n-}_t(B^{\nu,*}) = 2(1-\nu)L^{\partial\Omega^n}_t(B^{\nu,*}) \quad \text{and} \quad L^{\partial\Omega^n+}_t(B^{\nu,*}) = 2\nu L^{\partial\Omega^n}_t(B^{\nu,*}) \quad (5.21)$$

where $L^{\partial\Omega^n}_t(B^{\nu,*})$ is a symmetric local time (independent from ν). Since $t \mapsto L^{\partial\Omega^n}_t(B^{\nu,*})$ is a continuous additive functional, formulas in (5.21) define PCAFs. Indeed, for $\eta > 0$, $\eta L \in \mathbf{A}^+_c$ iff $L \in \mathbf{A}^+_c$ ([49, Proposition VI.45.10]). The representations (5.21) can be also abtained by considering excursions of $\widetilde{B^{\nu,*}}$ and suitable time changes. According with (5.8), for the sequence of probabilities $\nu(n)$, it holds that

$$\frac{1}{\nu(n)}\frac{c_n w^n}{1+c_n w^n} \to 1 \quad \text{uniformly as } n \to \infty. \quad (5.22)$$

Observe that we always have $c_n w^n \to 0$ (as $n \to \infty$) as basic assumption between conductivity and thickness of the fiber, the insulating fractal layer case. We use the fact that, for any $f \in \mathcal{B}_+$,

$$\begin{aligned} \int_\Lambda f(y)\ell^{n+}_t(y; B^{\nu,*})m^n_\varepsilon(dy) = &\int_\Lambda f(y)\ell^n_t(y; B^{\nu,*})\, \mathfrak{m}^n_\varepsilon(dy), \quad \text{if} \quad \Lambda \subseteq \partial\Omega^n \\ =&2\nu(n)\sigma_n \int_\Lambda f(y)\ell^n_t(y; B^{\nu,*})\, m^n_\varepsilon(dy) \end{aligned} \quad (5.23)$$

and

$$\int_\Lambda f(y)\ell^n_t(y; B^{\nu,*})m^n_\varepsilon(dy) = 2\nu(n)\int_\Lambda f(y)\ell^n_t(y; B^w)\, m^n_\varepsilon(dy), \quad \text{if} \quad \Lambda \subseteq \Sigma^n \quad (5.24)$$

under $\mathbb{E}_x^n$. Formulas (5.23) and (5.24) can be also obtained by considering (5.16) together with representation (5.9) and by following similar arguments as in [11]. Indeed, for $0 < t_1 < t_2 < \tau_{\Omega_\varepsilon^n}$, and $\Lambda = \text{supp}[\mu]$ where μ is the Revuz measure of A_t, we have that

$$\mathbb{E}_x^n \left[\int_{t_1}^{t_2} f(B_s^{\nu,*}) dA_s \right] = \int_{t_1}^{t_2} ds \int_{\Omega^*} f(y)\, p^{\nu,*}(s,x,y)\, \mu(dy). \tag{5.25}$$

If $A_t = L_t^{\Lambda+}$, then

$$\begin{aligned} \mathbb{E}_x^n \left[\int_0^t f(B_s^{\nu,*}) dA_s \right] &= \int_0^t ds \int_{\Omega^*} f(y)\, p^{w,*}(s,x,y)\, \mu(y) \mathfrak{m}_\varepsilon^n(dy) \\ &= 2\nu(n)\sigma_n \int_0^t ds \int_\Lambda f(y)\, p^{w,*}(s,x,y)\, m_\varepsilon^n(dy) \end{aligned} \tag{5.26}$$

where $p^{w,*}(s,x,y)$ is the transition kernel of $B^{\nu,*}$ on $\Omega_\varepsilon^n \setminus \Omega^n$ and formula (5.23) follows by (5.18) and (5.20). Notice that for $\Lambda \subseteq \Sigma^n$ (that is, $A_t = \Gamma_t^\Lambda$), the integral (5.24) vanishes as $n \to \infty$.

For the Neumann heat kernel p_N in an inner uniform domain, it holds that ([32])

$$c_1 t^{-1} e^{-\frac{d^2(x,y)}{c_2 t}} \le p_N(t,x,y) \le c_3 t^{-1} e^{-\frac{d^2(x,y)}{c_4 t}}. \tag{5.27}$$

In view of (5.25) and the Gaussian bound (5.27), there exists $C = C(t_1, t_2) > 0$ such that

$$\mathbb{E}_x^n \left[\int_{t_1}^{t_2} f(B_s^{\nu,*}) dA_s \right] \le C \int_{\Omega^*} f(y)\, \mu(dy). \tag{5.28}$$

It is known that the reflecting BM B^+ spends zero Lebesgue amount of time on the boundary $\partial\Omega$. On the other hand, we are interested in L_t^Ω obtained as a limit of $\Gamma_t^{\Omega_\varepsilon^n \setminus \Omega^n}$ for some $t < T$. Thus, we focus on $L^{\partial\Omega^n +}$ or equivalently on $\Gamma_t^{\Omega_\varepsilon^n \setminus \Omega^n}$ further in our analysis. In order to streamline the notation as much as possible we write ℓ_t^n in place of $\ell_t^n(B^{\nu,*})$ and ℓ_t instead of ℓ_t^∞ when no confusion arises.

5.2 The probabilistic framework

Here the aim is to provide a suitable framework to start with in the next section. We formalize some link between the previous sections and Brownian motions on trap domains, in particular on a domain with Koch interfaces. Hereafter, we assume that $d_n = 0$ and $d = 0$ without loss of generality. The problem in Theorem 2.1 can be formulated as follows.

Theorem 5.1. *The unique weak solution of problem* (2.9) *can be written as*

$$u_n(x) = \mathbb{E}_x^n \left[\int_0^{\tau_{\Omega_\epsilon^n}} e^{-s\delta_n} f_n(\widetilde{B_s^{\nu(n,*)}}) ds \right]. \tag{5.29}$$

The associated Dirichlet form, say $\mathcal{E}_0^{\mu_{A^n}}$ with $D(\mathcal{E}_0^{\mu_{A^n}}) = H_0^1(\Omega_\varepsilon^n)$, is obtained by (5.1) as perturbation of (2.8) by the Revuz measure of the additive functional A_t^n associated with the killing time $\tau_{\Omega_\varepsilon^n}$. We continue with the following representation of the solution in Theorem 2.2.

Theorem 5.2. *The unique weak solution of* (2.14) *can be written as*

$$u(x) = \mathbb{E}_x \left[\int_0^\infty e^{-s\delta_0 - c_0 L_s^{\partial\Omega}} f(B_s^+)\, ds \right] \tag{5.30}$$

where $B^+ = (\{B_t^+\}_{t\ge0}; \mathfrak{F}^+; \mathbb{P}_x, x \in \overline{\Omega})$ *is a reflecting BM on* $\overline{\Omega}$ *and* $L_t^{\partial\Omega} = L_t^{\partial\Omega}(B^+)$ *is the local time on the boundary* $\partial\Omega$.

The associated Dirichlet form, say $\mathcal{E}_0^{\mu_{A^n}}$, is therefore given by (2.11) with $D(\mathcal{E}_0^{\mu_{A^n}}) = H^1(\Omega)$. The solution (5.30) is obtained by considering the exponential random variable ζ with parameter

$c_0 > 0$ (independent from B^+) and $\zeta^\Omega = \inf\{s \geq 0 \,:\, L^{\partial\Omega}_s \notin [0,\zeta]\}$. Thus, the associated semigroup is written as

$$\begin{aligned}\mathbf{P}^+_t f(x) =& \mathbb{E}_x\left[e^{-\delta_0 t} f(B^+_t); t < \zeta^\Omega\right] = \mathbb{E}_x\left[e^{-\delta_0 t} f(B^+_t); \zeta > L^{\partial\Omega}_t\right]\\ =& \mathbb{E}_x\left[e^{-\delta_0 t} f(B^+_t)\, \mathbb{E}[\zeta > L^{\partial\Omega}_t | \mathfrak{F}^+]\right] = \mathbb{E}_x\left[f(B^+_t)\, e^{-\delta_0 t - c_0 L^{\partial\Omega}_t}\right].\end{aligned}$$

Let $A_t = \Gamma^\Lambda_t(B)$ and $A^{-1}_t = \inf\{s \geq 0 \,:\, A_s \notin [0,t]\}$. Since A_t is a non-decreasing process, $(A^{-1}_t < s) \equiv (A_s > t)$ and we say that A^{-1} is the inverse of A. Obviously we have that $(\zeta^\Omega > t) \equiv (L^{\partial\Omega}_t < \zeta)$. It is worth mentioning that B^+_t can not be written (for all $t > 0$) as $B_{(A_t)^{-1}}$. We can not consider the skew product representation as in Section 4 or in the recent paper [51] for instance. The reflecting BM has been investigated by many researchers and some different constructions have been also considered. Nevertheless, some technical problems can arise from the characterization of the domains. Here we consider a domain with fractal boundary and in particular, we exploit the fact that our pre-fractal and fractal Koch domains are non trap. This permits us to consider occupation measures even if the fractal nature of the boundary does not allow the study of the corresponding time changed processes. Theorem 2.3 can be formulated as follows.

Theorem 5.3. *The unique weak solution of problem* (2.18) *can be written as*

$$u(x) = \mathbb{E}_x\left[\int_0^{\tau_\Omega} e^{-s\delta_0} f(B_s) ds\right] \tag{5.31}$$

where τ_Ω is the first time the BM B hits the boundary $\partial\Omega$.

Let ∞_{D^c} be the measure which is $+\infty$ on the complement D^c of a Borel set D. Formula (5.11) becomes

$$\mathbf{P}^n_t f(x) = \mathbb{E}^n_x[f(B^{\nu,*}_t)] = \mathbb{E}^n_x[e^{-\overline{A^n_t}} f(\widetilde{B^{\nu,*}_t})] = \mathbb{E}^n_x[e^{-\delta_n t} f(\widetilde{B^{\nu,*}_t})\,;\, t < \tau_{\Omega^n_\varepsilon}] \tag{5.32}$$

where $\overline{A^n_t} = \delta_n\, t + A^n_t$ is a PCAF with drift δ_n ($\delta_n \geq 0$) and associated Revuz measure which can be written as $\mu_{\overline{A^n}} = \mu_n + \infty_{D^c}$ and $D = \Omega^n_\varepsilon$. The resolvent kernel is written as follows

$$R^n_\lambda f(x) = \mathbb{E}^n_x\left[\int_0^{\tau_{\Omega^n_\varepsilon}} e^{-\lambda t - \delta_n t} f(\widetilde{B^{\nu,*}_t}) dt\right] = \mathbb{E}^n_x\left[\int_0^\infty e^{-\lambda t - \overline{A^n_t}} f(\widetilde{B^{\nu,*}_t}) dt\right]. \tag{5.33}$$

For the sake of simplicity we consider $\delta_n = 0$ (if not otherwise specified) in the theorems below. The case $\delta_n > 0$ can be immediately obtained by considering $\overline{A^n_t}$ with $\mu_{\overline{A^n}}(dx) = \mu_{A^n}(dx) + \delta_n\, dx$ and following similar arguments. We rewrite (5.10) by considering that the semigroup (5.32) generates the Dirichlet form on $L^2(\Omega^n_\varepsilon)$ given by

$$\mathcal{E}^{\mu_{A^n}}_0(u,v) = a_n(u,v) + \langle u, v\rangle_{\mu_{A^n}}, \quad u,v \in D(\mathcal{E}^{\mu_{A^n}}_0) \cap L^2(\mu_{A^n}) \tag{5.34}$$

with $D(\mathcal{E}^{\mu_{A^n}}_0) = H^1(\Omega^n_\varepsilon)$ where a_n is given in (2.8). Observe that the part process of $\widetilde{B^{\nu,*}_t}$ is transient if and only if $\mathrm{Cap}(\mathbb{R}^2 \setminus \Omega^n_\varepsilon) > 0$ ([24, Proposition 3.5.10]). The lifetime is finite and the process is killed. The representation (5.32) says also that for our initial problem (2.9) can be given a variational formulation as in [16] by defining the measure

$$\mu^{\varepsilon(n)}(B) = \begin{cases} +\infty, & \text{if } \mathrm{Cap}_1(B \cap (\mathbb{R}^2 \setminus \Omega^n_\varepsilon)) > 0,\\ 0, & \text{otherwise}\end{cases} \tag{5.35}$$

and the corresponding Dirichlet form $\mathcal{E}^{\mu^\epsilon}_0$ with $D(\mathcal{E}^{\mu^\epsilon}_0) = H^1(\mathbb{R}^2)$. Thus, the Dirichlet condition is prescribed in the capacity sense and the modified BM moves on $\mathbb{R}^2$.

We shall approach the convergence in L^2 of the solutions we are interested in, by first considering convergence of measures. Let $\{\mathbb{P}^n\}_n$ be a sequence of probability measures on $(E,\mathfrak{E})$. We say that $\mathbb{P}^n$ converges weakly-$\star$ to $\mathbb{P}$ on $(E,\mathfrak{E})$ as $n \to \infty$ and write $\mathbb{P}^n \stackrel{w^\star}{\to} \mathbb{P}$, if $\mathbb{E}^n f(X^n) = \int_E f d\mathbb{P}^n \to \int_E f d\mathbb{P} = \mathbb{E} f(X)$, $\forall\, f \in C_b(E)$ where X^n and X are the random variables with probability measures

$\mathbb{P}^n$ and $\mathbb{P}$ (that is, X^n convergences in law to X and we also write $X^n \stackrel{law}{\to} X$). If a sequence of stochastic processes converges (weakly) in the sense of finite-dimensional laws (write $X^n \stackrel{f.d.}{\to} X$) we are in need of tightness in order to get convergence in law. Moreover, we write $X^n \stackrel{law}{\to} \infty$ (meaning also that $X^n \stackrel{a.s.}{\to} \infty$, that is almost surely or with probability one) if $\forall\, M,\ \exists n^* : \mathbb{P}(X^n > M) = 1,\ \forall\, n > n^*$. We use vague convergence arguments in this case, that is for a sequence of measures μ_n on $E \cup \{+\infty\}$ we have $\mu_n \stackrel{v}{\to} \mu$ if $\langle f, \mu_n \rangle \to \langle f, \mu \rangle$, $\forall\, f \in C_0^+$, the class of functions $f : \mathbb{R} \to \mathbb{R}_+$ with compact support.

Theorem 5.4. *([42])The Mosco convergence of the forms is equivalent to the strong convergence of the associated resolvents and semigroups.*

Convergence of semigroups, by the Markov property, provides convergence of finite dimensional laws. In particular (let the symbol "$\to$" denote strong convergence of semigroups), for the semigroup (5.32), under (2.25) and (2.26), consider that (see theorems 2.4 and (2.5)):

i) Neumann, under (2.27) with $c_0 = 0$ and $\delta_0 > 0$,

$$\mathbf{P}_t^n f_n(x) \to \mathbb{E}_x \left[e^{-\delta_0 t} f(B_t^+) \right]; \tag{5.36}$$

ii) Robin, under (2.27) with $c_0 > 0$ and $\delta_0 \geq 0$,

$$\mathbf{P}_t^n f_n(x) \to \mathbb{E}_x \left[e^{-\delta_0 t} f(B_t^+) \,;\, t < \zeta^\Omega \right]; \tag{5.37}$$

iii) Dirichlet, under (2.30) and $\delta_0 \geq 0$,

$$\mathbf{P}_t^n f_n(x) \to \mathbb{E}_x \left[e^{-\delta_0 t} f(B_t) \,;\, t < \tau_\Omega \right] = \mathbb{E}_x \left[e^{-\delta t} f(B_t^+) \,;\, t < \tau_\Omega \right]. \tag{5.38}$$

Thus, starting from the part process of $\widetilde{B_t^{\nu,*}}$ on Ω_ε^n (and therefore from (5.32)), we simply write

$$\mathbf{P}_t^n f_n(x) \to \mathbf{P}_t f(x) = \mathbb{E}_x \left[e^{-\delta_0 t} f(B_t^+);\, t < T_{c_\infty} \right] \tag{5.39}$$

where the stopping time depends on $\lim_{n\to\infty} c_n = c_\infty \in [0, \infty]$. We arrive at the reflecting BM on $\overline{\Omega}$ stopped by T_{c_∞}. However, the convergence in (5.39) follows once the convergence of a suitable sequence of stopping times to T_{c_∞} in an appropriate sense has been shown. If $m_n \to m$, for the Borel sets $\{\Lambda_j\}$ we have that

$$\mathbb{E}_{m_n}^n [\mathbf{1}_{\Lambda_1}(X_{t_1}^n) \cdots \mathbf{1}_{\Lambda_k}(X_{t_k}^n)] \to \mathbb{E}_m [\mathbf{1}_{\Lambda_1}(X_{t_1}) \cdots \mathbf{1}_{\Lambda_k}(X_{t_k})] \tag{5.40}$$

as $n \to \infty$. This is due to the Markov property and the fact that

$$\mathbb{E}_x^n [\mathbf{1}_{\Lambda_1}(X_{t_1}^n) \cdots \mathbf{1}_{\Lambda_k}(X_{t_k}^n)] = \mathbf{P}_{t_1}^n \mathbf{1}_{\Lambda_1} \mathbf{P}_{t_2 - t_1}^n \mathbf{1}_{\Lambda_2} \cdots \mathbf{P}_{t_k - t_{k-1}}^n \mathbf{1}_{\Lambda_k}(x) \tag{5.41}$$

where $\mathbf{P}_t^n f(x)$ is the transition (non conservative) semigroup (5.32). Thus we have convergence of finite dimensional laws. If in addition, $\mathbb{P}_{m_n}^n$ is tight, then $\mathbb{P}_{m_n}^n$ converges weakly-$\star$ to $\mathbb{P}_m$.

Definition 5.2. *The sequence of probability measures $\{\mathbb{P}^n\}_n$ on a metric space E is said to be tight if for every $\epsilon > 0$, there exists a compact set $K \subseteq E$ such that* $\sup_n \mathbb{P}^n(E \setminus K) \leq \epsilon$.

We use the (Kolmogorov-Chentsov) criterion based on the moments of increments, that is, the sequence X^n is tight if $X_0^n = 0$ and there exist $\alpha, \beta > 0$ and $C > 0$ such that, for $T > 0$,

$$\mathbb{E}[|X_t^n - X_s^n|^\alpha] \leq C\, |t - s|^{\beta + 1} \tag{5.42}$$

holds uniformly on $n \in \mathbb{N}$ and $0 \leq s, t \leq T$ (see [36, Corollary 14.9]). Thus, the sequence X^n is tight in the space of all continuous processes, equipped with the norm of locally uniform convergence.

6 Main results

We consider occupation measures on both Ω^n_ε and $\partial\Omega^n$ (local times) instead of planar Brownian motions. Let $\zeta^{\Omega^n_\varepsilon}$ be the lifetime of $B^{\nu,*}_t$ on Ω^n_ε and ζ^Ω be the lifetime of the limit process on Ω. Let us focus now on (5.39). Let $X^n_t = \{B^{\nu(n),*}_t,\ t < T_{c_n}\}$ be the process with transition semigroup $\mathbf{P}^n_t$ (associated with the form $\mathcal{E}^{\mu_{A^n}}_0$) and X_t be the process with transition semigroup $\mathbf{P}_t$ (associated with the form $\mathcal{E}^{\mu_{A^\infty}}_0$). Our aim is to prove the following theorem.

Theorem 6.1. *Let A^n_t be as in* (5.32). *We have:*

τ.R) $c_n \to c_0 \Leftrightarrow \zeta^{\Omega^n_\varepsilon} \stackrel{law}{\to} \zeta^\Omega \Leftrightarrow \mu_{A^n} \stackrel{w}{\to} \mu_{A^\infty} = c_0\,\mu_\alpha$. X_t is an elastic (or partially reflected) BM on $\overline{\Omega}$;

τ.N) $c_n \to 0 \Leftrightarrow \zeta^{\Omega^n_\varepsilon} \stackrel{a.s.}{\to} \infty \Leftrightarrow \mu_{A^n} \stackrel{w}{\to} \mu_{A^\infty} = 0$. X_t is a reflecting BM on $\overline{\Omega}$;

τ.D) $c_n w^n \to 0$, $c_n \to \infty \Leftrightarrow \zeta^{\Omega^n_\varepsilon} \stackrel{law}{\to} \tau_\Omega \Leftrightarrow \mu_{A^n} \stackrel{v}{\to} \mu_{A^\infty} = \infty$ (is locally infinite). X_t is an absorbing BM on Ω.

The main tools we deal with are stopping times. We first assume that a.s. $\zeta^{\Omega^n_\varepsilon} = T_{c_n}\ \forall\, n$, that is the lifetime is equivalent to a random time depending on $c_n \geq 0$. Then, we focus on the sequence of random times T_{c_n} with $c_n \to c_\infty \in [0,\infty]$ as $n \to \infty$ and we study the convergence $T_{c_n} \to T_{c_\infty}$. Thus, T_{c_∞} plays the role of lifetime for the limit BM on Ω (or $\overline{\Omega}$).

Remark 6.1. *Let ζ^n be a r.v. with density law $\mathbb{P}(\zeta^n \in dx) = c_n \exp(-c_n\, x)\mathbf{1}_{[0,\infty)}(x)\, dx$. We obviously have that $\mathbb{P}(\zeta^n \leq x) = 1 - \exp(-c_n x)$ and $\mathbb{E}\zeta^n = 1/c_n$. Let ζ^∞ be such that $\zeta^n \to \zeta^\infty$ as $n \to \infty$.*

If $c_n \to \infty$, then $\zeta^n \stackrel{\mathbb{P}}{\to} 0$. Indeed, by Markov's inequality, we have that $\mathbb{P}(|\zeta^n - \zeta^\infty| > \epsilon) \leq \epsilon^{-1}\mathbb{E}|\zeta^n| \to 0$ as $c_n \to \infty$. Moreover, $\zeta^\infty \geq 0$ with $\mathbb{E}\zeta^\infty = 0$. Thus, $\mathbb{P}(\zeta^\infty = 0) = 1$.

If $c_n \to 0$ we have that $\mathbb{P}(\zeta^\infty \leq x) = 0$ for all $x \in [0,\infty)$ with $\zeta^\infty \geq 0$ and, by observing that $\mathbb{E}\zeta^\infty = \infty$, we conclude that $\mathbb{P}(\lim_{n\to\infty} \zeta^n = \infty) = 1$.

If $c_n \to c_0 \in (0,\infty)$, we simply have that $\zeta^n \stackrel{law}{\to} \zeta$ as $n \to \infty$ where ζ is the exponential r.v. with parameter c_0.

We can also relate ζ^n to the time the process $B^{\nu,*}_t$ spends on (or cross) the pre-fractal $\partial\Omega^n$. The sequence ζ^n depends on the local time $L^{\partial\Omega^n}_t = L^{\partial\Omega^n}_t(B^{\nu,*})$. For a fixed n, denote by $\widehat{\zeta^{\Omega^n}}$ the r.v. written as

$$\widehat{\zeta^{\Omega^n}} = \inf\{0 < s \leq \zeta^{\Omega^n_\varepsilon} : L^{\partial\Omega^n}_s(\widetilde{B^{\nu,*}}) > \zeta^n\} = \inf\{s > 0 : L^{\partial\Omega^n}_s(B^{\nu,*}) > \zeta^n\} \tag{6.1}$$

assuming that ζ^n is independent from $B^{\nu,*}$ (and therefore, from the local time on the pre-fractal boundary). Obviously, $\widehat{\zeta^{\Omega^n}}$ is a sequence of Markov stopping times before absorption on $\partial\Omega^n_\varepsilon$. To be clear, $\widehat{\zeta^{\Omega^n}} \leq \zeta^{\Omega^n_\varepsilon}$ with probability one, $\forall\, n \in \mathbb{N}$. We also observe that (6.1) can be regarded as the lifetime of the process up to the last visit on $\overline{\Omega}^n$ assuming that ζ^n is the time the process spends on the pre-fractal boundary before absorption on $\partial\Omega^n_\varepsilon$. In this case we have the lifetime on Ω^n written as

$$\zeta^{\Omega^n} = \sup\{s > 0 : B^{\nu(n),*}_s \in \overline{\Omega}^n\} = \sigma_{\overline{\Omega}^n} \tag{6.2}$$

which is no longer Markovian. Moreover, $\mathbb{P}_x(\widehat{\zeta^{\Omega^n_\varepsilon}} > \zeta^{\Omega^n}) > 0$, m-a.e. x and

$$\mathbb{P}_x(\sigma_{\overline{\Omega}^n} > 0) = \mathbb{P}_x(\tau_{\Omega^n} < \infty).$$

We consider (6.1) with exponential threshold ζ^n.

Remark 6.2. *From the discussion in the previous remark, we have that*

$$\widehat{\zeta^{\Omega^n}} \stackrel{law}{\to} \inf\{s > 0 : L^{\partial\Omega}_s > 0\} = \tau_\Omega \quad \textit{as } c_n \to \infty \tag{6.3}$$

and

$$\widehat{\zeta^{\Omega^n}} \stackrel{law}{\to} \inf\{s > 0 : L^{\partial\Omega}_s > \infty\} = \infty \quad \textit{as } c_n \to 0 \tag{6.4}$$

provided that the occupation time sequence $L_t^{\partial\Omega^n}$ *converges in law to the local time* $L_t^{\partial\Omega}$ *of the corresponding limit process. If* $c_n \to c_0 \in (0,\infty)$*, then the lifetime of the limit process depends on the random variable* ζ *with parameter* c_0*. In particular, we have that*

$$\widehat{\zeta^{\Omega^n}} \stackrel{law}{\to} \zeta^{\Omega} = \inf\{s > 0 \,:\, L_s^{\partial\Omega}(B^+) > \zeta\} \tag{6.5}$$

provided the convergence in law of the occupation time process on the fractal boundary $\partial\Omega$*.*

For the sake of simplicity we write ζ^{Ω} in place of ζ^{Ω^∞}. We also focus on the PCAF

$$\widetilde{A_t^{n+}} = \Gamma_t^{\Sigma^n}(\widetilde{B^{\nu,*}}) + L_t^{\partial\Omega^n +}(\widetilde{B^{\nu,*}}) = \Gamma_t^{\Omega^n_\varepsilon \setminus \Omega^n}(\widetilde{B^{\nu,*}}) \tag{6.6}$$

and $\int_0^{t\wedge\tau_{\Omega^n_\varepsilon}} d\widetilde{A^n_s}$ is the occupation time process on $\Omega^n_\varepsilon \setminus \Omega^n$ of the skew BM $B^{\nu,*}$. Let us write

$$R^n_\lambda \mu_n^+(x) = 2(1+c_n)^{-1} c_n\, \sigma_n\, \mathbb{E}^n_x\left[\int_0^{\zeta^{\Omega^n_\varepsilon}} e^{-\lambda s - s\delta_n} \mathbf{1}_{\partial\Omega^n}(\widetilde{B_s^{\nu(n),*}}) ds\right]$$

where $\mu_n^+(dx) = 2(1+c_n)^{-1} c_n \sigma_n\, \mathbf{1}_{\partial\Omega^n}(x)\, m^n_\varepsilon(dx)$. Let $A_t^{n+} \in \mathbf{A}_c^+$ be in Revuz correspondence with the measure $\mu_n^+(\cdot)$.

Proposition 6.1. *Under* (2.13)*, the boundary local time* $\{L_t^{\partial\Omega +},\, t < \zeta^{\partial\Omega}\}$ *is the unique PCAF such that, for any* $x \in \Omega^n_\varepsilon$*,*

$$R^n_\lambda \mu_n^+(x) \to \mathbb{E}_x\left[\int_0^{\zeta^{\Omega}} e^{-\lambda s - \delta_0 s}\, dL_s^{\partial\Omega +}\right] \quad \textit{as} \quad n \to \infty. \tag{6.7}$$

Proof. First we notice that μ_n has finite energy integral. Indeed,

$$\int_{\Omega^*} v(x)\, \mu_n^+(dx) \le \|\mu_n^+\|_{L^2(\partial\Omega^n)}\, \|v\|_{L^2(\partial\Omega^n)} = 2(1+c_n)^{-1} c_n \sqrt{\sigma_n} \|v\|_{L^2(\partial\Omega^n)}.$$

From [20, Theorem 8.1] we know that

$$\|v\|^2_{L^2(\partial\Omega^n)} \le \frac{C}{\sigma_n} \|v\|^2_{H^1(\mathbb{R}^2)}$$

where C is independent of n. Since $(1+c_n)^{-1}c_n \le 1$, by extension theorem (see [20, Theorem A.3]) we obtain that $\langle \mu_n^+, v\rangle \le \sqrt{C}\, \|v\|_{H^1(\Omega^n_\varepsilon)}$ and $\mu_n^+ \in S_0$. Since, under (2.13), $R^n_\lambda \mu_n^+(x)$ is bounded (and in view of [24, Lemma 4.1.5]) we have that $\mu_n^+ \in S_{00}$. Let $c_\infty \ge 0$ be such that $c_n \to c_\infty$. From (2.29), for all $f \in \mathcal{B}_+$ we get that

$$\langle f, \mu_n^+ \rangle \to 2(1+c_\infty)^{-1} c_\infty \int_{\partial\Omega} f\, d\mu \quad \text{as} \quad n \to \infty.$$

For a fixed n, consider the occupation measure (6.6). In view of (5.18), (5.23) and (5.24), we write

$$\begin{aligned}\mathbb{E}^n_x\left[\int_0^{t\wedge\tau_{\Omega^n_\varepsilon}} f(\widetilde{B_s^{\nu(n),*}}) d\widetilde{A_s^{n+}}\right] =& \mathbb{E}^n_x\left[\int_{\Omega^*} f(y) \mathbf{1}_{\Omega^n_\varepsilon\setminus\Omega^n}(y) \ell^n_{t\wedge\tau_{\Omega^n_\varepsilon}}(y; \widetilde{B^{\nu,*}}) m^n_\varepsilon(dy)\right]\\ =& 2\nu(n)\, \mathbb{E}^n_x\left[\int_0^{t\wedge\tau_{\Omega^n_\varepsilon}} f(\widetilde{B_s^{\nu,*}}) d\Gamma_s^{\Sigma^n} + \int_0^{t\wedge\tau_{\Omega^n_\varepsilon}} f(\widetilde{B_s^{\nu,*}}) dL_s^{\partial\Omega^n}\right]\end{aligned}$$

where $\Gamma_t^{\Sigma^n} = \Gamma_t^{\Sigma^n}(B^w)$. Set $U^\lambda_n f(x) = \mathbb{E}^n_x\left[\int_0^\infty e^{-\lambda t - \delta_n t} f(\widetilde{B_t^{\nu,*}})\, \mathbf{1}_{(t<\zeta^{\Omega^n_\varepsilon})} d\widetilde{A_t^{n+}}\right]$. From (5.19) and the fact that $\mathbb{E}^n_x[\mathbf{1}_{(0<\zeta^{\Omega^n_\varepsilon})}] = \mathbb{P}^n_x(\zeta^{\Omega^n_\varepsilon} > 0) = 1$ for all $x \in \Omega^n_\varepsilon$ we obtain

$$\lim_{\lambda\to\infty} \langle \lambda U^\lambda_n f, m^n_\varepsilon \rangle = \langle f, \widetilde{\mu_n}^+ \rangle, \quad \forall\, f \in \mathcal{B}_+ \tag{6.8}$$

where $\widetilde{\mu_n}^+(dx) = 2\nu(n)(\mathbf{1}_{\Sigma^n} + \sigma_n \mathbf{1}_{\partial\Omega^n})\, m^n_\varepsilon(dx)$ on $\Omega^n_\varepsilon \setminus \Omega^n$ and therefore, by [30, Theorem 5.1.4] there exists a unique PCAF of $B_t^{\nu,*}$ in Revuz correspondence with $\widetilde{\mu_n}^+$, that is (6.6). Notice also that, since $\nu(n) \leq 1$, we can follow the same arguments as before in order to see that $\widetilde{\mu_n}^+ \in S_0$. Let us consider $\xi_n \in \Sigma^n$ and $0 \leq M_n < \infty$. Since condition (5.22) holds true, from Proposition 2.1, we get that

$$\langle f, \widetilde{\mu_n}^+ \rangle = M_n\, c_n w^n(\xi_n) f(\xi_0) + \int_{\partial\Omega^n} f(x) \widetilde{\mu_n}^+(dx) \to 2(1+c_\infty)^{-1} c_\infty \int_{\partial\Omega^n} f d\mu, \quad \textit{as } n \to \infty$$

where $\mu = \mu_\alpha$ is the only Borel measure which is in Revuz correspondence with the "symmetric" local time $L^{\partial\Omega}_t$. Indeed,

$$\lim_{\lambda\to\infty} \lambda \mathbb{E}_m \left[\int_0^\infty e^{-\lambda s} f(B_s) dL^{\partial\Omega}_s \right] = \int_{\Omega^*} f d\mu.$$

Notice that

$$R^n_\lambda \mu^+_n(x) \to 2(1+c_\infty)^{-1} c_\infty \, \mathbb{E}_x \left[\int_0^{\zeta^\Omega} e^{-\lambda s - \delta_0 s} \mathbf{1}_{\partial\Omega}(B^+_s) ds \right] < \infty$$

according with (5.39). From the one to one correspondence between (6.6) and its Revuz measure $\widetilde{\mu_n}^+$ we prove (6.7) and the claim. □

Let us consider $A^{n-}_t \in \mathbf{A}^+_c$ and

$$R^n_\lambda \mu^-_n(x) = 2(1+c_n)^{-1} \sigma_n \mathbb{E}^n_x \left[\int_0^{\zeta^{\Omega^n_\varepsilon}} e^{-\lambda s - s\delta_n} \mathbf{1}_{\partial\Omega^n}(\widetilde{B^{\nu(n),*}_s}) ds \right]$$

where μ^-_n is the Revuz measure of A^{n-}_t.

Proposition 6.2. *Under* (2.13)*, the boundary local time* $\{L^{\partial\Omega-}_t,\, t < \zeta^\Omega\}$ *is the unique PCAF such that, for any* $x \in \Omega^n_\varepsilon$*,*

$$R^n_\lambda \mu^-_n(x) \to \mathbb{E}_x \left[\int_0^{\zeta^\Omega} e^{-\lambda s - \delta_0 s} dL^{\partial\Omega-}_s \right], \quad \textit{as } n \to \infty. \tag{6.9}$$

Proof. We basically follows the proof of Proposition 6.1. Indeed, we can write the left local time by considering the density $\ell^{n-}_t(y) = 2(1-\nu(n))\sigma_n \ell^n_t(y)$ as indicated in Section 5.1, an increasing sequence of open sets D^n_j such that $D^n = \cup_j D^n_j \to \Omega$ and the sets $\Lambda_n = \overline{\Omega^n} \setminus D_n$. Thus, we consider

$$\widetilde{A^{n-}_t} = \Gamma^{\Omega^n \setminus D^n}_t(\widetilde{B^{\nu,*}}) + L^{\partial\Omega^n-}_t(\widetilde{B^{\nu,*}}) = \Gamma^{\Lambda_n}_t(\widetilde{B^{\nu,*}})$$

instead of (6.6). We get that $\widetilde{\mu_n}^-(dx) = 2(1-\nu(n))\sigma_n m^n_\varepsilon(dx)$ is the Revuz measure of $\widetilde{A^{n-}_t}$. The result follows from the previous proof of Proposition 6.1. □

From the previous results we have that μ^+_n is the Revuz measure of

$$A^{n+}_t = 2\frac{c_n}{1+c_n}\sigma_n \int_{\partial\Omega^n} \ell^n_t(y) m^n_\varepsilon(dy) = 2\frac{c_n}{1+c_n} L^{\partial\Omega^n}_t$$

and, analogously we can relate μ^-_n, the Revuz measure of A^{n-}_t, to $L^{\partial\Omega^n}_t$. Then, we can define the symmetric local time $L^{\partial\Omega^n}_t$ of the part process of $\widetilde{B^{\nu,*}_t}$ such that, according with Proposition 2.1, we get that

$$L^{\partial\Omega^n}_t = \sigma_n \int_{\partial\Omega^n} \ell^n_t(y)\, m^n_\varepsilon(dy) \to \int_{\partial\Omega} \ell_t(y)\, d\mu = L^{\partial\Omega}_t \quad \text{as } n \to \infty. \tag{6.10}$$

The convergence above follows by considering that the $L^{\partial\Omega}_t \in \mathbf{A}^+_c$. The connection between tightness on the line and continuity of the limit process has been pointed out starting from [2, 3]. Here we provide a detailed discussion about the convergence in (6.10). We begin with the following result concerning the sequence $L^{\partial\Omega^n}_t$, $t < T$.

Proposition 6.3. *The sequence* $\{L_t^{\partial\Omega^n}\}_n$ *is tight in* $C([0,T],[0,\infty))$.

Proof. Let us consider the set $\Lambda_n \subseteq \partial\Omega^n$. The process $L_t^{\Lambda_n}$, $t < T$ is a continuous additive functional of zero energy ([24, pag. 149]) and $\mathbb{E}_x[|L_t^{\Lambda_n}|, t < T] < \infty$ q.e. x. Indeed, from (5.21) and (5.28), we have that

$$\mathbb{E}_x^n\left[\int_s^t dL_u^{\Lambda_n -}\right] + \mathbb{E}_x^n\left[\int_s^t dL_u^{\Lambda_n +}\right] = 2\mathbb{E}_x^n\left[\int_s^t dL_u^{\Lambda_n}\right] \leq const \cdot (t-s)\cdot \sigma_n \int_{\Lambda_n} m_\varepsilon^n(dy). \tag{6.11}$$

We have that $L_0^{\Lambda_n} = 0$ for all n and, for $k > 1$ and $c_1 > 0$,

$$\mathbb{E}_x[|L_t^{\Lambda_n} - L_s^{\Lambda_n}|^k] \leq \sigma_n\, m_\varepsilon^n(\Lambda_n)\, c_1\, |t-s|^k < c_1\, |t-s|^k. \tag{6.12}$$

Indeed, for $s < t$,

$$\mathbb{E}_x\left[\int_s^t dL_z^{\Lambda_n}\right]^k = k! \int_s^{z_1} \dots \int_{z_{k-1}}^t \mathbb{E}_x\left[\mathbf{1}_{\Lambda_n}(B_{z_1}^{\nu(n),*})\dots \mathbf{1}_{\Lambda_n}(B_{z_k}^{\nu(n),*})\right] dz_1 \dots dz_k \tag{6.13}$$

$$\leq k! \int_s^t \dots \int_s^t \mathbb{E}_x\left[\mathbf{1}_{\Lambda_n}(B_{z_1}^{\nu(n),*})\dots \mathbf{1}_{\Lambda_n}(B_{z_k}^{\nu(n),*})\right] dz_1 \dots dz_k. \tag{6.14}$$

We recall (5.27) and the fact that $\mathbf{P}_t^D \leq \mathbf{P}_t^N$ a.e., the transition function of the reflecting BM dominates that of the absorbing BM. From (5.41) and (6.11) we can write (6.12). Since $k > 1$, a Kolmogorov-type criterion shows the tightness. □

Since $\widehat{\zeta^{\Omega^n}} \leq \zeta^{\Omega^n_\varepsilon}$ we can write $\mathbb{P}_x(\zeta^{\Omega^n_\varepsilon} \leq t) \leq \mathbb{P}_x(\widehat{\zeta^{\Omega^n}} \leq t) = \mathbb{P}_x(\zeta^n \leq L_t^{\partial\Omega^n}) = 1 - \mathbb{E}_x e^{-c_n L_t^{\partial\Omega^n}}$ and, equivalently $\mathbb{P}_x(\zeta^{\Omega^n_\varepsilon} > t) \geq \mathbb{E}_x e^{-c_n L_t^{\partial\Omega^n}}$. Assume that u is the bounded solution to $\partial_t u = \mathcal{L}_n\, u - \widehat{\kappa_n}(x) u$ with $u_0 = 1$ in $H^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2, \sigma_n \mathbf{1}_{\partial\Omega^n} ds)$ where $\widehat{\kappa_n} = c_n \mathbf{1}_{\partial\Omega^n}(x)$. Then, as $t \to 0$, $t^{-1}(1-u(x,t)) \to -\mathcal{L}_n\, u_0 + \widehat{\kappa_n}(x) u_0 = \widehat{\kappa_n}(x)$ gives the killing rate which can be also obtained as ([12])

$$\lim_{t\to 0} t^{-1}\mathbb{P}_x(B_t^{\nu,*} \text{ is killled in the time interval } (0,t]).$$

Since for $A_t = \int_0^t f(X_s)ds$, $de^{A_t}/dt = f(X_t)e^{A_t}$ and $e^{A_t} - 1 = \int_0^t f(X_s)e^{A_s}ds$, we get that

$$\lim_{t\to 0} t^{-1}(1 - \mathbb{E}_x e^{-c_n L_t^{\partial\Omega^n}}) = \lim_{t\to 0} \mathbb{E}_x\left[c_n \mathbf{1}_{\partial\Omega^n}(B_t^{\nu,*})\right] = \widehat{\kappa_n}(x).$$

Then, $u(x,t) = \mathbb{P}_x^n(\widehat{\zeta^{\Omega^n}} > t)$. We use the symbol $\widehat{\kappa_n}$ in order to underline the connection with the transition semigroup

$$\widehat{\mathbf{P}_t^n} f(x) = \mathbb{E}_x^n[f(\widetilde{B_t^{\nu(n),*}}); t < \widehat{\zeta^{\Omega^n}}] \tag{6.15}$$

with resolvent $\widehat{R_\lambda^n}$ where $\widehat{\zeta^{\Omega^n}}$ is defined as in (6.1). For δ_n identically zero,

$$\widehat{\mathbf{P}_t^n}\mathbf{1}_{\Omega^n_\varepsilon}(x) = \mathbb{E}_x[\mathbf{1}_{\Omega^n_\varepsilon}(\widetilde{B_t^{\nu(n),*}}); t < \widehat{\zeta^{\Omega^n}}] = \mathbb{P}_x^n(\widehat{\zeta^{\Omega^n}} > t).$$

If $\delta_n > 0$ we have that $\widehat{\mathbf{P}_t^n}\mathbf{1}_{\Omega^n_\varepsilon}(x) = e^{-\delta_n t}\, \mathbb{P}_x^n(\widehat{\zeta^{\Omega^n}} > t)$.

The heat equation solution with Robin boundary conditions has been studied using a Feynamn-Kac formula and a theorem of Ray and Knight on Brownian local time in [8]. In [28], the authors reviewed the Kac's method by underlining the connection between the higher-order moments of $A_\kappa = \int_0^T \kappa(X_t)dt$ and the Feynman-Kac formula. Thus, $\mathbb{E}_m[A_\kappa]^k = k! m G_\kappa^k \mathbf{1}$ where $G_\kappa w(x) = \int G(x,dy)\kappa(y)w(y)$ (here m is an arbitrary initial distribution). In particular,

$$f(x) = \mathbb{E}_x\left[\exp \int_0^T \kappa(B_s)ds\right] = \sum_{k=0}^\infty G_\kappa^k \mathbf{1}(x), \quad x \in \Lambda \tag{6.16}$$

is finite if and only if is the minimal solution to $f(x) = 1 + \int \kappa(y)f(y)G(x,dy)$ where $\kappa : \Lambda \mapsto [0,\infty)$ is a measurable Borel function, $T = \tau_\Lambda$ and G is the Green function of the killed BM on the boundary of the open set Λ.

Proposition 6.4. *For $x \in \Omega^1_\varepsilon$, $\forall\, t \geq 0$,*

$$\int_0^\infty e^{-c_n l}\, \mathbb{P}^n_x(L^{\partial\Omega^n}_t \in dl) \to \int_0^\infty e^{-c_\infty l}\, \mathbb{P}_x(L^{\partial\Omega}_t \in dl), \quad \text{as } n \to \infty$$

m-a.e. x, for every sequence $c_n \geq 0$ such that $c_n \to c_\infty \in [0, \infty]$.

Proof. The higher-order moment of the boundary local time can be written as in formula (6.13) with $s = 0$. Then, by applying (5.40) we have that $\mathbb{E}^n_x \left[L^{\partial\Omega^n}_t\right]^k \to \mathbb{E}_x \left[L^{\partial\Omega}_t\right]^k$ weakly (continuously on $t \geq 0$, m-a.e. x). Indeed, we have weak convergence of finite-dimensional distributions from Theorem 5.4 and tightness from Proposition 6.3. Consider the expansion (6.16). We can also write $v_{\widehat{\kappa_n}}(x,t) = \mathbb{E}^n_x \exp -c_n L^{\partial\Omega^n}_t$ for $t \leq T_{c_n}$ in terms of (6.13) with $s = 0$. For $c_n = \lambda \geq 0\ \forall n$, $v_{\widehat{\kappa_n}}$ is the Laplace transform of $\mathbb{P}^n_x(L^{\partial\Omega^n}_t \in dl)/dl$. Since, $\mathbb{P}^n_x \overset{w^\star}{\to} \mathbb{P}_x$, m-a.e. x (from the convergence of moments), then the convergence holds for every $c_n \geq 0$ (and therefore, for every $c_\infty \geq 0$). □

We can write $A^n_t = \int_0^t \kappa_n(B^{\nu(n),*}_s)ds$ in (5.32) where $\kappa_n(\cdot)$ is the state dependent rate for the Markov killing time T_{c_n}. Thus, we can focus on the multiplicative functional $M^n_t = e^{-A^n_t}$ associated with the stopping time T_{c_n} and therefore, with the sub-Markov semigroup (5.39). In particular, (5.39) characterizes M^n_t uniquely ([12, Proposition 1.9]). Also we write $\overline{M^n_t} = e^{-\overline{A^n_t}}$. Notice that (for $\lambda \geq 0$)

$$R^n_\lambda \mathbf{1}_{\Omega^n_\varepsilon}(x) = \mathbb{E}^n_x \left[\int_0^\infty e^{-\lambda t}\overline{M^n_t}dt\right] \quad \text{and} \quad \widehat{R^n_\lambda}\mathbf{1}_{\Omega^n_\varepsilon}(x) = \mathbb{E}^n_x \left[\int_0^\infty e^{-\lambda t}\widehat{M^n_t}dt\right]$$

where $\widehat{M^n_t} = \exp(-\delta_n t - c_n L^{\partial\Omega^n}_t)$ with δ_n, c_n as in (2.13). The fundamental principle in our investigation is to approximate M by $\widehat{M}$ and transfer the properties known for the approximating functional.

Theorem 6.2. *$\widehat{\zeta^{\Omega^n}} \overset{law}{\to} T_{c_\infty}$ and under (2.25), $\widehat{R^n_\lambda}f_n \to R_\lambda f$ strongly in $L^2(\Omega)$.*

Proof. First we show that $\widehat{\zeta^{\Omega^n}} \overset{law}{\to} T_{c_\infty}$ as $c_n \to c_\infty$. Let us write $\zeta^n = \varsigma/c_n$ with $\mathbb{P}(\varsigma > x) = e^{-x}$, $x \geq 0$. Thus, a.s. $\zeta^n \to \zeta^\infty$. Assume $\delta_n = 0$ and $\lambda > 0$. From $\widehat{\mathbf{P}^n_t}$ we write the associated resolvent

$$\widehat{R^n_\lambda}\mathbf{1}_{\Omega^n_\varepsilon}(x) = \int_0^\infty e^{-\lambda t}\widehat{\mathbf{P}^n_t}\mathbf{1}_{\Omega^n_\varepsilon}(x)dt = \int_0^\infty e^{-\lambda t}\mathbb{P}^n_x(\widehat{\zeta^{\Omega^n}} > t)dt$$

and, from the resolvent (5.33),

$$R^n_\lambda \mathbf{1}_{\Omega^n_\varepsilon}(x) = \int_0^\infty e^{-\lambda t}\mathbf{P}^n_t \mathbf{1}_{\Omega^n_\varepsilon}(x)dt = \int_0^\infty e^{-\lambda t}\mathbb{P}^n_x(\zeta^{\Omega^n_\varepsilon} > t)dt.$$

Recall that (5.39) holds true. That is, from M-convergence of forms ([20]), we have that (Theorem 5.4)

$$R^n_\lambda \mathbf{1}_{\Omega^n_\varepsilon}(x) \to \int_0^\infty e^{-\lambda t}\mathbb{P}_x(T_{c_\infty} > t)dt := \widetilde{v}_\lambda(x)$$

strongly in $L^2(\Omega)$. The true stopping time T_{c_∞} depends on c_∞ and corresponds to the lifetime of the limit process. By construction we have that a.s. $\widehat{\zeta^{\Omega^\infty}} = T_{c_\infty}$ and

$$\lim_{n\to\infty} R^n_\lambda \mathbf{1}_{\Omega^n_\varepsilon}(x) = \lim_{n\to\infty} \widehat{R^n_\lambda}\mathbf{1}_{\Omega^n_\varepsilon}(x), \quad \forall\, x \in \Omega^1_\varepsilon. \tag{6.17}$$

Now we show convergence in law.

Step 1) Let us consider a reflecting BM on Ω^n, say X^n, and the first hitting time τ_{Ω^n} on the pre-fractal boundary $\partial\Omega^n$. Let X be a reflecting BM on Ω with $\Omega^n \uparrow \Omega$. We have that $X^n \overset{law}{\to} X$ as $n \to \infty$ ([22, 14]). Consider now the killed process $X^n_t, t < \tau_{\Omega^n}$ started at $x \in \Omega^{(1)}$. Here we can follows the same arguments as in [10, Theorem 4.1 and Theorem 3.3]. In particular, we consider the additive functional A^n_t associated with τ_{Ω^n} with Revuz measure $\mu_{A^n} = \infty_{D_n}$ where

D_n is the complement of Ω^n. From stable convergence of multiplicative functionals we arrive at weak convergence of stopping times, that is $\tau_{\Omega^n} \stackrel{law}{\to} \tau_\Omega$. Therefore, we get a direct consequence of M-convergence of the associated form and tightness of $\mathbb{P}^n_x(\tau_{\Omega^n} > t)$ (indeed $\mathbb{E}^n_x \tau_{\Omega^n}$ is bounded by $\mathbb{E}^n_x \tau_{\Omega^{(1)}}$ uniformly on n). Notice also that we arrive at the same result by considering Dirichlet condition on $\partial\Omega^n$ and the corresponding (killed) process on Ω^n. The variational approach has been treated in [17].

Step 2) Now we focus on the family $\{\mathbb{P}^n_x\,;\, x \in \Omega^n_\varepsilon\}$. Let us write $X^{x,n}_t = \{B^{\nu(n),*}_t$ started from $x \in \Omega^n_\varepsilon = \overline{\Omega^n} \cup \Sigma^n\}$. We first observe that

$$\forall\, n, \quad X^{x,n}_t \in \partial \quad \text{if and only if} \quad (t > \zeta^{\Omega^n_\varepsilon}) \vee (x \in \partial).$$

Since $x \in \overline{\Omega^1_\varepsilon} \setminus \Omega^n_\varepsilon \Rightarrow x \in \partial$ as $n \to \infty$, we can consider only starting points $x \in \Omega^1_\varepsilon$. Notice also that, in the Neumann case, the cemetery point is assumed to be $\partial = \{\emptyset\}$ and such that $m(\partial) = 0$. This corresponds to $\mathrm{Cap}(\overline{\Omega^1_\varepsilon} \setminus \Omega^n) \to 0$. Thus, for $x \in \Omega^1_\varepsilon$ we write

$$\begin{aligned}
\mathbb{P}^n_x(\widehat{\zeta^{\Omega^n}} > t) =& \mathbb{P}^n_x(\widehat{\zeta^{\Omega^n}} > t, (t < \tau_{\Omega^n}) \cup (t \geq \tau_{\Omega^n}))\\
=& \mathbb{E}^n_x\left[e^{-c_n L^{\partial\Omega^n}_t} \big| t < \tau_{\Omega^n}\right] \mathbb{P}^n_x(t < \tau_{\Omega^n}) + \mathbb{E}^n_x\left[e^{-c_n L^{\partial\Omega^n}_t} \big| t \geq \tau_{\Omega^n}\right] \mathbb{P}^n_x(t \geq \tau_{\Omega^n})\\
=& \mathbb{P}^n_x(\tau_{\Omega^n} > t) + \mathbb{E}^n_x\left[e^{-c_n L^{\partial\Omega^n}_t} \big| L^{\partial\Omega^n}_t > 0\right] \mathbb{P}^n_x(t \geq \tau_{\Omega^n}).
\end{aligned}$$

From Proposition 6.4 and the fact that $\tau_{\Omega^n} \stackrel{law}{\to} \tau_\Omega$, we have that

$$\mathbb{P}^n_x(\widehat{\zeta^{\Omega^n}} > t) \stackrel{w^\star}{\to} \mathbb{P}_x(\tau_\Omega > t) + \mathbb{E}_x\left[e^{-c_\infty L^{\partial\Omega}_t} \big| L^{\partial\Omega}_t > 0\right] \mathbb{P}_x(t \geq \tau_\Omega) \tag{6.18}$$

and therefore we obtain:

i) if $c_n \to 0$, $\mathbb{P}^n_x(\widehat{\zeta^{\Omega^n}} > t) \stackrel{w^\star}{\to} \mathbb{P}_x(\tau_\Omega > t) + \mathbb{P}_x(t \geq \tau_\Omega) = 1$ for all $t \geq 0$, $x \in \overline{\Omega}$ and

$$\mathbb{P}_x(\omega\,:\, \lim_{n\to\infty} \widehat{\zeta^{\Omega^n}}(\omega) = \infty) = 1 = \mathbb{P}_x(\omega\,:\, T_0(\omega) = \infty);$$

ii) if $c_n \to c_0$, by considering

$$\mathbb{E}_x\left[e^{-c_0 L^{\partial\Omega}_t} \big| L^{\partial\Omega}_t = 0\right] \mathbb{P}_x(\tau_\Omega > t) + \mathbb{E}_x\left[e^{-c_0 L^{\partial\Omega}_t} \big| L^{\partial\Omega}_t > 0\right] \mathbb{P}_x(t \geq \tau_\Omega)$$

we get

$$\mathbb{P}^n_x(\widehat{\zeta^{\Omega^n}} > t) \stackrel{w^\star}{\to} \mathbb{E}_x\left[e^{-c_0 L^{\partial\Omega}_t}\right] = \mathbb{P}_x(\zeta^\Omega > t), \quad \forall\, t \geq 0,\, (x \in \overline{\Omega})$$

and thus, a.s. $T_{c_0} = \zeta^\Omega$;

iii) if $c_n \to \infty$, $\mathbb{P}^n_x(\widehat{\zeta^{\Omega^n}} > t) \stackrel{w^\star}{\to} \mathbb{P}_x(\tau_\Omega > t)$ for all $t > 0$, $(x \in \Omega)$ and we conclude that a.s. $T_\infty = \tau_\Omega$.

Thus, we obtain that $\widehat{\zeta^{\Omega^n}} \stackrel{law}{\to} T_{c_\infty}$.

With this at hand, we now continue the proof. Since $\widehat{\zeta^{\Omega^n}} \leq \zeta^{\Omega^n_\varepsilon}$ with probability one, we have that $\mathbb{P}^n_x(\zeta^{\Omega^n_\varepsilon} > t) \geq \mathbb{P}^n_x(\widehat{\zeta^{\Omega^n}} > t)$. Thus, we have that $\widehat{\mathbf{P}}^n_t \mathbf{1} \leq \mathbf{P}^n_t \mathbf{1}$ and $\widehat{R}^n_\lambda \mathbf{1} \leq R^n_\lambda \mathbf{1}$ m-a.e. x. Notice that, from this we obtain once again (6.17). Strong convergence of resolvents in $L^2(\Omega)$ implies that

$$\lim_{n\to\infty} \|R^n_\lambda \mathbf{1}\|_{L^2(\Omega^n_\varepsilon)} = \|\widetilde{v}_\lambda\|_{L^2(\Omega)}.$$

Since $R^n_\lambda \mathbf{1}$ is uniformly bounded, $\|\widehat{R}^n_\lambda \mathbf{1}\|_{L^2(\Omega^n_\varepsilon)} < C$ for all n. From this and convergence a.e. we conclude that $\widehat{R}^n_\lambda \mathbf{1} \to \widetilde{v}_\lambda$ weakly in $L^2(\Omega)$. Convergence of $\widehat{R}^n_\lambda \mathbf{1}$ implies that

$$\liminf_{n\to\infty} \|\widehat{R}^n_\lambda \mathbf{1}\|_{L^2(\Omega^n_\varepsilon)} \geq \|\widetilde{v}_\lambda\|_{L^2(\Omega)}.$$

Since we have that

$$\|\widetilde{v}_\lambda\|_{L^2(\Omega)} \leq \liminf \|\widehat{R^n_\lambda}\mathbf{1}\|_{L^2(\Omega^n_\varepsilon)} \leq \liminf \|R^n_\lambda \mathbf{1}\|_{L^2(\Omega^n_\varepsilon)} \leq \lim \|R^n_\lambda \mathbf{1}\|_{L^2(\Omega^n_\varepsilon)} = \|\widetilde{v}_\lambda\|_{L^2(\Omega)}$$

we conclude that

$$\lim_{n\to\infty} \|R^n_\lambda \mathbf{1}\|_{L^2(\Omega^n_\varepsilon)} = \lim_{n\to\infty} \|\widehat{R^n_\lambda}\mathbf{1}\|_{L^2(\Omega^n_\varepsilon)}. \tag{6.19}$$

Weak convergence of $\widehat{R^n_\lambda}\mathbf{1}$ together with (6.19) says that $\widehat{R^n_\lambda}\mathbf{1} \to \widetilde{v}_\lambda$ strongly in $L^2(\Omega)$.

From the contraction property of $\mathbf{P}^n_t$ we have that

$$\|\widehat{R^n_\lambda} f\|_{L^2(\Omega^n_\varepsilon)} \leq \|R^n_\lambda f\|_{L^2(\Omega^n_\varepsilon)} \leq \lambda^{-1/2}\|f\|_{L^2(\Omega^n_\varepsilon)}$$

for all measurable functions f. By repeating the previous steps, under (2.25), we get the strong convergence of $\widehat{R^n_\lambda} f_n$ (or $\lambda\widehat{R^n_\lambda} f_n$). □

In light of the previous result, we can study the convergence of (5.34) by considering the semigroup (6.15) and the corresponding form

$$\mathcal{E}_0^{\mu_{\widehat{A^n}}}(u,v) = a_n(u,v) + \langle u, v\rangle_{\mu_{\widehat{A^n}}}, \quad u,v \in D(\mathcal{E}_0^{\mu_{\widehat{A^n}}}) \cap L^2(\mu_{\widehat{A^n}}) \tag{6.20}$$

with $D(\mathcal{E}_0^{\mu_{\widehat{A^n}}}) = H^1(\mathbb{R}^2)$ where $\mu_{\widehat{A^n}}$ is supported on $\partial\Omega^n$. We recall that

$$\widehat{\mathbf{P}^n_t} f_n(x) = \mathbb{E}^n_x\left[f_n(\widetilde{B^{\nu(n),*}_t})\mathbf{1}_{(t<T_{c_n})}\right] = \mathbb{E}^n_x\left[f_n(\widetilde{B^{\nu(n),*}_t})\, e^{-c_n \int_0^t \mathbf{1}_{\partial\Omega^n}(\widetilde{B^{\nu(n),*}_s})ds}\right] \tag{6.21}$$

with $\mathbf{1}_{(t<T_{c_n})} \equiv \mathbf{1}_{(L^{\partial\Omega^n}_t < \zeta^n)}$.

Proof of Theorem 6.1. Let us write $D_n = \Omega^n_\varepsilon$ for the sake of simplicity. From Theorem 6.2 we have that

$$\mathbb{E}^n_x\left[\int_0^\infty e^{-\lambda t} f_n(\widetilde{B^{\nu,*}_t})\widehat{M^n_t}dt\right] \to R_\lambda f(x) := \mathbb{E}_x\left[\int_0^\infty e^{-\lambda t} f(B^+_t)M_t dt\right] \tag{6.22}$$

strongly in $L^2(\Omega)$ where the multiplicative functional $\widehat{M^n_t}$ is associated with the PCAF with Revuz measure $\widehat{\mu_n} = \delta_n + \mu_{\widehat{A^n}}$. Let us relate R^n_λ to $\overline{\mu_n} = \delta_n + \mu_{A^n}$ in the same sense. From (6.22) we have that, $\forall\, f \in C_0$,

$$\int f d\overline{\mu_n} \to \int f d\varrho \quad \Leftrightarrow \quad \int f d\widehat{\mu_n} \to \int f d\varrho.$$

Indeed, the resolvents identify uniquely the multiplicative functionals. Thus we get that

$$\overline{\mu_n} \overset{v}{\to} \widehat{\mu_\infty}. \tag{6.23}$$

Proposition 6.1 and Proposition 6.2 authorize us to study the asymptotic behaviour of $\mu_{\widehat{A^n}}$ where $\widehat{A^n_t} = c_n(A^{n+}_t + A^{n-}_t)/2$ and

$$\lim_{\lambda\to\infty} \langle \lambda U^n_\lambda f, m^n_\varepsilon\rangle = c_n\sigma_n \int f(x)\mathbf{1}_{\partial\Omega^n}(x) m^n_\varepsilon(dx) = \langle f, \mu_{\widehat{A^n}}\rangle.$$

Notice also that, if $L \in \mathbf{A}^+_c$ and $\eta > 0$, then $\eta L \in \mathbf{A}^+_c$. Thus, from (6.23), as $n \to \infty$, we get that $\mu_{A^n} \to \mu_{\widehat{A^\infty}} = c_\infty \mu$. This means equivalence between the corresponding additive functionals. In particular, we get that

$$\widehat{\mu_\infty} = \delta_0 + \begin{cases} 0 + \infty_{\overline{\Omega}^c}, & c_\infty = 0 & \text{(no kill)},\\ c_0\,\mu_\alpha\,\mathbf{1}_{\partial\Omega} + \infty_{\overline{\Omega}^c}, & c_\infty = c_0 \in (0,\infty) & \text{(elastic kill)},\\ 0 + \infty_{\Omega^c}, & c_\infty = \infty & \text{(kill on the boundary)}, \end{cases}$$

and $\overline{\mu_\infty} \to \delta_0 + c_\infty \mu$. With (5.8) in mind, we write

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{\mathbb{P}^n_{\partial\Omega^n}(B^{\nu(n),*}_t \in \Sigma^n)}{\mathrm{tick}(\Sigma^n)} \tag{6.24}$$

where

$$\mathbb{P}^n_{\partial\Omega^n}(B^{\nu(n),*}_t \in \Sigma^n) = \mathbb{P}^n(B^{\nu(n),*}_t \in \Sigma^n | B^{\nu(n),*}_0 \in \partial\Omega^n) = \nu(n)$$

is the probability for the modified BM started from $\partial\Omega^n$ to move outside Ω^n (toward Σ^n, that is the $\mathbb{P}^n$-measure of Σ^n) and $w^n/\mathrm{tick}(\Sigma^n) \to 1$ as $n \to \infty$ ($\mathrm{tick}(\Sigma^n)$ is the thickness of the fiber).

iii) If $c_n \to \infty$, then point iii) of the proof of Theorem 6.2 holds and is also in accord with (6.24), $\tau_{\Omega^n_\varepsilon} \to \tau_\Omega$ faster than $B^{\nu(n),*}_t \to B^+_t$. The BM is killed on the boundary. On the other hand, if $\tau_{\Omega^n_\varepsilon} \to \tau_\Omega$, the lifetime on Ω of the limit process is exactly τ_Ω. Then for all $x \in \Omega$ we have that $\mathbb{P}_x(L^{\partial\Omega}_t = 0) = 1$ for all $t < \tau_\Omega$ which means that

$$\forall\, x \in \Omega, \quad \mathbb{P}_x(e^{-c_n L^{\partial\Omega}_t} = 1, t < \tau_\Omega) = 1, \quad \text{for all } n \text{ (for all } c_n \geq 0).$$

This justifies the convection $\infty \cdot 0 = 0$. Moreover, if the lifetime is τ_Ω, from (6.3), it must be that $\zeta^n \to 0$, that is $c_n \to \infty$. Thus, $\tau_{\Omega^n_\varepsilon} \to \tau_\Omega$ if and only if condition (2.30) holds and $\partial\Omega$ becomes a Dirichlet boundary as $n \to \infty$. We get that $\tau_{\Omega^n_\varepsilon} \to \tau_\Omega \Leftrightarrow c_n \to \infty$.

ii) If $c_n \to 0$, then according with (6.24) we say that $B^{\nu(n),*}_t \to B^+_t$ on Ω for $t \geq 0$ and therefore, $\zeta^{\Omega^n} \to \infty$. On the other hand, Remark 6.1 and Remark 6.2 say that $c_n \to 0 \Rightarrow \widehat{\zeta^{\Omega^n}} \to \infty$. Since $\widehat{\zeta^{\Omega^n}} \leq \zeta^{\Omega^n_\varepsilon}$, $\widehat{\zeta^{\Omega^n}} \to \infty$ as well as the lifetime $\zeta^{\Omega^n_\varepsilon} \to \infty$. Moreover, $\zeta^{\Omega^n_\varepsilon} \to \infty \Rightarrow c_n \to 0$ by using Remark 6.1.

i) Since $\tau.N)$ and $\tau.D)$ hold true, if $c_n \to c_0$ and $c_0 \neq 0$ or $c_0 \neq \infty$, then $\zeta^{\Omega^n_\varepsilon} \to T$ and $T \neq \infty$ or $T \neq \tau_\Omega$. In particular, $T \in (\tau_\Omega, \infty)$ is a random variable (killing time) depending on $c_0 \in (0, \infty)$. Thus, $\zeta^{\Omega^n_\varepsilon} \to T$ where T depends on c_0. From Theorem 6.2 we have that $T = \zeta^\Omega$ is an exponential random variable with parameter c_0. Thus, $c_n \to c_0 \Leftrightarrow \zeta^{\Omega^n_\varepsilon} \to \zeta^\Omega$. In this case (and also the previous as particular cases) we recover the results about the asymptotic of Robin problems on pre-fractal domains ([17] by considering the forms (6.20)). □

From the Feynman-Kac representation (6.21) we write (5.34) by considering $\widehat{A^n_t} = c_n L^{\partial\Omega^n}_t$ where $\mu_{\widehat{A^n}}(dx) = c_n \sigma_n \mathbf{1}_{\partial\Omega^n}(x) ds$. Notice that, in the Dirichlet case, we use the convection $\infty \cdot 0 = 0$. We can also consider $\widehat{A^n_t} = K^n_t$ with the Revuz measure $\mu_{K^n}(dx)$ where the PCAF is associated with the killing time T_{c_n}. In particular, μ_{K^n} depends on c_n and can be related to μ^ε defined in (5.35). Let us focus on Remark 4.1. We observe that, as $n \to \infty$,

$$c_n \to c_0 > 0 \quad \Leftrightarrow \quad \mathrm{Cap}_1(\mathbb{R}^2 \setminus \Omega) > 0 \qquad \text{(transient case)}$$

where $\Rightarrow$ immediately follows and $\Leftarrow$ is obtained from $\tau.D)$ and $\tau.R)$. Moreover, from $\tau.N)$,

$$c_n \to 0 \quad \Leftrightarrow \quad \mathrm{Cap}_1(\mathbb{R}^2 \setminus \Omega) = 0 \qquad \text{(recurrent case).}$$

Indeed, the process X_t is transient iff $\mathrm{Cap}_1(\mathbb{R}^2 \setminus \Omega) > 0$ ([24, Proposition 3.5.10]). A vanishing capacity can be related with the hitting distribution and in particular with the recurrence of the corresponding process. Let us consider the hitting distribution $H^\alpha_{\Lambda^c} \mathbf{1}_E(x) = \mathbb{E}_x[e^{-\alpha\tau_\Lambda} \mathbf{1}_E(B_{\tau_\Lambda})]$ where the expected value is taken under (4.2). For $\alpha > 0$, $H^\alpha_{\Lambda^c}\mathbf{1}(x) = \mathbb{E}_x[e^{-\alpha\tau_\Lambda}]$. Since $\mathbb{R}^2 \setminus \Omega^n \downarrow \mathbb{R}^2 \setminus \Omega$ are sets of finite capacity, form [30, Theorem 4.2.1] we have that

$$\mathrm{Cap}_1(\mathbb{R}^2 \setminus \Omega^n) \to 0 \quad \text{iff} \quad H^1_{\mathbb{R}^2\setminus\Omega^n} \mathbf{1}(x) \to 0 \ \text{ q.e.}$$

and thus, for $x \in \Omega^1_\varepsilon$, $\mathbb{P}_x(\tau_{\Omega^n} < \infty) \to 0$ (see also (6.2)).

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