EXISTENCE AND HOMOGENIZATION FOR A SINGULAR PROBLEM THROUGH ROUGH SURFACES*

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Abstract. The paper deals with existence and homogenization for elliptic problems with lower order terms singular in the u-variable (u is the solution) in a cylinder Q in $\mathbb{R}^N$, so that the lower order term becomes infinite on the set $\{u = 0\}$. A rapidly oscillating interface inside Q separates the cylinder in two composite connected components. The interface has a periodic microstructure and it is situated in a small neighbourhood of a hyperplane which separates the two components of Q. At the interface we suppose the following transmission conditions: (i) the flux is continuous, (ii) the jump of a solution at the interface is proportional to the flux through the interface. This is a steady state model for the heat conduction in two heterogeneous electrically conducting materials with an imperfect contact between them. On the exterior boundary Dirichlet boundary conditions are prescribed.

We also derive a corrector result for every values of the two parameters γ and κ which are related respectively to the microstructure period and to the amplitude of the interface oscillations.

Key words. singular equations, homogenization, rough surfaces, interface conditions

AMS subject classifications. 35J75, 35J65, 35B27

1. Introduction.

In this paper we deal with a semilinear elliptic singular problem which models the stationary heat diffusion in a medium $Q = \omega\times]-l, l[$ made up of two connected composite components.
An interface Γ_ε, fixed for positive ε and rapidly oscillating as ε goes to zero, separates the two components, $Q_{\varepsilon 1}$ and $Q_{\varepsilon 2}$. The source term depends on the solution itself and becomes infinite when the solution vanishes.
Our model is the following

$$(P_\varepsilon)\qquad \begin{cases} -\mathrm{div}(A^\varepsilon\nabla u_\varepsilon) = f\,\zeta(u_\varepsilon) & \text{in } Q_{\varepsilon 1}\cup Q_{\varepsilon 2},\\ [A^\varepsilon\nabla u_\varepsilon]\cdot\nu_\varepsilon = 0 & \text{on } \Gamma_\varepsilon,\\ (A^\varepsilon\nabla u_\varepsilon)_1\cdot\nu_\varepsilon = -\varepsilon^\gamma h^\varepsilon[u_\varepsilon], & \text{on } \Gamma_\varepsilon,\\ u_\varepsilon = 0 & \text{on } \partial Q, \end{cases}$$

where $A^\varepsilon(x) = A(x/\varepsilon)$ with A bounded uniformly elliptic periodic matrix, $\zeta(s)$ is a nonnegative real function singular at $s = 0$, f is a nonnegative datum (not identically zero) whose summability depends on the growth θ of the singular function $\zeta(s)$ near the singularity $s = 0$ and ν_ε is the unit outward normal to $Q_{\varepsilon 1}$. $[\cdot]$ denotes the jump through Γ_ε.
The oscillating interface Γ_ε represent a rough surface which gives rise to an imperfect contact between the two components and this situation is modeled by a jump of the solution of the diffusion equation, which is proportional to the flux through the interface (see [13]).

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Our aim is to study the existence of a solution to problem (P_ε) for ε fixed and its macroscopic behaviour, that is the asymptotic behaviour as ε goes to zero of solutions for all values of the parameters appearing in the problem.

Singular lower order terms (sometimes as an absorption term) appears in problems which model boundary-layer phenomena for viscous fluids, non-Newtonian fluids (in particular pseudoplastic fluids) and in problems related to enzymatic kinetics or in the Langmuir-Hinshelwood model of heterogeneous chemical catalyst. Source terms depending in a singular way from the solution appear also in problems modelling heat transfer in electrical conductors.

We refer to Section 3 below for a description of some of these physical situations governed by (elliptic or parabolic) semilinear singular equations. We point out that if these phenomena take place in a region Q made of two composite materials having an imperfect contact between them, we are naturally led, at least in the stationary case, to problem (P_ε).

We refer to the early papers by [42], [50] for the theory of the H-convergence which allows to deal with general uniformly elliptic second order differential operators with oscillatory coefficients.

The homogenization of the linear problem with oscillating interface corresponding to P_ε (i.e. fixed right-hand side $f(x) \in L^2(Q)$) has been studied [22] and the case of perforated domains with jump was originally studied in [3] (see also [36], [20], [32] [40] and [21] for a wide bibliography). We refer to [1], [2], [14], [15], [38], [39], [41] (and references therein) for the homogenization in domains with an oscillating boundary when the amplitude of the oscillations goes to zero, and to [11], [12], [24] for the case of fixed amplitude. For transmission problem through an oscillating boundary of fixed amplitude see [11], [25] and for vanishing amplitude see [44]. Classical homogenization and corrector results can be found for instance in the books [6], [45] and [16].

Let us focus our attention on the main difficulties we have to deal with.

The first one is related to the presence of the singular term and we explain why below. We confine ourselves to the problem of the existence of a solution for ε fixed. Denoting by $v_{\varepsilon 1}$ and $v_{\varepsilon 2}$ the restrictions to $Q_{\varepsilon 1}$ and $Q_{\varepsilon 2}$ of a function v defined in Q, the framework space for problem (P_ε) is the following

$$W_0^\varepsilon := \{v \mid v_{\varepsilon 1} \in H^1(Q_{\varepsilon 1}),\ v_{\varepsilon 2} \in H^1(Q_{\varepsilon 2}) \quad \text{and} \quad v = 0 \text{ on } \partial Q\},$$

equipped with the norm

$$\|v\|_{W_0^\varepsilon} := \|\nabla v\|_{L_2(Q \setminus \Gamma_\varepsilon)},$$

where

$$\nabla v = \chi_{Q_{\varepsilon 1}} \nabla v_{\varepsilon 1} + \chi_{Q_{\varepsilon 2}} \nabla v_{\varepsilon 2}.$$

We approximate our problem through non singular problems (P_n) with solutions u_n (we omit here the parameter ε). Let us even assume the further condition that the function ζ appearing in the right-hand side is nonincreasing, which gives us the fact that $\{u_n\}$ is an increasing sequence, $u_n \geq u_{n-1} ... \geq u_1$. Even in this case no uniform bound from below on compact sets of Q is available on the sequence of the solutions $\{u_n\}$. Indeed we can apply strong maximum principle to the function u_1 in the upper part $Q_{\varepsilon 1}$ and in the lower part $Q_{\varepsilon 2}$ of Q but not in the whole Q since the function u_1 does not belong to $H_0^1(Q)$. Therefore, when we pass to the limit in the approximating problem (P_n) we are in trouble on the compact sets which cut the interface, which is in fact one of the main features of the problem.

This implies that we are naturally obliged to do an analysis of the behaviour of the singular terms near the singularity, which becomes one of the main tool in the proof. This technique is inspired by the similar one used in [27], [29] where existence and homogenization of singular problems in domains perforated by small holes is studied. We refer to [7], [9], [17], [34], [49] for existence results to singular elliptic problems in open sets Ω without interior interfaces, obtained by different techniques. Parabolic singular problems with general p-laplacian principal part, $p > 1$, are studied in [26]. Of course, a fortiori, the same kind of difficulties hold when studying the asymptotic behaviour as ε goes to zero. In this case we deal with the sequence $\{u_\varepsilon\}$ where u_ε is a solution for the problem (P_ε). Note that in any case this sequence does not have any monotonicity property even we assume that the function ζ is nonincreasing.
In the proofs of the main results stated in Theorem 4.1, Theorem 4.6 and Theorem 8.5 we split the integral of the singular term in two parts, the one on the set where the solution is close to the singularity and the one where it is far from it. Let us emphasize that in each proof we need to treat the two terms in a different way.
The second difficulty is the behaviour, as ε go to zero, of the boundary term which appear in the variational formulation of the problem. The different behaviour of this term depends on κ (the amplitude of the oscillation) and γ (which appears in the proportionality coefficient between the flux and the jump of the solution through the interface) and it gives rise to different limit problems.
The last difficulty is due to the fact that the assumption on the integrability of the datum f does not implies the boundedness of the solutions, so that we need often truncation arguments in the proofs. Note that in the existence and in the homogenization results we do not use any monotonicity assumption on the singular function $\zeta(s)$ which appears in the right-hand side. If we suppose in addition that $\zeta(s)$ is nonincreasing in s, we can prove the uniqueness of the solution.
A main tool for proving the homogenization result is a convergence result (Theorem 8.5) which proves that the gradient of the solution behaves like that of a suitable linear problem associated to a weak cluster point, as $\varepsilon \to 0$. Let us mention that this idea has been originally introduced in the literature for the homogenization of nonlinear problems with quadratic growth with respect to the gradient. The proof here is long and quite laborious, due to the difficulties mentioned above. We refer to [4], [5], for the case of a fixed domain and to [18] for periodically perforated domains (see also [19]).
Finally, we prove in Section 8 a corrector result for the corresponding linear problem, which completes the homogenization results proved in [22] (see Theorem 9.1). This implies, thanks to the convergence result of Theorem 8.5 mentioned above, that the linear corrector is also a corrector for the original nonlinear problem.
The paper is organized as follows:
In Section 2 we give the setting of the problem. In Section 3 we present some physical models governed by singular equations. In Section 4 we state the main results: existence, regularity, uniqueness, homogenization and correctors. Section 5 is devoted to the a priori estimates. In Section 6 we prove the existence result. In Section 7 we prove the regularity and the uniqueness results. Section 8 deals with the proof of the homogenization result. Section 9 is devoted to the proof of the corrector result. For completeness, in the Appendix we give the proof of the existence of solutions to the approximate nonsingular problems.

2. Setting of problem. We use here the framework introduced in [22] and, for simplicity, some notations therein.

Along this paper we suppose $N \geq 2$. If ω is a smooth bounded domain of $\mathbb{R}^{N-1}$ and l is a positive number, we will denote by Q the open bounded cylinder in $\mathbb{R}^N$ defined by $Q = \omega \times]-l, l[$.
We denote by $Y =]0,1[^N$ the volume reference cell and by $Y' =]0,1[^{N-1}$ the surface reference cell. Moreover, in the following, ε will be a positive parameter converging to zero.
Let $g : Y' \to \mathbb{R}$ a periodic positive Lipschitz continuous function, i.e. such that

$$(2.1) \qquad |g(y') - g(y'_1)| \leq L_g |y' - y'_1|, \quad \text{for every } y', y'_1 \in Y'.$$

If $\kappa > 0$ and $x' = (x_1, \ldots, x_{N-1})$ the graph

$$(2.2) \qquad \Gamma_\varepsilon = \left\{ x \in Q, x_N = \varepsilon^\kappa g(\frac{x'}{\varepsilon}) \right\}$$

represents an oscillating interface which divides the set Q in two subdomains

$$(2.3) \qquad Q_{\varepsilon 1} = \{ x \in Q, x_N > \varepsilon^\kappa g(\frac{x'}{\varepsilon}) \},$$

$$(2.4) \qquad Q_{\varepsilon 2} = \{ x \in Q, x_N < \varepsilon^\kappa g(\frac{x'}{\varepsilon}) \}$$

which are called the upper and the lower parts of Q, respectively.
Setting $\overline{g} = \max g$, by construction, the set $\omega \times [0, \varepsilon^\kappa \overline{g}]$ contains the oscillating interface, and the measure of this set goes to zero as $\varepsilon \to 0$ (see Figure 1).

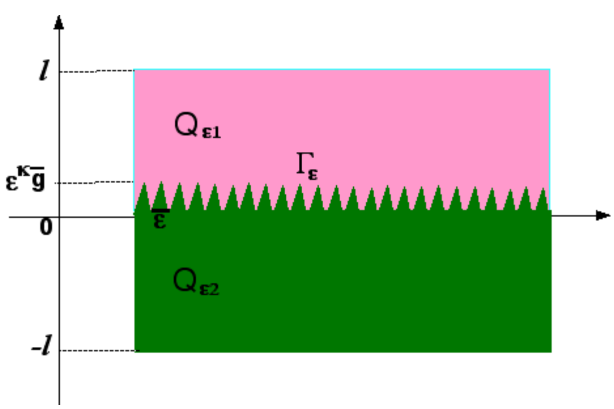

Figure 1: The upper and the lower parts of Q and the interface.

As observed in [22], the case $\kappa = 1$ presents a self-similar geometry because the interface Γ_ε can be obtained by homothetic dilatation of the fixed function $y_N = g(y')$ in $\mathbb{R}^N$. The case $\kappa > 1$ represents the flat case, while the case $0 < \kappa < 1$ describes a highly oscillating interface (see [22] for details).
We suppose that A is a Y-periodic matrix field satisfying, for $0 < \alpha < \beta$,

$$(2.5) \qquad (A(y)\lambda, \lambda) \geq \alpha |\lambda|^2, \qquad |A(y)\lambda| \leq \beta \lambda, \quad \text{a.e. in } Y \text{ and for any } \lambda \in \mathbb{R}^N.$$

Moreover, h will denote an Y'-periodic function such that, for some $h_0 \in \mathbb{R}^*_+$,

$$(2.6) \qquad h \in L^\infty(\Gamma), \text{ and } 0 < h_0 < h(y'), \text{ a.e. on } \Gamma,$$

where

$$\Gamma = \{y_N = g(y'),\ y' \in Y'\}. \tag{2.7}$$

We set, for any $\varepsilon > 0$,

$$A^\varepsilon(x) = A\Big(\frac{x}{\varepsilon}\Big), \qquad h^\varepsilon(x') = h\Big(\frac{x}{\varepsilon}\Big). \tag{2.8}$$

For any function v defined on Q we set

$$v_{\varepsilon 1} = v_{|Q_{\varepsilon 1}} \qquad v_{\varepsilon 2} = v_{|Q_{\varepsilon 2}} \tag{2.9}$$

and ν_ε stands for the unit outward normal to $Q_{\varepsilon 1}$.

Also, we use the notations:

- $\tilde{v}$ for the zero extension of a function v defined on a subset of Q,
- χ_E, the characteristic function of any set $E \subset \mathbb{R}^N$,
- $m_{Y'}(v) = \frac{1}{|Y'|}\int_{Y'} f\, dy'$, the average on Y' of any function $v \in L^1(Y')$.

Our aim is to prove some existence results (for fixed ε), and homogenization results as $\varepsilon \to 0$, of the following problem:

$$\begin{cases} -\mathrm{div}(A^\varepsilon \nabla u_\varepsilon) = f\,\zeta(u_\varepsilon) \qquad \text{in } Q \setminus \Gamma_\varepsilon, \\ (A^\varepsilon \nabla u_\varepsilon)_1 \cdot \nu_\varepsilon = (A^\varepsilon \nabla u_\varepsilon)_2 \cdot \nu_\varepsilon \qquad \text{on } \Gamma_\varepsilon, \\ (A^\varepsilon \nabla u_\varepsilon)_1 \cdot \nu_\varepsilon = -\varepsilon^\gamma h^\varepsilon (u_{\varepsilon 1} - u_{\varepsilon 2}), \qquad \text{on } \Gamma_\varepsilon. \\ u_\varepsilon = 0 \qquad \text{on } \partial Q, \end{cases} \tag{2.10}$$

where $\gamma \in \mathbb{R}$ and $\zeta\ :\ [0, +\infty[\to [0, +\infty]$ is a function such that

$$\zeta \in C^0([0, +\infty[), \qquad 0 \le \zeta(s) \le \frac{1}{s^\theta} \quad \text{for every } s \in]0, +\infty[, \text{ with } 0 < \theta \le 1. \tag{2.11}$$

and

$$f \ge 0, \text{ a.e. in } Q, \quad f \not\equiv 0, \quad \text{with } f \in L^r(Q) \text{ for } r \ge \frac{2}{1+\theta}\ (\ge 1). \tag{2.12}$$

We refer to Remark 4.4 for some comments on this assumption.

REMARK 2.1. *We want to stress that we do not assume any monotonicity property on the singular term $f\zeta(u)$. Note that no growth is required from below.*
A simple example of an oscillating function with singular behaviour which fits our assumptions is the following

$$f(x)\zeta(s) = \frac{f(x)}{s^\theta}\left(1 + \cos\frac{1}{s}\right),\ s > 0,$$

where $f(x)$ satisfies (2.12).
Let us also explain why we chose to assume that the function $f(x)$ appearing in the right-hand side of problem (2.10) belongs to a convenient Lebesgue space. This assumption allows to consider more general physical situations where possible infinite

concentrations appear in a point x_0, like $f(x) = \frac{1}{|x-x_0|^\alpha}$ with $\alpha < \alpha_0$, α_0 suitable positive real number.
This is also the case when we deal with the data f and u_0 of the classical model diffusion problem in a bounded cylinder $\Omega \times (0,T)$, without any dependence of the source term from the solution u, that is

$$(2.13)\qquad \begin{cases} u_t - \Delta_p u = f(x,t) & \text{in } \Omega \times (0,T) \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where Δ_p is the p-laplacian with $p > 1$(or its stationary version).
Looking for weak solutions, a large literature, starting from [33], [35], considers data f and u_0 like in the present paper, i.e. in convenient Lebesgue's spaces or, even worst, data f and u_0 measure (see [48] [8]).
On the other hand, confining to our stationary model in the domain Q, more regular data f, say $f \in C^0(\bar{Q})$, are obviously included in Lebesgue spaces. Let us point out that no advantage comes from such further regularity of the data in the proof of our existence result. Indeed our methods are "a priori estimate" methods which use, as a main tool, inequalities like Holder's and Young's ones and therefore the summability properties of the data. Of course more regularity on the data will induce more regularity on the solutions.

Through this paper, we suppose that ζ is singular in 0, which mean that $\zeta(0) = +\infty$, since otherwise ζ is bounded, which is a trivial case.
We introduce (under notation (2.9)) the space W_0^ε defined by

$$W_0^\varepsilon := \{v \in L^2(Q) \mid v_{\varepsilon 1} \in H^1(Q_{\varepsilon 1}),\ v_{\varepsilon 2} \in H^1(Q_{\varepsilon 2}) \quad \text{and} \quad v = 0 \text{ on } \partial Q\},$$

equipped with the norm

$$(2.14)\qquad \|v\|_{W_0^\varepsilon} := \|\nabla v\|_{L_2(Q\setminus\Gamma_\varepsilon)},$$

where

$$\nabla v = \widetilde{\nabla v_{\varepsilon 1}} + \widetilde{\nabla v_{\varepsilon 2}},$$

that is, we identify ∇v with the absolutely continuous part of the gradient of v.
In the same way we define

$$(2.15)\qquad Q_1 = \{x \in Q \,:\, x_N > 0\}, \quad Q_2 = \{x \in Q \,:\, x_N < 0\}, \quad \Gamma_0 = \{x \in Q \,:\, x_N = 0\}$$

and, for any function v defined on Q,

$$(2.16)\qquad v_1 = v_{|Q_1} \qquad v_2 = v_{|Q_2}.$$

Observe that

$$(2.17)\qquad \chi_{Q_{\varepsilon i}} \to \chi_{Q_i} \quad \text{strongly in } L^p(Q),\ 1 \le p < +\infty, \quad \text{and weakly * in } L^\infty(Q).$$

Then we introduce the space

$$W_0^0 := \{v \in L^2(Q) \mid v_1 \in H^1(Q_1),\ v_2 \in H^1(Q_2) \quad v = 0 \text{ on } \partial Q\},$$

equipped with the norm

$$\|v\|_{W_0^0} := \|\nabla v\|_{L_2(Q\setminus\Gamma_0)},$$

In the sequel we also use the notations

$$\Gamma_{\varepsilon,0} = \Gamma_\varepsilon \cup \Gamma_0. \tag{2.18}$$

and

$$Q_\varepsilon = Q \setminus \Gamma_\varepsilon, \qquad Q_0 = Q \setminus \Gamma_0, \qquad Q_{\varepsilon,0} = Q \setminus \Gamma_{\varepsilon,0}. \tag{2.19}$$

Let us observe that (2.14) is a norm, due to the following Poincaré inequality: there exists a constant $c_{_P}$ (independent of ε) such that, for any $v \in W_0^\varepsilon$

$$\|v\|_{L^2(Q)} \le c_{_P} \|\nabla v\|_{L^2(Q_\varepsilon)}. \tag{2.20}$$

Moreover, we have

Proposition 2.2. *([22]) If $\kappa \ge 1$ in (2.2), then there exist two families of linear continuous extensions operators $P_{\varepsilon 1} : H^1(Q_{\varepsilon 1}) \to H^1(Q)$ and $P_{\varepsilon 2} : H^1(Q_{\varepsilon 2}) \to H^1(Q)$ which are bounded uniformly in ε, that is*

$$\|P_{\varepsilon 1} v\|_{H^1(Q)} \le c\|v\|_{H^1(Q_{\varepsilon 1})}, \qquad \text{for every } v \in H^1(Q_{\varepsilon 1}),$$

$$\|P_{\varepsilon 2} v\|_{H^1(Q)} \le c\|v\|_{H^1(Q_{\varepsilon 2})}, \qquad \text{for every } v \in H^1(Q_{\varepsilon 2}),$$

where c only depend on the Lipschitz constant L_g of the function g (and is independent of ε).

Remark 2.3. *From Proposition 2.2, if $\kappa \ge 1$ we have the following uniform Sobolev-Poincaré inequality: there exists a constant c (independent of ε) such that, for any $v \in W_0^\varepsilon$*

$$\|v\|_{L^p(Q)} \le c\|\nabla v\|_{L^2(Q_\varepsilon)} \tag{2.21}$$

for every $p \in [2, 2^*]$ if $N > 2$ and for every $p \in [2, +\infty[$ if $N = 2$. The constant c depends on p, N and L_g. Note that, if $\kappa < 1$ the estimate is not uniform for $p > 2$, since the height of the cogs is much greater then its width, so that the constant c depends on the parameter ε and it blows up as ε goes to zero.

3. Physical meaning of the model. In this section we try to present some physical phenomena leading to mathematical models governed by semilinear elliptic equations with singular lower order terms. Some of them deal with non newtonian fluids and some others with diffusion in electrical conductors.
Of course, as pointed out in the introduction, if this kind of phenomena take place in composite materials possibly having inside rough interfaces we can have modelling problems which look like problem P_ε. Metamaterials, for example, are composite materials that "gain their properties from their structure, besides their composition; their precise shape, geometry, size, orientation and arrangement can affect the waves of light or sound in an unconventional manner, creating material properties which are unachievable with conventional materials." ([47])
Let us present a first class of phenomena described by a singular semilinear equation. Following [43], a non-Newtonian fluid is called pseudoplastic if the shear stress τ is a function of the strain rate $\dfrac{\partial u}{\partial y}$ via the expression

$$\tau = K\Big(\frac{\partial u}{\partial y}\Big)^n, \ 0 < n < 1,$$

where K is a positive constant, u is the velocity of the fluid along the boundary and y is the height above the boundary. Suppose that we look for an exact analytical solution to a basic problem in the boundary layer theory of these pseudoplastic fluids. Specifically, we are interested in the classical case of the incompressible flow of a uniform stream past a semi-infinite flat plate at zero incidence. Flows of this type are encountered in glacial advance [51], as well as in other geophysical contexts and in many industrial applications such as polymer or metal extrusion or continuous stretching of plastic films.
Following the discussion by [46], the boundary layer equations for steady flow over a semi-infinite flat plate may be written as

$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \dfrac{1}{\rho}\dfrac{\partial \tau}{\partial y}, \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \end{cases} \tag{3.1}$$

where ρ is the density, u and v are the velocity components parallel and normal to the plate and the shear stress is given by (3). The case n = 1 corresponds to a Newtonian fluid and for $0 < n < 1$ the "power law" relation (3) between shear stress and rate of strain has been proposed as a model for pseudoplastic non-Newtonian fluids. The standard boundary conditions are that the fluid have zero velocity on the plate and that the flow approach free stream conditions far from the plate. Thus $u(x,0) = v(x,0) = 0$, $u(x,\infty) = U_\infty$, where U_∞ is the uniform potential flow. Treating x and u as independent variables and τ as the dependent variable, it is possible to prove that system (3.1) can be transformed to

$$u\frac{\partial}{\partial x}\left(K^{\frac{1}{n}}\frac{\rho}{\tau^{\frac{1}{n}}}\right) + \frac{\partial^2 \tau}{\partial^2 u} = 0 \tag{3.2}$$

One seeks a solution to (3.2) of the form $\tau = \Phi(x)g(u)$. Substituting this into (3.2) leads to the results

$$\begin{cases} \Phi(x) = \left(-\dfrac{A(n+1)x}{\rho K^{\frac{1}{n}}}\right)^{-\frac{n}{n+1}} \\ g^{1/n}(u)g''(u) = Au, \end{cases} \tag{3.3}$$

where A is a arbitrary separation constant. The transformed boundary conditions become $g'(0) = 0$, $g(U_\infty) = 0$. Letting $u = \frac{u}{U_\infty}$ and choosing A appropriately leads to

$$\begin{cases} g^{1/n}(u)g''(u) + nu = 0, \\ g'(0) = 0, \quad g(1) = 0, \\ 0 < u < 1,\ 0 < n < 1. \end{cases} \tag{3.4}$$

which is infact a singular equation in the u variable.
Let us describe another concrete situation, described in [23] where singular terms appear in the model.
Suppose that we have a three dimensional region Q occupied by an electrical conductor. Each point becomes a source of heat as a current flows in Q. The function

$u(x,t)$ represents the temperature at the point x and at the time t, the function $V(x,t) = f^{\frac{1}{2}}(x,t)$ describes the local voltage drop in Q and $a(u) = \frac{1}{\zeta(u)}$ denotes the electrical resistivity. Then generation of heat occurs with a rate given by

$$\frac{V^2(x,t)}{a(u)} = f(x,t)\zeta(u),$$

so that the time dependent equation which models the phenomenon is

$$u_t - \Delta u = f(x,t)\zeta(u),$$

which in the stationary case reads

$$-\Delta u = f(x)\zeta(u).$$

In the case of a conductor material the electrical resistivity is a positive increasing funntion of the temperature u, which goes to zero as u goes to zero, (in some cases $a(u) = u^\alpha$ with $\alpha > 0$) so that the function $\zeta(u)$ in the right-hand side of the last equation is singular in the u variable on the set where the solution u is zero.

4. Statement of the main results.

4.1. The existence result. We state here the following existence result for problem (2.10), which is proved in Section 5:

THEOREM 4.1. *Under assumptions (2.5)-(2.8), (2.11) and (2.12), for every ε there exists at least a solution u_ε of problem (2.10), in the following sense:*

$$\text{(4.1)}\qquad \begin{cases} u_\varepsilon \in W_0^\varepsilon,\ u_\varepsilon > 0 \ \textit{a.e. in } Q, \\[2mm] \displaystyle\int_Q f\zeta(u_\varepsilon)\varphi\, dx < +\infty \quad \textit{and} \\[2mm] \displaystyle\int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla\varphi\, dx + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon (u_{\varepsilon 1} - u_{\varepsilon 2})(\varphi_1 - \varphi_2)\, d\sigma = \int_Q f\zeta(u_\varepsilon)\varphi\, dx, \\[2mm] \textit{for every } \varphi \in W_0^\varepsilon. \end{cases}$$

In the sequel any function u_ε satisfying (4.1) will be called solution to problem (2.10).

REMARK 4.2. *Observe that in the coordinates x' the boundary integral in the variational formulation reads*

$$\varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon (u_{\varepsilon 1} - u_{\varepsilon 2})(\varphi_1 - \varphi_2)\, d\sigma =$$

$$\varepsilon^\gamma \int_\omega h\big(\frac{x'}{\varepsilon}\big)\Big(u_{\varepsilon 1}\big(x', \varepsilon^\kappa g\big(\frac{x'}{\varepsilon}\big)\big) - u_{\varepsilon 2}\big(x', \varepsilon^\kappa g\big(\frac{x'}{\varepsilon}\big)\big)\Big)\Big(\varphi_1\big(x', \varepsilon^\kappa g\big(\frac{x'}{\varepsilon}\big)\big) - \varphi_2\big(x', \varepsilon^\kappa g\big(\frac{x'}{\varepsilon}\big)\big)\Big)$$

$$\times\Big(1 + \varepsilon^{2(\kappa-1)}(|\nabla_{y'} g(y')|_2)\big|_{y'=x'/\varepsilon}\Big)^{1/2}\, dx'.$$

4.2. Regularity and uniqueness results. In the theorem below we state that the solutions found in the previous Theorem 4.1 are bounded if the datum f is assumed more regular.

THEOREM 4.3. *Under assumptions* (2.5)-(2.8), (2.11) *and* (2.12), *assume in addition that*

$$f \in L^r(Q), \text{ for } r > \frac{N}{2}. \tag{4.2}$$

Then any solution u^ε of (4.1) *is bounded. Moreover, if $\kappa \geq 1$ any sequence of solutions $\{u^\varepsilon\}$ is bounded in $L^\infty(Q)$.*

REMARK 4.4. *Let us compare assumption* (2.12) *with assumption* (4.2). *For the case $N = 2$, if $0 < \theta < 1$ or if $\theta = 1$ and $r > 1$ in* (2.12), *assumption* (4.2) *is automatically satisfied. If $N = 3$ and* (2.12) *holds, the fact that* (4.2) *is satisfied or not depends on θ. For $N \geq 4$ assumption* (4.2) *is stronger that* (2.12).

The next result deals with the uniqueness of the solution found in Theorem 4.1. Here is the only point where we assume that the function $\zeta(s)$ defined in (2.11) has monotonicity properties, more precisely is non increasing.

THEOREM 4.5. *Let us assume* (2.5)-(2.8), (2.11) *and* (2.12) *and, in addition, that $\zeta(s)$ is non increasing in $]0, +\infty[$. Then, for every ε, there is a unique solution to problem* (4.1).

Theorems 4.3 and 4.5 are proved in Section 5.

4.3. Homogenization results. To state our homogenization results, let us introduce (see [6]) the homogenized tensor A^0, defined by

$$A^0\lambda = m_Y(A\nabla w_\lambda) \tag{4.3}$$

with $w_\lambda \in H^1(Y)$ the unique solution, for any $\lambda \in \mathbb{R}^N$, of

$$\begin{cases} \text{-div } (A\nabla w_\lambda) = 0 & \text{in } Y, \\ w_\lambda - \lambda \cdot y & Y\text{-periodic}, \\ m_Y(w_- \lambda \cdot y) = 0. \end{cases} \tag{4.4}$$

THEOREM 4.6. *Assume that* (2.5)-(2.8) *and* (2.11) *hold true; moreover if $\kappa \geq 1$ assume* (2.12) *while if $\kappa < 1$ suppose $f \in L^2(Q)$. Let u^ε be a solution of problem (4.1). Then, for every $\gamma \in \mathbb{R}$ there exists a subsequence (still denoted $\{\varepsilon\}$) and function u_0 such that*

$$u_0 \in W_0^0, \quad u_0 > 0 \ a.e. \ on \ Q, \quad \int_Q f\zeta(u_0)\varphi \, dx < +\infty \tag{4.5}$$

the following convergences hold true:

$$\begin{cases} i) \ u_\varepsilon \to u_0, & \text{strongly in } L_2(Q) \text{ and a.e. in } Q, \\ ii) \ \chi_{Q_{\varepsilon i}} \nabla u_\varepsilon \rightharpoonup \chi_{Q_i} \nabla u_0, & \text{weakly in } (L_2(Q))^N, \end{cases} \tag{4.6}$$

and

$$\chi_{Q_{\varepsilon i}} A^\varepsilon \nabla u^\varepsilon \rightharpoonup \chi_{Q_i} A^0 \ \nabla u_0, \qquad \text{weakly in } (L_2(Q))^N, \tag{4.7}$$

for $i = 1, 2$, where A^0 is given by (4.3).
Moreover, denoting

$$u_0 = \begin{cases} u_{01}(x), & x \in Q_1 \\ u_{02}(x), & x \in Q_2 \end{cases}$$

we have the limit problems below.

Suppose that one of the following assumptions holds

$$\kappa \geq 1 \text{ and } \gamma = 0 \tag{4.8}$$

or

$$0 < \kappa < 1 \text{ and } \gamma = 1 - \kappa. \tag{4.9}$$

Then, the function u_0 is a solution of the problem

$$\begin{cases} -\mathrm{div}(A^0 \nabla u_0) = f\zeta(u_0) \qquad \text{in } Q_0, \\ (A^0 \nabla u_0)_1 \cdot n = (A^0 \nabla u_0)_2 \cdot n \qquad \text{on } \Gamma_0, \\ (A^0 \nabla u_0)_2 \cdot n = -H(g,h)(u_{01} - u_{02}), \qquad \text{on } \Gamma_0, \\ u_0 = 0 \qquad \text{on } \partial Q, \end{cases} \tag{4.10}$$

where n is unit outward normal to Q_1 and

$$H(g,h) = \begin{cases} m_{Y'}\Big(h(1 + (|\nabla g|_2)^{1/2}\Big) \qquad \text{if } \kappa = 1 \text{ and } \gamma = 0, \\ m_{Y'}(h) \qquad \text{if } \kappa > 1 \text{ and } \gamma = 0, \\ m_{Y'}(h|\nabla g|) \qquad \text{if } 0 < \kappa < 1 \text{ and } \gamma = 1 - \kappa, \end{cases} \tag{4.11}$$

whose variational formulation is

$$\begin{cases} \displaystyle\int_{Q_0} A^0 \nabla u_0 \nabla \varphi \, dx + H(g,h) \int_{\Gamma_0} (u_{01} - u_{02})(\varphi_1 - \varphi_2) \, d\sigma \\ \displaystyle = \int_Q f\zeta(u_0)\varphi \, dx, \\ \textit{for every } \varphi \in W_0^\varepsilon. \end{cases} \tag{4.12}$$

Suppose now that one of the following assumptions holds

$$\kappa \geq 1 \text{ and } \gamma < 0 \tag{4.13}$$

or

$$0 < \kappa < 1 \text{ and } \gamma < 1 - \kappa. \tag{4.14}$$

Then, the function u_0 belongs to $H_0^1(Q)$ and is a solution of the problem

$$\begin{cases} -\mathrm{div}(A^0 \nabla u_0) = f\zeta(u_0) \qquad \text{in } Q, \\ u = 0 \qquad \text{on } \partial Q, \end{cases} \tag{4.15}$$

whose variational formulation is

$$\begin{cases} \displaystyle\int_Q A^0 \nabla u_0 \nabla \varphi \, dx = \int_Q f\zeta(u_0)\varphi \, dx, \\ \textit{for every } \varphi \in H_0^1(\Omega). \end{cases} \tag{4.16}$$

Finally, suppose that one of the following assumptions holds

$$\kappa \geq 1 \text{ and } \gamma > 0 \tag{4.17}$$

or

$$0 < \kappa < 1 \text{ and } \gamma > 1 - \kappa. \tag{4.18}$$

Then, u_{01} and u_{02} are solutions of the following two (independent) Neumann problems:

$$\begin{cases} -\mathrm{div}(A^0 \nabla u_{01}) = f\zeta(u_{01}) \quad \text{in } Q_1, \\ A^0 \nabla u_{01} \cdot n = 0 \quad \text{on } \Gamma_0, \\ u_{01} = 0 \qquad \text{on } \partial Q_1 \setminus \Gamma_0, \end{cases} \tag{4.19}$$

and

$$\begin{cases} -\mathrm{div}(A^0 \nabla u_{02}) = f\zeta(u_{02}) \quad \text{in } Q_2, \\ A^0 \nabla u_{02} \cdot n = 0 \quad \text{on } \Gamma_0, \\ u_{02} = 0 \qquad \text{on } \partial Q_2 \setminus \Gamma_0, \end{cases} \tag{4.20}$$

whose variational formulations are

$$\begin{cases} \displaystyle\int_{Q_1} A^0 \nabla u_{01} \nabla \varphi \, dx = \int_{Q_1} f\zeta(u_0)\varphi \, dx, \\ \text{for every } \varphi \in H^1(\Omega_1) \text{ such that } \varphi = 0 \text{ on } \partial Q_1 \setminus \Gamma_0 \end{cases} \tag{4.21}$$

and

$$\begin{cases} \displaystyle\int_{Q_2} A^0 \nabla u_{01} \nabla \varphi \, dx = \int_{Q_2} f\zeta(u_0)\varphi \, dx, \\ \text{for every } \varphi \in H^1(\Omega_2) \text{ such that } \varphi = 0 \text{ on } \partial Q_2 \setminus \Gamma_0, \end{cases} \tag{4.22}$$

respectively.
If, in addition, we suppose that the function $\zeta(s)$ defined in (2.11) is non decreasing, the solution u_0 of the above limit problems is unique and convergences (4.6) and (4.7) hold for the whole sequences.

The proof of this theorem is done in Section 7.

4.4. A corrector result. We complete here the convergences given in Theorem 4.6 by a corrector result, which shows that the corrector for the nonlinear problem (4.1) is the same as that of the associated linear problem.
We derive this result by a corrector result on the corresponding linear problem (Theorem 9.1), which is itself new and which will be proved in Section 8.
Then, the nonlinear corrector result stated in Theorem 4.7 below follows straightforward from Theorem 9.1 and Theorem 8.5 which is also an essential tool when proving Theorem 4.6.

Let us introduce the classical corrector matrix $C^\varepsilon = \left(C^\varepsilon_{ij}\right)_{1\le i,j\le n}$, given by

$$(4.23)\qquad \begin{cases} C^\varepsilon_{ij}(x) = C_{ij}\left(\dfrac{x}{\varepsilon}\right) & \text{a.e. on } Q,\\ C_{ij}(y) = \dfrac{\partial w_j}{\partial y_i}(y), \quad i,j = 1,...,n & \text{a.e. on } Y, \end{cases}$$

where $\{e_j\}_{j=1}^N$ is the canonical basis of R^N and w_j is the solution of problem (4.4), written for $\lambda = e_j$.

THEOREM 4.7. *Under the assumptions of Theorem 4.6, for every value of κ and γ, we have*

$$(4.24)\qquad \lim_{\varepsilon\to 0} ||\nabla u_\varepsilon - C^\varepsilon \nabla u_0||_{\left(L^1(Q_{\varepsilon,0})\right)^N} = 0.$$

where the corrector matrix C^ε is given by (4.23).

5. A priori estimates. In this section we give some a priori estimates for a solution w of problem (2.10), which are uniform with respect to ε and dependent on any function ζ satisfying (2.11) only through the constant θ.
This also provides uniform estimates with respect to n and ε for the solutions u^ε_n of the approximate problem (6.1), used in the next section to show (for fixed ε) the existence of a solution of problem (2.10). Indeed, the nonlinearity in the right-hand side of (6.1) still satisfies (2.11). These estimates are also used for the solution u_ε of problem (2.10) itself, when proving the homogenization result in Section 7.
Along this paper, we will denote by c different constants independent of ε.
For any function v in W^ε_0, we define

$$v^+ = \max\{v,\ 0\}, \quad v^- = -\min\{v,\ 0\}, \quad \text{a.e. on } Q,$$

which, by known results, still belong to W^ε_0. Clearly,

$$(5.1)\qquad v = v^+ - v^-.$$

REMARK 5.1. *Let us observe that for every $v \in W^\varepsilon_0$ one has*

$$(5.2)\qquad \begin{aligned} &(v_1 - v_2)(v_1^- - v_2^-) = (v_1^+ - v_2^+)(v_1^- - v_2^-) - (v_1^- - v_2^-)^2 =\\ &= -v_1^+ v_2^- - v_2^+ v_1^- - (v_1^- - v_2^-)^2 \le 0, \end{aligned}$$

as well as for their traces on Γ_ε.

PROPOSITION 5.2. *Under the assumptions* (2.5)-(2.8), (2.12) *and* (2.11), *let $w \in W^\varepsilon_0$ be a solution of problem* (2.10). *Then, the following a priori estimates hold:*

$$(5.3)\qquad \|w\|_{W^\varepsilon_0} \le c\|f\|^{\frac{1}{1+\theta}}_{L^{\frac{2}{1+\theta}}(Q)},$$

where $c = c(\alpha, c_P)$ and

$$(5.4)\qquad \|w_1 - w_2\|_{L^2(\Gamma_\varepsilon)} \le c\,\varepsilon^{-\frac{\gamma}{2}}\|f\|^{\frac{1}{1+\theta}}_{L^{\frac{2}{1+\theta}}(Q)},$$

where $c = c(\alpha, c_P, \theta)$.

Proof. Let us choose w as test function in the variational formulation (4.1) of problem (2.10). We use (2.5), (2.11), (2.12), Holder inequality and Poincaré inequality (2.20), getting

$$(5.5)\quad \begin{aligned} &\alpha\|\nabla w\|^2_{L^2(Q_\varepsilon)} + \varepsilon^\gamma \|w_1 - w_2\|^2_{L^2(\Gamma_\varepsilon)} \\ &\qquad \le \|f\|_{L^{\frac{2}{1+\theta}}(Q)} \|w\|^{1-\theta}_{L^2(Q)} \le c_P \|f\|_{L^{\frac{2}{1+\theta}}(Q)} \|\nabla w\|^{1-\theta}_{L^2(Q_\varepsilon)}. \end{aligned}$$

We first neglect the nonnegative boundary term in (5.5) and we get (5.3). Neglecting now the firs term in (5.5) and using (5.3), we easily get (5.4). ∎

Proposition 5.3. *Under the assumptions* (2.5)-(2.8)*,* (2.12) *and* (2.11)*, let* $w \in W_0^\varepsilon$ *be a solution of problem* (2.10)*. Then,*

$$(5.6)\quad \|f\zeta(w)\varphi\|_{L^1(Q)} \le c,$$

for every positive $\varphi \in W_0^\varepsilon$ *where* $c = c(\alpha, c_P, \|f\|_{L^r(Q)}, \theta, \beta, \|\nabla\varphi\|_{L^2(Q)})$.

Proof. We choose a nonnegative $\varphi \in H_0^1(Q)$ as test function in (4.1). Since the boundary term vanishes, from (2.5), estimate (5.3) and the Hölder inequality, it follows that

$$0 \le \int_Q f\zeta(w)\varphi \, dx \le c,$$

where $c = c(\alpha, \beta, \theta, c_P, \|f\|_{L^r(Q)}, \|\nabla\varphi\|_{L^2(Q)})$.

Let us take now a nonnegative $\varphi = (\varphi_{\varepsilon 1}, \varphi_{\varepsilon 2})$ in W_0^ε. Since Γ_ε is Lipschitz continuous, there exist still nonnegative ψ_1 and $\psi_2 \in H_0^1(Q)$ such that (see for instance [10], Ch. 9)

$$\varphi = (\varphi_{\varepsilon 1}, \varphi_{\varepsilon 2}) = (\psi_{1|Q_{\varepsilon 1}}, \psi_{2|Q_{\varepsilon 2}}).$$

Then we can write:

$$\begin{aligned} 0 \le \int_Q f\zeta(w)\varphi \, dx &= \int_{Q_{\varepsilon 1}} f\zeta(w)\psi_1 \, dx + \int_{Q_{\varepsilon 2}} f\zeta(w)\psi_2 \\ &\le \int_Q f\zeta(w)\psi_1 \, dx + \int_Q f\zeta(w)\psi_2 \, dx \le c, \end{aligned}$$

∎

The following proposition, which gives an estimate of the integral of the singular term close to the singular set $\{w = 0\}$, is crucial in the proof of our results, both existence and homogenization ones.

It makes use of similar techniques as those in [27], [29], which involve the auxiliary real function Z_δ defined by

$$(5.7)\quad Z_\delta(s) = \begin{cases} 1, & \text{if } 0 \le s \le \delta, \\ -\frac{s}{\delta} + 2, & \text{if } \delta \le s \le 2\delta, \\ 0, & \text{if } 2\delta \le s. \end{cases}$$

We also need for $k > 0$, the usual truncation function T_k at level k, defined by

$$(5.8)\quad T_k(s) = \begin{cases} -k, & \text{if } s < -k, \\ s, & \text{if } |s| \le k, \\ k, & \text{if } s > k. \end{cases}$$

PROPOSITION 5.4. *Under the assumptions* (2.5)-(2.8), (2.12) *and* (2.11), *let* $w \in W_0^\varepsilon$ *be a solution of problem* (2.10) *and* δ *a fixed positive real number. Then,*

$$\int_{\{0\le w\le\delta\}} f\,\zeta(w)\varphi\,dx \le \int_{Q_\varepsilon} A^\varepsilon\nabla w\,\nabla\varphi\,Z_\delta(w)\,dx \tag{5.9}$$
$$+\,\varepsilon^\gamma\int_{\Gamma_\varepsilon} h^\varepsilon(w_1-w_2)(Z_\delta(w_1)\varphi_1 - Z_\delta(w_2)\varphi_2)\,d\sigma$$
$$\le \int_{Q_\varepsilon} A^\varepsilon\nabla w\,\nabla\varphi\,Z_\delta(w)\,dx + 2\,\delta\,\varepsilon^\gamma\,||h||_{L^\infty(\Gamma)}\,||\varphi_1+\varphi_2||_{L^1(\Gamma_\varepsilon)},$$

for every $\varphi\in W_0^\varepsilon$, $\varphi\ge 0$, *where* Z_δ *is defined by* (5.7).

Proof. Let $\varphi\in W_0^\varepsilon$, $\varphi\ge0$. Taking, for $k>0$, $Z_\delta(w)T_k(\varphi)$ as test function in (4.1) where $T_k(s)$ is the truncation function given by (5.8), we obtain

$$\int_{Q_\varepsilon} A^\varepsilon\nabla w\nabla T_k(\varphi)Z_\delta(w)\,dx - \frac{1}{\delta}\int_{Q_\varepsilon\cap\{\delta<w<2\delta\}} A^\varepsilon\nabla w\nabla w T_k(\varphi)\,dx$$
$$+\,\varepsilon^\gamma\int_{\Gamma_\varepsilon} h^\varepsilon(w_1-w_2)(Z_\delta(w_1)T_k(\varphi)_1 - Z_\delta(w_2)T_k(\varphi)_2)\,d\sigma$$
$$=\int_Q f\zeta(w)\,Z_\delta(w)T_k(\varphi)\,dx.$$

Since w and φ are nonnegative, this implies

$$\int_{\{0\le w\le\delta\}} f\,\zeta(w)T_k(\varphi)\,dx \le \int_{Q_\varepsilon} A^\varepsilon\nabla w\,\nabla T_k(\varphi)\,Z_\delta(w)\,dx \tag{5.10}$$
$$+\,\varepsilon^\gamma\int_{\Gamma_\varepsilon} h^\varepsilon(w_1-w_2)(Z_\delta(w_1)T_k(\varphi)_1 - Z_\delta(w_2)T_k(\varphi)_2)\,d\sigma$$

and the following one:

$$\varepsilon^\gamma\int_{\Gamma_\varepsilon} h^\varepsilon(w_1-w_2)(Z_\delta(w_1)T_k(\varphi)_1 - Z_\delta(w_2)T_k(\varphi)_2)\,d\sigma$$
$$\le\;\varepsilon^\gamma\int_{\Gamma_\varepsilon} h^\varepsilon(w_1Z_\delta(w_1)T_k(\varphi)_1 + w_2Z_\delta(w_2)T_k(\varphi)_2)\,d\sigma$$
$$\le \varepsilon^\gamma\int_{\Gamma_\varepsilon} h^\varepsilon(w_1\chi_{\{w_1\le2\delta\}}T_k(\varphi)_1 + w_2\chi_{\{w_2\le2\delta\}}T_k(\varphi)_2)\,d\sigma$$
$$\le 2\,\delta\,\varepsilon^\gamma\,||h||_{L^\infty(\Gamma)}\,||\varphi_1+\varphi_2||_{L^1(\Gamma_\varepsilon)}.$$

where we used (5.7). This, together with (5.10) gives, for any $k>0$,

$$\int_{\{0\le w\le\delta\}} f\,\zeta(w)T_k(\varphi)\,dx \le \int_{Q_\varepsilon} A^\varepsilon\nabla w\,\nabla T_k(\varphi)\,Z_\delta(w)\,dx$$
$$+\,\varepsilon^\gamma\int_{\Gamma_\varepsilon} h^\varepsilon(w_1-w_2)(Z_\delta(w_1)T_k(\varphi)_1 - Z_\delta(w_2)T_k(\varphi_2)\,d\sigma$$
$$\le \int_{Q_\varepsilon} A^\varepsilon\nabla w\,\nabla T_k(\varphi)\,Z_\delta(w)\,dx + 2\,\delta\,\varepsilon^\gamma\,||h||_{L^\infty(\Gamma)}\,||\varphi_1+\varphi_2||_{L^1(\Gamma_\varepsilon)}.$$

To get the result, we pass now to the limit as k tends to infinity in the last inequalities, using Fatou's lemma (on the first integral) and the fact that $T_k(\varphi)$ strongly converges to φ in W_0^ε. ∎

REMARK 5.5. *We point out that estimate (5.9) near the singularity allows us to overcome a main difficulty. Indeed, due to the jump of the solutions on the interface, we cannot expect that they are uniformly bounded from below by a positive constant on compact sets ω of Q, which is a property often used in the literature for singular problems.*

The lack of bounds from below is essentially due to the fact that the strong maximum principle cannot be applied in the whole domain Q (since these functions do not belong to $H^1(Q)$), but only in $Q_{\varepsilon 1}$ and $Q_{\varepsilon 2}$. This concerns uniform estimates (with respect to n) for the solutions u_n of the approximating problems (6.1) introduced in Section 5 when proving of the existence result of u_ε for fixed ε. It concerns as well uniform estimates (with respect to ε) for the solutions u_ε of (4.1) itself, when studying the corresponding homogenization problem. Both were denoted by w above.

6. Proof of the existence (Theorem 4.1).

We define the following sequence of nonsingular problems, which approximates problem (2.10):

$$\text{(6.1)} \quad \begin{cases} -\operatorname{div}(A^\varepsilon\, \nabla u_n^\varepsilon) = T_n\big(f\,\zeta(|u_n^\varepsilon|)\big) & \text{in } Q_\varepsilon, \\ (A^\varepsilon \nabla u_n^\varepsilon)_1 \cdot \nu_\varepsilon = (A^\varepsilon \nabla u_n^\varepsilon)_2 \cdot \nu_\varepsilon & \text{on } \Gamma_\varepsilon, \\ (A^\varepsilon \nabla u_n^\varepsilon)_1 \cdot \nu_\varepsilon = -\varepsilon^\gamma\, h^\varepsilon (u_{n1}^\varepsilon - u_{n2}^\varepsilon), & \text{on } \Gamma_\varepsilon. \\ u_n^\varepsilon = 0 & \text{on } \partial Q, \end{cases}$$

where, for every $n \in \mathbb{N}$, $n \geq 1$, the function T_n is the truncation function given by (5.8).

Since in this proof ε is fixed, we denote A^ε, u_n^ε and h^ε simply by A, u_n and h omitting its dependence on ε.

Then, the variational formulation of problem (6.1) reads

$$\text{(6.2)} \quad \begin{cases} u_n \in W_0^\varepsilon, \\ \displaystyle\int_{Q_\varepsilon} A \nabla u_n \nabla \varphi \, dx + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h(u_{n1} - u_{n2})(\varphi_1 - \varphi_2)\, d\sigma = \int_Q T_n\big(f \zeta(|u_n|)\big) \varphi \, dx, \\ \text{for every } \varphi \in W_0^\varepsilon. \end{cases}$$

The existence of a solution of this problem, quite standard, is proved in the Appendix.

Let us show that

$$\text{(6.3)} \qquad u_n \geq 0, \quad \text{a.e. in } Q.$$

Choosing $\varphi = -u_n^-$ in (6.2) and using (2.12) we obtain

$$\text{(6.4)} \qquad \int_{Q_\varepsilon} A \nabla u_n^- \nabla u_n^- \, dx + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h(u_{n1} - u_{n2})(-u_{n1}^- + u_{n2}^-)\, d\sigma \leq 0.$$

The surface integral over Γ_ε is nonnegative, since from (5.1) one has

$$\begin{aligned} (u_{n1} - u_{n2})(-u_{n1}^- + u_{n2}^-) &= (-u_{n1}^- + u_{n2}^-)^2 + (-u_{n1}^- + u_{n2}^-)(u_{n1}^+ - u_{n2}^+) \\ &= (-u_{n1}^- + u_{n2}^-)^2 + (u_{n2}^- u_{n1}^+ + u_{n1}^- u_{n2}^+) \geq 0. \end{aligned}$$

Then (6.4) and the ellipticity of A imply that $u_n^- = 0$ almost everywhere, so that (6.3) holds and we can write $\zeta(u_n)$ instead of $\zeta(|u_n|)$ in the problem.
Observe now that $T_n(\zeta)$ satisfies the same assumptions as the function ζ, so that the a priori estimates given in Section 5 apply to the sequence $\{u_n\}$. Consequently, there exists $u_\varepsilon \in W_0^\varepsilon \cap L^2(Q)$ such that up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u_\varepsilon & \text{weakly in } W_0^\varepsilon \text{ and strongly in } L^2(Q), \\ u_n \to u_\varepsilon & \text{a.e. in } Q, \\ u_{n1} - u_{n2} \to u_{\varepsilon 1} - u_{\varepsilon 2} & \text{strongly in } L^2(\Gamma^\varepsilon). \end{cases} \tag{6.5}$$

This, together with (6.3) implies that $u_\varepsilon \geq 0$ almost everywhere in Q.
Let us now consider $\varphi \in W_0^\varepsilon$, $\varphi \geq 0$ and take the function $T_l(\varphi) \in W_0^\varepsilon \cap L^\infty(Q)$(see (5.8)) as test function in (6.2), for $l > 0$ fixed. We get

$$\begin{aligned} &\int_{Q_\varepsilon} A\nabla u_n \nabla T_l(\varphi)\, dx + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h(u_{n1} - u_{n2})(T_l(\varphi)_1 - T_l(\varphi)_2)\, d\sigma \\ &\quad = \int_Q T_n\big(f\zeta(u_n)\big) T_l(\varphi)\, dx. \end{aligned} \tag{6.6}$$

From Proposition 5.3 we have the uniform estimates

$$\|T_n(f\zeta(u_n))T_l(\varphi)\|_{L^1(Q)} \leq c,$$

with c independent of n. This together with (6.5), in view of Fatou's Lemma implies that

$$\int_Q f\zeta(u_\varepsilon) T_l(\varphi)\, dx < +\infty, \tag{6.7}$$

for any $\varphi \in W_0^\varepsilon$ and any fixed positive l.
Let us now pass to the limit in (6.6) for nonnegative φ, as $n \to \infty$ and for l fixed. Concerning the right-hand side of the equation, observe that we can apply the Lebesgue dominated convergence theorem only far from the singularity.
To overcome this difficulty, for every positive δ we split the right-hand side as

$$\begin{aligned} &\int_Q T_n\big(f\zeta(u_n)\big) T_l(\varphi)\, dx \\ &= \int_{\{0 \leq u_n \leq \delta\}} T_n\big(f\zeta(u_n)\big) T_l(\varphi)\, dx + \int_{\{\delta < u_n\}} T_n\big(f\zeta(u_n)\big) T_l(\varphi)\, dx \doteq I_n + J_n. \end{aligned} \tag{6.8}$$

From Proposition 5.4 it follows that

$$I_n \leq \int_{Q_\varepsilon} A\nabla u_n\, \nabla T_l(\varphi)\, Z_\delta(u_n)\, dx + 2\,\delta\,\varepsilon^\gamma\, ||h||_{L^\infty(\Gamma)}\, ||\varphi_1 + \varphi_2||_{L^1(\Gamma_\varepsilon)},$$

which using (6.5) and (5.7) yields

$$\limsup_{n\to\infty} I_n \leq \int_{Q_\varepsilon} A\nabla u_\varepsilon\, \nabla T_l(\varphi)\, Z_\delta(u_\varepsilon)\, dx + 2\,\delta\,\varepsilon^\gamma\, ||h||_{L^\infty(\Gamma)}\, ||\varphi_1 + \varphi_2||_{L^1(\Gamma_\varepsilon)}.$$

Since the gradient of H^1-functions vanishes on level sets,

$$\lim_{\delta\to 0}\int_{Q_\varepsilon} A\nabla u_\varepsilon\, \nabla T_l(\varphi)\, Z_\delta(u_\varepsilon)\, dx = \int_{Q_\varepsilon} A\nabla u_\varepsilon\, \nabla T_l(\varphi)\, \chi_{\{u_\varepsilon=0\}}\, dx = 0,$$

which gives

$$\lim_{\delta\to 0}\limsup_{n\to\infty} I_n = 0. \tag{6.9}$$

As far as it concerns the term J_n, we write it as

$$\begin{aligned} J_n = &\int_Q T_n\big(f\zeta(u_n)\big)T_l(\varphi)\, \chi_{\{u_n>\delta\}}\, \chi_{\{u_\varepsilon\neq\delta\}}\, dx \\ &+ \int_Q T_n\big(f\zeta(u_n)\big)\, T_l(\varphi)\chi_{\{u_n>\delta\}}\, \chi_{\{u_\varepsilon=\delta\}}\, dx. \end{aligned} \tag{6.10}$$

Due to assumption (2.12), $f\, T_l(\varphi)\in L^1(Q)$, so that

$$0\le T_n\big(f\zeta(u_n)\big)T_l(\varphi)\, \chi_{\{u_n>\delta\}}\, \chi_{\{u_\varepsilon\neq\delta\}} \le \frac{1}{\delta^\theta}\, f\, T_l(\varphi)\in L^1(Q)$$

and from (2.11) (6.5) and (6.7) we have, almost everywhere in Q,

$$\begin{aligned} &\lim_{n\to\infty} T_n\big(f\zeta(u_n)\big)T_l(\varphi)\, \chi_{\{u_n>\delta\}}\, \chi_{\{u_\varepsilon\neq\delta\}} = f\zeta(u_\varepsilon)T_l(\varphi)\, \chi_{\{u_\varepsilon>\delta\}}, \\ &\lim_{\delta\to 0}\chi_{\{u_\varepsilon>\delta\}} = \chi_{\{u_\varepsilon>0\}}. \end{aligned}$$

Then, applying twice the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\delta\to 0}\lim_{n\to\infty}\int_Q T_n\big(f\zeta(u_n)\big)T_l(\varphi)\, \chi_{\{u_n>\delta\}}\, \chi_{\{u_\varepsilon\neq\delta\}}\, dx = \int_Q f\zeta(u_\varepsilon)T_l(\varphi)\, \chi_{\{u_\varepsilon>0\}}\, dx. \tag{6.11}$$

To treat the second term of the right-hand side of (6.10), observe that for every $\delta>0$ except at most for a countable set C of values, one has $\operatorname{meas}\{x\in Q : u_\varepsilon(x)=\delta\}=0$, so that

$$\int_Q T_n\big(f\zeta(u_n)\big)\, T_l(\varphi)\chi_{\{u_n>\delta\}}\, \chi_{\{u_\varepsilon=\delta\}}\, dx = 0, \qquad \text{for every } \delta\in\mathbb{R}_+\setminus C.$$

This, together with (6.11) implies that

$$\lim_{\delta\to 0}\lim_{n\to\infty} J_n = \int_Q f\zeta(u_\varepsilon)T_l(\varphi)\, \chi_{\{u_\varepsilon>0\}}\, dx, \qquad \delta\in\mathbb{R}_+\setminus C. \tag{6.12}$$

Collecting (6.8)-(6.12) we can pass to the limit in the right-hand side of (6.6) getting

$$\limsup_{n\to\infty}\int_Q T_n\big(f\zeta(u_n)\big)T_l(\varphi)\, dx = \int_Q f\zeta(u_\varepsilon)T_l(\varphi)\, \chi_{\{u_\varepsilon>0\}}\, dx,$$

for every $\varphi\in W_0^\varepsilon$, $\varphi\ge 0$. This remains true for every $\varphi\in W_0^\varepsilon$ with any sign, using the fact that $\varphi=\varphi^+-\varphi^-$.

Consequently, since convergences (6.5) allow to easily pass to the limit in the left-hand side of (6.6), the function u_ε satisfies

$$(6.13)\qquad \begin{cases} u_\varepsilon \in W_0^\varepsilon, \quad u_\varepsilon \geq 0 \text{ a.e. on } Q, \quad \displaystyle\int_Q f\zeta(u_\varepsilon)T_l(\varphi)\,dx < +\infty \quad \text{and} \\ \displaystyle\int_{Q_\varepsilon} A\nabla u_\varepsilon \nabla T_l(\varphi)\,dx + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h(u_{\varepsilon 1} - u_{\varepsilon 2})(T_l(\varphi)_1 - T_l(\varphi)_2)\,d\sigma \\ = \displaystyle\int_Q f\zeta(u_\varepsilon)T_l(\varphi)\,\chi_{\{u_\varepsilon>0\}}\,dx, \quad \text{for every } \varphi \in W_0^\varepsilon. \end{cases}$$

Finally, from the strong maximum principle (see Theorem 8.19 of [30]) we deduce that $u_\varepsilon > 0$ a.e. in Q_ε, hence a.e. in Q, since the N-dimensional measure of Γ_ε is zero. Then problem (6.13) reads as

$$(6.14)\qquad \begin{cases} u_\varepsilon \in W_0^\varepsilon, \quad u_\varepsilon > 0 \text{ a.e. on } Q, \quad \displaystyle\int_Q f\zeta(u_\varepsilon)T_l(\varphi)\,dx < +\infty \quad \text{and} \\ \displaystyle\int_{Q_\varepsilon} A\nabla u_\varepsilon \nabla T_l(\varphi)\,dx + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h(u_{\varepsilon 1} - u_{\varepsilon 2})(T_l(\varphi)_1 - T_l(\varphi)_2)\,d\sigma \\ = \displaystyle\int_Q f\zeta(u_\varepsilon)T_l(\varphi)\,dx, \quad \text{for every } \varphi \in W_0^\varepsilon. \end{cases}$$

Finally, we easily pass to the limit in the left-hand side of (6.14) as l goes to $+\infty$. The right-hand side is then uniformly bounded in l, so that by Fatou's lemma we have $f\zeta(u_\varepsilon)\varphi \in L^1(Q)$. Then we can use Lebesgue theorem since we have for any positive l and any $\varphi \in W_0^\varepsilon$,

$$f\zeta(u_\varepsilon)T_l(\varphi) \leq f\zeta(u_\varepsilon)\varphi \in L^1(Q)$$

and this concludes the proof. □

7. Proofs of regularity (Theorem 4.3) and uniqueness (Theorem 4.5).

Proof of Theorem 4.3. Let us choose, for $\nu \in \mathbb{R}$, $\nu \geq 1$, the function

$$\varphi = G_\nu(u_\varepsilon) \doteq (u_\varepsilon - \nu)^+$$

as test function in (4.1), which is clearly in W_0^ε.
This gives

$$(7.1)\qquad \begin{aligned} &\int_{Q_\varepsilon} A^\varepsilon \nabla G_\nu(u_\varepsilon)\nabla G_\nu(u_\varepsilon)\,dx + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon(u_{\varepsilon 1} - u_{\varepsilon 2})(G_\nu(u_{\varepsilon 1}) - G_\nu(u_{\varepsilon 2}))\,d\sigma \\ &= \int_Q f\zeta(u_\varepsilon)G_\nu(u_\varepsilon)\,dx. \end{aligned}$$

Let us assume that $N > 2$. Since from (2.11) we have $\zeta(u_\varepsilon) \leq \dfrac{1}{k^\theta} \leq 1$ on the set where $G_\nu(u_\varepsilon) \neq 0$, taking into account the ellipticity of A and the fact that G_ν is not decreasing, we get using (2.21)

$$\int_Q (G_\nu(u_\varepsilon))^{2^*}\,dx\Big)^{\frac{2}{2^*}} \leq c\int_{Q_\varepsilon} |\nabla G_\nu(u_\varepsilon)|^2\,dx \leq \frac{c}{\alpha}\int_Q fG_\nu(u_\varepsilon)\,dx,$$

where $c = c(N, |Q_\varepsilon|)$.

This implies the result by classical arguments due to G. Stampacchia ([48]). The proof in the case $N = 2$ uses similar arguments and the fact that in this case the space $H_0^1(\Omega)$ is continuously embedded in the space $L^t(\Omega)$ for any $t > 1$.
The last statement follows from the fact that if $\kappa \geq 1$ the constant c above is independent of ε (see Remark 2.3). □

Proof of Theorem 4.5. Let u_ε and w_ε be two solutions to problem (4.1).
We choose $u_\varepsilon - w_\varepsilon$ as test function in both equations and we take the difference between the two equations, getting

$$\int_{Q_\varepsilon} A^\varepsilon \nabla(u_\varepsilon - w_\varepsilon)\nabla(u_\varepsilon - w_\varepsilon)\, dx + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon((u_{\varepsilon 1} - w_{\varepsilon 1}) - (u_{\varepsilon 2} - w_{\varepsilon 2}))^2\, d\sigma = \int_Q f(\zeta(u_\varepsilon) - \zeta(w_\varepsilon))(u_\varepsilon - w_\varepsilon)\, dx \leq 0,$$

where in the last inequality we have used the fact that the function $\zeta(s)$ is non increasing. By (2.5) and getting rid of the boundary term which is nonnegative, we get $u_\varepsilon = w_\varepsilon$ a.e. in Q.
□

8. Proof of Theorem 4.6 (homogenization). The main tool when proving Theorem 4.6 is Theorem 8.5, which shows that the gradient of the solution of problem (4.1) is equivalent (in the L^2-norm), as $\varepsilon \to 0$, to that of a suitable linear problem, given by (8.26). We present it in Section 7.2, after recalling some homogenization results for the linear problem in Section 7.1. Finally in Section 7.3 we prove Theorem 4.6.

8.1. Preliminaries. Let us introduce, for a given matrix field B in $L^\infty(Q)^{n^2}$ and for every ε, the map

$$\tau_B^\varepsilon : z \in W_0^0 \to \tau_B^\varepsilon(z) \in (W_0^\varepsilon)' \tag{8.1}$$

defined by

$$< \tau_B^\varepsilon(z), \varphi >_{W_0^\varepsilon,(W_0^\varepsilon)'} = \int_{Q_{\varepsilon,0}} B\nabla z \nabla\varphi\, dx, \tag{8.2}$$

where $Q_{\varepsilon,0}$ is given in (2.19).
In this section, using the notations of Section 2, we recall some homogenization results from [22], for the following linear problem:

$$\begin{cases} -\mathrm{div}(A^\varepsilon\nabla v_\varepsilon) = g - \tau_B^\varepsilon(z) & \text{in } Q_\varepsilon, \\ (A^\varepsilon\nabla v_\varepsilon)_2 \cdot n_\varepsilon = (A^\varepsilon\nabla v_\varepsilon)_1 \cdot n_\varepsilon & \text{on } \Gamma_\varepsilon, \\ (A^\varepsilon\nabla v_\varepsilon)_1 \cdot n_\varepsilon = -\varepsilon^\gamma h^\varepsilon(v_{\varepsilon 1} - v_{\varepsilon 2}), & \text{on } \Gamma_\varepsilon. \\ v_\varepsilon = 0 & \text{on } \partial Q, \end{cases} \tag{8.3}$$

whose variational formulation is

$$(8.4)\qquad \begin{cases} \text{Find } u_\varepsilon \in W_0^\varepsilon \text{ such that} \\ \displaystyle\int_{Q\setminus\Gamma_\varepsilon} A^\varepsilon \nabla v_\varepsilon \nabla\varphi\, dx + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon (v_{\varepsilon 1} - v_{\varepsilon 2})(\varphi_1 - \varphi_2)\, d\sigma \\ \displaystyle = \int_Q g\,\varphi\, dx + \int_{Q\setminus\Gamma_{\varepsilon,0}} B\nabla z \nabla\varphi\, dx, \quad \text{for every } \varphi \in W_0^\varepsilon. \end{cases}$$

where

$$(8.5)\qquad g \in L^2(Q), \quad z \in W_0^0, \quad B \text{ is a given matrix field in } L^\infty(Q)^{n^2}$$

and $\tau_B^\varepsilon(z)$ is defined by (8.1)-(8.2).
The matrix field A^ε and the function h^ε are given by (2.5)-(2.8).

THEOREM 8.1. *[22] Under assumptions (2.5)-(2.8) and (8.5) let v^ε be the solution of problem (8.3) and A^0 be given by (4.3)-(4.4). For every $\kappa > 0$ and $\gamma \in \mathbb{R}$ there exists a function $v_0 \in W_0^0$ such that the following convergences hold true:*

$$(8.6)\qquad \begin{cases} i)\ v_\varepsilon \to v_0, & \text{strongly in } L^2(Q), \\ ii)\ \chi_{Q_{\varepsilon i}} \nabla v_\varepsilon \rightharpoonup \chi_{Q_i}\nabla v_0, & \text{weakly in } (L^2(Q))^N, \end{cases}$$

and

$$(8.7)\qquad \chi_{Q_{\varepsilon i}} A^\varepsilon \nabla v_\varepsilon \rightharpoonup \chi_{Q_i} A^0\ \nabla v_0, \qquad \text{weakly in } (L^2(Q))^N,$$

for i=1,2. Moreover, denoting $v_{0i} = v_{0|_{Q_i}}$ for $i = 1, 2$, we have the limit problems below.
• Suppose that (4.8) or (4.9) holds. Then, the function v_0 is is the unique solution of the problem

$$(8.8)\qquad \begin{cases} -\mathrm{div}(A^0\nabla v_0) = g - \tau_B^0(z) & \text{in } Q_0, \\ (A^0\nabla v_0)_2 \cdot n = (A^0\nabla v_0)_1 \cdot n & \text{on } \Gamma_0, \\ (A^0\nabla v_0)_1 \cdot n = H(g,h)(v_{01} - v_{02}), & \text{on } \Gamma_0, \\ u = 0 & \text{on } \partial Q, \end{cases}$$

where $H(g,h)$ is given by (4.11) and $\tau_B^0 : W_0^0 \to (W_0^0)'$ is defined by

$$(8.9)\qquad < \tau_0(z), \varphi >_{W_0^0,(W_0^0)'} = -\int_{Q_0} B\nabla z \nabla\varphi\, dx.$$

• Suppose now that (4.13) or (4.14) holds. Then, the function v_0 belongs to $H_0^1(Q)$ and is the unique solution of the problem

$$(8.10)\qquad \begin{cases} -\mathrm{div}(A^0\nabla v_0) = g - \tau_B^0(z) & \text{in } Q, \\ u = 0 & \text{on } \partial Q. \end{cases}$$

• *Finally, suppose that* (4.17) *or* (4.18) *holds. Then,* v_{01} *and* v_{02} *are the unique solution of the following two (independent) Neumann problems:*

$$(8.11)\qquad \begin{cases} -\operatorname{div}(A^0\nabla v_{01}) = g - \operatorname{div}(B\nabla z) & \text{in } Q_1, \\ A^0\nabla v_{01}\cdot n = 0 & \text{on } \Gamma_0, \\ v_0 = 0 & \text{on } \partial Q_1 \setminus \Gamma_0, \end{cases}$$

and

$$(8.12)\qquad \begin{cases} -\operatorname{div}(A^0\nabla v_{02}) = g - \operatorname{div}(B\nabla z) & \text{in } Q_2, \\ A^0\nabla v_{02}\cdot n = 0 & \text{on } \Gamma_0, \\ v_0 = 0 & \text{on } \partial Q_2 \setminus \Gamma_0. \end{cases}$$

Remark 8.2. *The homogenization results proved in [22] deal with the case* $z = 0$*. It is easy to check that the proofs can be adapted without any significative modification when* $z \neq 0$*. Indeed, the test function used for passing to the limit in [22] is a function* φ *in* W_0^ε *such that* φ_1 *and* φ_2 *are restrictions of functions in* $H_0^1(Q)$*. Then for the additional term one has, as* $\varepsilon \to 0$*,*

$$\int_{Q_{\varepsilon,0}} B\nabla z\nabla\varphi\,dx = \int_{Q_{\varepsilon 1}} B\nabla z\nabla\varphi_1\,dx + \int_{Q_{\varepsilon 2}} B\nabla z\nabla\varphi_2\,dx \to \int_{Q_0} B\nabla z\nabla\varphi\,dx.$$

Observe also that if z *is in* $H_0^1(Q)$*, then the equation in* (8.10) *reads*

$$-\operatorname{div}(A^0\nabla v_0) = g - \operatorname{div}(B\nabla z).$$

The main difficulty when proving Theorem 8.1 in [22] concerns the way to pass to the limit in the boundary terms. We adapt the arguments used therein for the case where only one sequence depends on ε to show the proposition below, which deals with the case of two sequences depending on ε.

Proposition 8.3. *Let* $\{w_\varepsilon\}$ *be a sequence such that* $w_\varepsilon \in W_0^\varepsilon$ *for every* ε *and*

$$(8.13)\qquad \|w_\varepsilon\|_{W_0^\varepsilon} \le c, \quad \|w_{\varepsilon 1} - w_{\varepsilon 2}\|_{L^2(\Gamma_\varepsilon)} \le c\,\varepsilon^{-\frac{\gamma}{2}},$$

where c *is a constant independent on* ε*. Suppose that for some* $w \in W_0^0$ *one has*

$$(8.14)\qquad \begin{cases} i)\;\; w_\varepsilon \to w, & \textit{strongly in } L^2(Q), \\ ii)\;\; \chi_{Q_{\varepsilon i}}\nabla w_\varepsilon \rightharpoonup \chi_{Q_i}\nabla w, & \textit{weakly in } (L_2(Q))^N. \end{cases}$$

• *If* (4.13) *or* (4.14) *holds, then*

$$(8.15)\qquad w \textit{ belong to } H_0^1(Q).$$

Suppose now that $\{\psi_\varepsilon\}$ *is another sequence verifying the same estimates* (8.13) *such that for some* $\psi \in W_0^0$

$$(8.16)\qquad \begin{cases} i)\;\; \psi_\varepsilon \to \psi, & \textit{strongly in } L_2(Q), \\ ii)\;\; \chi_{Q_{\varepsilon i}}\nabla \psi_\varepsilon \rightharpoonup \chi_{Q_i}\nabla \psi, & \textit{weakly in } (L_2(Q))^N. \end{cases}$$

- *If* (4.8) *or* (4.9) *holds, under notation* (4.11),

$$\lim_{\varepsilon\to 0}\varepsilon^\gamma\int_{\Gamma_\varepsilon} h^\varepsilon(w_{\varepsilon 1}-w_{\varepsilon 2})(\psi_{\varepsilon 1}-\psi_{\varepsilon 2})d\sigma = H(g,h)\int_{\Gamma_0}(w_1-w_2)(\psi_1-\psi_2)d\sigma. \tag{8.17}$$

- *If* (4.17) *or* (4.18) *holds,*

$$\lim_{\varepsilon\to 0}\varepsilon^\gamma\int_{\Gamma_\varepsilon} h^\varepsilon(w_{\varepsilon 1}-w_{\varepsilon 2})(\psi_{\varepsilon 1}-\psi_{\varepsilon 2})d\sigma = 0. \tag{8.18}$$

Proof. We only explain how to derive the result from the argument introduced in [22], where one of the two sequence was fixed, that is independent of ε.
Suppose first that $\kappa \geq 1$. From Corollary 2.7 of [22] in (2.2) there exist two functions W_1 and W_2 in $H^1(Q)$ such that

$$P_{\varepsilon i}(w_{\varepsilon i}) \rightharpoonup W_i, \qquad \text{weakly in } H^1(Q), \quad i=1,2, \tag{8.19}$$

with

$$W_{1|Q_1} = w_1, \qquad W_{2|Q_2} = w_2. \tag{8.20}$$

Let us point out that in [22] convergence (8.19) is stated for a subsequence, but it actually holds for the whole sequence when (8.14) is supposed. Indeed, as usual in the literature, the extension operators in Proposition 2.2 can be chosen such that

$$\|P_{\varepsilon i} v\|_{L^2(Q)} \leq c\|v\|_{L^2(Q_{\varepsilon i})}, \qquad \text{for every } v \in H^1(Q_{\varepsilon i}), \quad i=1,2,$$

where c is independent of ε. Then, since $\{w_\varepsilon\}$ is a Cauchy sequence in $L^2(Q)$, the sequence $\{P_{\varepsilon i}(w_{\varepsilon i})\}$ is also a Cauchy sequence in $L^2(Q)$ for $i=1,2$. The same holds obviously for the sequence $\{\psi_\varepsilon\}$.
Then, we argue for the whole sequences $\{w_\varepsilon\}$ and $\{\psi_\varepsilon\}$ as in the proof of Theorems 4.1 and 5.1 of [22], observing that Lemma 3.2 used therein can be applied here to both sequences. We have

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} h^\varepsilon(w_{\varepsilon 1}-w_{\varepsilon 2})(\psi_{\varepsilon 1}-\psi_{\varepsilon 2})d\sigma = H(g,h)\int_{\Gamma_0}(w_1-w_2)(\psi_1-\psi_2)d\sigma, \tag{8.21}$$

which gives (8.18) and (8.17).
To prove (8.15), as in [22] it suffices to choose $\psi_\varepsilon = w_\varepsilon$ in (8.21). Indeed, since we are in the case $\gamma < 0$, the boundary a priori estimate in (8.13) implies that $\|w_{\varepsilon 1}-w_{\varepsilon 2}\|_{L^2(\Gamma_\varepsilon)} \to 0$; this, together with assumption (2.6), shows that the limit in the left-hand side of (8.21) is zero. Then $w_1 = w_2$ on Γ_0, which means that w belongs to $H^1_0(Q)$.
Finally, when $0 < \kappa < 1$, the result follows by the same arguments used in the proof of Theorem 6.1 of [22], observing again that the computation used therein for the sequence $\{u_\varepsilon\}$ can be applied here to both sequences. ∎

8.2. A main tool. In this section we state and prove Theorem 8.5, which plays an essential role in the proof of the homogenization result. Let us point out that one difficulty in order to prove the homogenization result stated in Theorem 4.6 is that in the variational formulation (4.1) the test functions belong to a space depending on ε and have a jump on Γ_ε, while in the limit problem we need functions in W^0_0.
To overcome this difficulty, along this paper we construct test functions as follow.

Let $\varphi \in W_0^0$. Then, there exist ψ_1 and $\psi_2 \in H_0^1(Q)$ such that

$$(\varphi_1, \varphi_2) = (\psi_{1|Q_1}, \psi_{2|Q_2}). \tag{8.22}$$

Observe that if φ is nonnegative, then ψ_1 and ψ_2 can be chosen nonnegative too. Then for every ε, we associate to φ the function $\varphi_\varepsilon \in W_\varepsilon^0$ defined by

$$\varphi_\varepsilon = (\psi_{1|Q_{\varepsilon 1}}, \psi_{2|Q_{\varepsilon 2}}) \in W_\varepsilon^0. \tag{8.23}$$

Observe that by construction and using (2.17), we have

$$\begin{cases} i) \ \ \varphi_\varepsilon \to \varphi, \qquad \text{strongly in } L^2(Q), \\ ii) \ \ \chi_{Q_{\varepsilon i}} \nabla \varphi_\varepsilon = \chi_{Q_{\varepsilon i}} \nabla \psi_i \to \chi_{Q_i} \nabla \psi_i, \qquad \text{weakly in } (L_2(Q))^N, \ i = 1, 2. \end{cases} \tag{8.24}$$

We have the following lemma:

Lemma 8.4. *Under the assumptions of Theorem 4.1 there exists a nonnegative function $u_0 \in W_0^\varepsilon$ and a subsequence (still denoted $\{\varepsilon\}$) such that convergences (4.6) hold. Also,*

$$\int_Q f\zeta(u_0)\varphi \, dx < +\infty, \qquad \textit{for every } \varphi \in W_0^0. \tag{8.25}$$

Moreover, if $\gamma < 0$, then u_0 belongs to $H_0^1(Q)$.

Proof. The convergences (for a subsequence) follow from the a priori estimates given in Section 5 applied to the sequence $\{u_\varepsilon\}$ of the solutions of (2.10), thanks to the compactness results given in [22] (Proposition 2.4).
Concerning (8.25), let φ be a nonnegative function in W_0^0 and φ_ε given by (8.23). Then, by Proposition 5.3,

$$\int_Q f\zeta(u_\varepsilon)\varphi_\varepsilon \, dx = \int_Q \chi_{Q_{\varepsilon 1}} f\zeta(u_{\varepsilon 1})\psi_1 \, dx + \int_Q \chi_{Q_{\varepsilon 2}} f\zeta(u_{\varepsilon 2})\psi_2 \, dx \le c$$

and from convergences (4.6) (2.3)-(2.4),

$$\chi_{Q_{\varepsilon i}} f\zeta(u_{\varepsilon i}) \to \chi_{Q_i} f\zeta(u_{0i}), \qquad \text{a.e in } Q, \quad i = 1, 2.$$

Then, the Fatou's Lemma gives (8.25) for nonnegative φ. This implies that $f\zeta(u_0)$ is finite almost everywhere. Then, if φ has any sign, it suffices to decompose it as $\varphi = \varphi^+ - \varphi^-$.
The last statement follows from Proposition 8.3 applied to the previous subsequence. ∎

From now on, we deal with the function u_0 and the subsequence given by Lemma 8.4. Let us introduce the solution v_ε of the linear problem

$$\begin{cases} -\text{div}(A^\varepsilon \nabla v_\varepsilon) = -\tau^\varepsilon_{A^0}(u_0) \qquad \text{in } Q_\varepsilon, \\ (A^\varepsilon \nabla v_\varepsilon)_1 \cdot n_\varepsilon = (A^\varepsilon \nabla v_\varepsilon)_2 \cdot n_\varepsilon \qquad \text{on } \Gamma_\varepsilon, \\ (A^\varepsilon \nabla v_\varepsilon)_1 \cdot n_\varepsilon = -\varepsilon^\gamma h^\varepsilon (v_{\varepsilon 1} - v_{\varepsilon 2}), \qquad \text{on } \Gamma_\varepsilon. \\ v_\varepsilon = 0 \qquad \text{on } \partial Q. \end{cases} \tag{8.26}$$

where $\tau^\varepsilon_{A^0}(u_0)$ is given by (8.2) (written for $B = A^0$ and $z = u_0$). Observe that from convergences (4.6) and Theorem 8.1 (with $g \equiv 0$), thanks to the uniqueness of the solution of the linear problems (8.8),(8.10) (8.11) and (8.12) it follows that

$$(8.27) \qquad \begin{cases} i) \;\; v_\varepsilon \to u_0, & \text{strongly in } L^2(Q), \\ ii) \;\; \chi_{Q_{\varepsilon i}} \nabla v_\varepsilon \rightharpoonup \chi_{Q_i} \nabla u_0, & \text{weakly in } (L^2(Q))^N, \\ iii) \;\; \chi_{Q_{\varepsilon i}} A^\varepsilon \nabla v^\varepsilon \rightharpoonup \chi_{Q_i} A^0 \; \nabla u_0, & \text{weakly in } (L^2(Q))^N, \end{cases}$$

for $i = 1, 2$.

Then, the main tool for proving Theorem 4.6, is the following result:

THEOREM 8.5. *Let u_ε and v_ε be solutions of problems (4.1) and (8.26), respectively. Under the assumption of Theorem 4.6 one has (for the subsequence given by Lemma 8.4)*

$$(8.28) \qquad \lim_{\varepsilon \to 0} \int_{Q_\varepsilon} |\nabla(u_\varepsilon - v_\varepsilon)|^2 \, dx = 0.$$

Proof. We need to distinguish the two cases $\kappa \geq 1$ and $\kappa < 1$.

Case 1 : $\kappa \geq 1$ and f satisfying (2.12).

Since the functions v_ε are not necessarily bounded, we approximate the nonnegative function u_0 by the sequence $\{u_m\}$ given by

$$u_m = T_m(u_0), \qquad \text{for every } m \in N, \; m \geq 1,$$

where T_m is the truncation function given by (5.8), so that

$$(8.29) \qquad 0 \leq u_m \leq u_0, \quad u_m \to u_0 \quad \text{strongly in } W^0_0 \quad \text{as } m \to +\infty.$$

Then, we define v^m_ε as the solution to

$$(8.30) \qquad \begin{cases} -\text{div}(A^\varepsilon \nabla v^m_\varepsilon) = -\tau^\varepsilon_{A^0}(u_m) & \text{in } Q_\varepsilon, \\ (A^\varepsilon \nabla v^m_\varepsilon)_1 \cdot n_\varepsilon = (A^\varepsilon \nabla v^m_\varepsilon)_2 \cdot n_\varepsilon & \text{on } \Gamma_\varepsilon, \\ (A^\varepsilon \nabla v^m_\varepsilon)_1 \cdot n_\varepsilon = -\varepsilon^\gamma h^\varepsilon (v^m_{\varepsilon 1} - v^m_{\varepsilon 2}), & \text{on } \Gamma_\varepsilon. \\ v^m_\varepsilon = 0 & \text{on } \partial Q. \end{cases}$$

Since we are assuming $\kappa \geq 1$ (this is the only point where we use this hypothesis), the uniform Sobolev-Poincaré inequality given by (2.21) holds. Then, since $u_m \in L^\infty(Q)$, by classical results from [48] (see also Proposition 4.3) for every m there exists a constant c_m such that

$$(8.31) \qquad \|v^m_\varepsilon\|_{L^\infty(Q)} \leq c_m, \qquad \text{for every } \varepsilon$$

and by Theorem 8.1,

$$(8.32) \qquad \begin{cases} i) \;\; v^m_\varepsilon \to u_m, & \text{strongly in } L^2(Q), \\ ii) \chi_{Q_{\varepsilon i}} \nabla v^m_\varepsilon \rightharpoonup \chi_{Q_i} \nabla u_m, & \text{weakly in } (L_2(Q))^N, \\ iii) \;\; \chi_{Q_{\varepsilon i}} A^\varepsilon \nabla v^m_\varepsilon \rightharpoonup \chi_{Q_i} A^0 \; \nabla u_m, & \text{weakly in } (L^2(Q))^N. \end{cases}$$

for $i = 1, 2$ as $\varepsilon \to 0$,

In this case ($\kappa \geq 1$) we prove the statement in three steps.

Step 1. Let us first prove that

$$\lim_{\varepsilon\to 0}\int_Q ((v_\varepsilon^m)^-)^2\,dx \leq \lim_{\varepsilon\to 0} c\int_{Q_\varepsilon} |\nabla(v_\varepsilon^m)^-|^2\,dx = 0 \quad \text{for any } m. \tag{8.33}$$

Choosing $-(v_\varepsilon^m)^- \in W_0^\varepsilon$ as test function in the variational formulation of (8.30) and using Remark 5.1 we obtain in view of (2.18),

$$\begin{aligned}
&\alpha\int_{Q_\varepsilon} |\nabla(v_\varepsilon^m)^-|^2 dx \\
&\leq -\int_{Q_\varepsilon} A^\varepsilon \nabla v_\varepsilon^m \nabla(v_\varepsilon^m)^- dx - \varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon (v_{\varepsilon 1}^m - v_{\varepsilon 2}^m)\big((v_{\varepsilon 1}^m)^- - (v_{\varepsilon 2}^m)^-\big) d\sigma \\
&= -\int_{Q_{\varepsilon,0}} A^0 \nabla u_m \nabla(v_\varepsilon^m)^-\,dx = -\int_{Q_{\varepsilon,0}} A^0\nabla u_m \nabla v_\varepsilon^m\ \chi_{\{v_\varepsilon^m\leq 0\}}\,dx \\
&\leq \frac{\beta^2}{2\alpha}\int_{Q_0} |\nabla u_m|^2 \chi_{\{v_\varepsilon^m\leq 0\}}\,dx + \frac{\alpha}{2}\int_{Q_\varepsilon} |\nabla(v_\varepsilon^m)^-|^2\,dx.
\end{aligned}$$

Using (8.32)(i) and the fact that u_m is nonnegative it results, up to a subsequence,

$$\chi_{\{v_\varepsilon^m \leq 0\}}\,\chi_{\{u_m\neq 0\}} \to \chi_{\{u_m<0\}} = 0 \qquad \text{a.e. in } Q, \quad \text{as } \varepsilon\to 0. \tag{8.34}$$

Moreover, $\nabla u_m = 0$ in the set where $u_m = 0$. Therefore

$$\int_{Q_0} |\nabla u_m|^2 \chi_{\{v_\varepsilon^m\leq 0\}}\,dx \to 0, \qquad \text{for every } m, \quad \text{as } \varepsilon \to 0,$$

which using (2.20) concludes the step.

Step 2. Let us prove that

$$\lim_{m\to\infty}\lim_{\varepsilon\to 0}\int_{Q_\varepsilon} |\nabla(u_\varepsilon - v_\varepsilon^m)|^2\,dx = 0. \tag{8.35}$$

We choose as test function in (4.1) and in the variational formulation of (8.30) the function

$$\Phi = u_\varepsilon - v_\varepsilon^m \in W_0^\varepsilon.$$

This gives, after subtraction of the two identities

$$\begin{aligned}
&\int_{Q_\varepsilon} A^\varepsilon \nabla(u_\varepsilon - v_\varepsilon^m)\nabla(u_\varepsilon - v_\varepsilon^m)\,dx \leq \int_{Q_\varepsilon} A^\varepsilon \nabla(u_\varepsilon - v_\varepsilon^m)\nabla(u_\varepsilon - v_\varepsilon^m)\,dx \\
&+\varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon (u_{\varepsilon 1} - u_{\varepsilon 2} - v_{\varepsilon 1}^m + v_{\varepsilon 2}^m)^2 d\sigma \\
&= \int_Q f\,\zeta(u_\varepsilon)(u_\varepsilon - v_\varepsilon^m)\,dx - \int_{Q_{\varepsilon,0}} A^0\nabla u_m\ \nabla(u_\varepsilon - v_\varepsilon^m)\,dx.
\end{aligned} \tag{8.36}$$

We take for the moment m fixed and pass to the limit on ε. From (4.6) and (8.32) we have

$$\lim_{\varepsilon\to 0}\int_{Q_{\varepsilon,0}} A^0\nabla u_m\ \nabla(u_\varepsilon - v_\varepsilon^m)\,dx = \int_{Q_0} A^0\nabla u_m\ \nabla(u_0 - u_m)\,dx. \tag{8.37}$$

Now, in order to pass to the limit in the term containing the singularity, we split it in two terms as below

$$\int_Q f\,\zeta(u_\varepsilon)(u_\varepsilon - v_\varepsilon^m)\,dx = \int_Q f\,\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+)\,dx + \int_Q f\,\zeta(u_\varepsilon)(v_\varepsilon^m)^-\,dx. \tag{8.38}$$

We will prove that

$$\lim_{\varepsilon\to 0}\int_Q f\,\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+)\,dx = \int_Q f\,\zeta(u_0)(u_0 - u^m)\chi_{\{u_0>0\}}\,dx \tag{8.39}$$

and

$$\lim_{\varepsilon\to 0}\int_Q f\,\zeta(u_\varepsilon)(v_\varepsilon^m)^-\,dx = 0. \tag{8.40}$$

We begin by proving (8.39). For any $\delta > 0$ we have

$$\begin{aligned}
&\int_Q f\,\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+)\,dx = \int_{\{\delta<u_\varepsilon\}} f\,\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+)\,dx\\
&+\int_{\{0<u_\varepsilon\le\delta\}} f\,\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+)\,dx \le \int_{\{\delta<u_\varepsilon\}} f\,\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+)\,dx\\
&+\int_{\{0<u_\varepsilon\le\delta\}} f\,\zeta(u_\varepsilon)u_\varepsilon\,dx \doteq J_\varepsilon^\delta + I_\varepsilon^\delta.
\end{aligned} \tag{8.41}$$

On the other hand, treating the term J_ε^δ as in (6.10), we can write

$$\begin{aligned}
J_\varepsilon^\delta &= \int_Q f\,\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+)\,\chi_{\{u_\varepsilon>\delta\}}\,\chi_{\{u_0\neq\delta\}}\,dx\\
&+\int_Q f\,\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+)\,\chi_{\{u_\varepsilon>\delta\}}\,\chi_{\{u_0=\delta\}}\,dx,
\end{aligned} \tag{8.42}$$

where (see the proof of Theorem 4.1)

$$\int_Q f\,\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+)\,\chi_{\{u_\varepsilon>\delta\}}\,\chi_{\{u_0=\delta\}}\,dx = 0, \tag{8.43}$$

except at most for a countable set of values of δ.

Concerning the first term, we have

$$\begin{aligned}
&|f\,\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+)\,\chi_{\{u_\varepsilon>\delta\}}\,\chi_{\{u_0\neq\delta\}}| \le f\,\zeta(u_\varepsilon)u_\varepsilon + f\,\zeta(u_\varepsilon)(v_\varepsilon^m)^+\chi_{\{u_\varepsilon>\delta\}}\\
&\le f\,\zeta(u_\varepsilon)u_\varepsilon + c_m\tfrac{1}{\delta^\theta} f\,,
\end{aligned} \tag{8.44}$$

where c_m is defined in (8.31) when $\kappa \ge 1$. This implies, using (2.11), (2.12), (5.3) and the Hölder inequality that

$$\int_E |f\,\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+)\,\chi_{\{u_\varepsilon>\delta\}}\,\chi_{\{u_0\neq\delta\}}| \le c\|f\|_{L^{\frac{2}{1+\theta}}(E)} + c_m\frac{1}{\delta^\theta}\|f\|_{L^1(E)}, \tag{8.45}$$

for any measurable set E in Q. Moreover from (2.11) and (4.6)

$$\lim_{\varepsilon\to 0} f\,\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+)\,\chi_{\{u_\varepsilon>\delta\}}\,\chi_{\{u_0\neq\delta\}} = f\,\zeta(u_0)(u_0 - u_m)\,\chi_{\{u_0>\delta\}} \quad \text{a.e. in } Q.$$

By the Vitali Theorem we obtain

$$\begin{aligned}(8.46)\qquad &\lim_{\varepsilon\to 0}\int_Q f\,\zeta(u_\varepsilon)(u_\varepsilon-(v_\varepsilon^m)^+)\,\chi_{\{u_\varepsilon>\delta\}}\,\chi_{\{u_0\neq\delta\}}\,dx\\ &=\int_Q f\,\zeta(u_0)(u_0-u_m)\,\chi_{\{u_0>\delta\}}\,dx.\end{aligned}$$

Note that this is the point where we need the bounded (with respect to ε) sequence v_ε^m defined by (8.30).

We can apply the Lebesgue dominated convergence theorem on the last integral of (8.46) as $\delta\to 0$ since, by Lemma 8.4, $f\,\zeta(u_0)(u_0-u_m)\in L^1(Q)$ getting

$$\begin{aligned}(8.47)\qquad &\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_Q f\,\zeta(u_\varepsilon)(u_\varepsilon-(v_\varepsilon^m)^+)\,\chi_{\{u_\varepsilon>\delta\}}\,\chi_{\{u_0\not\delta\}}\,dx\\ &=\int_Q f\,\zeta(u_0)(u_0-u_m)\,\chi_{\{u_0>0\}}\,dx.\end{aligned}$$

By (8.42), (8.43) and (8.47) we get

$$(8.48)\qquad \lim_{\delta\to 0}\lim_{\varepsilon\to 0}J_\varepsilon^\delta=\int_Q f\,\zeta(u_0)(u_0-u_m)\,\chi_{\{u_0>0\}}\,dx.$$

We estimate now the term I_ε^δ in (8.41). Observe that if $\theta<1$ from (2.11) we have

$$(8.49)\qquad I_\varepsilon^\delta\le\,\delta^{1-\theta}\int_{\{0<u_\varepsilon\le\delta\}}f\,dx\le c\,\delta^{1-\theta},$$

which gives

$$(8.50)\qquad \lim_{\delta\to 0}\lim_{\varepsilon\to 0}I_\varepsilon^\delta=0,$$

while if $\theta=1$,

$$(8.51)\qquad I_\varepsilon^\delta\le\int_{\{0<u_\varepsilon\le\delta\}}f\,dx=\int_Q f\chi_{\{0<u_\varepsilon\le\delta\}}\,\chi_{\{u_0\neq\delta\}}\,dx+\int_Q f\chi_{\{0<u_\varepsilon\le\delta\}}\,\chi_{\{u_0=\delta\}}\,dx.$$

Arguing as in the proof of Theorem 4.1, we deduce that except at most for a countable set of values of δ the second integral in the right-hand side of (8.51) is zero.

Hence, using (4.6), we have again (8.50) since

$$(8.52)\qquad \lim_{\delta\to 0}\lim_{\varepsilon\to 0}I_\varepsilon^\delta\le\int_Q f\chi_{\{u_0=0\}}\,dx=0,$$

as a consequence of (8.25) and the fact that the function $\zeta(s)$ is singular at $s=0$, which implies that

$$(8.53)\qquad \operatorname{meas}\{x\in Q\,|\,u_0=0\text{ and }f>0\}=0.$$

Hence, collecting (8.41), (8.48), (8.50) and (8.52) we get (8.39).

We are going to prove now (8.40). Let us choose δ_0 outside a convenient countable set so that

$$\int_{\{u_\varepsilon>\delta_0\}}f\,\zeta(u_\varepsilon)(v_\varepsilon^m)^-\chi_{\{u_0=\delta_0\}}\,dx=0$$

and split the integral in (8.40) as

$$\begin{aligned}&\int_Q f\,\zeta(u_\varepsilon)(v_\varepsilon^m)^-\,dx\\ &=\int_{\{u_\varepsilon\le\delta_0\}} f\,\zeta(u_\varepsilon)(v_\varepsilon^m)^-\,dx+\int_{\{u_\varepsilon>\delta_0\}} f\,\zeta(u_\varepsilon)(v_\varepsilon^m)^-\chi_{\{u_0\neq\delta_0\}}\,dx\\ &=A_\varepsilon+B_\varepsilon.\end{aligned}\tag{8.54}$$

By Proposition 5.4 (written for $\delta=\delta_0$) we have

$$\begin{aligned}0\le A_\varepsilon\le&\int_{Q_\varepsilon} A^\varepsilon\nabla u_\varepsilon\nabla(v_\varepsilon^m)^-\,Z_{\delta_0}(u_\varepsilon)\,dx\\ &+\varepsilon^\gamma\int_{\Gamma_\varepsilon} h^\varepsilon(u_{\varepsilon1}-u_{\varepsilon2})(Z_{\delta_0}(u_{\varepsilon1})(v_\varepsilon^m)_1^- - Z_{\delta_0}(u_{\varepsilon2})(v_\varepsilon^m)_2^-\,d\sigma\end{aligned}\tag{8.55}$$

We want to prove that

$$\lim_{\varepsilon\to0}A_\varepsilon=0.\tag{8.56}$$

As far as the first term in the right-hand side of (8.55) is concerned we use the Hölder inequality, estimates (5.3) and condition (8.33), so that it goes to zero as ε goes to zero.
Observe now that, for m fixed, thanks to (8.31) and the definition of Z_δ (see (5.7)), we can apply Proposition 8.3 to $w_\varepsilon=u_\varepsilon$ and $\psi_\varepsilon=Z_{\delta_0}(u_\varepsilon)(v_\varepsilon^m)^-$, for any $\gamma\in\mathbb{R}$.
Then, if $\gamma\ge0$, also the second term in the right-hand side goes to zero, since ψ_ε converges to $\psi=0$ strongly in $L^2(Q)$ by (8.33).
If $\gamma<0$ then u_0 belongs to $H^1_0(Q)$, so that the same holds true for the sequences u^m and v_ε^m defined by (8.29) and(8.30). This implies that also $(v_\varepsilon^m)^-$ belongs to $H^1_0(Q)$ so that $(v_\varepsilon^m)_1^-=(v_\varepsilon^m)_2^-$ and, since the function Z_{δ_0} is non increasing (see (5.7))

$$\begin{aligned}&\varepsilon^\gamma\int_{\Gamma_\varepsilon} h^\varepsilon(u_{\varepsilon1}-u_{\varepsilon2})(Z_{\delta_0}(u_{\varepsilon1})(v_\varepsilon^m)_1^- - Z_{\delta_0}(u_{\varepsilon2})(v_\varepsilon^m)_2^-)\,d\sigma\\ &=\varepsilon^\gamma\int_{\Gamma_\varepsilon} h^\varepsilon(u_{\varepsilon1}-u_{\varepsilon2})(Z_{\delta_0}(u_{\varepsilon1})-Z_{\delta_0}(u_{\varepsilon2})(v_\varepsilon^m)_1^-\,d\sigma\le0\end{aligned}$$

Therefore, for any value of γ (8.56) holds true.
We prove now that

$$\lim_{\varepsilon\to0}B_\varepsilon=0.\tag{8.57}$$

It is sufficient to observe that

$$0\le f\,\zeta(u_\varepsilon)(v_\varepsilon^m)^-\chi_{\{u_\varepsilon>\delta_0\}}\chi_{\{u_0\neq\delta_0\}}\le f\frac{c_m}{\delta_0^\theta}\in L^1(Q)$$

(where c_m is defined in (8.31)) and that, by (8.33),

$$f\,\zeta(u_\varepsilon)(v_\varepsilon^m)^-\chi_{\{u_\varepsilon>\delta_0\}}\chi_{\{u_0\neq\delta_0\}}\to0\text{ a.e. in }Q$$

This implies (8.57) by Lebesgue Theorem. Collecting (8.54), (8.56) and (8.57), we get (8.40) and therefore recalling (8.36)-(8.40),

$$
\begin{aligned}
&\alpha \limsup_{\varepsilon\to 0} \int_{Q_\varepsilon} |\nabla(u_\varepsilon - v_\varepsilon^m)|^2 \ dx \\
&\le \limsup_{\varepsilon\to 0} \int_{Q_\varepsilon} A^\varepsilon \nabla(u_\varepsilon - v_\varepsilon^m))\nabla(u_\varepsilon - v_\varepsilon^m)\, dx \qquad (8.58)\\
&\le - \int_{Q_0} A^0 \nabla u_m \, \nabla(u_0 - u_m)\, dx + \int_Q f\, \zeta(u_0)(u_0 - u_m)\, \chi_{\{u_0>0\}}\, dx.
\end{aligned}
$$

The first term of the right-hand side goes to zero as $m \to \infty$ since $u_m \to u_0$ (see (8.29)). For the same reason

$$
f\, \zeta(u_0)(u_0 - u_m)\, \chi_{\{u_0>0\}} \to 0 \ \text{ a.e. in } \ Q.
$$

Since, by Lemma 8.4

$$
0 \le f\, \zeta(u_0)(u_0 - u_m)\, \chi_{\{u_0>0\}} \le f\, \zeta(u_0) u_0 \in L^1(Q),
$$

the second term of the right-hand side of (8.58) also goes to zero as $m \to \infty$ by Lebesgue Theorem and this proves (8.35).

Step 3. In this step we prove that

$$
\lim_{m\to\infty} \lim_{\varepsilon\to 0} \int_{Q_\varepsilon} |\nabla(v_\varepsilon^m - v_\varepsilon)|^2 dx = 0,
$$

which concludes the proof in the case $\kappa \ge 1$, due to the previous step.
To this aim, we choose as test function in (8.26) and (8.30) the function $v_\varepsilon^m - v_\varepsilon$. This gives, after subtraction of the two identities and observing that the boundary term is nonnegative,

$$
\int_{Q_\varepsilon} A^\varepsilon \nabla(v_\varepsilon^m - v_\varepsilon)\, \nabla(v_\varepsilon^m - v_\varepsilon)\, dx \le \int_{Q_{\varepsilon,0}} A^0 \nabla(u_m - u_0)\, \nabla(v_\varepsilon^m - v_\varepsilon)\, dx,
$$

whose right-hand side goes to zero when passing to the limit first as $\varepsilon \to 0$ and then as $m \to \infty$, by convergences (8.32), (8.27) and (8.29). The ellipticity condition (2.5) allow to conclude this case.

Case 2 : $\kappa < 1$ and $f \in L^2(Q)$.

Note that in this case it is useless to introduce the sequence v_ε^m defined by (8.30) since it does not satisfies estimate (8.31) (see Remark 2.3 and the proof of Theorem 4.3). We recall that estimate (8.31) has been used in (8.44) and in (8.45). Here, since $f \in L^2(Q)$, we can simply use the sequence v_ε instead of the sequence v_ε^m throughout the proof. With the same argument used in the Step 1 we are able to prove that

$$
\lim_{\varepsilon\to 0} \int_{Q_\varepsilon} |\nabla v_\varepsilon^-|^2\, dx = 0 \quad \text{for any } m. \qquad (8.59)
$$

In the Step 2 we only have to replace (8.44) and (8.45) which do not hold anymore by

$$
|f\, \zeta(u_\varepsilon)(u_\varepsilon - v_\varepsilon^+)\, \chi_{\{u_\varepsilon>\delta\}}\, \chi_{\{u_0\neq\delta\}}| \le f\, \zeta(u_\varepsilon) u_\varepsilon + f \frac{1}{\delta^\theta} v_\varepsilon
$$

and

$$\text{(8.60)}\quad \int_E |f\,\zeta(u_\varepsilon)(u_\varepsilon - v_\varepsilon^+)\,\chi_{\{u_\varepsilon>\delta\}}\,\chi_{\{u_0\neq\delta\}}| \le c\|f\|_{L^{\frac{2}{1+\theta}}(E)} + \|f\|_{L^1(E)}\frac{1}{\delta^\theta}\|v_\varepsilon\|_{L^2(Q)}$$

for any measurable set E, respectively.
We note that by $(8.32)_i$ the sequence $\{\|v_\varepsilon\|_{L^2(Q)}\}$ is bounded. Then, since in view of (4.6) and (8.27) we have

$$\lim_{\varepsilon\to 0} f\,\zeta(u_\varepsilon)(u_\varepsilon - v_\varepsilon^+)\,\chi_{\{u_\varepsilon>\delta\}}\,\chi_{\{u_0\neq\delta\}} = 0,$$

by the Vitali Theorem

$$\lim_{\varepsilon\to 0}\int_Q f\,\zeta(u_\varepsilon)(u_\varepsilon - v_\varepsilon^+)\,\chi_{\{u_\varepsilon>\delta\}}\,\chi_{\{u_0\neq\delta\}} = 0.$$

Obviously we do not have anymore Step 3 and therefore the proof is completed also in the case $\kappa < 1$. □

REMARK 8.6. *The above proof would be simpler if in Step 2 we could take as test function* $u_\varepsilon - (v_\varepsilon^m)^+$ *instead of* $u_\varepsilon - v_\varepsilon^m$*. This it not possible due to the presence of the boundary term, which cannot be treated in this case.*

8.3. Proof of Theorem 4.6. We want to identify the problem satisfied by the function u^0 given by (4.6). To do that we need to pass to the limit in problem (4.1). Let $\varphi \in W_0^0$ and φ_ε be given by (8.22)-(8.23). It is not restrictive to assume that $\varphi \ge 0$. Indeed, if not, it suffices to decompose $\varphi = \varphi^+ - \varphi^-$ and we argue on each term.
For $l > 0$, let us choose $T_l(\varphi_\varepsilon) \in W_0^\varepsilon \cap L^\infty(Q)$ as test function in the variational formulation (4.1), with T_l given by (5.8).
Since $\varphi_{\varepsilon i} = \psi_i$ on Γ_ε for $i = 1, 2$, we obtain

$$\text{(8.61)}\quad \begin{aligned} &\int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla T_l(\varphi_\varepsilon)\,dx + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon(u_{\varepsilon 1} - u_{\varepsilon 2})(T_l(\psi_1) - T_l(\psi_2))d\sigma \\ &= \int_Q f\zeta(u_\varepsilon)T_l(\varphi_\varepsilon)dx, \end{aligned}$$

where we want to pass to the limit as $\varepsilon \to 0$. Let us observe that

$$\int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla T_l(\varphi_\varepsilon)\,dx = \int_{Q_\varepsilon} A^\varepsilon \nabla(u_\varepsilon - v_\varepsilon)\nabla T_l(\varphi_\varepsilon)\,dx + \int_{Q_\varepsilon} A^\varepsilon \nabla v_\varepsilon \nabla T_l(\varphi_\varepsilon)\,dx.$$

From Theorem 8.5, using (8.22)-(8.23) we have

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{Q_\varepsilon} A^\varepsilon \nabla(u_\varepsilon - v_\varepsilon)\nabla T_l(\varphi_\varepsilon)\,dx \\ &\le \beta(||\nabla\psi_1||_{L^2(Q)} + ||\nabla\psi_2||_{L^2(Q)}) \lim_{\varepsilon\to 0} ||\nabla(u_\varepsilon - v_\varepsilon||_{L^2(Q_\varepsilon)} = 0. \end{aligned}$$

On the other hand, from $(8.27)_{iii}$ and again using (8.22)-(8.23) we have

$$\int_{Q_\varepsilon} A^\varepsilon \nabla v_\varepsilon \nabla T_l(\varphi_\varepsilon)\,dx = \int_Q \chi_{Q_{\varepsilon 1}} A^\varepsilon \nabla v_\varepsilon \nabla T_l(\psi_1)\,dx + \int_Q \chi_{Q_{\varepsilon 2}} A^\varepsilon \nabla v_\varepsilon \nabla T_l(\psi_2)\,dx$$
$$\to \int_Q \chi_{Q_1} A^0 \nabla u_0 \nabla T_l(\psi_1)\,dx + \int_Q \chi_{Q_2} A^0 \nabla u_0 \nabla T_l(\psi_2)\,dx$$
$$= \int_{Q_0} A^0 \nabla u_0 \nabla T_l(\varphi)\,dx.$$

Hence,

$$(8.62)\qquad \begin{aligned} &\lim_{l\to+\infty}\lim_{\varepsilon\to 0}\int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla T_l(\varphi_\varepsilon)\,dx \\ &= \lim_{l\to+\infty}\int_{Q_0} A^0\nabla u_0 \nabla T_l(\varphi)\,dx = \int_{Q_0} A^0 \nabla u_0 \nabla\varphi\,dx, \end{aligned}$$

for any $\varphi \in W_0^0$, since

$$(8.63)\qquad T_l(\varphi) \to \varphi, \quad \text{strongly in } H^1(Q_i),\ i = 1,2.$$

Observe also that by a similar argument we obtain convergences (4.7), using again $(8.27)_{iii}$ and Theorem 8.5. Indeed,

$$\int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \Phi\,dx = \int_{Q_\varepsilon} A^\varepsilon \nabla(u_\varepsilon - v_\varepsilon)\Phi\,dx + \int_{Q_\varepsilon} A^\varepsilon \nabla v_\varepsilon \Phi\,dx$$

for every $\Phi \in L^2(Q)$.
Let us now pass to the limit in the right-hand side of (8.61).
In the spirit of the proof of Theorem 4.1, we split it in two terms like in (6.8) (see also (8.41)). We write

$$(8.64)\qquad \begin{aligned} &\int_Q f\zeta(u_\varepsilon)T_l(\varphi_\varepsilon)\,dx \\ &= \int_{\{0<u_\varepsilon\le\delta\}} f\zeta(u_\varepsilon)T_l(\varphi_\varepsilon)\,dx + \int_{\{u_\varepsilon>\delta\}} f\zeta(u_\varepsilon)T_l(\varphi_\varepsilon)\,dx \doteq \hat{I}_\varepsilon^\delta + \hat{J}_\varepsilon^\delta. \end{aligned}$$

The same arguments used to prove (6.12) (see also (8.48)), noting that

$$0 \le \hat{J}_\varepsilon^\delta \le \frac{l}{\delta^\theta}\, f \in L^1(Q),$$

give here

$$(8.65)\qquad \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \hat{J}_\varepsilon^\delta = \int_Q f\zeta(u_0)T_l(\varphi)\,\chi_{\{u_0>0\}}\,dx,$$

except at most for a countable set of values of δ.
From (5.9) and (8.23) we derive

$$0 < \hat{I}_\varepsilon^\delta \le \int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon\, \nabla T_l(\varphi_\varepsilon)\, Z_\delta(u_\varepsilon)\,dx$$
$$+ 4l\,\delta\,\varepsilon^\gamma ||h||_{L^\infty(\Gamma)},$$

with Z_δ defined in (5.7). On the other hand, as done when proving (6.9) we derive

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla T_l(\varphi_\varepsilon)\, Z_\delta(u_\varepsilon)\, dx = \lim_{\delta\to 0}\int_{Q_0} A^0 \nabla u_0 \nabla T_l(\varphi)\, Z_\delta(u_0)\, dx = 0,$$

since $Z_\delta(u_\varepsilon)$ converges a.e. to $\chi_{\{u_0=0\}}$ as δ tends to zero.
Consequently, if (4.17) or (4.18) or (4.8) or (4.9) holds (since $\gamma \geq 0$) it results

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \hat{I}^\delta_\varepsilon = 0, \tag{8.66}$$

which together with (8.64) and (8.65) gives, for the above cases,

$$\lim_{\varepsilon\to 0}\int_Q f\zeta(u_\varepsilon)T_l(\varphi_\varepsilon)\, dx = \int_Q f\zeta(u_0)T_l(\varphi)\,\chi_{\{u_0>0\}}\, dx. \tag{8.67}$$

Suppose now that (4.13) or (4.14) holds. Then, we can use the fact that from Proposition (8.3) the function u_0 belongs to $H^1_0(Q)$. As a consequence, we can choose $\varphi \in H^1_0(Q)$ in (8.61), which using inequality (5.9) stated in Proposition 5.4 gives

$$\begin{aligned} 0 < \hat{I}^\delta_\varepsilon &\leq \int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla T_l(\varphi_\varepsilon)\, Z_\delta(u_\varepsilon)\, dx \\ &+ \varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon T_l(\varphi)(u_{\varepsilon 1} - u_{\varepsilon 2})(Z_\delta(u_{\varepsilon 1}) - Z_\delta(u_{\varepsilon 2}))\, d\sigma \\ &\leq \int_{Q_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla T_l(\varphi_\varepsilon)\, Z_\delta(u_\varepsilon)\, dx, \end{aligned} \tag{8.68}$$

since Z_δ is non increasing.
Hence we still have (8.66), which together with (8.64) and (8.65) again give (8.67) for these last cases.
It remains to show that

$$\lim_{l\to+\infty}\int_Q f\zeta(u_0)T_l(\varphi)\,\chi_{\{u_0>0\}}\, dx = \int_Q f\zeta(u_0)\varphi\,\chi_{\{u_0>0\}}\, dx. \tag{8.69}$$

By (5.8) and Lemma 8.4 we deduce that $f\zeta(u_0)\varphi\chi_{\{u_0>0\}}$ is in $L^1(Q)$. Therefore, using again (8.63) and the Lebesgue dominated convergence theorem, we obtain (8.69) since for any $l > 0$,

$$0 \leq f\zeta(u_0)T_l(\varphi)\chi_{\{u_0>0\}} \leq f\zeta(u_0)\varphi\chi_{\{u_0>0\}} \in L^1(Q).$$

Finally, to pass to the limit in the boundary integral in (8.61) we use Proposition 8.3.
• If (4.8) or (4.9) holds, from (8.17) and (8.63) we have

$$\lim_{l\to+\infty}\lim_{\varepsilon\to 0}\varepsilon^\gamma\int_{\Gamma_\varepsilon} h^\varepsilon(u_{\varepsilon 1}-u_{\varepsilon 2})(T_l(\psi_1)-T_l(\psi_2))d\sigma = H(g,h)\int_{\Gamma_0}(u_{01}-u_{02})(\varphi_1-\varphi_2)d\sigma.$$

This together with (8.62), (8.67) and (8.69) allows to pass to the limit in (8.61), first as $\varepsilon \to 0$ then as $l \to +\infty$. We have that u_0 verifies

$$\int_{Q_0} A^0\nabla u_0 \nabla\varphi\, dx + H(g,h)\int_{\Gamma_0}(u_{01}-u_{02})(\varphi_1-\varphi_2)d\sigma = \int_Q f\zeta(u_0)\varphi\,\chi_{\{u_0>0\}}\, dx,$$

for every $\varphi \in W_0^0$. Using the maximum principle and Lemma 8.4 we obtain (4.5) and (4.10).

• If (4.17) or (4.18) holds, from Proposition 8.3 we deduce that the boundary integral in (8.61) goes to zero as ε goes to zero, so using (8.62), (8.67) we pass to the limit in (8.61), first as $\varepsilon \to 0$ then as $l \to +\infty$. We have that the limit function u_0 solves

$$\int_{Q_0} A^0 \nabla u_0 \nabla \varphi \, dx = \int_Q f\zeta(u_0)\varphi \, \chi_{\{u_0>0\}} \, dx, \qquad \text{for every } \varphi \in W_0^0.$$

Moreover, using here too the maximum principle and Lemma 8.4, we obtain (4.5) and the fact that u_{01} and u_{02} solve the two Neumann problems given by (4.19) and (4.20), respectively.

• Finally, suppose that (4.13) or (4.14) holds. Then u_0 belongs to $H_0^1(Q)$ and choosing in particular a test function φ in $H_0^1(Q)$, the boundary term in in (8.61) is zero. Then, we obtain

$$\int_Q A^0 \nabla u_0 \nabla \varphi \, dx = \int_Q f\zeta(u_0)\varphi \, \chi_{\{u_0>0\}} \, dx, \qquad \text{for every } \varphi \in H_0^1(Q).$$

Once again, by the strong maximum principle we deduce that the function u is strictly positive almost everywhere in Q, which together with Lemma 8.4 gives (4.5) and (4.16).

To conclude the proof, observe that the last statement is a straightforward consequence of Theorem 4.5. ☐

9. A Corrector result for the linear problem.

The main result of this section is a correctors for the linear problem (8.3), whose variational formulation is given in (8.4).

THEOREM 9.1. *Under the assumptions of Theorem 8.1, for every value of κ and γ, we have*

$$\lim_{\varepsilon \to 0} ||\nabla v_\varepsilon - C^\varepsilon \nabla v_0||_{\left(L^1(Q_{\varepsilon,0})\right)^N} = 0. \tag{9.1}$$

where the corrector matrix C^ε is given by (4.23).

This result will be proved at the end of this section. We adapt standard arguments (see for instance [16]) to our geometric situation. We first prove the following result:

PROPOSITION 9.2. *Under the assumptions of Theorem 9.1, there exists a positive constant $c = c(\alpha, \beta)$ such that, for every value of κ and γ,*

$$\limsup_{\varepsilon \to 0} \int_{Q_\varepsilon} |\nabla v_\varepsilon - C^\varepsilon \Phi|^2 \, dx \le c \int_{Q_0} |\nabla v_0 - \Phi|^2 \, dx, \tag{9.2}$$

for every $\Phi = (\Phi_1, \ldots, \Phi_N)$ such that the function $\Phi_i = (\Phi_{1i}, \ldots, \Phi_{ni})$ belong to $(\mathcal{D}(Q_i))^N$, for $i = 1, 2$.

Proof. Let $\Phi = (\Phi_1, \ldots, \Phi_N)$ such that the function $\Phi_i = (\Phi_{1i}, \ldots, \Phi_{ni})$ belong to $(\mathcal{D}(Q_i))^N$, for $i = 1, 2$. We have

$$\begin{aligned} \frac{1}{\alpha} \int_{Q_\varepsilon} |\nabla v_\varepsilon - C^\varepsilon \Phi|^2 \, dx &\le \int_{Q_\varepsilon} A^\varepsilon (\nabla v_\varepsilon - C^\varepsilon \Phi)(\nabla v_\varepsilon - C^\varepsilon \Phi) \, dx \\ &= \int_{Q_\varepsilon} A^\varepsilon \nabla v_\varepsilon \nabla v_\varepsilon \, dx - \int_{Q_\varepsilon} A^\varepsilon \nabla v_\varepsilon C^\varepsilon \Phi \, dx - \int_{Q_\varepsilon} A^\varepsilon C^\varepsilon \Phi \nabla v_\varepsilon \, dx \\ &+ \int_{Q_\varepsilon} A^\varepsilon C^\varepsilon \Phi C^\varepsilon \Phi \, dx \doteq I_1^\varepsilon - I_2^\varepsilon - I_3^\varepsilon + I_4^\varepsilon. \end{aligned} \tag{9.3}$$

Observe now that

(9.4) $\exists\, \varepsilon_0$ such that, for $\varepsilon \le \varepsilon_0$, supp $\Phi_{1i} \subset \omega\times]\varepsilon_0^\kappa \bar{g}, l[, \quad \forall\, i = 1, \ldots, n.$

Hence, from (4.4) and by a standard computation,

(9.5)
$$\begin{aligned}\lim_{\varepsilon\to 0} I_4^\varepsilon &= \lim_{\varepsilon\to 0}\int_{\omega\times]\varepsilon_0^\kappa\bar{g},l[} A^\varepsilon C^\varepsilon \Phi_1 C^\varepsilon \Phi_1 \, dx + \lim_{\varepsilon\to 0}\int_{Q_2} A^\varepsilon C^\varepsilon \Phi_2 C^\varepsilon \Phi_2 \, dx \\ &= \lim_{\varepsilon\to 0}\int_{Q_1} A^\varepsilon C^\varepsilon \Phi_1 C^\varepsilon \Phi_1 \, dx + \lim_{\varepsilon\to 0}\int_{Q_2} A^\varepsilon C^\varepsilon \Phi_2 C^\varepsilon \Phi_2 \, dx \\ &= \int_{Q_0} A^0 \Phi\Phi \, dx.\end{aligned}$$

Moreover, by the same argument for $\varepsilon \le \varepsilon_0$ it results

(9.6)
$$I_3^\varepsilon = \int_{Q_1} A^\varepsilon C^\varepsilon \Phi_1 \nabla v_{\varepsilon 1} \, dx + \int_{Q_2} A^\varepsilon C^\varepsilon \Phi_2 \nabla v_{\varepsilon 2} \, dx.$$

Let us recall now that if w_i is given by (4.4) for $\lambda = e_i$ and $w_i^\varepsilon(x) = \varepsilon w_i(\frac{x}{\varepsilon})$ a.e. in $\mathbb{R}^N$, then

(9.7)
$$\begin{cases} w_i^\varepsilon \rightharpoonup x_i, & \text{weakly in } H^1(Q), \\ w_i^\varepsilon \to x_i, & \text{strongly in } L_2(Q), \\ A^\varepsilon \nabla w_i^\varepsilon \rightharpoonup A^0, & \text{weakly in } (L_2(Q))^N \end{cases}$$

and a simple change of scale gives (see for instance [16])

(9.8)
$$\int_\omega A^\varepsilon \nabla w_i^\varepsilon \nabla v \, dx = 0, \quad \text{for every } v \in H_0^1(\omega),$$

for every open set $\omega \subset \mathbb{R}^N$. Hence, we have from (9.7), (9.8) and $(8.6)_i$

$$\begin{aligned}&\int_{Q_1} A^\varepsilon C^\varepsilon \Phi_1 \nabla v_{\varepsilon 1} \, dx \\ &= \sum_{i=1}^N \int_{Q_1} A^\varepsilon \nabla w_i^\varepsilon \nabla(\Phi_{1i} v_{\varepsilon 1}) \, dx - \sum_{i=1}^N \int_{Q_1} A^\varepsilon \nabla w_i^\varepsilon \nabla \Phi_{1i} v_{\varepsilon 1} \, dx \\ &= -\sum_{i=1}^N \int_{Q_1} A^\varepsilon \nabla w_i^\varepsilon \nabla \Phi_{1i} v_{\varepsilon 1} \, dx \to -\sum_{i=1}^N \int_{Q_1} A^0 e_i \nabla \Phi_{1i} v_1 \, dx.\end{aligned}$$

Treating in the same way the integral over Q_2 in (9.6), we have

(9.9)
$$\lim_{\varepsilon\to 0} I_3^\varepsilon = \int_{Q_0} A^0 \Phi \nabla v_0 \, dx.$$

On the other hand, choosing $\varphi = \Phi_i w_i^\varepsilon$ in (8.4) we have

$$
\begin{aligned}
I_2^\varepsilon &= \sum_{i=1}^N \int_{Q_\varepsilon} A^\varepsilon \nabla v_\varepsilon \Phi_i \nabla w_i^\varepsilon \, dx \\
&= \sum_{i=1}^N \int_{Q_\varepsilon} A^\varepsilon \nabla v_\varepsilon \nabla (\Phi_i w_i^\varepsilon) \, dx - \sum_{i=1}^N \int_{Q_\varepsilon} A^\varepsilon \nabla v_\varepsilon \nabla \Phi_i w_i^\varepsilon \, dx \\
&= \sum_{i=1}^N \int_Q g \Phi_i w_i^\varepsilon \, dx + \sum_{i=1}^N \int_{Q \setminus \Gamma_{\varepsilon,0}} B \nabla z \nabla (\Phi_i w_i^\varepsilon) \, dx - \sum_{i=1}^N \int_{Q_\varepsilon} A^\varepsilon \nabla v_\varepsilon \nabla \Phi_i w_i^\varepsilon \, dx,
\end{aligned}
$$

where we used the fact that

$$
\varepsilon^\gamma \sum_{i=1}^N \int_{\Gamma_\varepsilon} h^\varepsilon (v_{\varepsilon 1} - v_{\varepsilon 2})((\Phi w_i^\varepsilon)_1 - (\Phi w_i^\varepsilon)_2) \, d\sigma = 0,
$$

for $\varepsilon \leq \varepsilon_0$, since supp $(\Phi_2) \subset Q_2$ and (9.4) holds.
Consequently, in view of (9.7) and (8.7) and we obtain
(9.10)

$$
\begin{aligned}
\lim_{\varepsilon \to 0} I_2^\varepsilon &= \sum_{i=1}^N \int_Q g \Phi_i x_i \, dx + \sum_{i=1}^N \int_{Q_0} B \nabla z \nabla (\Phi_i x_i) \, dx - \sum_{i=1}^N \int_{Q_0} A^0 \nabla v_0 \nabla \Phi_i x_i \, dx \\
&= \sum_{i=1}^N \int_Q g \Phi_i x_i \, dx + \sum_{i=1}^N \int_{Q_0} B \nabla z \nabla (\Phi_i x_i) \, dx \\
&\quad - \sum_{i=1}^N \int_{Q_0} A^0 \nabla v_0 \nabla (\Phi_i x_i) \, dx + \int_{Q_0} A^0 \nabla v_0 \, \Phi \, dx.
\end{aligned}
$$

Observe now that for any case of κ and γ, since supp $(\Phi_i) \subset Q_2$ and (9.4) holds, using the limit problem satisfied by v_0 (according to the value of κ and γ) we get

$$
\sum_{i=1}^N \int_{Q_0} A^0 \nabla v_0 \nabla (\Phi_i x_i) \, dx = \sum_{i=1}^N \int_Q g \Phi_i x_i \, dx + \sum_{i=1}^N \int_{Q_0} B \nabla z \nabla (\Phi_i x_i) \, dx.
$$

Hence from (9.10) we deduce that

$$
\lim_{\varepsilon \to 0} I_2^\varepsilon = \int_{Q_0} A^0 \nabla v_0 \, \Phi \, dx. \tag{9.11}
$$

It remains to study the limit of the energy I_1^ε. Choosing v_ε as test function in (8.4) we have

$$
I_1^\varepsilon = -\varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon (v_{\varepsilon 1} - v_{\varepsilon 2})^2 \, d\sigma + \int_Q g v_\varepsilon \, dx + \int_{Q \setminus \Gamma_{\varepsilon,0}} B \nabla z \nabla v_\varepsilon \, dx. \tag{9.12}
$$

Observe first that from convergences (8.6) we deduce that

$$
\lim_{\varepsilon \to 0} \Big(\int_Q g v_\varepsilon \, dx + \int_{Q \setminus \Gamma_{\varepsilon,0}} B \nabla z \nabla v_\varepsilon \, dx \Big) = \int_Q g v_0 \, dx + \int_{Q \setminus \Gamma_0} B \nabla z \nabla v_0 \, dx. \tag{9.13}
$$

To treat the boundary term we apply Proposition 8.3 to $w_\varepsilon = \psi_\varepsilon = v_\varepsilon$. If (4.8) or (4.9) holds, we obtain

$$\lim_{\varepsilon\to 0} \varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon (v_{\varepsilon 1} - v_{\varepsilon 2})^2 \, d\sigma = \int_{\Gamma_0} H(g,h)(v_{01} - v_{02})^2 \, d\sigma, \tag{9.14}$$

while if (4.17) or (4.18) holds

$$\lim_{\varepsilon\to 0} \varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon (v_{\varepsilon 1} - v_{\varepsilon 2})^2 \, d\sigma = 0. \tag{9.15}$$

Hence, by (9.13), using v_0 as test function in the limit problem given by Theorem 4.6 for these cases (according to the value of κ and γ), we have

$$\lim_{\varepsilon\to 0} I_1^\varepsilon = \int_{Q_0} A^0 \nabla v_0 \nabla v_0 \, dx. \tag{9.16}$$

Suppose now that (4.13) or (4.14) holds. Then,

$$I_1^\varepsilon \leq \int_Q g v_\varepsilon \, dx + \int_{Q\setminus\Gamma_{\varepsilon,0}} B\nabla z \nabla v_\varepsilon \, dx.$$

which implies, using now (9.13) and the limit problem (8.10) from Theorem 8.1,

$$\limsup_{\varepsilon\to 0} I_1^\varepsilon \leq \int_Q g v_0 \, dx + \int_{Q\setminus\Gamma_0} B\nabla z \nabla v_0 \, dx = \int_Q A^0 \nabla v_0 \nabla v_0 \, dx. \tag{9.17}$$

Then, from (7.1), collecting (9.5)-(9.9), together with (9.12) or (9.17) (according to the different cases) we have

$$\limsup_{\varepsilon\to 0} \int_{Q_\varepsilon} |\nabla v_\varepsilon - C^\varepsilon \Phi|^2 \, dx \leq \frac{1}{\alpha} \int_{Q_0} A^0 (\nabla v_0 - \Phi)(\nabla v_0 - \Phi) \, dx, \tag{9.18}$$

where in the case that (4.13) or (4.14) holds we can choose $\Phi \in \mathcal{D}(Q)^N$, which gives the claimed result. ∎

REMARK 9.3. *Let us point out that when (4.13) or (4.14) holds, we are not able to prove that the energy I_1^ε converges to the energy of the homogenized problem (4.15). Nevertheless, inequality (9.17) is sufficient to prove the proposition above.*

Proof of Theorem 9.1 For fixed $\delta > 0$, let $\Phi^\delta = (\Phi_1^\delta, \dots, \Phi_n^\delta)$ be such that the function $\Phi_i^\delta = (\Phi_{1i}^\delta, \dots, \Phi_{ni}^\delta)$ belong to $(\mathcal{D}(Q_i))^N$, for $i = 1, 2$, and

$$\|\nabla v_0 - \Phi^\delta\|_{(L^2(Q_0))^N} \leq \delta. \tag{9.19}$$

Then, from Proposition 9.2 and the boundedness of C^ε in $L^2(Q)$, using (9.19) we have

$$\begin{aligned} \limsup_{\varepsilon\to 0} \|\nabla v_\varepsilon - C^\varepsilon \nabla v_0\|_{(L^1(Q_{\varepsilon,0}))^N} &\leq \limsup_{\varepsilon\to 0} \left\|\nabla v_\varepsilon - C^\varepsilon \Phi^\delta\right\|_{(L^2(Q_{\varepsilon,0}))^N} \\ &+ \left\|C^\varepsilon \Psi - C^\varepsilon \Phi^\delta\right\|_{(L^2(Q_0))^N} \leq c\delta. \end{aligned} \tag{9.20}$$

This concludes the proof, since δ is arbitrary.

∎

We also have

COROLLARY 9.4. *Under the assumptions of Theorem 8.1, let $\delta > 0$ and $\Psi : Q \to \mathbb{R}^N$ be a simple function such that*

$$\|\nabla v_0 - \Psi\|_{(L^2(Q_0))^N} \le \delta. \tag{9.21}$$

Then,

$$\limsup_{\varepsilon\to 0} \|\nabla v_\varepsilon - C^\varepsilon \Psi\|_{(L^2(Q_{\varepsilon,0}))^N} \le c\,\delta,$$

where c depends only on α, β and Y.

Proof. For fixed $\delta > 0$, let Ψ be a simple function satisfying and let $\Phi^\delta = (\Phi^\delta_1, \dots, \Phi^\delta_n)$ be such that $\Phi^\delta_i = (\Phi^\delta_{1i}, \dots, \Phi^\delta_{ni}) \in (\mathcal{D}(Q_i))^N$, for $i = 1, 2$, and

$$\|\nabla v_0 - \Phi^\delta\|_{(L^2(Q_0))^N} \le \delta. \tag{9.22}$$

Then,

$$\begin{aligned} &\|\nabla v_\varepsilon - C^\varepsilon \Psi\|_{(L^2(Q_{\varepsilon,0}))^N} \\ &\le \left\|\nabla v_\varepsilon - C^\varepsilon \Phi^\delta\right\|_{(L^2(Q_{\varepsilon,0}))^N} + \left\|C^\varepsilon \Psi - C^\varepsilon \Phi^\delta\right\|_{(L^2(Q_0))^N}. \end{aligned} \tag{9.23}$$

Since $\{C^\varepsilon\}$ is bounded in $L^2(Q))^N$, from (9.21) and (9.22) via the Hölder inequality,

$$\begin{aligned} &\left\|C^\varepsilon \Psi - C^\varepsilon \Phi^\delta\right\|_{(L^2(Q_0))^N} \le c_1 \left\|\Psi^\delta - \Psi\right\|_{(L^2(Q_0))^N} \le \\ &c_1(\|\nabla v_0 - \Psi\|_{(L^2(Q_0))^N} + \left\|\nabla v_0 - \Phi^\delta\right\|_{(L^2(Q_0))^N}) \le 2c_1\delta. \end{aligned} \tag{9.24}$$

On the other hand, from Proposition 9.2 and (9.22) we derive

$$\limsup_{\varepsilon\to 0} \left\|\nabla v_\varepsilon - C^\varepsilon \Phi^\delta\right\|_{(L^2(Q_{\varepsilon,0}))^N} \le c\delta,$$

which together with (9.23) and (9.24) concludes the proof. ∎

10. Appendix.

We prove here the existence of a solution of the approximate problem (6.1), where for simplicity we omit the dependence of the functions on n. To do that, we apply the Schauder's Theorem to the map

$$F : \; w \in L^2(Q) \longmapsto u \in L^2(Q),$$

where u is the unique solution in W^ε_0 of the problem

$$\begin{cases} -\mathrm{div}(A\nabla u) = T_n\big(f\,\zeta(|w|)\big) & \text{in } Q_\varepsilon, \\ (A\nabla u)_1 \cdot \nu = (A\nabla u)_2 \cdot \nu & \text{on } \Gamma_\varepsilon, \\ (A\nabla u)_1 \cdot \nu = -\varepsilon^\gamma\, h(u_1 - u_2), & \text{on } \Gamma_\varepsilon. \\ u = 0 & \text{on } \partial Q, \end{cases} \tag{10.1}$$

and T_n is the truncation at level n given by (5.8). The Lax-Milgram Theorem gives the existence and uniqueness of u and shows that $F(L^2(Q))$ is contained in a ball of W_0^ε, so that there exists a ball B of $L^2(Q)$ which is invariant for F. Since (see Proposition 2.4 of [22]) W_0^ε is compact in $L^2(Q)$, the set $F(B)$ is a compact.

It remains to show that F is continuous. Let us take a sequence $\{w_m\}$ which converges to some w in $L^2(Q)$. Then, $u_m = F(w_m)$ satisfies

$$
\begin{cases}
u_m \in W_0^\varepsilon, \\
\displaystyle\int_{Q_\varepsilon} A\nabla u_m \nabla\varphi \, dx + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h(u_{m1} - u_{m2})(\varphi_1 - \varphi_2)\, d\sigma = \\
\displaystyle\int_Q T_n\big(f\zeta(|w_m|)\big)\varphi \, dx, \quad \text{for every } \varphi \in W_0^\varepsilon,
\end{cases}
\tag{10.2}
$$

and, up to a subsequence, from what showed above it converges to some u_0 weakly in W_0^ε, strongly in $L^2(Q)$ and almost everywhere in Q.

Then, passing to the limit in (10.2), we obtain

$$
\int_{Q_\varepsilon} A\nabla u_0 \nabla\varphi \, dx + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h(u_{01} - u_{02})(\varphi_1 - \varphi_2)\, d\sigma = \int_Q T_n\big(f\zeta(|w|)\big)\varphi \, dx,
$$

for every $\varphi \in W_0^\varepsilon$. This implies, by uniqueness, that $u_0 = F(w)$ and concludes the proof. □

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