THE MODULI SCHEME OF AFFINE SPHERICAL VARIETIES WITH A FREE WEIGHT MONOID

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ABSTRACT. We study Alexeev and Brion's moduli scheme M_Γ of affine spherical varieties with weight monoid Γ under the assumption that Γ is free. We describe the tangent space to M_Γ at its 'most degenerate point' in terms of the combinatorial invariants of spherical varieties and deduce that the irreducible components of M_Γ, equipped with their reduced induced scheme structure, are affine spaces.

1. INTRODUCTION

As part of the classification problem of algebraic varieties equipped with a group action, spherical varieties, which include symmetric, toric and flag varieties, have received considerable attention; see, e.g., [Bri90, Kno96, Lun01, Los09b]. In [AB05], V. Alexeev and M. Brion introduced an important new tool for the study of affine spherical varieties over an algebraically closed field $\Bbbk$ of characteristic 0. We recall that an affine variety X equipped with an action of a connected reductive group G is called spherical if it is normal and its coordinate ring $\Bbbk[X]$ is multiplicity-free as a G-module. For such a variety a natural invariant, which completely describes the G-module structure of $\Bbbk[X]$, is its **weight monoid** $\Gamma(X)$. By definition, $\Gamma(X)$ is the set of isomorphism classes of irreducible representations of G that occur in $\Bbbk[X]$. In view of the classification problem, we have the following natural question: how 'good' an invariant is $\Gamma(X)$, or more explicitly: to what extent does $\Gamma(X)$ determine the multiplicative structure of $\Bbbk[X]$?

Alexeev and Brion brought geometry to this question as follows. After choosing a Borel subgroup B of G, and a maximal torus T in B, we can identify $\Gamma(X)$ with a finitely generated submonoid of the monoid Λ^+ of dominant weights. Let Γ be another such submonoid of Λ^+ and put

$$V(\Gamma) = \oplus_{\lambda\in\Gamma} V(\lambda),$$

where we used $V(\lambda)$ for the irreducible G-module corresponding to $\lambda \in \Lambda^+$. Let U be the unipotent radical of B and let $V(\Gamma)^U$ be the subspace of U-invariants, which is also the space of highest weight vectors in $V(\Gamma)$. By choosing an isomorphism $V(\Gamma)^U \to \Bbbk[\Gamma]$ of T-modules, where $\Bbbk[\Gamma]$ is the semigroup ring associated to Γ, we equip $V(\Gamma)^U$ with a T-multiplication law. Alexeev and Brion's moduli scheme M_Γ parametrizes the G-multiplication laws on $V(\Gamma)$ which extend the multiplication law on $V(\Gamma)^U$. For an introduction to this moduli scheme, we refer the reader to [Bri13, §4.3]. Examples of M_Γ have been computed in [Jan07, BCF08, PVS12].

Let Λ be the weight lattice of G, that is, Λ is the character group of T. Because X is normal, its weight monoid $\Gamma(X)$ also satisfies the following equality in $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$

$$\Gamma(X) = \mathbb{Z}\Gamma(X) \cap \mathbb{Q}_{\geq 0}\Gamma. \tag{1.1}$$

By definition, this makes $\Gamma(X)$ a **normal** submonoid of Λ^+.

In [Bri13], Brion conjectured that the irreducible components of M_Γ are affine spaces. A precise version of this conjecture is the following.

Conjecture 1.1. *If Γ is a normal submonoid of Λ^+, then the irreducible components of M_Γ, equipped with their reduced induced scheme structure, are affine spaces.*

This conjecture was verified for free and G-saturated monoids of dominant weights in [BCF08]. In fact, Bravi and Cupit-Foutou proved that under these assumptions, M_Γ is an affine space. In [PVS12, PVS14] it is shown that M_Γ is an affine space when Γ is the weight monoid of a spherical G-module. Luna provided the first non-irreducible example (unpublished): for $G = \mathrm{SL}(2) \times \mathrm{SL}(2)$ and $\Gamma = \langle 2\omega, 4\omega + 2\omega' \rangle$, the scheme M_Γ is the union of two lines meeting in a point. In this paper, we verify that Conjecture 1.1 holds when Γ is free.

Theorem (Corollary 5.3)**.** *If Γ is a free submonoid of Λ^+, then the irreducible components of M_Γ, equipped with their reduced induced scheme structure, are affine spaces.*

The bulk of this paper is devoted to the description of the tangent space to M_Γ at its 'most degenerate point' X_0 in terms of certain combinatorial invariants, called N-spherical roots. To be more precise, we introduce some more terminology and recall some facts. If X is an affine spherical G-variety X, then its **root monoid** $\mathscr{M}_X$ is the submonoid of Λ generated by the set

$$\{\lambda + \mu - \nu \mid \lambda, \mu, \nu \in \Lambda^+ \text{ such that } \langle \Bbbk[X]_{(\lambda)} \cdot \Bbbk[X]_{(\mu)} \rangle_{\Bbbk} \cap \Bbbk[X]_{(\nu)} \neq 0\}.$$

Here $\Bbbk[X]_{(\lambda)}$ is the isotypic component of $\Bbbk[X]$ of type $\lambda \in \Lambda^+$. Loosely speaking, $\mathscr{M}_X$ detects how far the decomposition $\Bbbk[X] = \oplus_{\lambda \in \Gamma(X)} \Bbbk[X]_{(\lambda)}$ is from being a grading by $\Gamma(X)$. A deep result by Knop [Kno96, Theorem 1.3] says that the saturation of $\mathscr{M}_X$, which is the intersection in $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ of the cone $\mathbb{Q}_{\geq 0}\mathscr{M}_X$ and the lattice $\mathbb{Z}\mathscr{M}_X$, is a freely generated monoid. Its basis $\Sigma^N(X)$ is called the set of **N-spherical roots** of X. By [AB05, Proposition 2.13] a formal consequence of our theorem above is that if X is an affine spherical G-variety with a free weight monoid, then its root monoid $\mathscr{M}_X$ is also free; see Corollary 5.2.

In their seminal paper [AB05], Alexeev and Brion equipped M_Γ with an action of the maximal torus T of G. For this action, M_Γ has a unique closed orbit, which is a fixed point X_0. Consequently, the tangent space $\mathrm{T}_{X_0}\mathrm{M}_\Gamma$ to M_Γ at the point X_0 is a finite-dimensional T-module. We describe this tangent space as follows.

Theorem (Theorem 4.1 and Corollary 4.2)**.** *If Γ is a free submonoid of Λ^+ , then $\mathrm{T}_{X_0}\mathrm{M}_\Gamma$ is a multiplicity-free T-module, and $\gamma \in \Lambda$ occurs as a weight in $\mathrm{T}_{X_0}\mathrm{M}_\Gamma$ if and only if there exists an affine spherical G-variety $X_{-\gamma}$ with weight monoid Γ and $\Sigma^N(X_{-\gamma}) = \{-\gamma\}$.*

To prove this we first use the combinatorial theory of spherical varieties [Kno91, Lun01, Los09b] to combinatorially characterize the weights γ for which such a variety $X_{-\gamma}$ exists; see Corollary 2.17. Such a characterization was sketched by Luna in 2005 in an unpublished note.

Notation. Except if explicitly stated otherwise, Γ will be a *free* submonoid of Λ^+ with basis $F = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$. We will use S for the set of simple roots of G (associated to B and T) and R^+ for the set of positive roots. The irreducible representation of G associated to the dominant weight $\lambda \in \Lambda^+$ is denoted by $V(\lambda)$ and we use v_λ for a highest weight vector in $V(\lambda)$. We use $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}, \mathfrak{n}$, etc. for the Lie algebra of G, B, T, U, etc., respectively. When α is a root, $X_\alpha \in \mathfrak{g}_\alpha$ is a root operator and $\alpha^\vee$ the coroot.

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on irreducible components of affine schemes. It alerted them to a mistake in an earlier version of this paper.

As this paper was being completed, R. Avdeev and S. Cupit-Foutou announced that they had independently obtained similar results [ACF14].

2. SPHERICAL ROOTS ADAPTED TO Γ

In this section Γ denotes a normal, but *not necessarily free*, submonoid of Λ^+. By combining results from [Lun01, Kno91, Los09b, BP11] we will describe when a set of spherical roots is 'adapted' or 'N-adapted' to Γ. In particular, in Corollaries 2.16 and 2.17 we give an explicit characterization for when an element σ of the root lattice is 'adapted' or 'N-adapted' to Γ.

Definition 2.1. We say that a subset Σ of $\mathbb{N}S$ is **N-adapted** to Γ if there exists an affine spherical G-variety X such that $\Gamma(X) = \Gamma$ and $\Sigma^N(X) = \Sigma$. By slight abuse of language, we say that an element σ of $\mathbb{N}S$ is N-adapted to Γ if $\{\sigma\}$ is N-adapted to Γ.

We will give the definition of 'adapted', which requires some more notions from the theory of spherical varieties, in Definition 2.11 below. After recalling some basic definitions concerning spherical varieties, we briefly discuss, in Section 2.2, the notion of 'spherically closed spherical systems', and the role they play in classifiying spherically closed spherical subgroups of G. We then, in Section 2.3 review Luna's 'augmentations'. They classify the subgroups of G which have a given spherical closure K. Finally, after recalling some basic results from the Luna-Vust theory of spherical embeddings in Section 2.4, we deduce the combinatorial characterization of adapted and N-adapted spherical roots.

2.1. **Basic definitions.** In this section we briefly recall the basic definitions of the theory of spherical varieties by freely quoting from [Lun01]. For more details on these notions the reader can also consult [Pez10, Tim11].

We recall that a (not necessarily affine) G-variety X is called **spherical** if it is normal and contains an open dense orbit for B. If X is affine, this is equivalent to the definition given before in terms of $\Bbbk[X]$.

The complement of the open B-orbit in X consists of finitely many B-stable prime divisors. Among those, the ones that are *not* G-stable are called the **colors** of X. The set of colors of X is denoted by Δ_X.

By the **weight lattice** $\Lambda(X)$ of X we mean the subgroup of Λ made up of the B-weights in the field of rational functions $\Bbbk(X)$. Since X has a dense B-orbit two rational B-eigenfunctions on X of the same weight are scalar multiples of one another.

Let P_X be the stabilizer of the open B-orbit and denote by S^p_X the subset of simple roots corresponding to P_X, which is a parabolic subgroup of G containing B.

Let $\mathcal{V}_X \subset \mathrm{Hom}(\Lambda(X), \mathbb{Q})$ be the so-called **valuation cone** of X, i.e. the set of $\mathbb{Q}$-valued G-invariant valuations on $\Bbbk(X)$ seen as functionals on $\Lambda(X)$. By [Bri90, Theorem 3.5] $\mathcal{V}_X$ is a simplicial cone. Let $\Sigma(X)$ be the set of linearly independent primitive elements in $\Lambda(X)$ such that

$$\mathcal{V}_X = \{v \in \mathrm{Hom}(\Lambda(X), \mathbb{Q}) : \langle v, \sigma \rangle \leq 0 \text{ for all } \sigma \in \Sigma(X)\},$$

i.e. the set of **spherical roots** of X.

Similarly, the discrete valuations on $\Bbbk(X)$ associated with colors give rise to functionals on $\Lambda(X)$. This yields the so-called **Cartan pairing** of X, a $\mathbb{Z}$-bilinear map denoted by

$$c_X \colon \mathbb{Z}\Delta_X \times \Lambda(X) \to \mathbb{Z}.$$

Since X has a dense B-orbit, it has a dense G-orbit. Let H be the stabilizer of a point in this orbit, which we can then identify with G/H. The group H is called a **spherical subgroup** of G because G/H is a spherical G-variety. To H, we can associate a larger group $\overline{H}$, called the **spherical closure** of H: the normalizer of H in G acts by G-equivariant automorphisms on G/H and $\overline{H}$ is the kernel of the induced action of this normalizer on Δ_X (see [Lun01, §6.1] or [BL11, §2.4.1]). We recall that it follows from [BL11, Lemma 2.4.2] that $\overline{\overline{H}} = \overline{H}$ (see [Pez13, Proposition 3.1] for a direct proof).

2.2. **Spherical systems.** Here we briefly recall the definition of spherical system and its role in the classification of spherical varieties, see [Lun01, BL11].

Recall that by [Los09b, BP11] wonderful G-varieties (or their open G-orbits) are classified by their so-called spherical system. This was known as Luna's conjecture, another proof of which was proposed in [Cup10]. In particular, because they have a unique wonderful embedding by [Kno96], spherical homogeneous spaces G/K with K **spherically closed** (that is, $\overline{K} = K$) correspond to spherically closed spherical G-systems (systems satisfying certain combinatorial conditions, as explained below):

$$G/K \longmapsto \mathscr{S}_{G/K} = (S^p_{G/K}, \Sigma(G/K), \mathbf{A}_{G/K}).$$

Let K be a spherically closed spherical subgroup of G. The set $\Sigma(G/K)$ of spherical roots of G/K is included in the root lattice $\mathbb{Z}S$ (because K contains the center of G) and it is a basis of $\Lambda(G/K)$. Let $\mathbf{A}_{G/K}$ be the set of colors that are not stable under some minimal parabolic containing B and corresponding to a simple root belonging to $\Sigma(G/K)$. The full Cartan pairing restricts to the $\mathbb{Z}$-bilinear pairing $c_{G/K}\colon \mathbb{Z}\mathbf{A}_{G/K} \times \mathbb{Z}\Sigma(G/K) \to \mathbb{Z}$, also called restricted Cartan pairing.

Definition 2.2. The set $\Sigma^{sc}(G)$ of **spherically closed spherical roots of** G is defined as

$$\Sigma^{sc}(G) := \{\sigma \in \mathbb{Z}S\colon \sigma \in \Sigma(G/K) \text{ for some spherically closed spherical subgroup } K \text{ of } G\}.$$

Let H be a spherical subgroup of G and let X be any spherical G-variety with open G-orbit G/H. Let $\overline{H}$ be the spherical closure of H. We define

$$\Sigma^{sc}(X) := \Sigma^{sc}(G/H) := \Sigma(G/\overline{H});$$
$$\Sigma^{N}(X) := \Sigma^{N}(G/H) := \Sigma(G/N_G(H)).$$

Remark 2.3. 1. It follows from [Kno96, Theorem 1.3] that for X affine, $\Sigma^N(X)$ given in Definition 2.2 agrees with the description in Section 1 of the set of N-spherical roots of X.
2. Thanks to [Los09b, Theorem 2] one can precisely describe the relationship between the three sets $\Sigma(X)$, $\Sigma^{sc}(X)$ and $\Sigma^N(X)$; see Proposition 2.9 and [VS13] for more information.
3. While $\Sigma^{sc}(X)$ and $\Sigma^N(X)$ are subsets of $\mathbb{N}S$, there exist wonderful varieties X such that $\Sigma(X) \not\subset \mathbb{Z}S$ (see [Was96]).
4. $\Sigma(X)$ is not always a basis of $\Lambda(X)$, but it is when X is wonderful.
5. The weight lattice, valuation cone and spherical roots are birational invariants of the spherical variety X since they only depend on its open G-orbit G/H. The same is true of the colors and the Cartan pairing once we (naturally) identify the colors of G/H with their closure in X.

The set $\Sigma^{sc}(G)$ is finite. More precisely, there is the next proposition, which follows from the classification of spherically closed spherical subgroups K of G with $\Lambda(G/K)$ of rank 1 [Ahi83, Los09b], see also [BL11, § 1.1.6 and § 2.4.1]. We recall that the **support** $\operatorname{supp}(\sigma)$ of $\sigma \in \mathbb{N}S$ is the set of simple roots which have a nonzero coefficient in the unique expression of σ as a linear combination of the simple roots.

Proposition 2.4. *An element σ of $\mathbb{N}S$ belongs to $\Sigma^{sc}(G)$ if and only if after numbering the simple roots in* $\operatorname{supp}(\sigma)$ *like Bourbaki (see* [Bou68]*) σ is listed in Table 1.*

Recall that K is a spherically closed spherical subgroup of G. As proved in [Lun01, §7.2], the triple $\mathscr{S}_{G/K} = (S^p_{G/K}, \Sigma(G/K), \mathbf{A}_{G/K})$ is a spherically closed Luna spherical system in the following sense.

TABLE 1. spherically closed spherical roots

Type of support	σ
A_1	α
A_1	2α
$\mathsf{A}_1 \times \mathsf{A}_1$	$\alpha + \alpha'$
A_n, $n \geq 2$	$\alpha_1 + \ldots + \alpha_n$
A_3	$\alpha_1 + 2\alpha_2 + \alpha_3$
B_n, $n \geq 2$	$\alpha_1 + \ldots + \alpha_n$
	$2(\alpha_1 + \ldots + 2\alpha_n)$
B_3	$\alpha_1 + 2\alpha_2 + 3\alpha_3$
C_n, $n \geq 3$	$\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n$
D_n, $n \geq 4$	$2\alpha_1 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$
F_4	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$
G_2	$4\alpha_1 + 2\alpha_2$
	$\alpha_1 + \alpha_2$

Definition 2.5. Let $(S^p, \Sigma, \mathbf{A})$ be a triple where S^p is a subset of S, Σ is a subset of $\Sigma^{sc}(G)$ and $\mathbf{A}$ is a finite set endowed with a $\mathbb{Z}$-bilinear pairing $c\colon \mathbb{Z}\mathbf{A} \times \mathbb{Z}\Sigma \to \mathbb{Z}$. For every $\alpha \in \Sigma \cap S$, let $\mathbf{A}(\alpha)$ denote the set $\{D \in \mathbf{A} : c(D, \alpha) = 1\}$. Such a triple is called a **spherically closed spherical** G**-system** if all the following axioms hold:

- (A1) for every $D \in \mathbf{A}$ and every $\sigma \in \Sigma$, we have that $c(D,\sigma) \leq 1$ and that if $c(D,\sigma) = 1$ then $\sigma \in S$;
- (A2) for every $\alpha \in \Sigma \cap S$, $\mathbf{A}(\alpha)$ contains two elements, which we denote by D_α^+ and D_α^-, and for all $\sigma \in \Sigma$ we have $c(D_\alpha^+, \sigma) + c(D_\alpha^-, \sigma) = \langle \alpha^\vee, \sigma \rangle$;
- (A3) the set $\mathbf{A}$ is the union of $\mathbf{A}(\alpha)$ for all $\alpha \in \Sigma \cap S$;
- (Σ1) if $2\alpha \in \Sigma \cap 2S$ then $\frac{1}{2}\langle \alpha^\vee, \sigma \rangle$ is a non-positive integer for all $\sigma \in \Sigma \setminus \{2\alpha\}$;
- (Σ2) if $\alpha, \beta \in S$ are orthogonal and $\alpha + \beta$ belongs to Σ then $\langle \alpha^\vee, \sigma \rangle = \langle \beta^\vee, \sigma \rangle$ for all $\sigma \in \Sigma$;
- (S) every $\sigma \in \Sigma$ is **compatible** with S^p, that is, for every $\sigma \in \Sigma$ there exists a spherically closed spherical subgroup K of G with $S^p_{G/K} = S^p$ and $\Sigma(G/K) = \{\sigma\}$.

Remark 2.6. 1. Condition (S) of Definition 2.5 can be stated in purely combinatorial terms as follows (see [BL11, §1.1.6]). A spherically closed spherical root σ is compatible with S^p if and only if:

- in case $\sigma = \alpha_1 + \ldots + \alpha_n$ with support of type B_n

$$\{\alpha \in \operatorname{supp}\sigma\colon \langle \alpha^\vee, \sigma \rangle = 0\} \setminus \{\alpha_n\} \subseteq S^p \subseteq \{\alpha \in S\colon \langle \alpha^\vee, \sigma \rangle = 0\} \setminus \{\alpha_n\},$$

- in case $\sigma = \alpha_1 + 2(\alpha_2 + \ldots + \alpha_{n-1}) + \alpha_n$ with support of type C_n

$$\{\alpha \in \operatorname{supp}\sigma\colon \langle \alpha^\vee, \sigma \rangle = 0\} \setminus \{\alpha_1\} \subseteq S^p \subseteq \{\alpha \in S\colon \langle \alpha^\vee, \sigma \rangle = 0\},$$

- in the other cases

$$\{\alpha \in \operatorname{supp}\sigma\colon \langle \alpha^\vee, \sigma \rangle = 0\} \subseteq S^p \subseteq \{\alpha \in S\colon \langle \alpha^\vee, \sigma \rangle = 0\}.$$

2. Definition 2.5 combines the standard definition of spherical system, see [Lun01, §2], with the requirement that it be spherically closed, see [Lun01, §7.1] and [BL11, §2.4].

As shown in [Lun01], the set $\Delta_{G/K}$ of colors and the Cartan pairing c of G/K are uniquely determined by $\mathscr{S}_{G/K}$, in the sense that they can be naturally identified with the set of colors of and the full Cartan pairing of $\mathscr{S}_{G/K}$, defined as follows. Let $\mathscr{S} = (S^p, \Sigma, \mathbf{A})$ be a (spherically closed) spherical G-system. The **set of colors of $\mathscr{S}$** is the finite set Δ obtained as the disjoint union $\Delta = \Delta^a \cup \Delta^{2a} \cap \Delta^b$ where:

- $\Delta^a = \mathbf{A}$,
- $\Delta^{2a} = \{D_\alpha : \alpha \in S \cap \frac{1}{2}\Sigma\}$,
- $\Delta^b = \{D_\alpha : \alpha \in S \setminus (S^p \cup \Sigma \cup \frac{1}{2}\Sigma)\}/\sim$, where $D_\alpha \sim D_\beta$ if α and β are orthogonal and $\alpha + \beta \in \Sigma$.

The **full Cartan pairing of $\mathscr{S}$** is the $\mathbb{Z}$-bilinear map $c\colon \mathbb{Z}\Delta \times \mathbb{Z}\Sigma \to \mathbb{Z}$ defined as:

$$c(D,\sigma) = \begin{cases} c(D,\sigma) & \text{if } D \in \Delta^a; \\ \frac{1}{2}\langle \alpha^\vee, \sigma\rangle & \text{if } D = D_\alpha \in \Delta^{2a}; \\ \langle \alpha^\vee, \sigma\rangle & \text{if } D = D_\alpha \in \Delta^b. \end{cases}$$

2.3. **Augmentations.** We continue to use K for a spherically closed spherical subgroup of G. By [Lun01, Proposition 6.4] spherical homogeneous spaces G/H such that $\overline{H}$, the spherical closure of H, is equal to K are classified by their weight lattice, which is an augmentation of $\mathscr{S}_{G/K}$.

Definition 2.7. Let $\mathscr{S} = (S^p, \Sigma, \mathbf{A})$ be a spherically closed spherical G-system with Cartan pairing $c : \mathbb{Z}\mathbf{A} \times \mathbb{Z}\Sigma \to \mathbb{Z}$. An **augmentation** of $\mathscr{S}$ is a lattice $\Lambda' \subset \Lambda$ endowed with a pairing $c' : \mathbb{Z}\mathbf{A} \times \Lambda' \to \mathbb{Z}$ such that $\Lambda' \supset \Sigma$ and

(a1) c' extends c;
(a2) if $\alpha \in S \cap \Sigma$ then $c'(D_\alpha^+, \xi) + c'(D_\alpha^-, \xi) = \langle \alpha^\vee, \xi\rangle$ for all $\xi \in \Lambda'$;
(σ1) if $2\alpha \in 2S \cap \Sigma$ then $\alpha \notin \Lambda'$ and $\langle \alpha^\vee, \xi\rangle \in 2\mathbb{Z}$ for all $\xi \in \Lambda'$;
(σ2) if α and β are orthogonal elements of S with $\alpha + \beta \in \Sigma$ then $\langle \alpha^\vee, \xi\rangle = \langle \beta^\vee, \xi\rangle$ for all $\xi \in \Lambda'$; and
(s) if $\alpha \in S^p$ then $\langle \alpha^\vee, \xi\rangle = 0$ for all $\xi \in \Lambda'$.

Let Δ be the set of colors of $\mathscr{S}$. The **full Cartan pairing** of the augmentation is the $\mathbb{Z}$-bilinear map $c' : \mathbb{Z}\Delta \times \Lambda' \to \mathbb{Z}$ given by

$$c'(D,\gamma) = \begin{cases} c'(D,\gamma) & \text{if } D \in \Delta^a; \\ \frac{1}{2}\langle \alpha^\vee, \gamma\rangle & \text{if } D = D_\alpha \in \Delta^{2a}; \\ \langle \alpha^\vee, \gamma\rangle & \text{if } D = D_\alpha \in \Delta^b. \end{cases} \tag{2.1}$$

Remark 2.8. By the definition of spherical closure, $\Delta_{G/H}$ and $\Delta_{G/\overline{H}}$ are naturally identified and the full Cartan pairing $\mathbb{Z}\Delta_{G/H} \times \Lambda(G/H) \to \mathbb{Z}$ on G/H is the full Cartan pairing of the augmentation corresponding to H (see Proposition 6.4 and the proof of Theorem 3 in [Lun01]).

We state here, for future reference, the following consequence of [Los09b, Theorem 2].

Proposition 2.9. *Let G/H be a spherical homogeneous space with $\Sigma^{sc}(G/H) = \Sigma$. Then*

$$\Sigma^N(G/H) = (\Sigma \setminus \Sigma_l) \cup 2\Sigma_l,$$

where $\Sigma_l = \{\alpha \in \Sigma \cap S\colon c_{G/H}(D_\alpha^+, \gamma) = c_{G/H}(D_\alpha^-, \gamma)$ for all $\gamma \in \Lambda(G/H)\}$.

Proof. This follows immediately from comparing [Los09b, Theorem 2], which describes the relationship between $\Sigma(G/H)$ and $\Sigma^N(G/H)$ with [Lun01, Lemma 7.1], which describes the relationship between $\Sigma(G/H)$ and $\Sigma^{sc}(G/H)$. Note that [Lun01, Lemma 7.1] can be deduced from [Los09b] without appealing to Luna's conjecture. □

2.4. **Colored cones and weight monoids of affine spherical varieties.** An embedding of a spherical homogeneous space G/H is called **simple** if it has only one closed orbit. Affine spherical varieties are simple. We recall from [Kno91, Theorem 3.1] that simple embeddings X of the spherical homogoneous space G/H are classified by their colored cone:

$$X \longmapsto (\mathcal{C}(X), \mathcal{F}(X)).$$

$\mathcal{F}(X)$ is a subset of $\Delta_{G/H}$ such that the subset $c(\mathcal{F}(X), \cdot)$ of $\mathrm{Hom}(\Lambda(G/H), \mathbb{Q})$ does not contain 0, and $\mathcal{C}(X)$ is a strictly convex polyhedral cone in $\mathrm{Hom}(\Lambda(G/H), \mathbb{Q})$ generated by $c(\mathcal{F}(X), \cdot)$ and finitely many elements of $\mathcal{V}_{G/H}$, of which the relative interior intersects $\mathcal{V}(G/H)$. By [Kno91, Theorem 6.7] the simple embedding X is affine if and only if there exists a character $\chi \in \Lambda(G/H)$ that is non-positive on $\mathcal{V}_{G/H}$, zero on $\mathcal{C}(X)$ and $c(\cdot, \chi)$ is strictly positive on $\Delta_{G/H} \setminus \mathcal{F}(X)$.

We gather some known results about the weight monoid of affine spherical varieties.

Proposition 2.10. *If X is an affine sperical G-variety with weight monoid $\Gamma(X)$ and open orbit G/H, then*

(a) the weight lattice of X (or of G/H) is $\mathbb{Z}\Gamma(X)$;
(b) the set S^p_X (which is the same as $S^p_{G/H}$) is equal to $\{\alpha \in S \colon \langle \alpha^\vee, \gamma\rangle = 0$ for all $\gamma \in \Gamma(X)\}$;
(c) the dual cone $\Gamma^\vee(X) := \{v \in \mathrm{Hom}(\mathbb{Z}\Gamma(X), \mathbb{Q}) : \langle v, \gamma\rangle \geq 0$ for all $\gamma \in \Gamma(X)\}$ to $\Gamma(X)$ is a strictly convex polyhedral cone;
(d) every ray of $\Gamma^\vee(X)$ contains an element of $c(\Delta_{G/H}, \cdot)$ or of $\mathcal{V}_{G/H}$;
(e) $\Gamma^\vee(X)$ contains $c(\Delta_{G/H}, \cdot)$.

Proof. These statements are well-known to experts. A proof of (a) can be found in [Bri10, Proposition 2.8(i)]. Assertion (b) is [Cam01, Lemme 10.2]. Assertion (c) is a standard fact in convex geometry. Parts (d) and (e) follow from the fact that the weight monoid $\Gamma(X)$ consists exactly of those elements of $\Lambda(G/H)$ that are non-negative on $\mathcal{C}(X)$ and on $c(\Delta_{G/H}, \cdot)$. This, in turn, is so because the boundary of the open B-orbit in X is of pure codimension 1 in X and because, by [Kno91, Lemma 2.4], each ray of $\mathcal{C}(X)$ which does not contain an element of $c(\Delta_{G/H}, \cdot)$ is spanned by the valuation of a G-stable prime divisor of X. □

2.5. **Adapted spherical roots.** Recall that Γ is a normal submonoid of Λ^+. Combining the results recalled above, one derives the condition on a set of spherical roots Σ for being adapted to Γ.

Definition 2.11. We say that a subset Σ of $\Sigma^{sc}(G)$ is **adapted** (or **N-adapted**) to Γ if there exists an affine spherical G-variety X such that $\Gamma(X) = \Gamma$ and $\Sigma^{sc}(X) = \Sigma$ (respectively, $\Sigma^N(X) = \Sigma$).

Remark 2.12. Let Σ be a subset of $\Sigma^{sc}(G)$. Losev's Theorem [Los09a, Theorem 1.2] asserts that there is *at most one* affine spherical G-variety X with $\Gamma(X) = \Gamma$ and $\Sigma^N(X) = \Sigma$. Because $\Sigma^{sc}(X)$ determines $\Sigma^N(X)$ (see Proposition 2.9) there is also at most one affine spherical G-variety Y with $\Sigma^{sc}(Y) = \Sigma$ and $\Gamma(Y) = \Gamma$.

The dual cone to Γ is

$$\Gamma^\vee := \{v \in \mathrm{Hom}(\mathbb{Z}\Gamma, \mathbb{Q}) : \langle v, \gamma\rangle \geq 0 \text{ for all } \gamma \in \Gamma\}.$$

It is a strictly convex polyhedral cone. We denote the set of primitive vectors on its rays by $E(\Gamma)$:

$$E(\Gamma) := \{\delta \in (\mathbb{Z}\Gamma)^* \colon \delta \text{ spans a ray of } \Gamma^\vee \text{ and } \delta \text{ is primitive}\}. \tag{2.2}$$

Observe that

(2.3)

$$E(\Gamma) = \{\delta \in (\mathbb{Z}\Gamma)^* \colon \delta \text{ is primitive}, \delta(\Gamma) \subset \mathbb{Z}_{\geq 0}, \delta \text{ is the equation of a face of codim 1 of } \mathbb{Q}_{\geq 0}\Gamma\}.$$

Moreover, for $\alpha \in S \cap \mathbb{Z}\Gamma$, we define

$$a(\alpha) := \{\delta \in (\mathbb{Z}\Gamma)^* \colon \delta(\alpha) = 1 \text{ and } \big(\delta \in E(\Gamma) \text{ or } \alpha^\vee - \delta \in E(\Gamma)\big)\}.$$

Finally, we put

$$S^p(\Gamma) := \{\alpha \in S : \langle \alpha^\vee, \gamma \rangle = 0 \text{ for all } \gamma \in \Gamma\}.$$

Proposition 2.13. *Let Γ be a normal submonoid of Λ^+. A subset Σ of $\Sigma^{sc}(G)$ is adapted to Γ if and only if there exists a spherically closed spherical system $\mathscr{S} = (S^p, \Sigma, \mathbf{A})$ such that*

(1) $S^p = S^p(\Gamma)$; and
(2) $\mathbb{Z}\Gamma$ is an augmentation of $\mathbb{Z}\Sigma$; and
(3) if $\delta \in E(\Gamma)$, then $\langle \delta, \sigma \rangle \leq 0$ for all $\sigma \in \Sigma$ or there exists $D \in \Delta$ such that $c(D, \cdot)$ is a positive multiple of δ; where Δ is the set of colors of $\mathscr{S}$ and $c : \mathbb{Z}\Delta \times \mathbb{Z}\Gamma \to \mathbb{Z}$ is the full Cartan pairing of the augmentation; and
(4) $c(D, \cdot) \in \Gamma^\vee$ for all $D \in \Delta$.

Proof. This is a consequence of the results we reviewed in Sections 2.2 through 2.4. We begin with the *necessity* of the conditions. Let X be an affine spherical G-variety with $\Sigma^{sc}(X) = \Sigma$ and $\Gamma(X) = \Gamma$. Let G/H be the open orbit of X and let $\overline{H}$ be the spherical closure of H. Then $\Sigma^{sc}(X) = \Sigma(G/\overline{H})$ by definition, and $S^p_{G/H} = S^p(\Gamma)$ by Proposition 2.10(b). Moreover $S^p_{G/H} = S^p_{G/\overline{H}}$. It follows from §5.1 and Lemma 7.1 in [Lun01] that $(S^p(\Gamma), \Sigma, \mathbf{A}_{G/\overline{H}})$ is a spherically closed spherical system. Since H has spherical closure $\overline{H}$, (2) follows from [Lun01, Proposition 6.4]. Conditions (3) and (4) follow from (d) and (e) of Proposition 2.10.

We now show that the conditions are *sufficient* for Σ to be adapted to Γ. By [BP11] there exists a spherically closed spherical subgroup K of G with spherical system $\mathscr{S}$. Condition (2) implies by [Lun01, Proposition 6.4] that there exists a spherical subgroup H of G with $\overline{H} = K$ and $\Lambda(G/H) = \mathbb{Z}\Gamma$. What remains is to prove that G/H has an affine embedding X with weight monoid Γ. That is, by [Kno91, Theorems 3.1 and 6.7] we have to show that there exists a colored cone $(\mathcal{C}, \mathcal{F})$ in $\mathrm{Hom}(\mathbb{Z}\Gamma, \mathbb{Q})$, with respect to $\mathcal{V} = \{v \in \mathrm{Hom}(\mathbb{Z}\Gamma, \mathbb{Q}) : \langle v, \sigma \rangle \leq 0 \text{ for all } \sigma \in \Sigma\}$ and the set of colors Δ of $\mathscr{S}$, such that:

(i) there exists $\chi \in \mathbb{Z}\Gamma$ that is non-positive on $\mathcal{V}$, zero on $\mathcal{C}$ and strictly positive on $\Delta \setminus \mathcal{F}$; and
(ii) $\Gamma = \{\gamma \in \mathbb{Z}\Gamma : \langle v, \gamma \rangle \geq 0 \text{ for all } v \in \mathcal{C} \cup \Delta\}$.

We claim that if (1), (3) and (4) hold, then the desired colored cone exists. Indeed, take $\mathcal{C}$ to be the maximal face of $\Gamma^\vee$ whose relative interior meets $\mathcal{V}$ with $\mathcal{F}$ the set of colors contained in $\mathcal{C}$ (such a maximal face exists since the zero face actually meets $\mathcal{V}$). Then $c(\mathcal{F}, \cdot)$ does not contain 0 by (1) and $\mathcal{C}$ is contained in a hyperplane that separates $\mathcal{V}$ and $\Delta \setminus \mathcal{F}$. This yields χ. The inclusion "$\subset$" of (ii) holds because $\mathcal{C} \subset \Gamma^\vee$ and because $c(\Delta, \cdot) \subset \Gamma^\vee$ by (4). The other inclusion follows from (3) and the maximality of $\mathcal{C}$. ☐

Remark 2.14. It follows from equation (2.4) below that the spherical system $\mathscr{S}$ and the Cartan pairing of the augmentation in Proposition 2.13 are uniquely determined by Γ and Σ.

Corollary 2.15. *Let Γ be a normal submonoid of Λ^+. A subset Σ of $\Sigma^{sc}(G)$ is N-adapted to Γ if and only if there exists a subset $\widetilde{\Sigma}$ of $\Sigma^{sc}(G)$ which is adapted to Γ and such that $\Sigma = (\widetilde{\Sigma} \setminus \widetilde{\Sigma}_l) \cup 2\widetilde{\Sigma}_l$, where $\widetilde{\Sigma}_l = \{\alpha \in \widetilde{\Sigma} \cap S \colon a(\alpha) \text{ has one element}\}$.*

Proof. This is a consequence of Proposition 2.13 and Proposition 2.9 once we show the following: if c is the full Cartan pairing of an augmentation $\mathbb{Z}\widetilde{\Sigma} \subset \mathbb{Z}\Gamma$ of a spherical system $\mathscr{S} = (S^p(\Gamma), \widetilde{\Sigma}, \mathbf{A})$ as in Proposition 2.13, then

$$a(\alpha) = \{c(D^+_\alpha, \cdot), c(D^-_\alpha, \cdot)\} \tag{2.4}$$

for all $\alpha \in \widetilde{\Sigma} \cap S$. To prove the inclusion "$\subset$" in (2.4), let $\delta \in a(\alpha)$. Then, $\langle \delta, \alpha \rangle = \langle \alpha^\vee - \delta, \alpha \rangle = 1 > 0$ and at least one of δ and $\alpha^\vee - \delta$ is in $E(\Gamma)$. By (3) in Proposition 2.13 it follows that $\{\delta, \alpha^\vee - \delta\}$ contains a positive rational multiple of $c(D, \cdot)$ for some color D. By axiom (A1) of the spherical system

$\mathscr{S}$, and the description (2.1) of c, the color D must be D_α^+ or D_α^-. Since $c(D_\alpha^+,\alpha) = c(D_\alpha^-,\alpha) = 1$, this implies that the two sets $\{\delta, \alpha^\vee - \delta\}$ and $\{c(D_\alpha^+,\cdot), c(D_\alpha^-,\cdot)\}$ intersect, and so by axiom (a2) of the augmentation, they are equal. For the reverse inclusion in (2.4) we have to show that $c(D_\alpha^+,\cdot)$ or $c(D_\alpha^-,\cdot)$ belongs to $E(\Gamma)$. If neither belongs to $E(\Gamma)$, then by (3) and (4) in Proposition 2.13 and the description (2.1) of c, each of them is a linear combination with positive rational coefficients of elements of $\mathrm{Hom}(\mathbb{Z}\Gamma, \mathbb{Q})$ which are nonpositive on α. This contradicts the fact that $c(D_\alpha^+,\alpha) = 1$ and finishes the proof of equation (2.4). $\square$

As the next two corollaries show, one can characterize very explicitly whether a single spherical root is adapted (Corollary 2.16) or N-adapted (Corollary 2.17) to Γ. In a 2005 working document, Luna had proposed a statement like Corollary 2.16. We remark that while Proposition 2.13 and Corollary 2.15 depend on the full classification of wonderful varieties by spherical systems (Luna's conjecture), the next two results only use the combinatorial classification of rank 1 wonderful varieties, which was obtained in [Bri89] and also in [Ahi83].

Corollary 2.16. *Let Γ be a normal submonoid of Λ^+. If $\sigma \in \Sigma^{sc}(G)$, then σ is adapted to Γ if and only if all of the following conditions hold:*

(1) $\sigma \in \mathbb{Z}\Gamma$;
(2) σ is compatible with $S^p(\Gamma)$;
(3) if $\sigma \notin S$ and $\delta \in E(\Gamma)$ such that $\langle \delta, \sigma \rangle > 0$ then there exists $\beta \in S \setminus S^p(\Gamma)$ such that $\beta^\vee$ is a positive multiple of δ;
(4) if $\sigma \in S$ then
   (a) $a(\sigma)$ has one or two elements; and
   (b) $\langle \delta, \gamma \rangle \geq 0$ for all $\delta \in a(\sigma)$ and all $\gamma \in \Gamma$; and
   (c) $\langle \delta, \sigma \rangle \leq 1$ for all $\delta \in E(\Gamma)$;
(5) if $\sigma = 2\alpha \in 2S$, then $\alpha \notin \mathbb{Z}\Gamma$ and $\langle \alpha^\vee, \gamma \rangle \in 2\mathbb{Z}$ for all $\gamma \in \Gamma$;
(6) if $\sigma = \alpha + \beta$ with $\alpha, \beta \in S$ and $\alpha \perp \beta$, then $\alpha^\vee = \beta^\vee$ on Γ.

Proof. Let $\sigma \in \Sigma^{sc}(G)$. Define the triple $\mathscr{S}$ by

$$\mathscr{S} := \begin{cases} (S^p(\Gamma), \{\sigma\}, \emptyset) & \text{if } \sigma \notin S; \\ (S^p(\Gamma), \{\sigma\}, \{D_\sigma^+, D_\sigma^-\}) & \text{if } \sigma \in S. \end{cases}$$

Let Δ be the set of colors of $\mathscr{S}$ (see Section 2.2) and let $c : \mathbb{Z}\Delta \times \mathbb{Z}\Gamma$ be the bilinear pairing given by equation (2.1) if $\sigma \notin S$ and by

$$c(D,\gamma) = \langle \alpha^\vee, \gamma \rangle \text{ if } D = D_\alpha \in \Delta^b; \tag{2.5}$$

$$\{c(D_\sigma^+,\cdot), c(D_\sigma^-,\cdot)\} = a(\sigma), \tag{2.6}$$

if $\sigma \in S$. By Remark 2.14, we have to show that the conditions of the corollary hold if and only if $\mathscr{S}$ is a spherically closed spherical system of which $\mathbb{Z}\Gamma$ together with c is an augmentation such that conditions (3) and (4) of Proposition 2.13 hold. We briefly describe the straightforward verification.

We begin with the case $\sigma \notin S$. Then we have that $\mathscr{S}$ is a spherically closed spherical G-system if and only if (2) holds. Then c gives an augmentation of $\mathscr{S}$ if and only if (1), (5) and (6) hold. Condition (4) of Proposition 2.13 is vacuous since $\Gamma \subset \Lambda^+$ and every $c(D,\cdot)$ is a positive multiple of a coroot. Condition (3) in the corollary is the same as condition (3) of Proposition 2.13 by the definition of c.

We proceed to the case $\sigma \in S$. Now $\mathscr{S}$ is a spherically closed spherical G-system if and only if (2) and (4a) hold. Next, by construction, c gives an augmentation of $\mathscr{S}$ if and only if we have (1). Condition (4) of Proposition 2.13 is equivalent to (4b). Finally, condition (3) of Proposition 2.13 is equivalent to (4c), again by the definition of c. $\square$

Corollary 2.17. *Let Γ be a normal submonoid of Λ^+. If $\sigma \in \Sigma^{sc}(G)$, then σ is N-adapted to Γ if and only if all of the following conditions hold:*

(1) $\sigma \in \mathbb{Z}\Gamma$;
(2) σ is compatible with $S^p(\Gamma)$;
(3) if $\sigma \notin S$ and $\delta \in E(\Gamma)$ such that $\langle \delta, \sigma \rangle > 0$ then there exists $\beta \in S \setminus S^p(\Gamma)$ such that $\beta^\vee$ is a positive multiple of δ;
(4) if $\sigma \in S$ then
   (a) $a(\sigma)$ has two elements; and
   (b) $\langle \delta, \gamma \rangle \geq 0$ for all $\delta \in a(\sigma)$ and all $\gamma \in \Gamma$; and
   (c) $\langle \delta, \sigma \rangle \leq 1$ for all $\delta \in E(\Gamma)$;
(5) if $\sigma = 2\alpha \in 2S$, then $\langle \alpha^\vee, \gamma \rangle \in 2\mathbb{Z}$ for all $\gamma \in \Gamma$;
(6) if $\sigma = \alpha + \beta$ with $\alpha, \beta \in S$ and $\alpha \perp \beta$, then $\alpha^\vee = \beta^\vee$ on Γ.

Proof. By Corollary 2.15, if $\sigma \notin S \cup 2S$, then σ is adapted to Γ if and only if it is N-adapted to Γ. From the same corollary it follows that $\sigma \in S$ is N-adapted to Γ if and only if it is adapted to Γ and $a(\sigma)$ has two elements. The only remaining case is $\sigma = 2\alpha$ for some $\alpha \in S$. Again by Corollary 2.15, 2α is N-adapted to Γ if and only if either

(i) 2α is adapted to Γ; or
(ii) α is adapted to Γ and $a(\alpha)$ has one element.

We assume that (1) and (2) hold and claim that (3) and (5) hold if and only if (i) or (ii) is true. Indeed, it is clear from Corollary 2.16 that if 2α is adapted to Γ then we have (3) and (5). On the other hand, if α is adapted to Γ and $a(\alpha)$ has one element, then that element is $\frac{1}{2}\alpha^\vee$ and so (5) holds. Moreover, condition (4c) of Corollary 2.16 implies (3) of this corollary. Conversely, suppose that we have (3) and (5). Since the restricion of $\alpha^\vee$ to $\mathbb{Z}\Gamma$ belongs to $\Gamma^\vee$ and $\langle \alpha^\vee, 2\alpha \rangle > 0$, there exists $\delta \in E(\Gamma)$ such that $\langle \delta, 2\alpha \rangle > 0$. It follows from (3) that $\delta = q\beta^\vee$ for some $\beta \in S \setminus S^p(\Gamma)$ and $q \in \mathbb{Q}_{>0}$. Clearly, $\beta = \alpha$, which proves that δ is the only element of $E(\Gamma)$ that takes a positive value on 2α. Now, suppose that 2α is not adapted to Γ, i.e. that (i) does not hold. Then α must be an element of $\mathbb{Z}\Gamma$. By (5), $\frac{1}{2}\alpha^\vee$ takes integer values on $\mathbb{Z}\Gamma$, and since it takes value 1 on α, it is primitive in $(\mathbb{Z}\Gamma)^*$ and therefore an element of $E(\Gamma)$ and the only element of $a(\alpha)$. It follows from Corollary 2.16 that (ii) is true. This finishes the proof. ☐

3. The T_{ad}-weights in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$

For the remainder of the paper, Γ will be a free monoid with basis $F \subset \Lambda^+$. In this section, we begin by recalling that the moduli scheme M_Γ is an open subscheme of a certain invariant Hilbert scheme H_Γ. This allows one to realize the tangent space $\mathrm{T}_{X_0}\mathrm{M}_\Gamma$ as a T-submodule of a certain vector space $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. In Section 3.2 we prove that the if γ is a T-weight in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$, then it is a spherical root of spherically closed type. In Section 3.3 we further show that γ is compatible with $S^p(\Gamma)$. We also show that if $\gamma \notin S$, then the weight space $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}_{(\gamma)}$ has dimension at most 1. For notational and computational convenience, we actually work with the opposite of Alexeev and Brion's T-action on M_Γ and with a twist of their action on H_Γ (see Section 3.1).

3.1. **The invariant Hilbert scheme and its tangent space.** We briefly review some known facts regarding M_Γ and its relation to a certain invariant Hilbert scheme H_Γ. For more details we refer to [AB05], [Bri13, Section 4.3] and to [PVS12, §2.1 and §2.2]. Recall that Γ is a free monoid of dominant weights with basis $F = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$, and put

$$V := V(\lambda_1) \oplus V(\lambda_2) \oplus \ldots \oplus V(\lambda_r);$$
$$x_0 := v_{\lambda_1} + v_{\lambda_2} + \ldots + v_{\lambda_r}.$$

We denote by H_Γ the Hilbert scheme $\mathrm{Hilb}^G_h(V)$ of [AB05], where h is the characteristic function of $\Gamma^* := -w_0\Gamma$ (where w_0 is the longest element in the Weyl group of G). The scheme H_Γ parametrizes the G-stable ideals I of $\Bbbk[V]$ such that $\Bbbk[V]/I \simeq \oplus_{\lambda\in\Gamma^*}V(\lambda)$ as G-modules. We equip H_Γ with the action of T described in [PVS12, §2.2]. This is the same action as in [BCF08], and is a 'twist' of the action in [AB05] and in [Bri13, p. 101]. The center $Z(G)$ of G belongs to the kernel of the action, which therefore descends to an action of $T_{\mathrm{ad}} := T/Z(G)$. We will refer to our action as the "T_{ad}-action" on H_Γ. As was reviewed in [PVS12, §2.2] it follows from [AB05, Corollary 1.17 and Lemma 2.2] that since Γ^* is free, we can view M_{Γ^*} as a T_{ad}-stable open subscheme of H_Γ. Under this identification, the T_{ad}-fixed point X_0 of M_{Γ^*} corresponds to a certain subvariety of V which we also denote by X_0, namely

$$X_0 = \text{ the closure of the } G\text{-orbit of } x_0 \text{ in } V. \tag{3.1}$$

The next proposition relates M_Γ to H_Γ.

Proposition 3.1. *Let Γ be a free monoid of dominant weights. If we equip M_Γ with the opposite of the T_{ad}-action in* [AB05] *and H_Γ with the T_{ad}-action in* [PVS12, §2.2]*, then there is a T_{ad} equivariant open embedding*

$$\mathrm{M}_\Gamma \hookrightarrow \mathrm{H}_\Gamma$$

which sends the unique T_{ad}-fixed point of M_Γ to the point X_0 in equation (3.1).

Proof. This a matter of "formal bookkeeping." Composing the action of G on $V(\Gamma)$ with the Chevalley involution of G induces an isomorphism $\mathrm{M}_\Gamma \simeq \mathrm{M}_{\Gamma^*}$. Composing this isomorphism with the open T_{ad}-equivariant embedding $\mathrm{M}_{\Gamma^*} \hookrightarrow \mathrm{H}_\Gamma$ chosen above gives an open embedding $\mathrm{M}_\Gamma \hookrightarrow \mathrm{H}_\Gamma$. Comparing the definition of the action in [AB05] with that of the action in [PVS12, §2.2] one shows that this open embedding is T_{ad}-equivariant for the actions as given in the proposition. □

Remark 3.2. The action Alexeev and Brion defined on M_Γ is conceptually the most natural, while the action we are using on H_Γ is computationally more convenient.

By [AB05, Proposition 1.13], there is a canonical isomorphism

$$\mathrm{T}_{X_0}\mathrm{H}_\Gamma \simeq H^0(X_0, \mathcal{N}_{X_0|V})^G$$

where $H^0(X_0, \mathcal{N}_{X_0|V})^G$ is the space of G-invariant global sections of the normal sheaf $\mathcal{N}_{X_0|V}$ of X_0 in V. Moreover, by [Bri13, Proposition 3.10], there is an inclusion of T_{ad}-modules

$$H^0(X_0, \mathcal{N}_{X_0|V})^G \hookrightarrow (V/\mathfrak{g}\cdot x_0)^{G_{x_0}} \simeq H^0(G\cdot x_0, \mathcal{N}_{X_0|V})^G,$$

where the T_{ad}-action on $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ is induced by the following action of T_{ad} on V. For $t \in T_{\mathrm{ad}}$ and v a T-weight vector of weight δ in $V(\lambda) \subset V$, we put $t\cdot v := \lambda(t)\delta(t)^{-1}v$.

3.2. The T_{ad}-weights in $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ are spherical roots of G.

In this section, we prove the following theorem.

Theorem 3.3. *If γ is a T_{ad}-weight in $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ then γ is a spherically closed spherical root of G.*

Proof. Corollary 3.8 and Corollary 3.14. □

For future use, we recall the following elementary and well-known facts regarding $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$. We include proofs for convenience. Before stating them we define

$$F^\perp := \{\beta \in R^+ \colon \langle \lambda, \beta^\vee \rangle = 0 \text{ for all } \lambda \in F\}.$$

Proposition 3.4. *(a) A basis of T_{ad}-eigenvectors of $\mathfrak{g}\cdot x_0$ is given by $\{v_\lambda \colon \lambda \in F\} \cup \{X_{-\beta}\cdot x_0 \colon \beta \in R^+ \setminus F^\perp\}$.*

(b) If $[v]$ is a T_{ad}-eigenvector in $V/\mathfrak{g}\cdot x_0$ of weight γ, then $[v] \in (V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ if and only if $\gamma \in \mathbb{Z}\Gamma$ and $X_\beta \cdot v \in \mathfrak{g}\cdot x_0$ for all $\beta \in S \cup -(S\cap F^\perp)$.

Proof. Assertion (a) follows from the fact that $\mathfrak{g}\cdot x_0 = \mathfrak{b}^-\cdot x_0 = \mathfrak{t}\cdot x_0 + \mathfrak{n}^-\cdot x_0$ and that F is linearly independent. Assertion (b) follows from [PVS12, Lemma 2.16] and the fact that $\mathfrak{g}_{x_0}$ is generated as a Lie algebra by $\mathfrak{t}_{x_0}$ and the root spaces $\mathfrak{g}_\beta$ with $\beta \in S \cup -(S\cap F^\perp)$ (see, e.g., [Hum75, Theorem 30.1]). ☐

In the remainder of this section, γ is a T_{ad}-weight occuring in $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ and $v\in V$ a T_{ad}-eigenvector of weight γ such that $[v]$ is a nonzero element of $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$. By Propostion 3.4 (and the choice of our T_{ad}-action), the weight γ belongs to $\mathbb{N}S \cap \mathbb{Z}\Gamma$.

Lemma 3.5 ([BCF08, Lemma 3.3])**.**

(1) There exists at least one simple root α such that $X_\alpha v \neq 0$.
(2) If α is a simple root such that $X_\alpha v \neq 0$ and $\gamma \neq \alpha$, then $\gamma - \alpha$ is a positive root.
(3) If α is a simple root such that $\gamma-\alpha$ is a root then there exists $z \in \Bbbk$ such that $X_\alpha v = z\, X_{-\gamma+\alpha} x_0$.

Proof. The vector v cannot be a linear combination of the highest weight vectors v_{λ_i}, otherwise (since the weights λ_i are linearly independent) it would belong to $\mathfrak{t}\cdot x_0 \subset \mathfrak{g}\cdot x_0$. Moreover, since $X_\alpha \in \mathfrak{g}_{x_0}$ for all $\alpha \in S$, $X_\alpha v$ is a T_{ad}-eigenvector of weight $\gamma-\alpha$ in $\mathfrak{g}\cdot x_0$. ☐

We first deal with the case where γ is a root. Notice that since $\gamma \in \mathbb{N}S$, it is then a positive root. As is well known, we then also have that $\mathrm{supp}(\gamma)$ is a connected subset of the Dynkin diagram of G.

Lemma 3.6. *If γ is a root, which is not simple, then there exist at least two distinct simple roots α such that $\gamma-\alpha$ is a root.*

Proof. Assume that there exists only one simple root α such that $\gamma - \alpha$ is a root. By Lemma 3.5, there exists $z\in\Bbbk$ such that $X_\alpha v = z\, X_{-\gamma+\alpha}x_0$. Moreover, there exists $z' \in \Bbbk^\times$ such that $[X_\alpha, X_{-\gamma}] = z'\, X_{-\gamma+\alpha}$. Therefore, if we put $z'' = z/z'$ then $X_\alpha(v + z''\, X_{-\gamma}x_0) = 0$. Since $[v] = [v + z''\, X_{-\gamma}x_0]$ in $V/\mathfrak{g}\cdot x_0$ we can assume that $X_\alpha v = 0$. Since $\gamma - \alpha'$ is not a positive root for all $\alpha' \in S \setminus \{\alpha\}$, it then follows that $X_\alpha v = 0$ for all $\alpha \in S$, which contradicts Lemma 3.5(1). ☐

Proposition 3.7. *If γ is a root, of which the support is not of type G_2, then it is a locally dominant short root, i.e. the dominant short root in the root subsystem generated by the simple roots of its support.*

Proof. **I**. Let α_1 and α_2 be two orthogonal simple roots such that $\gamma-\alpha_1$ and $\gamma-\alpha_2$ are roots. Notice that $\gamma-\alpha_1-\alpha_2$ is also a root. We claim that if there exists $\lambda\in F$ not orthogonal to $\gamma-\alpha_1-\alpha_2$, then we can assume

$$X_{\alpha_1}v = X_{\alpha_2}v = 0. \tag{3.2}$$

Indeed, there exist $z_1, z_2 \in \Bbbk^\times$ such that

$$[X_{\alpha_1}, X_{-\gamma}] = z_1\, X_{-\gamma+\alpha_1};$$
$$[X_{\alpha_2}, X_{-\gamma}] = z_2\, X_{-\gamma+\alpha_2}.$$

Moreover, using the Jacobi identity and the fact that $[X_{\alpha_1}, X_{\alpha_2}] = 0$ one finds that

$$[X_{\alpha_2}, X_{-\gamma+\alpha_1}] = \frac{z_2}{z_1}[X_{\alpha_1}, X_{-\gamma+\alpha_2}].$$

By Lemma 3.5(3), there exist $z_1', z_2' \in \Bbbk$ such that

$$X_{\alpha_1}v = z_1'\, X_{-\gamma+\alpha_1}x_0;$$
$$X_{\alpha_2}v = z_2'\, X_{-\gamma+\alpha_2}x_0.$$

Since $X_{\alpha_2} X_{\alpha_1} v = X_{\alpha_1} X_{\alpha_2} v$ we obtain that

$$(\frac{z_2}{z_1} z_1' - z_2')[X_{\alpha_1}, X_{-\gamma+\alpha_2}]x_0 = 0.$$

Using that there exists $\lambda \in F$ not orthogonal to the root $\gamma - \alpha_1 - \alpha_2$ it follows that $\frac{z_2}{z_1} z_1' - z_2' = 0$, that is

$$\frac{z_1'}{z_1} = \frac{z_2'}{z_2}.$$

This implies that by replacing v by $v - \frac{z_1'}{z_1} X_{-\gamma} x_0 = v - \frac{z_2'}{z_2} X_{-\gamma} x_0$, we can assume (3.2).

II. The same can be done if we have $\alpha_1, \alpha_2, \ldots, \alpha_k$ simple roots with α_j orthogonal to α_{j+1} for all $j \in \{1, 2, \ldots, k-1\}$ and such that $\gamma - \alpha_j$ is a root for all $j \in \{1, 2, \ldots, k\}$. More precisely, we claim that if there exists $\lambda \in F$ not orthogonal to $\gamma - \alpha_1 - \ldots - \alpha_k$, then we can assume that for all $j \leq k$

$$X_{\alpha_j} v = 0. \tag{3.3}$$

Indeed, for every $j \in \{1, 2, \ldots, k\}$ there exists, as in part I, $z_j \in \Bbbk^\times$ and $z_j' \in \Bbbk$ such that $[X_{\alpha_j}, X_{-\gamma}] = z_j X_{-\gamma+\alpha_j}$ and $X_{\alpha_j} v = z_j' X_{-\gamma+\alpha_j} x_0$. Let λ be an element of F that is not orthogonal to $\gamma - \alpha_1 - \ldots - \alpha_k$. Then λ is not orthogonal to $\gamma - \alpha_j - \alpha_{j+1}$ for all $j \in \{1, 2, \ldots, k-1\}$. By applying part I $(k-1)$ times to the pairs α_j, α_{j+1} we obtain that

$$\frac{z_1'}{z_1} = \frac{z_2'}{z_2} = \ldots = \frac{z_k'}{z_k}.$$

This implies that by replacing v by $v - \frac{z_1'}{z_1} X_{-\gamma} x_0$, we can assume (3.3).

III. Assume that there exist more than two simple roots, say $\alpha_1, \ldots, \alpha_k$, such that $\gamma - \alpha_j$ is a root for all $j \in \{1, 2, \ldots, k\}$. We claim that they can be reordered such that α_j is orthogonal to α_{j+1} for all $j < k$ as in part II.

This can be verified by making use of the classification of root systems, checking case-by-case all the positive roots, noticing along the way (although we will not need this) that k is at most 3. This is straightforward for the classical types. To avoid the large number of case-by-case checkings in the exceptional types E_6, E_7, E_8 and F_4 one can use for example the following argument. If it were not possible to reorder the simple roots $\alpha_1, \ldots, \alpha_k$ as required, then there would exist three roots among them, say $\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}$, such that α_{j_2} is not orthogonal to both α_{j_1} and α_{j_3}. We will now show that this is impossible for each exceptional type using well-known properties of root systems of rank 2 and 3. Notice, in particular, that if the support of γ is not of type G_2 and if $\gamma - \alpha$ is a root for some simple root α, then

$$\langle \alpha^\vee, \gamma \rangle \geq 0$$

since otherwise there would exist a root string of length greater than 3.

In types E_6, E_7 and E_8 all the roots have the same length so we would necessarily have $\langle (\alpha_{j_m})^\vee, \gamma \rangle = 1$ for $m \in \{1, 2, 3\}$, but this is absurd since it would mean that $\langle (\alpha_{j_1} + \alpha_{j_2} + \alpha_{j_3})^\vee, \gamma \rangle = 3$. In type F_4 the three simple roots would generate a root subsytem of type B_3 or of type C_3. In the former case (type B_3) we would necessarily have $\langle (\alpha_{j_1})^\vee, \gamma \rangle = \langle (\alpha_{j_2})^\vee, \gamma \rangle = 1$ assuming α_{j_1} and α_{j_2} are long, but this is absurd since it would mean $\langle (\alpha_{j_1} + \alpha_{j_2} + \alpha_{j_3})^\vee, \gamma \rangle \geq 4$. In the latter case (type C_3) we would necessarily have $\langle (\alpha_{j_1})^\vee, \gamma \rangle = 1$ assuming α_{j_1} is long. If $\langle (\alpha_{j_3})^\vee, \gamma \rangle$ is positive, then $\langle (\alpha_{j_1} + \alpha_{j_2} + \alpha_{j_3})^\vee, \gamma \rangle$ is greater than 2, which is not possible in type F_4. If $\langle (\alpha_{j_3})^\vee, \gamma \rangle = 0$, then $\gamma + \alpha_{j_3}$ is a root, and $\langle (\alpha_{j_1} + \alpha_{j_2} + \alpha_{j_3})^\vee, \gamma + \alpha_3 \rangle$ is greater than 2, which is again absurd.

IV. We now want to prove that γ is locally dominant (if the support of γ is not of type G_2). The fact that γ is locally short then follows. Indeed, if the support of γ is not simply laced, then the highest root in the root system generated by that support does not satisfy Lemma 3.6:

- in type B_n, $n \geq 2$, the highest root is $\alpha_1 + 2(\alpha_2 + \ldots + \alpha_n) = \omega_2$;

- in type C_n, $n \geq 3$, the highest root is $2(\alpha_1 + \ldots + \alpha_{n-1}) + \alpha_n = 2\omega_1$;
- in type F_4 the highest root is $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = \omega_1$.

To obtain a condradiction we assume that γ is not locally dominant, that is, we assume that there exists $\beta \in \mathrm{supp}(\gamma)$ such that $\langle \beta^\vee, \gamma \rangle < 0$. Recall from part III that in type different from G_2 if $\gamma - \alpha$ is a root for a simple root α, then $\langle \alpha^\vee, \gamma \rangle \geq 0$.

Suppose first that there are exactly $k > 2$ simple roots, say $\alpha_1, \ldots, \alpha_k$, such that $\gamma - \alpha_j$ is a root for all $j \leq k$. From the assumption that γ is not locally dominant, it follows that there exists $\lambda \in F$ not orthogonal to $\gamma - \alpha_1 - \ldots - \alpha_k$. By parts II and III we can then assume that $X_{\alpha_j} v = 0$ for all $j \leq k$. This contradicts Lemma 3.5(1).

If there are exactly two simple roots α_1 and α_2 such that $\gamma - \alpha_1$ and $\gamma - \alpha_2$ are roots, and α_1 and α_2 are orthogonal, then by part I we get the same contradiction with Lemma 3.5(1).

Furthermore, if the support of γ has cardinality ≤ 2, then the proposition follows by Lemma 3.6. Indeed, the only roots with support of cardinality ≤ 2 satisfying Lemma 3.6 are:

- with support of type A_1, α_1,
- with support of type A_2, $\alpha_1 + \alpha_2$,
- with support of type B_2, $\alpha_1 + \alpha_2$.

Therefore, we now restrict to the case of support of γ of cardinality > 2, and assume that there are only two simple roots α_1 and α_2, such that $\gamma - \alpha_1$ and $\gamma - \alpha_2$ are roots, and that α_1 and α_2 are not orthogonal. Notice that $\alpha_1 + \alpha_2$ is a root. Up to exchanging α_1 and α_2 we can assume that

$$\langle \alpha_2^\vee, \gamma \rangle > 0 \text{ and } \alpha_1 + 2\alpha_2 \notin R. \tag{3.4}$$

Indeed, at least one of the two $\langle \alpha_1^\vee, \gamma \rangle$ and $\langle \alpha_2^\vee, \gamma \rangle$ must be positive (otherwise γ would be antidominant), and not both $2\alpha_1 + \alpha_2$ and $\alpha_1 + 2\alpha_2$ can be roots. If say $2\alpha_1 + \alpha_2$ is a root, then $\|\alpha_1\| < \|\alpha_2\|$, hence α_2 is long and therefore $\langle \alpha_2^\vee, \gamma \rangle$ must be > 0.

Under (3.4) we have

$$\langle \alpha_2^\vee, \gamma - \alpha_1 \rangle \geq 1 + 1$$

hence $\gamma - \alpha_1 - \alpha_2$ and $\gamma - \alpha_1 - 2\alpha_2$ are roots. Since γ is not locally dominant, there is an element λ of F such that $\langle (\gamma - \alpha_1 - 2\alpha_2)^\vee, \lambda \rangle \neq 0$.

To conclude the proof of the proposition, we use once again an argument similar to that of part I. Indeed, we will show in part V that we can assume that $X_{\alpha_1} v = X_{\alpha_2} v = 0$, which contradicts Lemma 3.5(1).

V. We finish by proving the following claim: if α_1 and α_2 are simple roots such that

- $\alpha_1 + 2\alpha_2$ is not a root;
- $\gamma - \alpha_1$, $\gamma - \alpha_2$, $\gamma - \alpha_1 - \alpha_2$, and $\gamma - \alpha_1 - 2\alpha_2$ are roots; and
- $\langle (\gamma - \alpha_1 - 2\alpha_2)^\vee, \lambda \rangle \neq 0$ for some $\lambda \in F$; then

we can assume that $X_{\alpha_1} v = X_{\alpha_2} v = 0$.

Since $\alpha_1 + 2\alpha_2$ is not a root we have that $[X_{\alpha_2}, X_{\alpha_1+\alpha_2}] = 0$. By the third assumption of the claim,

$$X_{-(\gamma-\alpha_1-2\alpha_2)} x_0 \neq 0. \tag{3.5}$$

We first show that we can assume that

$$X_{\alpha_2} v = X_{\alpha_1+\alpha_2} v = 0. \tag{3.6}$$

There exist $z_1', z_2' \in \Bbbk$ such that

$$X_{\alpha_2} v = z_1' X_{-(\gamma-\alpha_2)} x_0;$$
$$X_{\alpha_1+\alpha_2} v = z_2' X_{-(\gamma-\alpha_1-\alpha_2)} x_0.$$

Next, there exist $z_1, z_2 \in \Bbbk^\times$ such that

$$[X_{\alpha_2}, X_{-\gamma}] = z_1 X_{-(\gamma-\alpha_2)};$$
$$[X_{\alpha_1+\alpha_2}, X_{-\gamma}] = z_2 X_{-(\gamma-\alpha_1-\alpha_2)}.$$

As in part I, one deduces from $X_{\alpha_2}X_{\alpha_1+\alpha_2}v = X_{\alpha_1+\alpha_2}X_{\alpha_2}v$ that

$$(\frac{z_2}{z_1} z_1' - z_2')[X_{\alpha_2}, X_{-(\gamma-\alpha_1-\alpha_2)}]x_0 = 0.$$

Using (3.5), it follows that

$$\frac{z_1'}{z_1} = \frac{z_2'}{z_2}. \tag{3.7}$$

Hence, if we replace v by $v - \frac{z_1'}{z_1}X_{-\gamma}x_0 = v - \frac{z_2'}{z_2}X_{-\gamma}x_0$, then equations (3.6) hold.

We now complete the proof by showing that (3.6) implies that

$$X_{\alpha_1}v = 0. \tag{3.8}$$

There exists $z \in \Bbbk$ such that $X_{\alpha_1}v = zX_{-(\gamma-\alpha_1)}x_0$. From (3.6) we have that

$$0 = X_{\alpha_1+\alpha_2}v = X_{\alpha_2}X_{\alpha_1}v = zX_{\alpha_2}X_{-(\gamma-\alpha_1)}x_0 = zX_{-(\gamma-\alpha_1-\alpha_2)}x_0,$$

where the second equality uses that $X_{\alpha_2}v = 0$ and the fourth one uses that $X_{\alpha_2}x_0 = 0$. Since equation (3.5) implies that $X_{\alpha_2}X_{-(\gamma-\alpha_1-\alpha_2)}x_0 \neq 0$, we have that $X_{-(\gamma-\alpha_1-\alpha_2)}x_0 \neq 0$, and therefore that $z = 0$ which proves equation (3.8), the claim at the start of part V and the proposition. □

The following is Theorem 3.3 for the case that γ is a root.

Corollary 3.8. *Let γ be a T_{ad}-weight in $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$. If γ is a root, then γ is a spherically closed spherical root of G.*

Proof. If the support of γ is not of type G_2, then by Proposition 3.7 we have only to check the locally dominant short roots. The following roots do not satisfy Lemma 3.6.

- With support of type D_n, $n \geq 4$: $\alpha_1 + 2(\alpha_2 + \ldots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n = \omega_2$.
- With support of type E_6: $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = \omega_2$.
- With support of type E_7: $2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = \omega_1$.
- With support of type E_8: $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 = \omega_8$.

Therefore, we are left with all spherically closed spherical roots.

- With support of type A_n, $n \geq 1$: $\alpha_1 + \ldots + \alpha_n$.
- With support of type B_n, $n \geq 2$: $\alpha_1 + \ldots + \alpha_n$.
- With support of type C_n, $n \geq 3$: $\alpha_1 + 2(\alpha_2 + \ldots + \alpha_{n-1}) + \alpha_n$.
- With support of type F_4: $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$.

If the support of γ is of type G_2 the only positive root satisfying Lemma 3.6 is $\alpha_1 + \alpha_2$, which is a spherically closed spherical root. □

Let us now consider the case where γ is not a root. In contrast to the root case, here we notice the following general fact.

Proposition 3.9. *Let α be a simple root and let β be a non-simple positive root such that $\alpha + \beta$ is not a root. Then there exists no simple root $\alpha' \neq \alpha$ such that $(\alpha + \beta) - \alpha'$ is a root.*

Proof. Assume that there exists a simple root $\alpha' \neq \alpha$ such that $\alpha + \beta - \alpha'$ is a root. Since $\beta - \alpha'$ is nonzero, it is a root. This follows from the fact that $\alpha + \beta$ is not a root, whence$\langle \alpha^\vee, \beta \rangle \geq 0$, and so $\langle \alpha^\vee, \alpha + \beta - \alpha' \rangle > 0$. Finally, to deduce that $\alpha + \beta$ is a root (i.e. a contradiction), one can use for example a saturation argument (see [Hum72, Lemma 13.4.B]) as follows.

Restrict the adjoint representation to the Levi subalgebra associated with α and α'. Since $\beta - \alpha'$ is a root, both β and $\alpha + \beta - \alpha'$ occur as weights in the same irreducible summand, say of highest weight λ. From $\langle \alpha^\vee, \beta \rangle \geq 0$, we get that $\langle \alpha^\vee, \alpha + \beta \rangle > 0$, and since $\alpha + \beta$ is not a root, $\langle (\alpha')^\vee, \alpha + \beta - \alpha' \rangle \geq 0$, and so $\langle (\alpha')^\vee, \alpha + \beta \rangle > 0$. Consequently, $\alpha + \beta$ is dominant with respect to α and α'. Moreover $\lambda - \alpha - \beta$ is a sum of simple roots, because $\lambda - \beta$ and $\lambda - (\alpha + \beta - \alpha')$ both belong to $\mathrm{span}_{\mathbb{N}}\{\alpha, \alpha'\}$. This implies that $\alpha + \beta$ is a root. ☐

Let γ be a T_{ad}-weight in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ which is not a root. Until Proposition 3.13, *we assume that γ is not the sum of two orthogonal simple roots,* so that we can speak of the unique simple root α such that $\gamma - \alpha$ is a root.

Lemma 3.10. *Let α be the simple root such that $\gamma - \alpha$ is a root. If $\gamma \neq 2\alpha$ then α is orthogonal to $\gamma - \alpha$.*

Proof. We can choose a basis of $\mathfrak{g}$

$$\{X_\beta : \beta \text{ root}\} \cup \{\alpha^\vee : \alpha \text{ simple root}\}$$

such that $[X_\beta, X_{-\beta}] = \beta^\vee$ for all positive roots β, and then for all roots β_1, β_2 denote by c_{β_1,β_2} the scalar such that $[X_{\beta_1}, X_{\beta_2}] = c_{\beta_1,\beta_2} X_{\beta_1+\beta_2}$. For example, a Chevalley basis does the job (see [Hum72, Theorem 25.2]).

We can assume that $X_\alpha v = X_{-\gamma+\alpha} x_0 \neq 0$. Assume also, to obtain a contradiction, that $\langle \alpha^\vee, \gamma - \alpha \rangle > 0$. Hence $\gamma - 2\alpha$ is a positive root. Since γ is not a root, we have that $X_{\gamma-\alpha} X_\alpha v = X_\alpha X_{\gamma-\alpha} v$. From the following identities

$$\begin{aligned} X_{\gamma-\alpha} X_\alpha v &= \frac{1}{c_{\gamma-2\alpha,\alpha}} [X_{\gamma-2\alpha}, X_\alpha] X_\alpha v = \frac{1}{c_{\gamma-2\alpha,\alpha}} [X_{\gamma-2\alpha}, X_\alpha] X_{-\gamma+\alpha} x_0 = \\ &= \frac{1}{c_{\gamma-2\alpha,\alpha}} \big(X_{\gamma-2\alpha} [X_\alpha, X_{-\gamma+\alpha}] - X_\alpha [X_{\gamma-2\alpha}, X_{-\gamma+\alpha}] \big) x_0 = \\ &= \frac{c_{\alpha,-\gamma+\alpha}}{c_{\gamma-2\alpha,\alpha}} [X_{\gamma-2\alpha}, X_{-\gamma+2\alpha}] x_0 - \frac{c_{\gamma-2\alpha,-\gamma+\alpha}}{c_{\gamma-2\alpha,\alpha}} [X_\alpha, X_{-\alpha}] x_0 \end{aligned}$$

$$\begin{aligned} X_\alpha X_{\gamma-\alpha} v &= \frac{1}{c_{\gamma-2\alpha,\alpha}} X_\alpha [X_{\gamma-2\alpha}, X_\alpha] v = \frac{1}{c_{\gamma-2\alpha,\alpha}} X_\alpha [X_{\gamma-2\alpha}, X_{-\gamma+\alpha}] x_0 = \\ &= \frac{c_{\gamma-2\alpha,-\gamma+\alpha}}{c_{\gamma-2\alpha,\alpha}} [X_\alpha, X_{-\alpha}] x_0 \end{aligned}$$

it then follows that

$$\frac{c_{\alpha,-\gamma+\alpha}}{c_{\gamma-2\alpha,\alpha}} (\gamma - 2\alpha)^\vee - 2 \frac{c_{\gamma-2\alpha,-\gamma+\alpha}}{c_{\gamma-2\alpha,\alpha}} \alpha^\vee \tag{3.9}$$

takes value zero on all $\lambda \in F$. Since $\gamma \in \mathbb{Z}F$, the expression (3.9) takes value zero on γ, too.

Actually, the linear combination (3.9) of coroots does not depend on the choice of the basis of $\mathfrak{g}$. Indeed,

$$\begin{aligned} c_{\gamma-2\alpha,\alpha} (\gamma - \alpha)^\vee &= [[X_{\gamma-2\alpha}, X_\alpha], X_{-\gamma+\alpha}] = \\ &= [X_{\gamma-2\alpha}, [X_\alpha, X_{-\gamma+\alpha}]] - [X_\alpha, [X_{\gamma-2\alpha}, X_{-\gamma+\alpha}]] = \\ &= c_{\alpha,-\gamma+\alpha} (\gamma - 2\alpha)^\vee - c_{\gamma-2\alpha,-\gamma+\alpha} \alpha^\vee \end{aligned}$$

and

$$(\gamma - \alpha)^\vee = \frac{\|\gamma - 2\alpha\|^2}{\|\gamma - \alpha\|^2} (\gamma - 2\alpha)^\vee + \frac{\|\alpha\|^2}{\|\gamma - \alpha\|^2} \alpha^\vee.$$

Therefore, since $(\gamma - 2\alpha)^\vee$ and $\alpha^\vee$ are linearly independent, (3.9) becomes

$$\frac{\|\gamma - 2\alpha\|^2}{\|\gamma - \alpha\|^2}(\gamma - 2\alpha)^\vee + 2\frac{\|\alpha\|^2}{\|\gamma - \alpha\|^2}\alpha^\vee \tag{3.10}$$

which is proportional to $\gamma^\vee$. Since $\|\gamma\|^2$ is not zero, the expression in (3.9) cannot take value zero on γ, and we have obtained the desired contradiction. $\square$

Lemma 3.11 ([BCF08, Lemma 3.6]). *Let α be the simple root such that $\gamma - \alpha$ is a positive root. If $\gamma - \alpha = \beta_1 + \beta_2$ with β_1 and β_2 positive roots then $\alpha + \beta_1$ or $\alpha + \beta_2$ is a root.*

Proof. We can assume that $X_\alpha v = X_{-\gamma+\alpha} x_0 \neq 0$. If $\alpha + \beta_1 \notin R^+$ then $X_{\beta_2} v$, which belongs to $\mathfrak{g} \cdot x_0$, is equal to 0. Similarly, if $\alpha + \beta_2 \notin R^+$ then $X_{\beta_1} v = 0$. Therefore, if neither $\alpha + \beta_1$ nor $\alpha + \beta_2$ is a root, then $X_{\gamma-\alpha} v = 0$. Since $\gamma \notin R^+$, this implies

$$0 = X_\alpha X_{\gamma-\alpha} v = X_{\gamma-\alpha} X_\alpha v = X_{\gamma-\alpha} X_{-\gamma+\alpha} x_0$$

which means $X_{-\gamma+\alpha} x_0 = 0$, a contradiction. $\square$

Lemma 3.12. *Let α be the simple root such that $\gamma - \alpha$ is a root. Let δ be a simple root and k an integer $2 \leq k \leq 4$ such that $\gamma - j\alpha - \delta$ is a root for $1 \leq j \leq k$, $j\alpha + \delta$ is a root for $1 \leq j < k$, but $k\alpha + \delta$ is not a root. Then $\gamma - k\alpha$ is orthogonal to every $\lambda \in F$; and in particular*

$$\|\gamma - \alpha\|^2 = (k-1)\|\alpha\|^2. \tag{3.11}$$

Proof. We can choose a basis as in the proof of Lemma 3.10, and assume $X_\alpha v = X_{-\gamma+\alpha} x_0$, nonzero.

First, let us assume also, for simplicity, that $k = 2$. Then one has the following identities.

$$\begin{aligned}
X_{\gamma-\alpha} X_\alpha v &= \frac{1}{c_{\gamma-\alpha-\delta,\delta}} [X_{\gamma-\alpha-\delta}, X_\delta] X_{-\gamma+\alpha} x_0 = \\
&= \frac{1}{c_{\gamma-\alpha-\delta,\delta}} \big(X_{\gamma-\alpha-\delta} [X_\delta, X_{-\gamma+\alpha}] - X_\delta [X_{\gamma-\alpha-\delta}, X_{-\gamma+\alpha}] \big) x_0 = \\
&= \frac{c_{\delta,-\gamma+\alpha}}{c_{\gamma-\alpha-\delta,\delta}} [X_{\gamma-\alpha-\delta}, X_{-\gamma+\alpha+\delta}] x_0 - \frac{c_{\gamma-\alpha-\delta,-\gamma+\alpha}}{c_{\gamma-\alpha-\delta,\delta}} [X_\delta, X_{-\delta}] x_0
\end{aligned}$$

$$\begin{aligned}
X_\alpha X_{\gamma-\alpha} v &= \frac{1}{c_{\gamma-\alpha-\delta,\delta}} X_\alpha [X_{\gamma-\alpha-\delta}, X_\delta] v = -\frac{1}{c_{\gamma-\alpha-\delta,\delta}} X_\alpha X_\delta X_{\gamma-\alpha-\delta} v = \\
&= -\frac{1}{c_{\gamma-\alpha-\delta,\delta} c_{\gamma-2\alpha-\delta,\alpha}} X_\alpha X_\delta [X_{\gamma-2\alpha-\delta}, X_\alpha] v = \\
&= -\frac{1}{c_{\gamma-\alpha-\delta,\delta} c_{\gamma-2\alpha-\delta,\alpha}} X_\alpha X_\delta [X_{\gamma-2\alpha-\delta}, X_{-\gamma+\alpha}] x_0 = \\
&= -\frac{c_{\gamma-2\alpha-\delta,-\gamma+\alpha}}{c_{\gamma-\alpha-\delta,\delta} c_{\gamma-2\alpha-\delta,\alpha}} X_\alpha [X_\delta, X_{-\alpha-\delta}] x_0 = \\
&= -\frac{c_{\gamma-2\alpha-\delta,-\gamma+\alpha} c_{\delta,-\alpha-\delta}}{c_{\gamma-\alpha-\delta,\delta} c_{\gamma-2\alpha-\delta,\alpha}} [X_\alpha, X_{-\alpha}] x_0
\end{aligned}$$

We thus find a linear combination of co-roots

$$\frac{c_{\delta,-\gamma+\alpha}}{c_{\gamma-\alpha-\delta,\delta}} (\gamma - \alpha - \delta)^\vee - \frac{c_{\gamma-\alpha-\delta,-\gamma+\alpha}}{c_{\gamma-\alpha-\delta,\delta}} \delta^\vee + \frac{c_{\gamma-2\alpha-\delta,-\gamma+\alpha} c_{\delta,-\alpha-\delta}}{c_{\gamma-\alpha-\delta,\delta} c_{\gamma-2\alpha-\delta,\alpha}} \alpha^\vee \tag{3.12}$$

which must take value zero on all $\lambda \in F$. We now compute the coefficients in the above linear combination of coroots, showing they do not depend on the choice of the basis of $\mathfrak{g}$. Indeed,

$$\begin{aligned} c_{\gamma-\alpha-\delta,\delta}(\gamma-\alpha)^\vee &= [[X_{\gamma-\alpha-\delta}, X_\delta], X_{-\gamma+\alpha}] = \\ &= [X_{\gamma-\alpha-\delta}, [X_\delta, X_{-\gamma+\alpha}]] - [X_\delta, [X_{\gamma-\alpha-\delta}, X_{-\gamma+\alpha}]] = \\ &= c_{\delta,-\gamma+\alpha}(\gamma-\alpha-\delta)^\vee - c_{\gamma-\alpha-\delta,-\gamma+\alpha}\delta^\vee \end{aligned}$$

and, since

$$\begin{aligned} & c_{\gamma-\alpha-\delta,\delta}c_{\gamma-2\alpha-\delta,\alpha}(\gamma-\alpha)^\vee + c_{\gamma-\alpha-\delta,-\gamma+\alpha}c_{\gamma-2\alpha-\delta,\alpha}\delta^\vee = \\ &= [[[X_{\gamma-2\alpha-\delta}, X_\alpha], X_\delta], X_{-\gamma+\alpha}] - [[[X_{\gamma-2\alpha-\delta}, X_\alpha], X_{-\gamma+\alpha}], X_\delta] = \\ &= [[X_{\gamma-2\alpha-\delta}, X_\alpha], [X_\delta, X_{-\gamma+\alpha}]] = \\ &= [[X_{\gamma-2\alpha-\delta}, [X_\delta, X_{-\gamma+\alpha}]], X_\alpha] - [X_{\gamma-2\alpha-\delta}, [[X_\delta, X_{-\gamma+\alpha}], X_\alpha]] = \\ &= [[X_\delta, [X_{\gamma-2\alpha-\delta}, X_{-\gamma+\alpha}]], X_\alpha] - [X_{\gamma-2\alpha-\delta}, [[X_\delta, X_{-\gamma+\alpha}], X_\alpha]] = \\ &= -c_{\delta,-\alpha-\delta}c_{\gamma-2\alpha-\delta,-\gamma+\alpha}\alpha^\vee - c_{-\gamma+\alpha+\delta,\alpha}c_{\delta,-\gamma+\alpha}(\gamma-2\alpha-\delta)^\vee, \end{aligned}$$

also

$$c_{-\gamma+\alpha+\delta,\alpha}c_{\delta,-\gamma+\alpha}(\gamma-2\alpha-\delta)^\vee = -c_{\gamma-2\alpha-\delta,\alpha}c_{\delta,-\gamma+\alpha}(\gamma-\alpha-\delta)^\vee - c_{\delta,-\alpha-\delta}c_{\gamma-2\alpha-\delta,-\gamma+\alpha}\alpha^\vee.$$

On the other hand,

$$(\gamma-\alpha)^\vee = \frac{\|\gamma-\alpha-\delta\|^2}{\|\gamma-\alpha\|^2}(\gamma-\alpha-\delta)^\vee + \frac{\|\delta\|^2}{\|\gamma-\alpha\|^2}\delta^\vee$$

and

$$(\gamma-2\alpha-\delta)^\vee = \frac{\|\gamma-\alpha-\delta\|^2}{\|\gamma-2\alpha-\delta\|^2}(\gamma-\alpha-\delta)^\vee - \frac{\|\alpha\|^2}{\|\gamma-2\alpha-\delta\|^2}\alpha^\vee.$$

Therefore, since $(\gamma-\alpha-\delta)^\vee$ is neither proportional to $\delta^\vee$ nor to $\alpha^\vee$, (3.12) becomes

$$\frac{\|\gamma-\alpha-\delta\|^2}{\|\gamma-\alpha\|^2}(\gamma-\alpha-\delta)^\vee + \frac{\|\delta\|^2}{\|\gamma-\alpha\|^2}\delta^\vee - \frac{\|\alpha\|^2}{\|\gamma-\alpha\|^2}\alpha^\vee \tag{3.13}$$

which is proportional to $(\gamma-2\alpha)^\vee$.

For $k > 2$ the proof is similar. If $k = 3$, the analog of (3.12) is

$$\begin{aligned} & \frac{c_{\delta,-\gamma+\alpha}}{c_{\gamma-\alpha-\delta,\delta}}(\gamma-\alpha-\delta)^\vee - \frac{c_{\gamma-\alpha-\delta,-\gamma+\alpha}}{c_{\gamma-\alpha-\delta,\delta}}\delta^\vee + \\ + \quad & \frac{c_{\gamma-2\alpha-\delta,-\gamma+\alpha}c_{\delta,-\alpha-\delta}}{c_{\gamma-\alpha-\delta,\delta}c_{\gamma-2\alpha-\delta,\alpha}}\alpha^\vee + \\ - \quad & \frac{c_{\gamma-3\alpha-\delta,-\gamma+\alpha}c_{\delta,-\alpha-\delta}c_{\alpha,-2\alpha-\delta}}{c_{\gamma-\alpha-\delta,\delta}c_{\gamma-2\alpha-\delta,\alpha}c_{\gamma-3\alpha-\delta,\alpha}}\alpha^\vee \end{aligned}$$

which is proportional to $(\gamma-3\alpha)^\vee$. If $k = 4$, we get

$$\begin{aligned} & \frac{c_{\delta,-\gamma+\alpha}}{c_{\gamma-\alpha-\delta,\delta}}(\gamma-\alpha-\delta)^\vee - \frac{c_{\gamma-\alpha-\delta,-\gamma+\alpha}}{c_{\gamma-\alpha-\delta,\delta}}\delta^\vee + \\ + \quad & \frac{c_{\gamma-2\alpha-\delta,-\gamma+\alpha}c_{\delta,-\alpha-\delta}}{c_{\gamma-\alpha-\delta,\delta}c_{\gamma-2\alpha-\delta,\alpha}}\alpha^\vee + \\ - \quad & \frac{c_{\gamma-3\alpha-\delta,-\gamma+\alpha}c_{\delta,-\alpha-\delta}c_{\alpha,-2\alpha-\delta}}{c_{\gamma-\alpha-\delta,\delta}c_{\gamma-2\alpha-\delta,\alpha}c_{\gamma-3\alpha-\delta,\alpha}}\alpha^\vee + \\ + \quad & \frac{c_{\gamma-4\alpha-\delta,-\gamma+\alpha}c_{\delta,-\alpha-\delta}c_{\alpha,-2\alpha-\delta}c_{\alpha,-3\alpha-\delta}}{c_{\gamma-\alpha-\delta,\delta}c_{\gamma-2\alpha-\delta,\alpha}c_{\gamma-3\alpha-\delta,\alpha}c_{\gamma-4\alpha-\delta,\alpha}}\alpha^\vee \end{aligned}$$

which is proportional to $(\gamma-4\alpha)^\vee$.

Finally, since $\gamma - k\alpha$ is orthogonal to every $\lambda \in F$, we have $(\gamma - k\alpha, \gamma) = 0$, which yields (3.11). Indeed, the assumption implies that $\gamma \neq 2\alpha$, hence $(\alpha, \gamma - \alpha) = 0$ by Lemma 3.10, and

$$0 = (\gamma - k\alpha, \gamma) = \|\gamma - \alpha\|^2 - (k-1)\|\alpha\|^2.$$

□

Proposition 3.13. *Suppose γ is not a root and let α be a simple root such that $\gamma - \alpha$ is a root. Then $\gamma - \alpha$ is locally the highest root, i.e. the highest root in the root subsystem generated by the simple roots of its support.*

Proof. **I**. First we want to prove that $\gamma - \alpha$ is locally dominant. We can assume that $\gamma - \alpha$ is not simple. Hence, by Lemma 3.10, α is orthogonal to $\gamma - \alpha$.

There exists a simple root δ (different from α) such that $\gamma - \alpha - \delta$ is a root. By Proposition 3.9 and Lemma 3.11 $\alpha + \delta$ is a root.

Since $\alpha + \delta$ is a root, $\langle \alpha^\vee, \delta \rangle < 0$. Therefore, $\langle \alpha^\vee, \gamma - \alpha - \delta \rangle > 0$ hence $\gamma - 2\alpha - \delta$ is a root. If moreover $2\alpha + \delta$ is a root, then by $\mathfrak{sl}(2)$-theory, $\langle \alpha^\vee, \alpha + \delta \rangle \leq 0$ and so $\langle \alpha^\vee, \gamma - 2\alpha - \delta \rangle \geq 0$, whence $\gamma - 3\alpha - \delta$ is a root. If $3\alpha + \delta$ is also a root, then α and δ span a root system of type G_2. Consequently, $\langle \alpha^\vee, \gamma - 3\alpha - \delta \rangle = -1$ and $\gamma - 4\alpha - \delta$ is a root.

Therefore we can apply Lemma 3.12 and obtain that, for some $k \geq 1$, $\gamma - k\alpha$ is orthogonal to every $\lambda \in F$. This implies that $\langle (\alpha')^\vee, \gamma \rangle = 0$ for all $\alpha' \in \mathrm{supp}(\gamma) \setminus \{\alpha\}$, whence $\langle (\alpha')^\vee, \gamma - \alpha \rangle \geq 0$ for all such α'. Since α is orthogonal to $\gamma - \alpha$, it follows that $\gamma - \alpha$ is locally dominant.

II. To obtain a contradiction, we now assume that $\gamma - \alpha$ is not locally the highest root, that is, a locally short dominant root with support of non-simply-laced type:

- in type B_n, $n \geq 2$, the short dominant root is $\alpha_1 + \ldots + \alpha_n = \omega_1$;
- in type C_n, $n \geq 3$, the short dominant root is $\alpha_1 + 2(\alpha_2 + \ldots + \alpha_{n-1}) + \alpha_n = \omega_2$;
- in type F_4 the short dominant root is $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 = \omega_4$;
- in type G_2 the short dominant root is $2\alpha_1 + \alpha_2 = \omega_1$.

By equation (3.11), α is also short and $k = 2$, in particular the support of γ is not of type G_2. Moreover, by Lemma 3.10, α is orthogonal to $\gamma - \alpha$. In type B_n and in type F_4 this implies that γ is a root.

We are left with the case where the support of $\gamma - \alpha$ is of type C_n. Since α is short, α is orthogonal to $\gamma - \alpha$, γ is not a root, and moreover there exists a simple root $\delta \neq \alpha$ satisfying the hypothesis of Lemma 3.12 for $k = 2$, we have that $n > 3$, $\delta = \alpha_2$ and $\alpha = \alpha_3$. This contradicts Lemma 3.11, because α_1 and $\gamma - \alpha - \alpha_1$ are roots, but neither $\alpha_1 + \alpha$ nor $\gamma - \alpha_1$ is a root. □

The following is Theorem 3.3 for the case that γ is not a root.

Corollary 3.14. *Let γ be a T_{ad}-weight in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. If γ is not a root, then γ is a spherically closed spherical root of G.*

Proof. We list all the locally highest roots β and deduce which are the only possible non-roots γ (obtained by adding to β a simple root) satisfying Lemmas 3.10, 3.11 and 3.12.

In general, $\langle \alpha^\vee, \beta \rangle$ must be ≥ 0 otherwise $\alpha + \beta \in R^+$. If α is not in the support of β it must be orthogonal to β, and in this case, by Lemma 3.11, β must necessarily be simple.

Let us start with β simple, i.e., with support of type A_1: $\beta = \alpha_1 = 2\omega_1$ gives only

$$2\alpha_1 \quad \text{or} \quad \alpha_1 + \alpha_1'.$$

Let us now pass to β not simple and recall that α must necessarily belong to the support of β, moreover by Lemma 3.10 $\langle \alpha^\vee, \beta \rangle = 0$ and by Lemma 3.12, for all $\alpha' \in S \setminus \{\alpha\}$, $\langle (\alpha')^\vee, \alpha + \beta \rangle = 0$.

With support of type A_n, $n \geq 2$: $\beta = \alpha_1 + \ldots + \alpha_n = \omega_1 + \omega_n$ gives only, for $n = 3$,

$$\alpha_1 + 2\alpha_2 + \alpha_3.$$

With support of type B_n, $n \geq 2$: $\beta = \alpha_1 + 2(\alpha_2 + \ldots + \alpha_n) = \omega_2$ if $n \geq 3$ (it equals $2\omega_2$ if $n = 2$) gives only

$$2(\alpha_1 + \ldots + \alpha_n)$$

or, for $n = 3$,

$$\alpha_1 + 2\alpha_2 + 3\alpha_3.$$

With support of type D_n, $n \geq 4$: $\beta = \alpha_1 + 2(\alpha_2 + \ldots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n = \omega_2$ gives only

$$2(\alpha_1 + \ldots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$$

or, for $n = 4$,

$$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$$

and

$$\alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4$$

which are equal to $2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ up to an automorphism of the Dynkin diagram.

With support of type G_2: $\beta = 3\alpha_1 + 2\alpha_2 = \omega_2$ gives only

$$4\alpha_1 + 2\alpha_2.$$

The remaining cases give no other possibilities:

- with support of type C_n, $n \geq 3$, $\beta = 2(\alpha_1 + \ldots + \alpha_{n-1}) + \alpha_n = 2\omega_1$;
- with support of type E_6, $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = \omega_2$;
- with support of type E_7, $\beta = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = \omega_1$;
- with support of type E_8, $\beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 = \omega_8$;
- with support of type F_4, $\beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = \omega_1$.

□

3.3. **Further properties of T_{ad}-weights in $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$.** After Theorem 3.3 the only possible T_{ad}-weights in $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ are spherically closed spherical roots of G, but each of them occur only under special conditions which we are going to describe.

The first statement is indeed a refinement of Theorem 3.3. Recall the notion of compatibility with S^p (see axiom (S) of Definition 2.5 and Remark 2.6.1).

Theorem 3.15. *If γ is a T_{ad}-weights in $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ then γ is a spherically closed spherical root of G compatible with $S^p(\Gamma)$.*

Proof. If $\gamma = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ with support of type A_n, then $\{\alpha_2, \alpha_3, \ldots, \alpha_{n-1}\} \subset S^p(\Gamma)$. This follows from part I of the proof of Proposition 3.7.

If $\gamma = \alpha_1 + 2\alpha_2 + \alpha_3$ with support of type A_3, then $\{\alpha_1, \alpha_3\} \subset S^p(\Gamma)$. This follows by Lemma 3.12 ($\alpha = \alpha_2$, $\delta = \alpha_1$ and $k = 2$).

If $\gamma = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ with support of type B_n, then $\{\alpha_2, \alpha_3, \ldots, \alpha_{n-1}\} \subset S^p(\Gamma)$ and $\alpha_n \notin S^p(\Gamma)$. The former follows from part I of the proof of Proposition 3.7. For the latter, we can assume that $X_{\alpha_n}v = 0$ and $X_{\alpha_1}v = X_{-\gamma+\alpha_n}x_0$ nonzero, which implies $\alpha_n \notin S^p$.

If $\gamma = 2(\alpha_1 + \ldots + \alpha_n)$ with support of type B_n, then $\{\alpha_2, \ldots, \alpha_n\} \subset S^p(\Gamma)$. This follows by Lemma 3.12 ($\alpha = \alpha_1$, $\delta = \alpha_2$ and $k = 2$).

If $\gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3$ with support of type B_3, then $\{\alpha_1, \alpha_2\} \subset S^p(\Gamma)$. This follows by Lemma 3.12 ($\alpha = \alpha_3$, $\delta = \alpha_2$ and $k = 3$).

If $\gamma = \alpha_1 + 2(\alpha_2 + \ldots + \alpha_{n-1}) + \alpha_n$ with support of type C_n, then $\{\alpha_3, \alpha_4, \ldots, \alpha_n\} \subset S^p(\Gamma)$. This follows from part V of the proof of Proposition 3.7.

If $\gamma = 2(\alpha_1 + \ldots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$ with support of type D_n, then $\{\alpha_2, \ldots, \alpha_n\} \subset S^p(\Gamma)$. This follows by Lemma 3.12 ($\alpha = \alpha_1$, $\delta = \alpha_2$ and $k = 2$).

If $\gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ with support of type F_4, then $\{\alpha_1, \alpha_2, \alpha_3\} \subset S^p(\Gamma)$. This follows from part V of the proof of Proposition 3.7.

If $\gamma = 4\alpha_1 + 2\alpha_2$ with support of type G_2, then $\alpha_2 \in S^p(\Gamma)$. This follows by Lemma 3.12 ($\alpha = \alpha_1$, $\delta = \alpha_2$ and $k = 4$). ☐

Proposition 3.16. *If γ is not a simple root then the T_{ad}-eigenspace $(V/\mathfrak{g}\cdot x_0)^{G_{x_0}}_{(\gamma)}$ has dimension ≤ 1.*

Proof. If γ is a root (not simple), recall that there exist two simple roots, say α_1 and α_2, such that $\gamma - \alpha_1$ and $\gamma - \alpha_2$ is a root, and $\gamma - \alpha$ is not a root for all $\alpha \in S \setminus \{\alpha_1, \alpha_2\}$. In particular, for all $\alpha \in S \setminus \{\alpha_1, \alpha_2\}$, we necessarily have $X_\alpha v = 0$. By adding to v a suitable scalar multiple of $X_{-\gamma}x_0$, we can assume that also $X_{\alpha_2}v = 0$. Moreover, by choosing a suitable scalar multiple, we can assume that $X_{\alpha_1}v = X_{-\gamma+\alpha_1}x_0$.

If γ is neither a root nor the sum of two orthogonal simple roots, recall that there exists a simple root α_1 such that $\gamma - \alpha_1$ is a root, and $\gamma - \alpha$ is not a root for all $\alpha \in S \setminus \{\alpha_1\}$. In particular, for all $\alpha \in S \setminus \{\alpha_1\}$, we necessarily have $X_\alpha v = 0$. Therefore, by choosing a suitable scalar multiple, we can assume that $X_{\alpha_1}v = X_{-\gamma+\alpha_1}x_0$.

In both cases we claim that under the above assumptions v is uniquely determined. Indeed, if v_1 and v_2 are two vectors in V of T_{ad}-weight γ fulfilling the above conditions, then $X_\alpha(v_1 - v_2) = 0$ for all $\alpha \in S$, which implies $v_1 = v_2$.

We are left with only one case: the spherical root $\gamma = \alpha + \alpha'$ with support of type $\mathsf{A}_1 \times \mathsf{A}_1$. We can assume $X_\alpha v = X_{-\alpha'}x_0$. For all $i \in \{1, \ldots, r\}$, $\dim V(\lambda_i)_{(\gamma)} \leq 1$, and the condition $X_\alpha v = X_{-\alpha'}x_0$ uniquely determines every component $v_i \in V(\lambda_i)$ of v. ☐

4. The weight spaces of $\mathrm{T}_{X_0}\mathrm{H}_\Gamma$

In this section we prove the following.

Theorem 4.1. *If Γ is a free monoid of dominant weights, then $\mathrm{T}_{X_0}\mathrm{H}_\Gamma$ is a multiplicity-free T_{ad}-module of which all the weights belong to $\Sigma^{sc}(G)$. Moreover, if $\gamma \in \Sigma^{sc}(G)$ occurs as a T_{ad}-weight in $\mathrm{T}_{X_0}\mathrm{H}_\Gamma$ then γ is N-adapted to Γ.*

Proof. The assertion that all T_{ad}-weights of $\mathrm{T}_{X_0}\mathrm{H}_\Gamma$ belong to $\Sigma^{sc}(G)$ follows from the inclusion $\mathrm{T}_{X_0}\mathrm{H}_\Gamma \hookrightarrow (V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$ and Theorem 3.3, while the assertion that the weight space $(\mathrm{T}_{X_0}\mathrm{H}_\Gamma)_{(\gamma)}$ has dimension at most one follows from Proposition 3.16 if $\gamma \notin S$, and from Proposition 4.6 below if $\gamma \in S$. The statement that if $\gamma \in \Sigma^{sc}(G)$ is a T_{ad}-weight in $\mathrm{T}_{X_0}\mathrm{H}_\Gamma$, then γ is N-adapted to Γ, is contained in Proposition 4.6 for $\gamma \in S$ and is shown in Section 4.3 for $\gamma \notin S$. ☐

Recall from Proposition 3.1 that M_Γ is T_{ad}-equivariantly isomorphic to an open subscheme of H_Γ. Because every T_{ad}-weight in $\mathrm{T}_{X_0}\mathrm{M}_\Gamma \simeq \mathrm{T}_{X_0}\mathrm{H}_\Gamma$ is an element of $\Sigma^{sc}(G)$ (see Theorem 3.3) we obtain the following converse to the second statement in Theorem 4.1.

Corollary 4.2. *Let Γ be a free monoid of dominant weights and let $\sigma \in \Sigma^{sc}(G)$. If σ is N-adapted to Γ, then σ is a T_{ad}-weight in $\mathrm{T}_{X_0}\mathrm{M}_\Gamma$.*

Proof. Let X be an affine spherical G-variety with $\Gamma(X) = \Gamma$ and $\Sigma^N(X) = \{\sigma\}$, and let $\mathscr{M}_X$ be its root monoid. Recall that $\Sigma^N(X)$ is the basis of the saturation of $\mathscr{M}_X$. Let $\{a_1, a_2, \ldots, a_k\}$ be a subset of $\mathbb{N}$ such that $\{a_1\sigma, a_2\sigma, \ldots, a_k\sigma\}$ is the minimal set of generators of $\mathscr{M}_X$. By [AB05, Proposition 2.13], the T_{ad}-orbit closure of X, seen as a closed point of M_Γ, is $\mathrm{Spec}(\Bbbk[-\mathscr{M}_X])$. A straightforward computation using the basic theory of semigroup rings (see, e.g., [MS05, §7.1]) shows that

$$\mathrm{T}_{X_0}(\overline{T_{\mathrm{ad}}\cdot X}) \simeq V(a_1\sigma) \oplus V(a_2\sigma) \oplus \ldots \oplus V(a_k\sigma)$$

as T_{ad}-modules. We claim that one of the a_i is equal to 1 (and consequently that $\mathscr{M}_X$ is generated by $\{\sigma\}$). We show this by contradiction. Suppose that all of the a_i are at least 2. Then $k \geq 2$, since otherwise σ would not be in $\mathbb{Z}\mathscr{M}_X$. Since $\mathrm{T}_{X_0}(\overline{T_{\mathrm{ad}}\cdot X}) \subset \mathrm{T}_{X_0}\mathrm{M}_\Gamma \subset (V/\mathfrak{g}\cdot x_0)^{G_{x_0}}$, it then follows from Theorem 3.3 that $\{\sigma, a_1\sigma, a_2\sigma\} \subset \Sigma^{sc}(G)$. By the classification of spherically closed spherical roots (cf. Proposition 2.4) this is impossible: only the double or half of a spherically closed spherical root can be a spherically closed spherical root, and never both. □

As before, Γ will be a free monoid of dominant weights with basis $F = \{\lambda_1, \lambda_2, \dots, \lambda_r\}$. If $\lambda \in F$, then we will write $\lambda^\#$ for the corresponding element of the dual basis of $(\mathbb{Z}\Gamma)^*$; in other words, for all $\mu \in F \setminus \{\lambda\}$ we have $\langle \lambda^\#, \mu \rangle = 0$, whereas $\langle \lambda^\#, \lambda \rangle = 1$. Recall that $E(\Gamma)$ is defined in (2.3). Because Γ is free, we have that $E(\Gamma)$ is the dual basis to F:

$$E(\Gamma) = \{\lambda^\# \in (\mathbb{Z}\Gamma)^* \colon \lambda \in F\}.$$

For $\lambda \in F$ we put

$$z_\lambda := x_0 - v_\lambda.$$

4.1. **The extension criterion.** We recall from [PVS14] a criterion which allows to decide whether a T_{ad}-eigenvector $[v] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}} \simeq H^0(G \cdot X_0, \mathcal{N}_{X_0|V})^G$ belongs to the subspace $\mathrm{T}_{X_0}\mathrm{H}_\Gamma \simeq H^0(X_0, \mathcal{N}_{X_0|V})^G$.

We denote by $X_0^{\leq 1} \subset X_0$ the union of $G \cdot x_0$ with all G-orbits of X_0 that have codimension 1. By [Bri10, Lemma 1.14] $X_0^{\leq 1}$ is an open subset of X_0. The following proposition is a special case of [Bri13, Lemma 3.9]. Together with Theorem 4.5 it gives the aforementioned criterion.

Proposition 4.3. *A section $s \in H^0(G \cdot X_0, \mathcal{N}_{X_0|V})$ extends to X_0 if and only if it extends to $X_0^{\leq 1}$.*

We recall that the orbit structure of X_0 is well-understood [VP72, Theorem 8]. It is easy to describe the orbits of codimension 1 (see, e.g., [PVS12, Proposition 3.1] for details).

Proposition 4.4. *The G-orbits of codimension 1 in X_0 are exactly the orbits $G \cdot z_\lambda$ where λ is an element of F that satisfies the following property:*

for every $\alpha \in S$ such that $\langle \alpha^\vee, \lambda \rangle \neq 0$ there exists $\mu \in F \setminus \{\lambda\}$ such that $\langle \alpha^\vee, \mu \rangle \neq 0$.

Theorem 4.5 ([PVS14])**.** *Let $v \in V$ be a T_{ad}-eigenvector of weight γ such that $0 \neq [v] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. Let $\lambda \in F$. Recall that $z_\lambda = x_0 - v_\lambda$. Assume that $z_\lambda \in X_0^{\leq 1}$ and put $Z := G \cdot x_0 \cup G \cdot z_\lambda$. Put $a := \langle \lambda^\#, \gamma \rangle$. Denote by $s \in H^0(G \cdot x_0, \mathcal{N}_{X_0|V})^G$ the G-equivariant section such that $s(x_0) = [v]$.*

A) If $a \leq 0$, then s extends to an element of $H^0(Z, \mathcal{N}_{X_0|V})^G$.
B) If $a > 1$, then s does not extend to an element of $H^0(Z, \mathcal{N}_{X_0|V})^G$.
C) If $a = 1$, then the following are equivalent:
  i) s extends to an element of $H^0(Z, \mathcal{N}_{X_0|V})^G$;
  ii) there exist $\hat{v} \in V(\lambda)$ such that $[v] = [\hat{v}]$ as elements of $V/\mathfrak{g} \cdot x_0$.

4.2. **The spherical root $\gamma = \alpha \in S$.** In this section, we discuss the T_{ad}-weight space $(\mathrm{T}_{X_0}\mathrm{H}_\Gamma)_{(\alpha)}$, where α is a simple root. Specifically, we will prove the following proposition, which is a special case of Theorem 4.1.

Proposition 4.6. *If α is a simple root then $\dim(\mathrm{T}_{X_0}\mathrm{H}_\Gamma)_{(\alpha)} \leq 1$. Moreover, if $\dim(\mathrm{T}_{X_0}\mathrm{H}_\Gamma)_{(\alpha)} = 1$ then α is N-adapted to Γ.*

The proof of Proposition 4.6 will be given on page 23. We first need a few lemmas and introduce notation we will use for the remainder of this section. Put $F(\alpha) := \{\lambda \in F \colon \langle \alpha^\vee, \lambda \rangle \neq 0\}$. We order the elements of F such that for $F(\alpha) = \{\lambda_1, \lambda_2, \dots, \lambda_p\}$ for some $p \leq r$. Then $F \setminus F(\alpha) = \{\lambda_{p+1}, \lambda_{p+2}, \dots, \lambda_r\}$.

Lemma 4.7. *For every $i \in \{1, 2, \dots, p\}$, put $v_i = X_{-\alpha} v_{\lambda_i}$. Then $v_1 + v_2 + \dots + v_p$ spans the T_{ad}-weight space of weight α in $\mathfrak{g} \cdot x_0$. If $\alpha \in \mathbb{Z}\Gamma$, then*

$$(V/\mathfrak{g} \cdot x_0)_{(\alpha)}^{G_{x_0}} = \langle [v_1], [v_2], \dots, [v_{p-1}] \rangle_{\Bbbk}.$$

Proof. By elementary highest weight theory, the T_{ad}-weight space in V of weight α is spanned by $\{v_1, v_2, \ldots, v_p\}$, and the intersection of this weight space with $\mathfrak{g} \cdot x_0$ is the line spanned by $X_{-\alpha}x_0 = v_1 + v_2 + \ldots + v_p$. A straightforward application of Proposition 3.4 shows that $[v_i] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ for each every $i \in \{1, 2, \ldots, p-1\}$. ☐

Lemma 4.8. *Suppose $\alpha \in \mathbb{Z}\Gamma$ and $|F(\alpha)| \geq 2$. Let $\lambda \in F$. If $\langle \lambda^{\#}, \alpha \rangle > 0$, then $G \cdot z_\lambda$ has codimension* 1 *in X_0.*

Proof. We will apply Proposition 4.4. Since $\alpha \in \mathbb{Z}\Gamma$ and Γ is free, there exists a partition $F = F_1 \cup F_2$ of F and for every $\mu \in F$ a unique nonnegative integer a_μ such that

$$\alpha = \sum_{\mu \in F_1} a_\mu \mu - \sum_{\mu \in F_2} a_\mu \mu. \tag{4.1}$$

By assumption $\lambda \in F_1$ and $a_\lambda = \langle \lambda^{\#}, \alpha \rangle > 0$. Let $\beta \in S \setminus \{\alpha\}$ such that $\langle \beta^\vee, \lambda \rangle \neq 0$. Then, since $F \subset \Lambda^+$ and $\langle \beta^\vee, \alpha \rangle \leq 0$, it follows from the expression (4.1) that there exists $\mu \in F_2$ such that $a_\mu \langle \beta^\vee, \mu \rangle \geq a_\lambda \langle \beta^\vee, \lambda \rangle$. In particular, $\langle \beta^\vee, \mu \rangle \neq 0$. Furthermore, by the assumption that $|F(\alpha)| \geq 2$, there exists $\mu \in F \setminus \{\lambda\}$ such that $\langle \alpha^\vee, \mu \rangle \neq 0$. This finishes the proof. ☐

Lemma 4.9. *Let α be a simple root. Recall that $F(\alpha) = \{\lambda \in F \colon \langle \alpha^\vee, \lambda \rangle \neq 0\}$ and put $E(\alpha) := \{\delta \in E(\Gamma) \colon \langle \delta, \alpha \rangle = 1\}$. Then $\dim(\mathrm{T}_{X_0}\mathrm{H}_\Gamma)_{(\alpha)} \leq 1$ and if $\dim(\mathrm{T}_{X_0}\mathrm{H}_\Gamma)_{(\alpha)} = 1$ then*

(i) $\alpha \in \mathbb{Z}\Gamma$;
(ii) $|F(\alpha)| \geq 2$;
(iii) $\langle \delta, \alpha \rangle \leq 1$ *for all* $\delta \in E(\Gamma)$;
(iv) $|E(\alpha)| \leq 2$;
(v) If $|E(\alpha)| = 2$ *then* $E(\alpha) = \{\lambda^{\#} \in E(\Gamma) \colon \lambda \in F(\alpha)\}$.

Proof. Let us assume that $\dim(\mathrm{T}_{X_0}\mathrm{H}_\Gamma)_{(\alpha)} \geq 1$. Let $[v]$ be a nonzero element of $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}_{(\alpha)}$ such that the G-equivariant section $s \in H^0(G \cdot x_0, \mathcal{N})^G$ defined by $s(x_0) = [v]$ extends to X_0. By Proposition 3.4 and Lemma 4.7, conditions (i) and (ii) hold. Lemma 4.8 and Theorem 4.5 then imply (iii). We now prove (iv). If $|E(\alpha)| \geq 3$, then by Theorem 4.5 and Lemma 4.8, there exist at least three elements $\lambda, \mu, \nu \in F(\alpha)$ such that there exist $y_\lambda \in V(\lambda)$, $y_\mu \in V(\mu)$ and $y_\nu \in V(\nu)$ for which $[v] = [y_\lambda] = [y_\mu] = [y_\nu] \in V/\mathfrak{g} \cdot x_0$. This is impossible by Lemma 4.7 and (iv) is proved. We turn to (v). Suppose $E(\lambda) = \{\lambda, \mu\}$. By Lemma 4.8 and Theorem 4.5, there exist $y_\lambda \in V(\lambda)$ and $y_\mu \in V(\mu)$ such that $[v] = [y_\lambda] = [y_\mu] \in V/\mathfrak{g} \cdot x_0$. Using Lemma 4.7 again, (v) follows.

Finally, we show that $\dim(\mathrm{T}_{X_0}\mathrm{H}_\Gamma)_{(\alpha)} \leq 1$. Since $\alpha \in \mathbb{Z}\Gamma$, there is at least one $\lambda \in E(\alpha)$. Lemma 4.8 and Theorem 4.5 again imply that $[v] = [y_\lambda]$ for some $y_\lambda \in V(\lambda)$, which finishes the proof. ☐

Remark 4.10. By Corollary 4.2 and the proof of Proposition 4.9 below, the preceding lemma gives alternative conditions for α to be N-adapted to Γ when Γ is free. We list them as a separate lemma, since they seem easier to check then those in Corollary 2.17.

Proof of Proposition 4.6. Lemma 4.9 says that $\dim(\mathrm{T}_{X_0}\mathrm{H}_\Gamma)_{(\alpha)} \leq 1$. We assume conditions (i) – (v) in Lemma 4.9 and deduce conditions (1), (2), (4a), (4b) and (4c) in Corollary 2.17. For (1) and (4c), there is nothing to show. For the spherical root α, (2) follows from (1). Since $\alpha \in \mathbb{Z}\Gamma$, we have that $E(\alpha)$ contains at least one element by (iii). Suppose $\lambda^{\#} \in E(\alpha)$. Clearly $\lambda^{\#} \in a(\alpha)$. We claim that $\alpha^\vee - \lambda^{\#} \neq \lambda^{\#}$. Otherwise, we would have $\lambda^{\#} = \frac{1}{2}\alpha^\vee$, which would contradict (ii). This shows $|a(\alpha)| \geq 2$. Now, if $a(\alpha)$ had a third element, then $E(\alpha)$ would have two elements, say $\lambda^{\#}$ and $\mu^{\#}$, with $\alpha^\vee - \lambda^{\#} \neq \mu^{\#}$. But this yields a contradiction: by (v), we have that $\langle \alpha^\vee, \lambda \rangle = \langle \alpha^\vee, \mu \rangle = 1$ and then that $\alpha^\vee - \lambda^{\#}$ takes the same values as $\mu^{\#}$ on F. We have deduced (4a). Finally, (4b) is clear since $a(\alpha) = \{\lambda^{\#}, \alpha^\vee - \lambda^{\#}\}$ for some $\lambda \in F(\alpha)$. ☐

4.3. **The non-simple spherical roots.** To complete the proof of Theorem 4.1, we show in this section that if γ is a spherically closed spherical root, which is not a simple root and which occurs as a T_{ad}-weight in $\mathrm{T}_{X_0}\mathrm{H}_\Gamma$, then γ is N-adapted to Γ.

We recall that conditions (1) and (2) of Corollary 2.17 follow from Theorem 3.15. We now verify condition (3): if $\delta \in E(\Gamma)$ such that $\langle \delta, \gamma \rangle > 0$ then there exists $\beta \in S \setminus S^p(\Gamma)$ such that $\beta^\vee$ is a positive multiple of δ. The argument is the same for all the non-simple spherical roots γ.

Let $v \in V$ be a T_{ad}-eigenvector of weight γ such that $0 \neq [v] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. Let $\lambda \in F$. Recall that $z_\lambda = x_0 - v_\lambda$ and put $a = \langle \lambda^\#, \gamma \rangle$. Assume $a > 0$.

We claim that under this assumption, $\operatorname{codim}_{X_0} G \cdot z_\lambda \geq 2$. Indeed, if $\operatorname{codim}_{X_0} G \cdot z_\lambda$ were 1, then by Theorem 4.5(B) $a = 1$ and by Theorem 4.5(C) there would exist $\hat{v} \in V(\lambda)$ such that $[v] = [\hat{v}]$ as elements of $V/\mathfrak{g} \cdot x_0$. Therefore, there would exist $\alpha \in S$ such that $\gamma - \alpha \in R^+$, and such that $X_\alpha \hat{v}$ is nonzero and is equal to $X_{-\gamma+\alpha} x_0$ up to a nonzero scalar multiple. This would imply that there exists $\alpha' \in S$ such that $\langle (\alpha')^\vee, \lambda \rangle > 0$ and $\langle (\alpha')^\vee, \mu \rangle = 0$ for all $\mu \in F \setminus \{\lambda\}$, which gives a contradiction and proves the claim.

The fact that $\operatorname{codim}_{X_0} G \cdot z_\lambda \geq 2$ means that there exists $\beta \in S$ such that $\langle \beta^\vee, \lambda \rangle > 0$ and $\langle \beta^\vee, \mu \rangle = 0$ for all $\mu \in F \setminus \{\lambda\}$. This says exactly that the restriction of $\beta^\vee$ to $\mathbb{Z}\Gamma$ is a positive multiple of $\lambda^\#$, which is condition (3).

We continue with the remaining conditions of Corollary 2.17. Condition (4) does not apply to non-simple spherical roots.

Condition (5) follows using the analysis of Section 3. Indeed, we have shown that if $[v]$ is a nonzero T_{ad}-eigenvector of weight 2α in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$, with $\alpha \in S$, then $X_\alpha v$ is a (nonzero) scalar multiple of $X_{-\alpha} x_0$. Since $2\alpha \in \mathbb{Z}\Gamma$, there exists $\lambda \in F$ such that $\langle \alpha^\vee, \lambda \rangle > 0$ and $\langle \lambda^\#, 2\alpha \rangle > 0$. By the argument we used for condition (3), λ is the unique element of F which is non-orthogonal to α. It follows that we actually have that $X_\alpha v$ is a nonzero scalar multiple of $X_{-\alpha} v_\lambda$. This implies that the T-eigenspace of weight $\lambda - 2\alpha$ in $V(\lambda)$ is nonzero, hence $\langle \alpha^\vee, \lambda \rangle \geq 2$. Consequently $\langle \alpha^\vee, \lambda \rangle \in \{2, 4\}$ and $\langle \alpha^\vee, \mu \rangle = 0$ for all $\mu \in F \setminus \{\lambda\}$, hence $\alpha^\vee$ takes an even value on every element of $\mathbb{Z}\Gamma$.

Condition (6) follows analogously from Section 3. Indeed, we have shown that if $[v]$ is a nonzero T_{ad}-eigenvector of weight $\alpha + \alpha'$ in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$, with α and α' orthogonal simple roots, then $X_\alpha v$, if nonzero, is a scalar multiple of $X_{-\alpha'} x_0$, and $X_{\alpha'} v$, if nonzero, is a scalar multiple of $X_\alpha x_0$.

Since $\alpha + \alpha' \in \mathbb{Z}\Gamma$, there exists $\lambda \in F$ such that $\langle \alpha^\vee, \lambda \rangle > 0$ and $\langle \lambda^\#, \alpha + \alpha' \rangle > 0$. By the argument we used for condition (3), λ is the unique element of F which is non-orthogonal to α. Then $X_\alpha v \neq 0$. Indeed if it were 0, then $X_{\alpha'} v$ would be nonzero, hence scalar multiple of $X_{-\alpha} v_\lambda$, which yields a contradiction:

$$0 = X_{\alpha'} X_\alpha v = X_\alpha X_{\alpha'} v = X_\alpha X_{-\alpha} v_\lambda \neq 0,$$

Therefore $X_\alpha v = X_{-\alpha'} x_0$, and if $\langle (\alpha')^\vee, \mu \rangle \neq 0$ then the T-eigenspace of weight $\mu - \alpha - \alpha'$ in $V(\mu)$ is nonzero, hence also $\langle \alpha^\vee, \mu \rangle \neq 0$. This implies that α' is non-orthogonal to λ and orthogonal to μ for all $\mu \in F \setminus \{\lambda\}$. Therefore $\alpha^\vee$ and $(\alpha')^\vee$ are equal on every element of $\mathbb{Z}\Gamma$. This completes the proof of Theorem 4.1.

Remark 4.11. The information given in this remark is not needed for our results. We include it because it gives explicit conditions on F for each spherically closed spherical root γ, which is not a simple root, to occur as a T_{ad}-weight in $\mathrm{T}_{X_0}\mathrm{H}_\Gamma$, that is, to be N-adapted to Γ.

For each spherically closed spherical root γ, there exists $\alpha \in S$ such that $\langle \alpha^\vee, \gamma \rangle > 0$. If γ is a T_{ad}-weight in $\mathrm{T}_{X_0}\mathrm{H}_\Gamma$, then $\gamma \in \mathbb{Z}\Gamma$, and so there exits $\lambda \in F$ such that $\langle \alpha^\vee, \lambda \rangle > 0$ and $\langle \lambda^\#, \gamma \rangle > 0$. If γ is not a simple root, then by the argument above showing that γ satifies condition (3) of Corollary 2.17, we have that λ is the only element of F which is not orthogonal to α, that is, $b\lambda^\# = \alpha^\vee$ on $\mathbb{Z}\Gamma$ for some positive integer b.

We now list, for each γ, the possibilities for $\lambda^\#$.

(1) If $\gamma = 2\alpha$, with α a simple root, then locally $\gamma = 4\omega$. In this case $\alpha^\vee = b\lambda^\#$ with $b \in \{2,4\}$.
(2) If $\gamma = \alpha + \alpha'$, with α and α' two orthogonal simple roots, then locally $\gamma = 2\omega + 2\omega'$. In this case $\alpha^\vee = (\alpha')^\vee = b\lambda^\#$ with $b \in \{1,2\}$.
(3) If $\gamma = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ with support of type A_n with $n \geq 2$, then locally $\gamma = \omega_1 + \omega_n$. In this case, $\alpha^\vee = \lambda^\#$ with $\alpha \in \{\alpha_1, \alpha_n\}$.
(4) If $\gamma = \alpha_1 + 2\alpha_2 + \alpha_3$ with support of type A_3, then locally $\gamma = 2\omega_2$. In this case, we have $\alpha_2^\vee = b\lambda^\#$ with $b \in \{1,2\}$.
(5) If $\gamma = \alpha_1 + \ldots + \alpha_n$ with support of type B_n with $n \geq 2$, then locally $\gamma = \omega_1$. Here $\alpha_1^\vee = \lambda^\#$.
(6) If $\gamma = 2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_n$ with support of type B_n with $n \geq 2$, then locally $\gamma = 2\omega_1$. Here $\alpha_1^\vee = b\lambda^\#$, with $b \in \{1,2\}$.
(7) If $\gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3$ with support of type B_3, then locally $\gamma = 2\omega_3$. Here $\alpha_3^\vee = b\lambda^\#$ with $b \in \{1,2\}$.
(8) If $\gamma = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \ldots + 2\alpha_{n-1} + \alpha_n$ with support of type C_n with $n \geq 3$, then locally $\gamma = \omega_2$. Here $\alpha_2^\vee = \lambda^\#$.
(9) If $\gamma = 2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ with support of type D_n with $n \geq 4$, then locally $\gamma = 2\omega_1$. Here $\alpha_1^\vee = b\lambda^\#$ with $b \in \{1,2\}$.
(10) If $\gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ with support of type F_4, then locally $\gamma = \omega_4$. Here $\alpha_4^\vee = \lambda^\#$.
(11) If $\gamma = 4\alpha_1 + 2\alpha_2$ with support of type G_2, then locally $\gamma = 2\omega_1$. Here $\alpha_1^\vee = b\lambda^\#$ with $b \in \{1,2\}$.
(12) If $\gamma = \alpha_1 + \alpha_2$ with support of type G_2, then locally $\gamma = -\omega_1 + \omega_2$. Here $\alpha_2^\vee = \lambda^\#$.

5. The irreducible components of M_Γ

In this section we prove the following.

Theorem 5.1. *Let Γ be a free monoid of dominant weights. Then the T_{ad}-orbit closures in M_Γ, equipped with their reduced induced scheme structure, are affine spaces.*

The proof is given below. By [AB05, Proposition 2.13] this theorem has the following formal consequence.

Corollary 5.2. *If X is an affine spherical variety with free weight monoid, then its root monoid $\mathscr{M}_X$ is free too.*

Another consequence is that Conjecture 1.1 holds for free monoids.

Corollary 5.3. *If Γ is a free monoid of dominant weights then the irreducible components of M_Γ, equipped with their reduced induced scheme structure, are affine spaces.*

Proof. Since the T_{ad}-orbits in M_Γ are in bijection with isomorphism classes of affine spherical G-varieties, by [AB05, Theorem 1.12] and there are only finitely many such isomorphism classes, by [AB05, Corollary 3.4], we have that every irreducible component Z of M_Γ contains a dense T_{ad}-orbit. It then follows from Theorem 5.1 that Z, equipped with its reduced induced scheme structure, is an affine space. □

Proof of Theorem 5.1. Let X be an affine spherical G-variety of weight monoid Γ, seen as a (closed) point in M_Γ. By [AB05, Corollary 2.14], we know that the normalization of $\overline{T_{\mathrm{ad}} \cdot X}$ is an affine space. It is therefore enough to show that $\overline{T_{\mathrm{ad}} \cdot X}$ is smooth at X_0. We do this by showing that

$$\dim \mathrm{T}_{X_0}(\overline{T_{\mathrm{ad}} \cdot X}) = \dim \overline{T_{\mathrm{ad}} \cdot X}. \tag{5.1}$$

Recall that $\Sigma^N(X)$ is the basis of the monoid obtained by saturation of the root monoid $\mathscr{M}_X$. To deduce (5.1) we make use of Theorem 4.1: the T_{ad}-weights in $\mathrm{T}_{X_0}(\overline{T_{\mathrm{ad}} \cdot X}) \subseteq \mathrm{T}_{X_0}\mathrm{M}_\Gamma$ are spherical roots N-adapted to Γ, each one occurring with multiplicity 1. This, together with the fact that every T_{ad}-weight in $\mathrm{T}_{X_0}(\overline{T_{\mathrm{ad}} \cdot X})$ has to be an element of the root monoid $\mathscr{M}_X$, and hence a nonnegative integer linear combination of elements of $\Sigma^N(X)$, gives (5.1) once we prove Proposition 5.4 below. Indeed, applying this proposition with $\Sigma = \Sigma^N(X)$ yields that the T_{ad}-weights in $\mathrm{T}_{X_0}(\overline{T_{\mathrm{ad}} \cdot X})$ belong to $\Sigma^N(X)$, while $\dim \overline{T_{\mathrm{ad}} \cdot X} = |\Sigma^N(X)|$ by [AB05, Proposition 2.13]. ☐

Proposition 5.4. *Let Σ be a subset of $\Sigma^{sc}(G)$ such that every $\gamma \in \Sigma$ is N-adapted to Γ. If $\sigma \in \Sigma^{sc}(G) \cap \mathbb{N}\Sigma$ is N-adapted to Γ, then $\sigma \in \Sigma$.*

Proof. First of all, σ (of spherically closed type) must be compatible with $S^p(\Gamma)$ and is a nonnegative integer linear combination of other elements of $\Sigma^{sc}(G)$ that satisfy the same compatibility condition. This gives strong restrictions. Indeed, σ can only be the sum of two simple roots (equal or not, orthogonal or not). All the other types of spherical roots have support that nontrivially intersects $S^p(\Gamma)$, and they can be excluded by a straighforward if somewhat lengthy case-by-case verification.

Moreover, σ cannot be the double of a simple root, say 2α, with $\alpha \in \Sigma$, since α and 2α cannot both be N-adapted to Γ. Indeed, if 2α is N-adapted and $\alpha \in \mathbb{Z}\Gamma$, then $\alpha^\vee = 2\delta$ for some $\delta \in E(\Gamma)$. Hence $a(\alpha)$ has only 1 element and α is not N-adapted to Γ.

Analogously, σ cannot be the sum of two orthogonal simple roots, say $\alpha + \alpha'$, with α and α' in Σ. Indeed, since $\alpha + \alpha'$ is adapted to Γ and $\langle \alpha^\vee, \alpha \rangle \neq \langle (\alpha')^\vee, \alpha \rangle$, α cannot belong to $\mathbb{Z}\Gamma$.

Finally, let σ be the sum of two nonorthogonal simple roots, say $\alpha_1 + \alpha_2$, with α_1 and α_2 in Σ. For all $\delta \in E(\Gamma)$ such that $\delta(\sigma) > 0$ there exists a simple coroot which is an integer multiple of δ. Take $\delta \in E(\Gamma)$ with $\langle \delta, \sigma \rangle > 0$. Such a δ exists because $\langle \alpha_1^\vee, \sigma \rangle$ or $\langle \alpha_2^\vee, \sigma \rangle$ is positive, $\sigma \in \mathbb{Z}\Gamma$ and $\Gamma \subset \Lambda^+$. Then δ must be positive on at least one of the two simple roots α_1 or α_2. Suppose it is positive on α_1. Then $\delta \in a(\alpha_1)$, since α_1 is N-adapted to Γ, hence δ takes the value 1 on α_1. Therefore $\alpha_1^\vee = 2\delta$, which is not possible if α_1 is N-adapted to Γ. ☐

Remark 5.5. While the reduced induced scheme structure is the only natural scheme structure on the T_{ad}-orbit closures of Theorem 5.1, there is at least one other natural scheme structure on the irreducible components of M_Γ, namely the one given by the primary ideals of $\Bbbk[\mathrm{M}_\Gamma]$ associated to minimal primes. One can ask whether Conjecture 1.1 remains true for that scheme structure. Another natural question is whether or when M_Γ is in fact a reduced scheme. We note that the tangent space $\mathrm{T}_{X_0}\mathrm{M}_\Gamma$ might fail to detect the "non-reducedness" of M_Γ. For example, the two affine schemes $\mathrm{Spec}(\Bbbk[x,y]/\langle xy \rangle)$ and $\mathrm{Spec}(\Bbbk[x,y]/\langle x^2 y \rangle)$ have the same tangent space at the point corresponding to the maximal ideal $\langle x, y \rangle$.

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