# Bäcklund transformations: a tool to study Abelian and non-Abelian nonlinear evolution equations 

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#### Abstract

The KdV eigenfunction equation is considered: some explicit solutions are constructed. These, to the best of the authors' knowledge, new solutions represent an example of the powerfulness of the method devised. Specifically, Bäcklund transformation are applied to reveal algebraic properties enjoyed by nonlinear evolution equations they connect. Indeed, Bäcklund transformations, well known to represent a key tool in the study of nonlinear evolution equations, are shown to allow the construction of a net of nonlinear links, termed Bäcklund chart, connecting Abelian as well as non Abelian equations. The present study concerns third order nonlinear evolution equations which are all connected to the KdV equation. In particular, the Abelian wide Bäcklund chart connecting these nonlinear evolution equations is recalled. Then, the links, originally established in the case of Abelian equations, are shown to conserve their validity when non Abelian counterparts are considered. In addition, the non-commutative case reveals a richer structure related to the multiplicity of non-Abelian equations which correspond to the same Abelian one. Reduction from the nc to the commutative case allow to show the connection of the KdV equation with KdV eigenfunction equation, in the scalar case. Finally, recently obtained matrix solutions of the mKdV equations are recalled.


Keywords: Nonlinear Evolution Equations; Bäcklund Transformations; Recursion Operators; Korteweg deVries-type equations; Invariances; Cole-Hopf Transformations.

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## 1 Introduction

The crucial role played by Bäcklund transformation in investigating soliton equations is well known as testified by [1, 7, 56, 53, 57, 32] referring only to some well known books on the subject. Third order nonlinear evolution equations, termed KdV-type, are studied. Indeed, both in the Abelian case [10, 9] as well as in the non-Abelian one [11, 14, 16, 17], a wide net of nonlinear evolution equations turns out to be connected to
the Korteweg-de Vries (KdV) equation or, respectively, to its nc counterpart. The net of links, termed Bäcklund chart, allows to reveal many interesting properties enjoyed by the nonlinear evolution equations it connects. Specifically, all algebraic properties which are preserved under Bäcklund transformations can be transferred from one equation to all the others within the Bäcklund chart. The nonlinear evolution equations under investigation, further to the KdV equation are the potential Korteweg-de Vries ( pKdV ), the modified Korteweg-de Vries (mKdV), and the KdV eigenfunction (KdV eig.). In addition, in the commutative case [10, 9] also the Dym equation is included in the Bäcklund chart. On the other hand, in the non commutative one, as pointed out in [17], there are two different modified Korteweg-de Vries equations which are connected to each other via two further nonlinear evolution equations: both of them reduce to the KdV eigenfunction when the non commutativity condition is removed.

Section 2 is devoted to briefly recall the definition of Bäcklund transformation together with some remarkable properties. In the subsequent Section 3 the Bäcklund charts connecting Abelian KdV-type equations is provided. In particular, the Bäcklund chart in [10, 9] is recalled. The subsequent Section 4 the non trivial invariance exhibited by the KdV eigenfunction equation is used to construct explicit solution it admits. These solutions are independent on time, but show an explicit dependence on the space variable $x$. The extension of the Bäcklund chart, induced by the Möbius invariance enjoyed by the KdV singularity equation, is given in Section 5. The Section 6 reconsiders the nonAbelian Bäcklund chart, constructed in [11, 14, 16]. The last Section briefly discusses further remarkable results Bäcklund transformations allow to achieve. Matrix solutions, generalisation of results to hierarchies of nonlinear evolution equations are mentioned.

## 2 Bäcklund transformation \& Bäcklund charts

This Section is devoted to a brief review on Bäcklund transformations aiming to provide the definitions needed in the following restriction the attention on the definitions used in the following The general definition of Bäcklund transformation in implicit form, according to [53] and references therein, given two non linear evolution equations of the type

$$
\begin{equation*}
u_{t}=K(u) \quad, \quad v_{t}=G(v) \tag{1}
\end{equation*}
$$

when $\beta \in \mathbb{R} \backslash\{0\}$ denotes the Bäcklund parameter, then the Bäcklund Relations read

$$
\begin{cases}\frac{\partial v}{\partial x^{\prime}}=: \mathbb{B}_{1}^{\prime}\left(u, u_{x} ; v ; \beta\right), & , \quad x^{\prime}=x^{\prime}(x, t) \\ \frac{\partial v}{\partial t^{\prime}}=: \mathbb{B}_{2}^{\prime}\left(u, u_{t} ; v ; \beta\right) \quad, & t^{\prime}=t^{\prime}(x, t)\end{cases}
$$

Then, compatibility conditions $\frac{\partial \mathbb{B}_{1}^{\prime}}{\partial t^{\prime}}=\frac{\partial \mathbb{B}_{2}^{\prime}}{\partial x^{\prime}}$ followed by elimination of $x^{\prime}, t^{\prime}, v$ from the latter give $u_{t}=K(u)$. On the other hand, when we write the Bäcklund Relations in terms
of $(x, t)$, compatibility condition combined with elimination of $x, t, v$ produce $v_{t}=G(v)$. Throughout, the following definition, see [24], is adopted.
Definition Given two evolution equations,

$$
\begin{aligned}
u_{t} & =K(u), K: M_{1} \rightarrow T M_{1}, u:(x, t) \in \mathbb{R}^{n} \times \mathbb{R} \rightarrow u(x, t) \in \mathbb{R}^{m} \subset M_{1} \\
v_{t} & =G(v), G: M_{2} \rightarrow T M_{2}, v:(x, t) \in \mathbb{R}^{n} \times \mathbb{R} \rightarrow v(x, t) \in \mathbb{R}^{m} \subset M_{2}
\end{aligned}
$$

then $B(u, v)=0$ represents a Bäcklund transformation between them whenever given two solutions of such equations, say, respectively, $u(x, t)$ and $v(x, t)$ such that

$$
\begin{equation*}
\left.B(u(x, t), v(x, t))\right|_{t=0}=0 \tag{2}
\end{equation*}
$$

it follows that,

$$
\begin{equation*}
\left.B(u(x, t), v(x, t))\right|_{t=\bar{t}}=0, \quad \forall \bar{t}>0, \quad \forall x \in \mathbb{R} \tag{3}
\end{equation*}
$$

As usual choice when soliton solutions are considered, it is assumed $M:=M_{1} \equiv M_{2}$ and, in addition, the generic fiber $T_{u} M$, at $u \in M$, is identified with $M$ itself As a consequence, solutions of such two equations are linked via the Bäcklund transformation which establishes a correspondence between them: it can graphically represented as can be depicted by the following fugure

$$
\begin{equation*}
u_{t}=K(u) \quad B v_{t}=G(v) \tag{4}
\end{equation*}
$$

which shows that the Bäcklund transformation relates the two nonlinear evolution equations.

## 3 Abelian Bäcklund chart

The net of links connecting the many different nonlinear evolution equations can be summarised in a Bäcklund chart; here the latest [17] is reported. Indeed, the construction is directly related to results in [28, 9] further developments, in the case of nc nonlinear evolution equations are comprised [11, [12, 14, 16] while a comparison between the two different cases is studied in [17]. The links among the various KdV-type equations are summarised in the following Bäcklund chart:

[^0]$$
\operatorname{KdV}(u){ }^{(a)} \operatorname{mKdV}(v){ }^{(b)} \operatorname{KdV} \text { eig. }(w) \frac{(c)}{} \operatorname{KdV} \text { sing.( }()^{(d)} \operatorname{int.} \text { sol } \operatorname{KdV}(s) \text { (e) } \operatorname{Dym}(\rho)
$$

Figure 1: KdV-type equtions Bäcklund chart
where, in turn, the linked nonlinear evolution equations are:

$$
\begin{array}{ll}
u_{t}=u_{x x x}+6 u u_{x} & \\
v_{t}=v_{x x x}-6 v^{2} v_{x} & (\mathrm{KdV}) \\
w_{t}=w_{x x x}-3 \frac{w_{x} w_{x x}}{w} & (\mathrm{mKdV}) \\
\varphi_{t}=\varphi_{x}\{\varphi ; x\}, \quad \text { where }\{\varphi ; x\}:=\left(\frac{\varphi_{x x}}{\varphi_{x}}\right)_{x}-\frac{1}{2}\left(\frac{\varphi_{x x}}{\varphi_{x}}\right)^{2} & \\
s^{2} s_{t}=s^{2} s_{x x x}-3 s s_{x} s_{x x}+\frac{3}{2} s_{x}^{3} & \\
\rho_{t}=\rho^{3} \rho_{\xi \xi \xi} & \\
\text { (indV soig.) } \\
\text { (Dym). }
\end{array}
$$

that is, in order, the Korteweg-de Vries (KdV), the modified Korteweg-de Vries (mKdV), and the KdV eigenfunction (KdV eig.), the Korteweg deVries interacting soliton (int.sol.KdV), the Korteweg deVries singuarity manifold (KdV sing.) and the Dym equations. Respectively, in the Bäcklund chart, $(a),(b),(c),(d),(e)$ denote the following Bäcklund transformations
(a) $u+v_{x}+v^{2}=0$,
(b) $v-\frac{w_{x}}{w}=0$,
(c) $w^{2}-\varphi_{x}=0$,
(d) $s-\varphi_{x}=0$,
and

$$
\begin{equation*}
\text { (e) } \bar{x}:=D^{-1} s(x), \rho(\bar{x}):=s(x), \quad \text { where } \quad D^{-1}:=\int_{-\infty}^{x} d \xi \text {, } \tag{7}
\end{equation*}
$$

so that $\bar{x}=\bar{x}(s, x)$ and, hence, $\rho(\bar{x}):=\rho(\bar{x}(s, x))$. The transformation (e) is termed reciprocal transformation since it interchanges the role of the dependent and independent variables ${ }^{2}$.

## 4 The KdV eigenfunction equation: invariance properties and solutions' construction

The KdV eigenfunction equation, for sake of brevity denoted as KdV eig., is included in a wide study by Konopelchenko in [38] where, among many other ones, it is proved to

[^1]be integrable via the inverse spectral transform (IST) method. Indeed, this equation was firstly derived in a founding article of the IST method [48] and also [62], later further investigated in [38, 44] wherein a wide variety of nonlinear evolution equations is studied. Nevertheless, the KdV eigenfunction equation does not appear in subsequent classification studies of integrable nonlinear evolution equations, such as [5, 51, 63, 45, 46] until very recently when, in [4], linearizable nonlinear evolution equations are classified. The $K d V$ eigenfunction equation is a third order nonlinear equation of KdV-type since it is connected via Bäcklund transformations with the Korteweg deVries (KdV), the modified Korteweg deVries (mKdV), the Korteweg deVries interacting soliton (int.sol.KdV) [27] and the Korteweg deVries singuarity manifold (KdV sing.), introduced by Weiss in [64] via the Painlevè test of integrability.

### 4.1 Invariance

This section is concerned only about the an invariance property enjoyed by the KdV eigenfunction equation. It can be trivially checked to be scaling invariant since on substitution of $\alpha w, \forall \alpha \in \mathbb{C}$, to $w$ it remains unchanged. In addition, according to [9], see prop. 4 therein, the following proposition shows further nontrivial invariances.

## Proposition 4.1

The $K d V$ eigenfunction equation $w_{t}=w_{x x x}-3 \frac{w_{x} w_{x x}}{w}$ is invariant under the transformation

$$
\begin{equation*}
\text { I : } \quad \hat{w}^{2}=\frac{a d-b c}{\left(c D^{-1}\left(w^{2}\right)+d\right)^{2}} w^{2}, \quad a, b, c, d \in \mathbb{C} \text { s.t. } a d-b c \neq 0, \tag{8}
\end{equation*}
$$

where

$$
D^{-1}:=\int_{-\infty}^{x} d \xi
$$

is well defined since so called soliton solutions are looked for in the Schwartz space $S\left(\mathbb{R}^{n}\right)$ 3.

The proof, according to [9], is based on the invariance under the Möbius group of transformations

$$
\begin{equation*}
\mathrm{M}: \quad \hat{\varphi}=\frac{a \varphi+b}{c \varphi+d}, \quad a, b, c, d \in \mathbb{C} \quad \text { such that } \quad a d-b c \neq 0 \tag{9}
\end{equation*}
$$

of the KdV singularity manifold equation

$$
\begin{equation*}
\varphi_{t}=\varphi_{x}\{\varphi ; x\}, \quad \text { where }\{\varphi ; x\}:=\left(\frac{\varphi_{x x}}{\varphi_{x}}\right)_{x}-\frac{1}{2}\left(\frac{\varphi_{x x}}{\varphi_{x}}\right)^{2} . \tag{10}
\end{equation*}
$$

Combination of such an invariance with the connection between the KdV eigenfunction and the KdV singularity manifold equation allows to prove the proposition. Indeed, let

$$
\begin{equation*}
\mathrm{M}: \hat{\varphi}=\frac{a \varphi+b}{c \varphi+d}, \quad \forall a, b, c, d \in \mathbb{C} \mid a d-b c \neq 0 \tag{11}
\end{equation*}
$$

$$
\begin{gathered}
\begin{array}{|c}
w_{t}=w_{x x x}-3 \frac{w_{x} w_{x x}}{w} \\
\downarrow \mathrm{I} \\
\varphi_{t}=\varphi_{x}\{\varphi ; x\} \\
\hat{w}_{t}=\hat{w}_{x x x}-3 \frac{\hat{w}_{x} \hat{w}_{x x}}{\hat{w}}-\hat{\mathrm{B}} \\
\hat{\varphi}_{t}=\hat{\varphi}_{x}\{\hat{\varphi} ; x\} \\
\hline
\end{array}
\end{gathered}
$$

Figure 2: Induced invariance Bäcklund chart.
the following Bäcklund chart where the Bäcklund transformations B and $\widehat{\mathrm{B}}$ are, respectively:

$$
\text { B : } \quad w^{2}-\varphi_{x}=0 \quad \text { and } \quad \widehat{\mathrm{B}}: \quad \hat{w}^{2}-\hat{\varphi}_{x}=0 .
$$

The invariance I follows via combination of the Möbius transformation M with the two Bäcklund transformations B and $\widehat{\mathrm{B}}$. An application of the invariance I is indicates how to construct solutions of the KdV eigenfunction equation.

### 4.2 Explicit Solutions: an example

In this subsection an example of solutions admitted be the KdV eigenfunction equation is constructed on the basis of the invariance in the previous subsection. Indeed, it is easily checked that $w(x, t)=k_{1}, \forall k_{1} \in \mathbb{R}$ represents a solution of the KdV eigenfunction equation

$$
\begin{equation*}
w_{t}=w_{x x x}-3 \frac{w_{x} w_{x x}}{w} . \tag{12}
\end{equation*}
$$

When, in the Möbius group the parameters are set to be

$$
\begin{equation*}
a=d=0, b=1, c=-1 \tag{13}
\end{equation*}
$$

the invariance $I$ indicates that also

$$
\begin{equation*}
\hat{w}(x, t)=\left(k_{1}^{2} x+k_{2}\right)^{-1}, \quad \forall k_{2} \in \mathbb{R} \tag{14}
\end{equation*}
$$

represent solutions of the KdV eigenfunction equation.
Further solutions can be obtained in the same way. Remarkably, also in the nc case solutions can be constructed.

## 5 Extension of the Abelian Bäcklund chart

Note that the whole Bäcklund chart in Fig. 1 can be extended, as indicated in the following figure.

[^2]\[

$$
\begin{aligned}
& A B_{1} \downarrow \quad A B_{2} \downarrow \quad A B_{3} \downarrow \quad M \downarrow \quad A B_{4} \downarrow \quad A B_{5} \downarrow \\
& \operatorname{KdV}(\tilde{u})^{(a)} \operatorname{mKdV}(\tilde{v}){ }^{(b)} \operatorname{KdV} \text { eig. }(\tilde{w}){ }^{(c)} \operatorname{KdV} \text { sing.( }(\tilde{\varphi}){ }^{(d)} \operatorname{int} . \text { sol } \operatorname{KdV}(\tilde{s}){ }^{(e)} \operatorname{Dym}(\tilde{\rho})
\end{aligned}
$$
\]

Figure 3: Abelian KdV-type hierarchies Bäcklund chart: induced invariances.
that is, since the KdV singularity manifold equation (KdV sing.) is invariant under the Möbius group of transformations, all the the auto-Bäcklund transformations $A B_{k}, k=1 \ldots 5$ follow. Note that $\mathrm{AB}_{1}$ and $\mathrm{AB}_{2}$ are, respectively, the well known KdV and the mKdV auto-Bäcklund transformations [47, 7, 28]. The invariance of the KdV eigenfunction equation is $\mathrm{I} \equiv A B_{3}$ [9] and, according to [28], auto-Bäcklund transformations of the the int. sol. KdV and Dym equations are also obtained, denoted as $\mathrm{AB}_{4}$, and $A B_{5}$.

## 6 Non-Abelian Bäcklund chart

In this section the attention is focussed on non-Abelian equations. Specifically, according to [43, 2, 21] nonlinear evolution equations in which the unknown is an operator on a Banach space are studied. These, for short, are termed operator equations. Crucial to the present study, both in the Abelian as well as in the non-Abelian setting, is that the algebraic properties of interest nonlinear evolution equations enjoy are preserved under Bäcklund transformations. To stress the distinction between scalar and operator unknown functions, they are, respectively, denoted via lower and upper-case letters. Taking into account the results in [3, 11, 14, 16, a Bäcklund chart which connects operator KdV-type equations, can be constructed: it is depicted in the following Fig. 4.


Figure 4: KdV-type equations Bäcklund chart: the non-Abelian case.
In Fig. 4. the third order nonlinear operator evolution equations where, as pointed out, all unknowns are denoted via capital case letters with the only exception of the KdV
sing. equation. The KdV-type operator equations are, in turn,

$$
\begin{array}{ll}
U_{t}=U_{x x x}+3\left\{U, U_{x}\right\} & \text { (KdV), } \\
V_{t}=V_{x x x}-3\left\{V^{2}, V_{x}\right\} & (\mathrm{mKdV}), \\
Q_{t}=Q_{x x x}-3 Q_{x x} Q^{-1} Q_{x} & \text { (meta-mKdV), } \\
\widetilde{Q}_{t}=\widetilde{Q}_{x x x}-3 \widetilde{Q}_{x} \widetilde{Q}^{-1} \widetilde{Q}_{x x} & \text { (mirror meta-mKdV), } \\
\widetilde{V}_{t}=\widetilde{V}_{x x x}+3\left[\widetilde{V}, \widetilde{V}_{x x}\right]-6 \widetilde{V} \widetilde{V}_{x} \widetilde{V} & \text { (amKdV), } \\
\phi_{t}=\phi_{x}\{\phi ; x\}, \quad \text { where }\{\phi ; x\}=\left(\phi_{x}^{-1} \phi_{x x}\right)_{x}-\frac{1}{2}\left(\phi_{x}^{-1} \phi_{x x}\right)^{2} & \text { (KdV sing.), } \\
S_{t}=S_{x x x}-\frac{3}{2}\left(S_{x} S^{-1} S_{x}\right)_{x} & \text { (int. sol KdV). }
\end{array}
$$

where the Bäcklund transformations $(a),(b),(c)$ linking the KdV with the mKdV, the amKdV with the int.sol KdV and the latter with the KdV sing. are the following, ones, the non-commutative counterparts of those in the commutative Bäcklund chart, see Fig. 1 and Fig. 3 .

$$
\begin{align*}
U & =-\left(V^{2}+V_{x}\right)  \tag{a}\\
\widetilde{V} & =\frac{1}{2} S^{-1} S_{x}  \tag{b}\\
S & =\phi_{x} \tag{c}
\end{align*}
$$

Notably, both the two equation named mirror meta-mKdV and meta-mKdV [16] coincide with the KdV eigenfunction equation when the nc condition is removed. Similarly, also the mKdV and amKdV equations when commutativity is assumed reduce to the usual $m K d V$ equation and, therefore the box in Fig. 4 collapses to a line and, therefore the Bäcklund chart in Fig. 1 can be recovered. The following Fig. 6 is esplicative.

where note that, in the commutative case, the composition of the transformations $V=$ $Q_{x} Q^{-1}$ and $\widetilde{V}=-Q^{-1} Q_{x}$ reduce to $V \rightarrow-V$, trivial sign invariance admitted by the mKdV equation; correspondingly, the two forms of modified KdV equations (amKdV and $m K d V$ ) coincide with the (Abelian) mKdV equation.

## 7 Conclusions and perspectives

This closing section aims to give a brief overview on two different lines of results concerning further developments that is, on one side the determination of explicit solutions
and on te other one, the extension of the Bäcklund chart from the considered nonlinear evolution equations to corresponding hierarchies. Further perspectives are finally mentioned.

### 7.1 Matrix solutions of soliton equations

This subsection is devoted to the special case when the operator is finite dimensional so that it admits a matrix representation. Thus, the aim is to emphasise the importance of Bäcklund transformations also when solutions admitted by non-Abelian soliton equations are looked for. Solutions admitted by the matrix equations are a subject of interest in the literature. The study presented, based on previous results [11, 12] further developed in [18, 19], take into account multisoliton solutions of the matrix KdV equation obtained by Goncharenko [31], via a generalisation of the Inverse Scattering Method. According to [12], and in particular Theorem 3 therein, generalises Goncharenko's multisoliton solutions which follow as special ones. Solutions of matrix mKdV equation are discussed and obtained in [18], where some $2 \times 2$ and some $3 \times 3$ examples are provided; in [19] the solution formula, which in 12 is obtained in the general operator case, is discussed referring to the case of a $d \times d, d \in \mathbb{N}$ and further solutions are produced to give an idea of the results in this direction and currently under investigation [20]. Further matrix solutions are obtained in [23, 31, 41, 60, 61, 35, 59].

### 7.2 Generalisations: hierarchies \& further perspectives

Finally, some further observations deserve attention. First of all, one of the properties of Bäcklund transformations crucial in the present research project, which involves not only the two authors, but also further collaborators, is the notion of recursion operator. Indeed the existence of a hereditary recursion operator admitted by a nonlinear evolution equation is a remarkable algebraic property [25]. Such a property, on one side, allows to construct a whole hierarchy of nonlinear evolution equations associated, for instance, to the KdV equation. On another side, the algebraic properties which characterise a hereditary recursion operator are preserved under Bäcklund transformations. Hence, since the KdV equation admits a hereditary recursion operator, the Bäcklund chart in Fig. 1 indicates the way to construct the recursion operators of all the nonlinear evolution equations it connects. In addition, such a Bäcklund chart can be naturally extended to the whole hierarchies of all the nonlinear evolution equations therein [28], in the Abelian case. The corresponding non-Abelian Bäcklund chart is constructed in [11, 12] and further extended in [14, 16]. Also the case of non-Abelian Burgers equation [40, 34, 36, 13, 15] shows a richer structure with respect to the corresponding Abelian one.

Furthermore, a $2+1$ - dimensional [49] Bäcklund chart which links the KadomtsevPetviashvili (KP), the modified Kadomtsev-Petviashvili (mKP) and other $2+1$ - dimensional soliton equations, such as the KP singularity manifold equation and the $2+1$ dimensional version of the Dym equation. The link, obtained by Rogers [52], between the KP and the $2+1$-dimensional Dym equation indicates the way to the construction of solutions of the Dym equation in $2+1$ - dimensions. Notably, as shown in [49], the

BC in Fig. 3 follows to represent a constrained version of the KP Bäcklund chart [50]. In addition, as shown in [28], the Hamiltonian and bi-Hamiltonian structure admitted by the KdV equation, since these properties are preserved under Bäcklund transformations [25], all the nonlinear evolution KdV-type equations in the Bäcklund chart, in Fig. 3. are all proved to admit a bi-Hamiltonian structure [42, 26, 29, 30]. As discussed in [8], a Bäcklund chart connects the Caudrey-Dodd-Gibbon-Sawata-Kotera and KaupKupershmidt hierarchies [22, 58, 37]. All the involved equations are 5th order nonlinear evolution equations; notably, the Bäcklund chart linking them all shows an impressive resemblance to the one connecting KdV-type equations. Again, such Bäcklund chart can be extended to the corresponding whole hierarchies [54, 10]. The study on $2+1$ dimensional non-Abelian equations seems of interest.

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[^0]:    ${ }^{1}$ It is generally assumed that $M$ is the space of functions $u(x, t)$ which, for each fixed $t$, belong to the Schwartz space $S$ of rapidly decreasing functions on $\mathbb{R}^{n}$, i.e. $S\left(\mathbb{R}^{n}\right):=\left\{f \in C^{\infty}\left(\mathbb{R}^{n}\right):\|f\|_{\alpha, \beta}<\right.$ $\left.\infty, \forall \alpha, \beta \in \mathbb{N}_{0}^{n}\right\}$, where $\|f\|_{\alpha, \beta}:=\sup _{x \in \mathbb{R}^{n}}\left|x^{\alpha} D^{\beta} f(x)\right|$, and $D^{\beta}:=\partial^{\beta} / \partial x^{\beta}$; throughout this article $n=1$.

[^1]:    ${ }^{2}$ see, for instance, [56] where reciprocal transformations are defined and applications are provided. The transformation $(e)$ is analysed in [10, 28] where it is shown to represent a Bäcklund transformation between the extended manifold consisting of the both the dependent and the independent variables.

[^2]:    ${ }^{3}$ see footnote on page 2 .

