

# FATIGUE EFFECTS IN ELASTIC MATERIALS WITH VARIATIONAL DAMAGE MODELS: A VANISHING VISCOSITY APPROACH

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ABSTRACT. We study the existence of quasistatic evolutions for a family of gradient damage models which take into account fatigue, that is the process of weakening in a material due to repeated applied loads. The main feature of these models is the fact that damage is favoured in regions where the cumulation of the elastic strain (or other relevant variables, depending on the model) is higher. To prove the existence of a quasistatic evolution, we follow a vanishing viscosity approach based on two steps: we first let the time-step  $\tau$  of the time-discretisation and later the viscosity parameter  $\varepsilon$  go to zero. As  $\tau \rightarrow 0$ , we find  $\varepsilon$ -approximate viscous evolutions; then, as  $\varepsilon \rightarrow 0$ , we find a rescaled approximate evolution satisfying an energy-dissipation balance.

**Keywords:** Fatigue; Gradient-damage models; Variational methods; Vanishing-viscosity approach

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## 1. INTRODUCTION

In Material Science, fatigue refers to the process which leads to the weakening of a material due to repeated applied loads, which individually would be too small to cause the direct failure of the material itself. Macroscopic fatigue fractures appear as a consequence of the interaction of many and complicated material phenomena occurring at the micro-scale, such as, for instance, plastic slip systems and coalescence of micro-voids, [39, 35, 37]. Fatigue failure is extremely dangerous, since it often occurs without forewarning resulting in devastating events, and is responsible for up to the 90% of all mechanical failures [38]. The main reason is that it is very difficult, in real situations, to identify the fatigue degradation state of a material. Therefore, its prediction still represents an open challenge for modeling and simulation at the cutting edge of mechanics.

Fatigue favours the occurrence of damage and fracture in different types of materials, both brittle and ductile. When the stress level is high enough to induce plastic deformations, the material is usually subjected to a so-called low-cycle fatigue regime; instead, high-cycle fatigue occurs if the stress is below the yield stress such that the strains are primarily elastic. Models where fatigue effects are induced by the cumulation of plastic deformations have been recently studied in [3, 4, 2, 1] and [9, 11, 12].

In this paper we study a phenomenological material model where damage is the only inelastic phenomenon and the fatigue weakening of the material is a consequence of repeated cycles of elastic deformations. Our work is inspired by the recent paper [5], where the authors propose a similar model in the one-dimensional setting and to which the reader is invited to refer to for further mechanical details.

As usual, damage is expressed in terms of a scalar variable which affects the elastic response of the material and may be interpreted as the local percentage of sound interatomical bonds. In contrast to many previous damage models [20, 28, 7, 41, 40, 24, 25, 26], in this paper the dissipation depends not only on the damage variable itself, but also on the history of the evolution. Indeed, damage is favoured in regions where a suitable history variable has a higher value. This history variable is defined pointwise in the body as the cumulation in time of a given function  $\zeta$  that may be the strain, or the stress, or the energy density, according to the model. As a consequence, the material may undergo a damage process even if the variable  $\zeta$  remains small during the evolution.

We are here interested in proving the existence of quasistatic evolutions for this model in a two-dimensional antiplane shear setting, following a vanishing viscosity approach. To present in detail our result, before expressing the strong formulation of the model in terms of differential inclusions, we introduce the time-incremental minimisation problem corresponding to a time discretisation  $t_k^i := i\tau_k^T = i\tau_k$  for the unknowns  $\alpha: \Omega \rightarrow [0, 1]$  (the damage variable) and  $u: \Omega \rightarrow \mathbb{R}$  (the displacement) assuming that the previous states  $(\alpha_k^j, u_k^j)_{j=0}^{i-1}$  are known:

$$(\alpha_k^i, u_k^i) \in \operatorname{argmin}_{\alpha \leq \alpha_k^{i-1}} \left\{ \frac{1}{2} \int_{\Omega} \mu(\alpha) |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla \alpha|^2 dx + \int_{\Omega} f(V_k^{i-1})(\alpha_k^{i-1} - \alpha) dx + \frac{\varepsilon}{2\tau_k} \|\alpha - \alpha_k^{i-1}\|_{L^2}^2 \right\}.$$

The functional minimised above consists of three parts: the internal energy

$$\mathcal{E}(\alpha, u) = \frac{1}{2} \int_{\Omega} \mu(\alpha) |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla \alpha|^2 dx,$$

given by the sum of the elastic energy and the damage regularisation term; the energy dissipated from the previous state

$$\int_{\Omega} f(V_k^{i-1})(\alpha_k^{i-1} - \alpha) dx = \mathcal{R}(\alpha - \alpha_k^{i-1}; V_k^{i-1}), \quad \text{with} \quad \mathcal{R}(\beta; V) = \begin{cases} -\int_{\Omega} f(V)\beta dx & \text{if } \beta \leq 0, \\ +\infty & \text{otherwise;} \end{cases} \quad (1.1)$$

and the viscosity term, depending on a small parameter  $\varepsilon$ . The elastic response is affected by the factor  $\mu(\alpha) > 0$ , where  $\mu$  is nondecreasing in  $\alpha$ , according to the fact that  $\alpha = 1$  represents a sound material and  $\alpha = 0$  a completely damaged one. (Notice that the constraint  $\alpha \leq \alpha_k^{i-1}$  enforces the irreversibility of the damage process.) The  $L^2$  norm of  $\nabla \alpha$  is the usual regularising term in gradient damage models (see the aforementioned works and [17, 10, 13] for coupling with plasticity). The dissipation term characterises the present model in comparison to other damage models, since the fatigue term  $f(V_k^{i-1})$  weights the damage increment. For every  $j$ , the history variable  $V_k^j$  is defined by

$$V_k^j := \sum_{h=1}^j |\zeta_k^h - \zeta_k^{h-1}|,$$

where  $\zeta_k^h$  represents the elastic strain, or the stress, or the density of the elastic energy at time  $t_k^h$ . Notice that  $V_k^j = \int_0^{t_k^j} |\dot{\zeta}_k(s)| ds$ , where  $\zeta_k(s)$  is the piecewise affine interpolation of  $\zeta_k^h$ . The function  $f$  is nonincreasing, so that in the minimisation it is more convenient to take  $\alpha$  lower where the cumulation  $V_k^{i-1}$  is larger. The viscosity term prevents  $\alpha_k^i$  to be too far (in  $L^2$ ) from the previous damage state  $\alpha_k^{i-1}$ .

The approach that we follow consists of two main steps: as, e.g., in [28, 7, 41, 40, 24, 25], we let first the time-step of the discretisation  $\tau_k$  and later the viscosity parameter  $\varepsilon$  tend to 0. More precisely, the starting point is to define for every  $k$  the discrete-time evolution  $(\alpha_{\varepsilon,k}(t), u_{\varepsilon,k}(t))$  as the piecewise affine interpolation of  $(\alpha_k^i, u_k^i)$  and to derive *a priori* estimates (cf. Proposition 3.5) which guarantee that  $\|\alpha_{\varepsilon,k}\|_{H^1(0,T;H^1(\Omega))}$ ,  $\|u_{\varepsilon,k}\|_{H^1(0,T;W^{1,p}(\Omega))}$  are bounded uniformly with respect to  $k$  (not with respect to  $\varepsilon$ ) and  $\|\alpha_{\varepsilon,k}\|_{W^{1,1}(0,T;H^1(\Omega))}$ ,  $\|u_{\varepsilon,k}\|_{W^{1,1}(0,T;W^{1,p}(\Omega))}$  are bounded uniformly with respect to  $k$  and  $\varepsilon$ , for some  $p > 2$ . We exploit the *a priori* estimates  $H^1$  in time to pass to the limit as  $k \rightarrow +\infty$ : for every  $\varepsilon$  we obtain an  $\varepsilon$ -approximate viscous evolution  $(\alpha_\varepsilon(t), u_\varepsilon(t))$  characterised by an equilibrium condition in  $u_\varepsilon(t)$ , a unilateral stability condition in  $\alpha_\varepsilon(t)$  (Karush-Kuhn-Tucker inequality), and an energy-dissipation balance (cf. Definition 4.1). This evolution may be expressed in terms of the differential inclusions (cf. (ev1) $_\varepsilon$ –(ev2) $_\varepsilon$  in Definition 4.1 and (ev3') $_\varepsilon$  in Lemma 4.12)

$$\begin{aligned} \partial_u \mathcal{E}(\alpha_\varepsilon(t), u_\varepsilon(t)) &= 0 \quad \text{in } H_{\partial_D \Omega}^{-1}(\Omega), \\ \partial_\alpha \mathcal{R}(\dot{\alpha}_\varepsilon(t); V_\varepsilon(t)) + \varepsilon \dot{\alpha}_\varepsilon(t) + \partial_\alpha \mathcal{E}(\alpha_\varepsilon(t), u_\varepsilon(t)) &\ni 0 \quad \text{in } (H^1(\Omega))', \end{aligned}$$

for a.e.  $t \in (0, T)$ , where  $V_\varepsilon(t)$  is the history variable associated to the evolution  $(\alpha_\varepsilon, u_\varepsilon)$ ,  $H_{\partial_D \Omega}^{-1}(\Omega)$  is the dual of  $\{v \in H^1(\Omega) : v = 0 \text{ on } \partial_D \Omega\}$ , and  $\partial_\alpha \mathcal{R}(\beta; V)$  is the (convex analysis) subdifferential of  $\mathcal{R}(\cdot; V)$ , i.e.  $\xi \in \partial_\alpha \mathcal{R}(\beta; V)$  if and only if  $\mathcal{R}(\beta; V) + \langle \xi, \beta - \bar{\beta} \rangle \leq \mathcal{R}(\beta; V)$  for every  $\beta \in H^1(\Omega)$ . (For the expression of  $\partial_u \mathcal{E}$  and  $\partial_\alpha \mathcal{E}$  we refer to Lemma 2.1.)

The *a priori* estimates  $W^{1,1}$  in time allow us to reparametrise the  $\varepsilon$ -approximate viscous evolutions and to obtain a family of equi-Lipschitz evolutions  $(\alpha_\varepsilon^\circ(s), u_\varepsilon^\circ(s))$  in a slower time scale  $s$ . At this stage we let  $\varepsilon \rightarrow 0$  and obtain an evolution  $(\alpha^\circ(s), u^\circ(s))$  together with a reparametrisation function  $t^\circ(s)$  that permits the passage from the slow to the original fast time scale  $t$ . In Theorem 5.1 we prove that  $(\alpha^\circ, u^\circ)$  still satisfies an equilibrium condition in  $u^\circ(s)$ , a unilateral stability condition in  $\alpha^\circ(s)$  (Karush-Kuhn-Tucker inequality), and an energy-dissipation balance. However, the dissipation in the energy balance weights the rate of damage with a function  $\tilde{f}^\circ(s) \leq f(V^\circ(s))$ , where  $V^\circ(s)$  is the history variable associated to the evolution  $(\alpha^\circ, u^\circ)$ . In terms of differential inclusions, this reads as (cf. (ev1)–(ev2) in Theorem 5.1 and (5.11) in Remark 5.3)

$$\begin{aligned} \partial_u \mathcal{E}(\alpha^\circ(s), u^\circ(s)) &= 0 \quad \text{in } H_{\partial_D \Omega}^{-1}(\Omega), \\ \partial_\alpha \mathcal{R}(\dot{\alpha}^\circ(s); \tilde{f}^\circ(s)) + \partial_\alpha \mathcal{E}(\alpha^\circ(s), u^\circ(s)) &\ni 0 \quad \text{in } (H^1(\Omega))', \end{aligned}$$

for a.e.  $s \in (0, S) \setminus U^\circ$ , where  $\mathcal{R}(\cdot; \tilde{f}^\circ(s))$  is defined as in (1.1) with  $\tilde{f}^\circ(s)$  in place of  $f(V^\circ(s))$ . The set  $U^\circ$  corresponds to jump instants (of the evolution in the fast time scale) reparametrised in the slow time scale: therein the limit evolution is governed by a variational inequality of viscous type, representing a fast unstable propagation in the original time scale. An interesting issue, that we were not able to solve, is to determine whether there are explicit examples where this inequality is strict and  $\tilde{f}^\circ(s)$  is actually the correct weight to consider in the energy-dissipation balance.

In the mathematical treatment of the present model some technical difficulties arise. Here we discuss the main issues in the *a priori* estimates and in the limits as  $k \rightarrow +\infty$  and  $\varepsilon \rightarrow 0$ .

The proof of the *a priori* estimates rests upon the manipulation of the Discrete Karush-Kuhn-Tucker conditions (3.9) and (3.10) evaluated at two subsequent times  $t_k^{i-1}$  and  $t_k^i$ , respectively, as e.g. in [33, 24, 30, 25, 11, 26]. The resulting estimate (3.17) contains in the right-hand side also discrete-time derivatives at time  $t_k^{i-1}$ , in contrast to the aforementioned works, where only discrete-time derivatives at time  $t_k^i$  appear. These additional terms are due to the presence of the fatigue weight  $f(V_k^{i-1})$  in the dissipation for the  $i$ -th incremental minimisation problem and prevent the immediate application of the discrete Gronwall estimate used in the previous works. We refine the usual technique to overcome this issue in (3.19)–(3.22).

The main difficulty in deriving the properties of the  $\varepsilon$ -approximate viscous evolutions  $(\alpha_\varepsilon(t), u_\varepsilon(t))$  consists in passing to the limit as  $k \rightarrow +\infty$  in the dissipation term containing the fatigue weight  $f(V_{\varepsilon,k}(t))$ . The *a priori* estimate on  $\|u_{\varepsilon,k}\|_{H^1(0,T;W^{1,p}(\Omega))}$  only guarantees that  $\nabla \dot{u}_{\varepsilon,k} \rightharpoonup \nabla \dot{u}_\varepsilon$  weakly in  $L^2(0,T;L^p(\Omega;\mathbb{R}^2))$ , and this convergence is not sufficient to deduce the convergence of  $V_{\varepsilon,k}$  to  $V_\varepsilon$ , even in the paradigmatic case where  $\zeta$  is the elastic strain, namely when the history variable is  $V(t) = \int_0^t |\nabla \dot{u}(s)| ds$ . To circumvent this problem we first let  $f(V_{\varepsilon,k}(t))$  converge to some  $\tilde{f}_\varepsilon(t)$  weakly\* in  $L^\infty(\Omega)$  for every  $t$  by an Helly-type theorem (cf. Lemma 4.6), to get an evolution  $(\alpha_\varepsilon(t), u_\varepsilon(t))$  satisfying the Karush-Kuhn-Tucker inequality, and the energy-dissipation balance with  $\tilde{f}_\varepsilon(t)$  in place of  $f(V_\varepsilon(t))$  (cf. Propositions 4.8 and 4.10). At this stage, we exploit the convergence of all the terms of the discrete-time energy-dissipation balance to the corresponding ones in the continuous-time energy-dissipation balance. This improves the convergence of  $\dot{\alpha}_{\varepsilon,k}$  to  $\dot{\alpha}_\varepsilon$  (Proposition 4.11), allowing us to deduce that  $\nabla \dot{u}_{\varepsilon,k} \rightarrow \nabla \dot{u}_\varepsilon$  strongly in  $L^2(0,T;L^p(\Omega;\mathbb{R}^2))$  and thus that  $\tilde{f}_\varepsilon(t) = f(V_\varepsilon(t))$ . Eventually, we obtain the existence of an  $\varepsilon$ -approximate evolution.

The scenario when  $\varepsilon \rightarrow 0$  is radically different. Indeed, here the energy-dissipation balance does not help to improve the weak convergence  $\nabla \dot{u}_\varepsilon^\circ \rightharpoonup \nabla \dot{u}^\circ$  for the rescaled evolutions  $(\alpha_\varepsilon^\circ(s), u_\varepsilon^\circ(s))$ , due to the rate-independence of the system as  $\varepsilon \rightarrow 0$ . As a consequence, the limit evolution is formulated with  $\tilde{f}^\circ(s)$ , the weak\*- $L^\infty$  limit of the fatigue weight reparametrisations  $f(V_\varepsilon^\circ(s))$ , in place of  $f(V^\circ(s))$ . This motivates why we pass to the limit in two steps, rather than directly taking a simultaneous limit  $\tau_k/\varepsilon_k \rightarrow 0$ ,  $k \rightarrow +\infty$ , as in the framework developed in [31, 27] and followed in [26].

## 2. ASSUMPTIONS ON THE MODEL

**Vector-valued functions.** In this paragraph we let  $X$  be a Banach space. We will often consider the Bochner integral of measurable functions  $v: [0, T] \rightarrow X$ . For the definition of this notion of integral and its main properties we refer to [8, Appendix] or to the textbook [19]. The Lebesgue space  $L^p(0, T; X)$  is defined accordingly. We recall that, if  $p \in [1, \infty)$  and  $X$  is separable, the dual of  $L^p(0, T; X)$  is  $L^{p'}(0, T; X')$ , where  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $X'$  is the dual of  $X$ .

For the definition and the main properties of absolute continuous functions  $AC([0, T]; X)$  and Sobolev functions  $W^{1,p}(0, T; X)$ , the reader is referred to [8, Appendix]. We recall here the Aubin-Lions Lemma [6, 36] about the compactness property enjoyed by  $W^{1,p}(0, T; X)$ . Let  $Y$  be a Banach space compactly embedded in  $X$ , and let  $1 \leq p, q \leq \infty$ . Then the space  $W = \{v \in L^p(0, T; Y) : \dot{v} \in L^q(0, T; X)\}$  is: 1) compact in  $L^p(0, T; X)$  if  $p < \infty$ ; 2) compact in  $C([0, T]; X)$  if  $p = \infty$  and  $q > 1$ .

In this paper, the Banach space  $X$  will be either a Lebesgue space  $L^q(U; \mathbb{R}^m)$  or a Sobolev space  $W^{1,q}(U)$ , where  $U$  is an open set of  $\mathbb{R}^n$ . Given an element  $v \in L^p(0, T; L^q(U; \mathbb{R}^m))$ ,  $p, q \in [1, \infty)$ , we identify it with the function  $v: [0, T] \times \Omega \rightarrow \mathbb{R}^m$  defined by  $v(t; x) := (v(t))(x)$ .

The norms  $\|\cdot\|_{L^p}$  and  $\|\cdot\|_{W^{1,p}}$  without any further notation will always denote the  $L^p$ -norm and the  $W^{1,p}$ -norm with respect to the space variable  $x$ , respectively.

**The reference configuration.** Throughout the paper,  $\Omega$  is a bounded, Lipschitz, open set in  $\mathbb{R}^2$  representing the cross-section of a cylindrical body in the reference configuration. The deformation  $v: \Omega \times \mathbb{R} \rightarrow \Omega \times \mathbb{R}$  takes the form  $v(x_1, x_2, x_3) = (x_1, x_2, x_3 + u(x_1, x_2))$ , where  $u: \Omega \rightarrow \mathbb{R}$  is the vertical displacement. In this antiplane shear framework, the two dimensional setting is the physical relevant one. This assumption gives the compact embedding  $H^1(\Omega)$  in  $L^p(\Omega)$  for every  $p \in [1, \infty)$ , which we employ in the *a priori* estimates.

We assume that  $\partial\Omega = \overline{\partial_D\Omega} \cup \overline{\partial_N\Omega}$ , where  $\partial_D\Omega$  and  $\partial_N\Omega$  are relatively open sets in  $\partial\Omega$  with  $\partial_D\Omega \cap \partial_N\Omega = \emptyset$  and  $\mathcal{H}^1(\partial_D\Omega) > 0$ . A Dirichlet boundary datum will be prescribed on the set  $\partial_D\Omega$ .

In order to apply the integrability result [22] to our problem (see Remark 3.2 below), we assume that  $\Omega \cup \partial_N\Omega$  is regular in the sense of [22, Definition 2]. (Notice that in dimension 2 this regularity assumption on  $\Omega \cup \partial_N\Omega$  is satisfied, e.g., when the relative boundary  $\partial(\partial_N\Omega)$  in  $\partial\Omega$  consists of a finite number of points.)

It is convenient to introduce the notation  $W_{\partial_D\Omega}^{-1,p}(\Omega)$  for the dual of the space  $\{u \in W^{1,p'}(\Omega) : u = 0 \text{ on } \partial_D\Omega\}$ , where  $\frac{1}{p} + \frac{1}{p'} = 1$ .

**The total energy.** Following [21], the damage state of the body is represented by an internal variable  $\alpha: \Omega \rightarrow [0, 1]$ . The value  $\alpha = 1$  corresponds to a sound state, whereas  $\alpha = 0$  corresponds to the maximum possible damage. As usual in gradient damage models [34], the system in analysis comprises a regularizing term  $\|\nabla\alpha\|_{L^2}^2$ . In particular, the damage variable  $\alpha$  belongs to the Sobolev space  $H^1(\Omega)$ .

For every  $\alpha \in H^1(\Omega)$  and  $u \in H^1(\Omega)$ , the *stored elastic energy* is defined by

$$\frac{1}{2} \int_{\Omega} \mu(\alpha) |\nabla u|^2 \, dx.$$

We make the following assumptions on the dependence of the shear modulus  $\mu$  on the damage variable  $\alpha$ :

$$\begin{aligned} \mu: \mathbb{R} \rightarrow [0, +\infty) \text{ is a } C^{1,1}(\mathbb{R}), \text{ nondecreasing function with } \mu(0) > 0, \\ \mu(\beta) = \mu(0) \text{ for } \beta \leq 0, \quad \mu(\beta) = \mu(1) \text{ for } \beta \geq 1. \end{aligned} \tag{2.1}$$

The regularity assumption on  $\mu$  is needed in the proof of Proposition 3.5 (see (3.14)). The condition (2.1) on  $\mu$  forces  $\alpha$  to take values in  $[0, 1]$  in the evolution (see Remark 3.1).

The *total energy* corresponding to a damage state  $\alpha$  and to a displacement  $u$  is

$$\mathcal{E}(\alpha, u) := \frac{1}{2} \int_{\Omega} \mu(\alpha) |\nabla u|^2 \, dx + \frac{1}{2} \int_{\Omega} |\nabla \alpha|^2 \, dx. \tag{2.2}$$

Notice that the constant  $\frac{1}{2}$  in the gradient damage regularisation term does not play a role in the mathematical treatment and may be replaced by any positive constant.

We compute here the derivatives of the total energy. Note that an integrability strictly higher than 2 is required on  $\nabla u$  to guarantee the differentiability of the energy with respect to  $\alpha$ .

**Lemma 2.1.** *The following statements hold true:*

*i) Let  $u \in W^{1,p}(\Omega)$ , with  $p > 2$ . Then the functional  $\alpha \in H^1(\Omega) \mapsto \mathcal{E}(\alpha, u)$  is differentiable and*

$$\langle \partial_{\alpha} \mathcal{E}(\alpha, u), \beta \rangle = \frac{1}{2} \int_{\Omega} \mu'(\alpha) |\nabla u|^2 \beta \, dx + \int_{\Omega} \nabla \alpha \cdot \nabla \beta \, dx, \tag{2.3}$$

*for every  $\alpha, \beta \in H^1(\Omega)$ .*

ii) Let  $\alpha \in H^1(\Omega)$ . Then the functional  $u \in H^1(\Omega) \mapsto \mathcal{E}(\alpha, u)$  is differentiable and

$$\langle \partial_u \mathcal{E}(\alpha, u), v \rangle = \int_{\Omega} \mu(\alpha) \nabla u \cdot \nabla v \, dx,$$

for every  $v \in H^1(\Omega)$ .

*Proof.* We only prove *i)*, the proof of *ii)* being trivial. The derivative of  $\frac{1}{2} \|\nabla \alpha\|_{L^2}^2$  simply gives the second integral in (2.3). As for the differentiability of  $\int \mu(\alpha) |\nabla u|^2 \, dx$ , let us fix  $\alpha, \beta \in H^1(\Omega)$ , and  $\delta > 0$ . By Young's inequality we have

$$\left| \frac{\mu(\alpha + \delta\beta) - \mu(\alpha)}{\delta} |\nabla u|^2 \right| \leq \|\mu'\|_{L^\infty} |\beta| |\nabla u|^2 \leq C \left[ |\beta|^q + |\nabla u|^p \right],$$

where  $q = \frac{p}{p-2} < \infty$ . Thanks to the embedding  $H^1(\Omega) \Subset L^q(\Omega)$ , we can apply the Dominated Convergence Theorem to deduce that the functional  $\alpha \in H^1(\Omega) \mapsto \mathcal{E}(\alpha, u)$  is Gâteaux-differentiable and its Gâteaux-differential is expressed by (2.3). Moreover, since  $u \in W^{1,p}(\Omega)$ , with  $p > 2$ , and  $H^1(\Omega) \Subset L^r(\Omega)$ , for any  $r \in [1, \infty)$ , it is immediate that the functionals in *i)* and *ii)* are Fréchet-differentiable.  $\square$

**Fatigue and damage dissipation.** The damage dissipation is affected by the cumulation of a suitable variable of the system during the history of the evolution. This variable may be for instance the elastic strain, the stress, or the density of the elastic energy, according to the material model. In the general case, we consider a function depending on the damage variable  $\alpha$  and on the elastic strain  $\nabla u$ : we take, for given evolutions  $\alpha \in AC([0, T]; L^q(\Omega; [0, 1]))$ ,  $u \in AC([0, T]; W^{1,p}(\Omega))$ , with  $p > 2$ ,  $\frac{1}{q} + \frac{1}{p} < \frac{1}{2}$ , the function

$$\zeta(t) := g(\alpha(t)) \nabla u(t), \tag{2.4}$$

where  $g \in C^{1,1}([0, 1])$ . (In the following we will guarantee that the damage variable takes values in  $[0, 1]$ , see Remark 3.1; one could also assume  $g \in C^{1,1}(\mathbb{R})$  and constant in  $(-\infty, 0]$  and  $[1, \infty)$  as done for  $\mu$ , the difference is that the terms involving  $g$  are constant in the incremental minimisation, see (3.1).) For instance, if  $g(\alpha) \equiv 1$ , then  $\zeta$  is simply the elastic strain; if  $g(\alpha) = \mu(\alpha)$ , then  $\zeta$  is the stress.

By our assumption on the evolutions  $\alpha, u$ , we have that  $\zeta \in AC([0, T]; L^2(\Omega; \mathbb{R}^2))$ , so we consider the corresponding cumulation

$$V^\zeta(t; x) \equiv V(t; x) := \int_0^t |\dot{\zeta}(s; x)| \, ds, \quad x \in \Omega, \tag{2.5}$$

defined as the Bochner integral in  $L^2(\Omega)$ .

In (2.5) the notation  $\equiv$  represents the fact that we do not write in the following the dependence of the cumulated variable from  $\zeta$ . We shall also use the notation  $V_k, V_\varepsilon$ , etc. for the cumulated variable corresponding to  $\zeta_k, \zeta_\varepsilon$ , etc. given by (2.4) for  $\alpha_k$  and  $u_k, \alpha_\varepsilon$  and  $u_\varepsilon$ , etc., respectively, specifying the correspondence in each case.

We notice that one could consider other possible choices for the variable  $\zeta$ , for which the results of this paper still hold. For instance, one could take  $\zeta(t) = g(\alpha) |\nabla u|^\theta$ , with  $\theta \in [1, p)$ , so  $\zeta \in AC([0, T]; L^{p/\theta}(\Omega))$  (see also the observations in Proposition 3.5 and Lemmas 4.3 and 4.4). This covers, e.g., the case where  $\zeta$  is the density of the elastic energy, i.e., when  $g(\alpha) = \mu(\alpha)$  and  $\theta = 2$ .

We denote by  $H_-^1(\Omega)$  the functions  $\beta \in H^1(\Omega)$  with  $\beta \leq 0$  a.e. in  $\Omega$ . For every measurable function  $V: \Omega \rightarrow [0, +\infty)$ , playing the role of the cumulation of  $\zeta$ , and for every  $\beta \in H_-^1(\Omega)$ , representing the damage

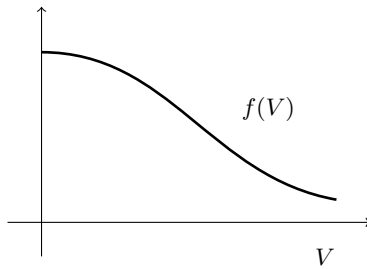
rate, we define the corresponding dissipation potential by

$$\mathcal{R}(\beta; V) := - \int_{\Omega} f(V)\beta \, dx, \quad (2.6)$$

where

$$f: [0, +\infty) \rightarrow [0, +\infty) \text{ is a Lipschitz, nonincreasing function with } f(0) > 0.$$

The regularity assumptions on  $f$ ,  $g$  are used in the proof of Proposition 3.5 (see (3.14)), and in Lemmas 4.3 and 4.4.



**Figure 1.** Graph of the function  $f$  appearing in the dissipation potential. The higher the value of  $V$ , the smaller the weight  $f(V)$  in the damage dissipation. Recall that  $V$  plays the role of the cumulation of the variable  $\zeta$ .

According to the general theory of Rate-Independent systems [29],  $\mathcal{R}$  naturally induces the following dissipation between two damage states  $\alpha_1, \alpha_2 \in H^1(\Omega)$  with  $0 \leq \alpha_1 \leq \alpha_2 \leq 1$  a.e. in  $\Omega$

$$\mathcal{D}(\alpha_1, \alpha_2; V) := \mathcal{R}(\alpha_2 - \alpha_1; V). \quad (2.7)$$

*Remark 2.2.* The dissipation potential  $\mathcal{R}$  that we choose here slightly differs from the one proposed in the model of [5]. In that paper, the dissipation potential features an additional term depending on the gradient of the damage variable. More precisely, using the notation of our paper, a choice more coherent with [5] would be  $\mathcal{R}(\dot{\alpha}, \nabla \alpha, \nabla \dot{\alpha}; V) = \int_{\Omega} f(V)(-\dot{\alpha} + \nabla \alpha \cdot \nabla \dot{\alpha}) \, dx$ . We explain here two reasons that lead us to the decision of not including the term  $\nabla \alpha \cdot \nabla \dot{\alpha}$  in the dissipation potential.

The first reason is a mathematical one. Note that a generic evolution  $\alpha(t)$  may not satisfy the inequality  $-\dot{\alpha} + \nabla \alpha \cdot \nabla \dot{\alpha} \geq 0$ ; the validity of this condition is however crucial for a physically consistent notion of dissipation potential. Our approach to the problem does not guarantee the *a priori* fulfilment of this condition.

The second reason is a modelling one. The model proposed in [5] is an approach to fatigue fracture via a phase-field model. In a classical phase-field model (without fatigue), the energy dissipated by a fracture is approximated by an energy of the form  $\int_{\Omega} (1 - \alpha) + \frac{1}{2} |\nabla \alpha|^2 \, dx$ , and in that case the term  $\int_{\Omega} \frac{1}{2} |\nabla \alpha|^2 \, dx$  should be interpreted as part of the dissipation. This explains why in [5] the rate of  $\frac{1}{2} |\nabla \alpha|^2$  appears in the definition of  $\mathcal{R}(\dot{\alpha}, \nabla \alpha, \nabla \dot{\alpha}; V)$  and the fatigue weight  $f(V)$  also affects this term. In contrast, our aim is to study damage models, whereas the approximation of fracture via damage is not in the scope of this paper. For this reason (as already done in other papers about damage models [28, 7, 41, 40, 24, 25, 26]) we interpret  $\int_{\Omega} \frac{1}{2} |\nabla \alpha|^2 \, dx$  as part of the internal energy of the system. In particular, the rate of  $\frac{1}{2} |\nabla \alpha|^2$  does not appear in the definition of the dissipation potential.

**Boundary conditions and initial data.** For every  $\bar{\alpha} \in H^1(\Omega)$  with  $0 \leq \bar{\alpha} \leq 1$  a.e. in  $\Omega$  and for every  $w \in H^1(\Omega)$ , the set of admissible pairs  $(\alpha, u)$  with respect to the damage variable  $\bar{\alpha}$  and the boundary datum  $w$  is defined by:

$$\mathcal{A}(\bar{\alpha}, w) := \{(\alpha, u) \in H^1(\Omega) \times H^1(\Omega) : 0 \leq \alpha \leq \bar{\alpha} \text{ a.e. in } \Omega, u = w \text{ on } \partial_D \Omega\}.$$

The quasistatic evolution will be driven by a boundary datum satisfying

$$w \in H^1(0, T; W^{1, \tilde{p}}(\Omega)), \quad (2.8)$$

where  $\tilde{p} > 2$  is a suitable exponent that is chosen according to Lemma 3.3. The  $\tilde{p}$  integrability of  $\nabla w$  is needed to control the increments of the displacement with those of the damage variable (cf. Lemma 3.3). The  $H^1$  regularity in time of the boundary datum is needed for the proof of the *a priori* bounds in Proposition 3.5 (see (3.26)).

We prescribe initial conditions  $\alpha_0 \in H^1(\Omega)$  and  $u_0 \in W^{1, \tilde{p}}(\Omega)$  at time  $t = 0$ . We assume, consistently with (2.5), that the initial cumulation  $V_0 = 0$  for notation simplicity. Taking a generic initial cumulation  $V_0 \in L^2(\Omega)$  with  $V_0 \geq 0$  a.e. in  $\Omega$  does not entail any mathematical difficulty. (Note that, in that case, definition (2.5) should be modified accordingly by adding the initial cumulation  $V_0$ .)

We require

$$\partial_\alpha \mathcal{E}(\alpha_0, u_0) \in L^2(\Omega). \quad (2.9)$$

Notice that one could also assume that  $\alpha_0$  and  $u_0$  are stable with  $V_{-1} = 0$ , so that the Euler conditions in Lemma 3.4 hold for  $i = 0$  too. The assumption (2.9) is slightly more general, since, for instance, the initial condition  $\alpha_0 = 0, u_0 = 0$  is always admissible, no matter whether it is stable or not.

### 3. INCREMENTAL MINIMUM PROBLEMS

**Construction of discrete-time evolutions.** We fix a sequence of subdivisions  $(t_k^i)_{i=0}^k$  of the interval  $[0, T]$ , where  $t_k^i := \frac{i}{k}T$  are equispaced nodes. We denote the step of the time discretisation by  $\tau_k = \frac{1}{k}$ . For notational simplicity, we omit the dependence of  $\tau_k$  on  $k$  and we use the symbol  $\tau$ . Moreover, we fix  $\varepsilon > 0$ .

We define the discretisation of the boundary datum  $w$  by  $w_k^i := w(t_k^i)$ ,  $i = 0, \dots, k$ .

Let  $\alpha_k^0 := \alpha_0$ ,  $u_k^0 := u_0$ ,  $\zeta_k^0 := g(\alpha_0)\nabla u_0$ , and  $V_k^0 := V_0 = 0$ . Assuming that we know  $\alpha_k^{i-1}$  and  $V_k^{i-1}$ , we define  $(\alpha_k^i, u_k^i)$  as a solution to the incremental minimisation problem (cf. (2.2), (2.6), (2.7) for the definition of  $\mathcal{E}$  and  $\mathcal{D}$ )

$$\min \{ \mathcal{E}(\alpha, u) + \mathcal{D}(\alpha, \alpha_k^{i-1}; V_k^{i-1}) + \frac{\varepsilon}{2\tau} \|\alpha - \alpha_k^{i-1}\|_{L^2}^2 : (\alpha, u) \in \mathcal{A}(\alpha_k^{i-1}, w_k^i) \} \quad (3.1)$$

and we set  $\zeta_k^i := g(\alpha_k^i)\nabla u_k^i$  and

$$V_k^i := V_k^{i-1} + |\zeta_k^i - \zeta_k^{i-1}| = \sum_{j=1}^i |\zeta_k^j - \zeta_k^{j-1}|.$$

The existence of a solution to (3.1) is obtained by employing the direct method of the Calculus of Variations.

*Remark 3.1.* It is immediate to see that  $\alpha_k^i$  is a solution to the problem

$$\min \{ \mathcal{E}(\alpha, u_k^i) + \mathcal{D}(\alpha, \alpha_k^{i-1}; V_k^{i-1}) + \frac{\varepsilon}{2\tau} \|\alpha - \alpha_k^{i-1}\|_{L^2}^2 : \alpha \in H^1(\Omega), 0 \leq \alpha \leq \alpha_k^{i-1} \leq 1 \}, \quad (3.2)$$

where  $u = u_k^i$  is fixed. Notice that  $\alpha_k^i$  is also a solution to the problem

$$\min \{ \mathcal{E}(\alpha, u_k^i) + \mathcal{D}(\alpha, \alpha_k^{i-1}; V_k^{i-1}) + \frac{\varepsilon}{2\tau} \|\alpha - \alpha_k^{i-1}\|_{L^2}^2 : \alpha \in H^1(\Omega), \alpha \leq \alpha_k^{i-1} \}, \quad (3.3)$$



where also competitors  $\alpha$  with negative values are taken into account. Indeed, let us fix a competitor for the problem (3.3), namely  $\alpha \in H^1(\Omega; \mathbb{R})$  with  $\alpha \leq \alpha_k^{i-1}$  and let us set  $\alpha^+ := \max\{\alpha, 0\}$ . We employ the fact that  $\alpha^+$  is a competitor for (3.2), the assumption (2.1), and the fact that  $\alpha_k^{i-1} \geq 0$  to obtain

$$\begin{aligned} \mathcal{E}(\alpha_k^i, u_k^i) + \mathcal{D}(\alpha_k^i, \alpha_k^{i-1}; V_k^{i-1}) + \frac{\varepsilon}{2\tau} \|\alpha_k^i - \alpha_k^{i-1}\|_{L^2}^2 \\ \leq \mathcal{E}(\alpha^+, u_k^i) + \mathcal{D}(\alpha^+, \alpha_k^{i-1}; V_k^{i-1}) + \frac{\varepsilon}{2\tau} \|\alpha^+ - \alpha_k^{i-1}\|_{L^2}^2 \\ \leq \mathcal{E}(\alpha, u_k^i) + \mathcal{D}(\alpha, \alpha_k^{i-1}; V_k^{i-1}) + \frac{\varepsilon}{2\tau} \|\alpha - \alpha_k^{i-1}\|_{L^2}^2. \end{aligned}$$

This proves the equivalence between (3.2) and (3.3).

We define the upper and lower piecewise constant interpolations by

$$\begin{aligned} \bar{t}_k(t) &:= t_k^i, & \bar{\alpha}_k(t) &:= \alpha_k^i, & \bar{u}_k(t) &:= u_k^i, & \bar{\zeta}_k(t) &:= \zeta_k^i, & \bar{w}_k(t) &:= w_k^i, \\ \underline{t}_k(t) &:= t_k^{i-1}, & \underline{\alpha}_k(t) &:= \alpha_k^{i-1}, & \underline{u}_k(t) &:= u_k^{i-1}, & \underline{\zeta}_k(t) &:= \zeta_k^{i-1}, & \underline{w}_k(t) &:= w_k^{i-1}, \end{aligned}$$

and

$$\underline{V}_k(t) := V_k^{i-1} \quad \text{for } t \in (t_k^{i-1}, t_k^i],$$

for  $i = 1, \dots, k$  and  $\bar{\alpha}_k(0) := \alpha_0$ ,  $\bar{u}_k(0) := u_0$ ,  $\underline{V}_k(0) := V_0 = 0$ , while  $\underline{t}_k(T) := T$ ,  $\underline{\alpha}_k(T) := \alpha_k^k$ ,  $\underline{u}_k(T) := u_k^k$ . Moreover, we consider the piecewise affine interpolations defined by

$$\begin{aligned} \alpha_k(t) &:= \alpha_k^{i-1} + (t - t_k^{i-1}) \dot{\alpha}_k^i, \\ u_k(t) &:= u_k^{i-1} + (t - t_k^{i-1}) \dot{u}_k^i, \\ \zeta_k(t) &:= \zeta_k^{i-1} + (t - t_k^{i-1}) \dot{\zeta}_k^i, \quad \text{for } t \in [t_k^{i-1}, t_k^i], \end{aligned}$$

for  $i = 1, \dots, k$ , where

$$\dot{\alpha}_k^i := \frac{\alpha_k^i - \alpha_k^{i-1}}{\tau}, \quad \dot{u}_k^i := \frac{u_k^i - u_k^{i-1}}{\tau}, \quad \dot{\zeta}_k^i := \frac{\zeta_k^i - \zeta_k^{i-1}}{\tau},$$

and define  $w_k$  as the affine interpolation in time of  $w$ . We set also

$$V_k(t) := \underline{V}_k(t) + \frac{t - \underline{t}_k(t)}{\tau} |\dot{\zeta}_k(t) - \zeta_k(\underline{t}_k(t))|. \quad (3.4)$$

It is not difficult to verify that Proposition A.4 yields

$$V_k(t) = \int_0^t |\dot{\zeta}_k(s)| \, ds \quad (3.5)$$

in the sense of Bochner integral in  $L^2(\Omega)$ .

Note that in the above definitions we dropped the dependence on  $\varepsilon$  for notation simplicity.

**A priori bounds on discrete-time evolutions.** We start the analysis of the discrete evolutions by deducing higher integrability properties of the strain. Following the idea of previous papers (see, e.g., [24]), we apply a result proved in [22, Theorem 1] (see also [23, Theorem 1.1] for an extension to the case of elliptic systems with the symmetric gradient in place of  $\nabla u$ ) regarding the integrability of solutions to elliptic systems with measurable coefficients and with mixed boundary conditions.

*Remark 3.2.* By [22, Theorem 1], there exist a constant  $C > 0$  and  $\tilde{p} > 2$  depending on  $\|\mu\|_{L^\infty}$  such that the following property is satisfied: for every  $\alpha \in H^1(\Omega)$ , for every  $p \in [2, \tilde{p}]$ , and for every  $\ell \in W_{\partial_D \Omega}^{-1,p}(\Omega)$ , the weak solution  $v \in W^{1,p}(\Omega)$  to the problem

$$\begin{cases} \operatorname{div}(\mu(\alpha)\nabla v) = \ell & \text{in } \Omega, \\ v = 0 & \text{on } \partial_D \Omega \end{cases}$$

satisfies

$$\|v\|_{W^{1,p}} \leq C\|\ell\|_{W_{\partial_D \Omega}^{-1,p}}.$$

In the following lemma we apply the regularity given by Remark 3.2 to deduce higher integrability of  $\nabla u_k(t)$  and to control the increments of the displacement  $u$  with the increments of the damage variable  $\alpha$ .

**Lemma 3.3** (Higher integrability of the strain). *There exist  $\tilde{p} > 2$  (depending only on  $\|\mu\|_{L^\infty}$ ) and a constant  $C > 0$  (depending only on  $\|\mu\|_{L^\infty}$ ,  $\|\mu'\|_{L^\infty}$ , and  $\|w\|_{L^\infty(0,T;W^{1,\tilde{p}}(\Omega))}$ ) such that*

$$\|\bar{u}_k(t)\|_{W^{1,\tilde{p}}} + \|\underline{u}_k(t)\|_{W^{1,\tilde{p}}} + \|u_k(t)\|_{W^{1,\tilde{p}}} \leq C, \quad \text{for } t \in [0, T] \quad (3.6a)$$

$$\|\dot{u}_k(t)\|_{W^{1,p}} \leq C \left[ \|\dot{\alpha}_k(t)\|_{L^q} + \|\dot{w}_k(t)\|_{W^{1,\tilde{p}}} \right], \quad \text{for } t \in (0, T) \setminus \{t_k^1, \dots, t_k^{k-1}\}, \quad (3.6b)$$

for every  $p \in [2, \tilde{p})$ , where  $q = \frac{p\tilde{p}}{\tilde{p}-p}$ .

*Proof.* Let  $\tilde{p} > 2$  be the exponent given in Remark 3.2. To prove (3.6a), let us fix  $t \in (t_k^{i-1}, t_k^i]$  for  $i \in \{1, \dots, k\}$  (notice that the inequality is trivial for  $t = 0$ ). By (3.1), the function  $u_k^i$  minimises  $\mathcal{E}(\alpha_k^i, u)$  among all  $u \in H^1(\Omega)$  with  $u = w_k^i$  on  $\partial_D \Omega$ . Therefore  $u_k^i$  is a weak solution to the problem

$$\begin{cases} \operatorname{div}(\mu(\alpha_k^i)\nabla u_k^i) = 0 & \text{in } \Omega, \\ u_k^i = w_k^i & \text{on } \partial_D \Omega. \end{cases} \quad (3.7)$$

By Remark 3.2, we have that

$$\|u_k^i - w_k^i\|_{W^{1,\tilde{p}}} \leq C \|\operatorname{div}(\mu(\alpha_k^i)\nabla w_k^i)\|_{W_{\partial_D \Omega}^{-1,\tilde{p}}} \leq C \|\mu\|_{L^\infty} \|w_k^i\|_{W^{1,\tilde{p}}},$$

which implies (3.6a) (recall the definition of  $\bar{u}_k$ ,  $\underline{u}_k$ ,  $u_k$  in terms of the family of  $u_k^i$ ).

To prove (3.6b), let us fix  $p \in [2, \tilde{p})$  and  $t \in (t_k^{i-1}, t_k^i)$  for  $t \in \{1, \dots, k\}$ . By (3.7) for  $i$  and  $i-1$  we get that the function  $v := u_k^i - u_k^{i-1} - w_k^i + w_k^{i-1}$  is a weak solution to the problem

$$\begin{cases} \operatorname{div}(\mu(\alpha_k^{i-1})\nabla v) = \ell & \text{in } \Omega, \\ v = 0 & \text{on } \partial_D \Omega, \end{cases} \quad (3.8)$$

where  $\ell := \operatorname{div}((\mu(\alpha_k^{i-1}) - \mu(\alpha_k^i))\nabla u_k^i) - \operatorname{div}(\mu(\alpha_k^{i-1})(\nabla w_k^i - \nabla w_k^{i-1}))$ . Notice that  $\ell \in W_{\partial_D \Omega}^{-1,p}(\Omega)$  by (3.6a). By Remark 3.2 and by Hölder's inequality we deduce that

$$\begin{aligned} \|v\|_{W^{1,p}} &\leq C\|\ell\|_{W_{\partial_D \Omega}^{-1,p}} \leq C \left[ \|(\mu(\alpha_k^{i-1}) - \mu(\alpha_k^i))\nabla u_k^i\|_{L^p} + \|\mu(\alpha_k^{i-1})(\nabla w_k^i - \nabla w_k^{i-1})\|_{L^p} \right] \\ &\leq C \left[ \|\mu'\|_{L^\infty} \|\alpha_k^i - \alpha_k^{i-1}\|_{L^q} \|\nabla u_k^i\|_{L^{\tilde{p}}} + \|\mu\|_{L^\infty} \|w_k^i - w_k^{i-1}\|_{W^{1,\tilde{p}}} \right], \end{aligned}$$

since  $\frac{1}{q} = \frac{1}{p} - \frac{1}{\tilde{p}}$ . By (3.6a) and dividing by  $\tau$  we conclude that

$$\|\dot{u}_k^i\|_{W^{1,p}} \leq C \left[ \|\dot{\alpha}_k^i\|_{L^q} + \|\dot{w}_k^i\|_{W^{1,\tilde{p}}} \right],$$

hence the thesis.  $\square$

We are now in a position to derive the Euler conditions satisfied by the damage variable in the discrete evolutions. These conditions are also called Discrete Karush-Kuhn-Tucker conditions, since we have a constraint of unidirectionality on the damage variable. They are a fundamental ingredient to deduce the *a priori* bounds in Proposition 3.5.

**Lemma 3.4** (Euler conditions). *For every  $t \in (0, T) \setminus \{t_k^1, \dots, t_k^{k-1}\}$  we have*

$$\langle \partial_\alpha \mathcal{E}(\bar{\alpha}_k(t), \bar{u}_k(t)), \beta \rangle + \mathcal{R}(\beta; \underline{V}_k(t)) + \varepsilon \langle \dot{\alpha}_k(t), \beta \rangle_{L^2} \geq 0 \quad (3.9)$$

for every  $\beta \in H^1(\Omega)$  such that  $\beta \leq 0$  a.e. in  $\Omega$ . Moreover

$$\langle \partial_\alpha \mathcal{E}(\bar{\alpha}_k(t), \bar{u}_k(t)), \dot{\alpha}_k(t) \rangle + \mathcal{R}(\dot{\alpha}_k(t); \underline{V}_k(t)) + \varepsilon \|\dot{\alpha}_k(t)\|_{L^2}^2 = 0. \quad (3.10)$$

*Proof.* Let us fix  $t \in (t_k^{i-1}, t_k^i)$  for some  $i \in \{1, \dots, k\}$ . Let  $\beta \in H^1(\Omega)$  with  $\beta \leq 0$  a.e. in  $\Omega$  and let  $\delta > 0$ . Since  $\alpha_k^i$  solves (3.3) and  $\alpha_k^i + \delta\beta \leq \alpha_k^{i-1}$ , we get

$$\begin{aligned} 0 &\leq \mathcal{E}(\alpha_k^i + \delta\beta, u_k^i) + \mathcal{D}(\alpha_k^i + \delta\beta, \alpha_k^{i-1}; V_k^{i-1}) + \frac{\varepsilon}{2\tau} \|\alpha_k^i + \delta\beta - \alpha_k^{i-1}\|_{L^2}^2 + \\ &\quad - \mathcal{E}(\alpha_k^i, u_k^i) - \mathcal{D}(\alpha_k^i, \alpha_k^{i-1}; V_k^{i-1}) - \frac{\varepsilon}{2\tau} \|\alpha_k^i - \alpha_k^{i-1}\|_{L^2}^2. \end{aligned}$$

Dividing by  $\delta$  and letting  $\delta \rightarrow 0^+$ , by (2.3) we get

$$\frac{1}{2} \int_{\Omega} \mu'(\alpha_k^i) |\nabla u_k^i|^2 \beta \, dx + \int_{\Omega} \nabla \alpha_k^i \cdot \nabla \beta \, dx - \int_{\Omega} f(V_k^{i-1}) \beta \, dx + \varepsilon \int_{\Omega} \dot{\alpha}_k^i \beta \, dx \geq 0.$$

This concludes the proof of (3.9).

To prove (3.10), notice that  $\alpha_k^i - \delta\dot{\alpha}_k^i \leq \alpha_k^{i-1}$  for  $0 < \delta < \tau$ . Since  $\alpha_k^i$  solves (3.3) we get that

$$\begin{aligned} 0 &\leq \mathcal{E}(\alpha_k^i - \delta\dot{\alpha}_k^i, u_k^i) + \mathcal{D}(\alpha_k^i - \delta\dot{\alpha}_k^i, \alpha_k^{i-1}; V_k^{i-1}) + \frac{\varepsilon}{2\tau} \|\alpha_k^i - \delta\dot{\alpha}_k^i - \alpha_k^{i-1}\|_{L^2}^2 + \\ &\quad - \mathcal{E}(\alpha_k^i, u_k^i) - \mathcal{D}(\alpha_k^i, \alpha_k^{i-1}; V_k^{i-1}) - \frac{\varepsilon}{2\tau} \|\alpha_k^i - \alpha_k^{i-1}\|_{L^2}^2. \end{aligned}$$

Dividing by  $\delta$  and letting  $\delta \rightarrow 0^+$ , by (2.3) this implies (3.10).  $\square$

The following proposition ensures that the evolution of  $\alpha$  and  $u$  is  $H^1$  in time uniformly in  $k$  for fixed  $\varepsilon$ , and  $AC$  in time uniformly in  $k$  and  $\varepsilon$ , with values in the target spaces  $H^1(\Omega)$  and  $W^{1,p}(\Omega)$ .

**Proposition 3.5** (*A priori bounds*). *Let  $\tilde{p}$  be as in Lemma 3.3. There exists a positive constant  $C$  independent of  $\varepsilon$ ,  $k$ , and  $t$  such that for every  $\varepsilon > 0$ ,  $k \in \mathbb{N}$ ,  $t \in (0, T) \setminus \{t_k^1, \dots, t_k^{k-1}\}$ ,  $p < \tilde{p}$ , it holds that*

$$\varepsilon \|\dot{\alpha}_k(t)\|_{L^2} \leq C \exp\left(C \frac{\bar{\tau}_k(t)}{\varepsilon}\right), \quad (3.11)$$

$$\varepsilon \int_0^{\bar{\tau}_k(t)} \|\dot{\alpha}_k(s)\|_{H^1}^2 \, ds + \varepsilon \int_0^{\bar{\tau}_k(t)} \|\dot{u}_k(s)\|_{W^{1,p}}^2 \, ds \leq C \exp\left(C \frac{\bar{\tau}_k(t)}{\varepsilon}\right), \quad (3.12)$$

$$\int_0^T \|\dot{\alpha}_k(s)\|_{H^1} \, ds + \int_0^T \|\dot{u}_k(s)\|_{W^{1,p}} \, ds \leq C. \quad (3.13)$$

*Proof.* We only need to show the estimates on  $\alpha_k(t)$ , since the estimates on  $u_k(t)$  simply follow from (3.6b).

We start with computations which are common in the proofs of all the three inequalities in the statement. The starting point is to obtain an estimate on the time increments of  $\dot{\alpha}_k(t)$  by testing the Euler equations at two subsequent times of the time discretisation. To do so, we fix  $i \in \{2, \dots, k\}$ . The case  $i = 1$  requires slightly different arguments. By (3.10) evaluated at a time  $t \in (t_k^{i-1}, t_k^i)$  we get that

$$\langle \partial_\alpha \mathcal{E}(\alpha_k^i, u_k^i), \dot{\alpha}_k^i \rangle + \mathcal{R}(\dot{\alpha}_k^i; V_k^{i-1}) + \varepsilon \|\dot{\alpha}_k^i\|_{L^2}^2 = 0.$$

On the other hand, by testing (3.9) with  $\beta = \dot{\alpha}_k^i$  at a time  $t \in (t_k^{i-2}, t_k^{i-1})$ , we get

$$\langle \partial_\alpha \mathcal{E}(\alpha_k^{i-1}, u_k^{i-1}), \dot{\alpha}_k^i \rangle + \mathcal{R}(\dot{\alpha}_k^i; V_k^{i-2}) + \varepsilon \langle \dot{\alpha}_k^{i-1}, \dot{\alpha}_k^i \rangle_{L^2} \geq 0.$$

Subtracting the second inequality from the first one, we infer that

$$\langle \partial_\alpha \mathcal{E}(\dot{\alpha}_k^i, u_k^i) - \partial_\alpha \mathcal{E}(\alpha_k^{i-1}, u_k^{i-1}), \dot{\alpha}_k^i \rangle + \mathcal{R}(\dot{\alpha}_k^i; V_k^{i-1}) - \mathcal{R}(\dot{\alpha}_k^i; V_k^{i-2}) + \varepsilon \langle \dot{\alpha}_k^i - \dot{\alpha}_k^{i-1}, \dot{\alpha}_k^i \rangle_{L^2} \leq 0,$$

namely,

$$\begin{aligned} & \varepsilon \langle \dot{\alpha}_k^i - \dot{\alpha}_k^{i-1}, \dot{\alpha}_k^i \rangle_{L^2} + \int_{\Omega} (\nabla \alpha_k^i - \nabla \alpha_k^{i-1}) \cdot \nabla \dot{\alpha}_k^i \, dx \\ & \leq \frac{1}{2} \int_{\Omega} [\mu'(\alpha_k^{i-1}) |\nabla u_k^{i-1}|^2 - \mu'(\alpha_k^i) |\nabla u_k^i|^2] \dot{\alpha}_k^i \, dx + \int_{\Omega} [f(V_k^{i-1}) - f(V_k^{i-2})] \dot{\alpha}_k^i \, dx \\ & \leq \frac{1}{2} \int_{\Omega} \mu'(\alpha_k^{i-1}) [|\nabla u_k^{i-1}|^2 - |\nabla u_k^i|^2] \dot{\alpha}_k^i \, dx + \frac{1}{2} \int_{\Omega} [\mu'(\alpha_k^{i-1}) - \mu'(\alpha_k^i)] |\nabla u_k^i|^2 \dot{\alpha}_k^i \, dx \\ & \quad + \|f'\|_{L^\infty} \int_{\Omega} |V_k^{i-1} - V_k^{i-2}| |\dot{\alpha}_k^i| \, dx \\ & \leq \frac{1}{2} \|\mu'\|_{L^\infty} \int_{\Omega} |\nabla u_k^i + \nabla u_k^i| |\nabla u_k^i - \nabla u_k^{i-1}| |\dot{\alpha}_k^i| \, dx + \frac{1}{2} \|\mu''\|_{L^\infty} \int_{\Omega} |\alpha_k^i - \alpha_k^{i-1}| |\nabla u_k^i|^2 |\dot{\alpha}_k^i| \, dx \\ & \quad + \|f'\|_{L^\infty} \int_{\Omega} |\zeta_k^{i-1} - \zeta_k^{i-2}| |\dot{\alpha}_k^i| \, dx \\ & \leq C \tau \left[ \|\nabla u_k^i + \nabla u_k^{i-1}\|_{L^2} \|\nabla \dot{u}_k^i\|_{L^p} \|\dot{\alpha}_k^i\|_{L^{q_1}} + \|\nabla u_k^i\|_{L^p}^2 \|\dot{\alpha}_k^i\|_{L^{q_1}}^2 \right. \\ & \quad \left. + \|\nabla u_k^{i-1}\|_{L^p} \|\dot{\alpha}_k^{i-1}\|_{L^{q_1}} \|\dot{\alpha}_k^i\|_{L^{q_1}} + \|\nabla \dot{u}_k^{i-1}\|_{L^p} \|\dot{\alpha}_k^i\|_{L^{q_1}} \right]. \end{aligned} \tag{3.14}$$

In the last inequality we have chosen  $p \in (2, \tilde{p})$  and  $q_1 \in (2, \infty)$  such that  $\frac{1}{p} + \frac{1}{q_1} = \frac{1}{2}$ , and we have employed the identity

$$\zeta_k^{i-1} - \zeta_k^{i-2} = [g(\alpha_k^{i-1}) - g(\alpha_k^{i-2})] \nabla u_k^{i-1} + g(\alpha_k^{i-2}) [\nabla u_k^{i-1} - \nabla u_k^{i-2}]$$

that gives

$$\int_{\Omega} |\zeta_k^{i-1} - \zeta_k^{i-2}| |\dot{\alpha}_k^i| \, dx \leq \tau \left( \|g'\|_{L^\infty} \|\nabla u_k^{i-1}\|_{L^p} \|\dot{\alpha}_k^{i-1}\|_{L^{q_1}} \|\dot{\alpha}_k^i\|_{L^{q_1}} + \|g\|_{L^\infty} \|\nabla \dot{u}_k^{i-1}\|_{L^p} \|\dot{\alpha}_k^i\|_{L^{q_1}} \right). \tag{3.15}$$

We remark that, taking  $\zeta_k^i := g(\alpha_k^i) |\nabla u_k^i|^\theta$ , with  $\theta \in [1, p)$  we could also get the conclusion in (3.14) with  $q'_1 \geq q_1$  such that  $\frac{\theta}{p} + \frac{1}{q'_1} = \frac{1}{2}$ , in place of  $q_1$ . Indeed

$$\zeta_k^{i-1} - \zeta_k^{i-2} = [g(\alpha_k^{i-1}) - g(\alpha_k^{i-2})] |\nabla u_k^{i-1}|^\theta + g(\alpha_k^{i-2}) [|\nabla u_k^{i-1}|^\theta - |\nabla u_k^{i-2}|^\theta],$$

and since, by the Mean Value Theorem,

$$||\nabla u_k^{i-1}|^\theta - |\nabla u_k^{i-2}|^\theta| \leq \theta (|\nabla u_k^{i-1}| + |\nabla u_k^{i-2}|)^{\theta-1} (|\nabla u_k^{i-1}| - |\nabla u_k^{i-2}|),$$

we have that

$$\begin{aligned} \int_{\Omega} |\zeta_k^{i-1} - \zeta_k^{i-2}| |\dot{\alpha}_k^i| \, dx & \leq \tau \left( \|g'\|_{L^\infty} \|\nabla u_k^{i-1}\|_{L^p} \|\dot{\alpha}_k^{i-1}\|_{L^{q'_1}} \|\dot{\alpha}_k^i\|_{L^{q'_1}} \right. \\ & \quad \left. + \|g\|_{L^\infty} \theta (\|\nabla u_k^{i-1}\|_{L^p} + \|\nabla u_k^{i-2}\|_{L^p}) \|\nabla \dot{u}_k^{i-1}\|_{L^p} \|\dot{\alpha}_k^i\|_{L^{q'_1}} \right). \end{aligned} \tag{3.16}$$

Using the fact that  $\langle \dot{\alpha}_k^i - \dot{\alpha}_k^{i-1}, \dot{\alpha}_k^i \rangle_{L^2} \geq \|\dot{\alpha}_k^i\|_{L^2} \left( \|\dot{\alpha}_k^i\|_{L^2} - \|\dot{\alpha}_k^{i-1}\|_{L^2} \right)$  and by Lemma 3.3 we infer that

$$\begin{aligned} & \varepsilon \|\dot{\alpha}_k^i\|_{L^2} \left( \|\dot{\alpha}_k^i\|_{L^2} - \|\dot{\alpha}_k^{i-1}\|_{L^2} \right) + \tau \|\nabla \dot{\alpha}_k^i\|_{L^2}^2 \\ & \leq C\tau \left[ \|\dot{\alpha}_k^i\|_{L^{q_1}} \|\dot{\alpha}_k^i\|_{L^{q_2}} + \|\dot{\alpha}_k^i\|_{L^{q_1}}^2 + \|\dot{\alpha}_k^i\|_{L^{q_1}} \|\dot{\alpha}_k^{i-1}\|_{L^{q_2}} + \|\dot{\alpha}_k^i\|_{L^{q_1}} \left( \|\dot{w}_k^i\|_{W^{1,\bar{p}}} + \|\dot{w}_k^{i-1}\|_{W^{1,\bar{p}}} \right) \right] \\ & \leq c_1\tau \left[ \|\dot{\alpha}_k^i\|_{L^r}^2 + \|\dot{\alpha}_k^i\|_{L^r} \|\dot{\alpha}_k^{i-1}\|_{L^r} + \|\dot{w}_k^i\|_{W^{1,\bar{p}}}^2 + \|\dot{w}_k^{i-1}\|_{W^{1,\bar{p}}}^2 \right], \end{aligned} \quad (3.17)$$

where  $q_2 := \frac{p\bar{p}}{p-p} \in (2, \infty)$  and  $r = \max\{q_1', q_2\} \in (2, \infty)$ . We labelled the constant in the last inequality with  $c_1$  in order to keep track of it in the sequel. By the compact embedding  $H^1(\Omega) \Subset L^r(\Omega)$  (notice that  $\Omega \subset \mathbb{R}^2$ ), we have that for every  $\delta > 0$  there exists a constant  $C_\delta > 0$  such that for every  $\beta \in H^1(\Omega)$

$$\|\beta\|_{L^r}^2 \leq \delta \|\nabla \beta\|_{L^2}^2 + C_\delta \|\beta\|_{L^1}^2 \leq \delta \|\nabla \beta\|_{L^2}^2 + C_\delta \|\beta\|_{L^1} \|\beta\|_{L^2}. \quad (3.18)$$

Adding  $c_1\tau \|\dot{\alpha}_k^i\|_{L^1}^2 - c_1\tau \|\dot{\alpha}_k^i\|_{L^r} \|\dot{\alpha}_k^{i-1}\|_{L^r} + \frac{\tau}{2} \|\dot{\alpha}_k^i\|_{L^2}^2$  to both sides of (3.17), choosing  $\delta$  suitably small in the previous inequality applied to  $\beta = \dot{\alpha}_k^i$ , and multiplying by  $\frac{2\tau}{\varepsilon}$  we have that

$$\begin{aligned} & 2\|\dot{\alpha}_k^i\|_{L^2} \left( \|\dot{\alpha}_k^i\|_{L^2} - \|\dot{\alpha}_k^{i-1}\|_{L^2} \right) + 2c_1\frac{\tau}{\varepsilon} \|\dot{\alpha}_k^i\|_{L^r} \left( \|\dot{\alpha}_k^i\|_{L^r} - \|\dot{\alpha}_k^{i-1}\|_{L^r} \right) + \frac{\tau}{\varepsilon} \|\dot{\alpha}_k^i\|_{H^1}^2 \\ & \leq c_2\frac{\tau}{\varepsilon} \left( \|\dot{w}_k^i\|_{W^{1,\bar{p}}}^2 + \|\dot{w}_k^{i-1}\|_{W^{1,\bar{p}}}^2 \right) + 2c_2\frac{\tau}{\varepsilon} \|\dot{\alpha}_k^i\|_{L^1} \|\dot{\alpha}_k^i\|_{L^2}. \end{aligned} \quad (3.19)$$

Let us set

$$\begin{aligned} A_i &:= \left[ \|\dot{\alpha}_k^i\|_{L^2}^2 + c_1\frac{\tau}{\varepsilon} \|\dot{\alpha}_k^i\|_{L^r}^2 \right]^{\frac{1}{2}}, & B_i &:= \sqrt{\frac{\tau}{2\varepsilon}} \|\dot{\alpha}_k^i\|_{H^1}, \\ C_i &:= \sqrt{c_2\frac{\tau}{\varepsilon}} \left[ \|\dot{w}_k^i\|_{W^{1,\bar{p}}}^2 + \|\dot{w}_k^{i-1}\|_{W^{1,\bar{p}}}^2 \right]^{\frac{1}{2}}, & D_i &:= c_2\frac{\tau}{\varepsilon} \|\dot{\alpha}_k^i\|_{L^1}. \end{aligned}$$

The quantities above are actually defined for every  $i = 1, \dots, k$ . When  $i = 1$ , we define  $C_1 := \sqrt{c_2\frac{\tau}{\varepsilon}} \|\dot{w}_k^1\|_{W^{1,\bar{p}}}$ .

Denoting by  $a_i := \left( \|\dot{\alpha}_k^i\|_{L^2}, \sqrt{c_1\frac{\tau}{\varepsilon}} \|\dot{\alpha}_k^i\|_{L^r} \right)$ , we get that

$$\begin{aligned} & \|\dot{\alpha}_k^i\|_{L^2} \left( \|\dot{\alpha}_k^i\|_{L^2} - \|\dot{\alpha}_k^{i-1}\|_{L^2} \right) + c_1\frac{\tau}{\varepsilon} \|\dot{\alpha}_k^i\|_{L^r} \left( \|\dot{\alpha}_k^i\|_{L^r} - \|\dot{\alpha}_k^{i-1}\|_{L^r} \right) = a_i \cdot (a_i - a_{i-1}) \\ & \geq |a_i| (|a_i| - |a_{i-1}|) = A_i (A_i - A_{i-1}). \end{aligned} \quad (3.20)$$

Since  $\tau \leq \varepsilon$  we have that

$$\begin{aligned} \frac{\tau}{2\varepsilon} \|\dot{\alpha}_k^i\|_{H^1}^2 &= \frac{\tau}{4\varepsilon} \|\dot{\alpha}_k^i\|_{H^1}^2 + \frac{\tau}{4\varepsilon} \|\dot{\alpha}_k^i\|_{H^1}^2 \geq \frac{\tau}{4\varepsilon} \|\dot{\alpha}_k^i\|_{L^2}^2 + \frac{C\tau}{4\varepsilon} \|\dot{\alpha}_k^i\|_{L^r}^2 \geq \frac{\tau}{4\varepsilon} \|\dot{\alpha}_k^i\|_{L^2}^2 + \frac{C\tau^2}{4\varepsilon^2} \|\dot{\alpha}_k^i\|_{L^r}^2 \\ &\geq \frac{C\tau}{\varepsilon} \left[ \|\dot{\alpha}_k^i\|_{L^2}^2 + c_1\frac{\tau}{\varepsilon} \|\dot{\alpha}_k^i\|_{L^r}^2 \right] = 2c_3\frac{\tau}{\varepsilon} A_i^2. \end{aligned} \quad (3.21)$$

Collecting (3.19)–(3.21) and setting  $\gamma := c_3\frac{\tau}{\varepsilon}$ , we obtain that

$$2A_i(A_i - A_{i-1}) + 2\gamma A_i^2 + B_i^2 \leq C_i^2 + 2A_i D_i, \quad (3.22)$$

for every  $i = 2, \dots, k$ .

Proof of estimate (3.11). Here we prove a slightly stronger inequality with an additional term on the left-hand side. Specifically, we show that

$$\varepsilon \left[ \|\dot{\alpha}_k(t)\|_{L^2}^2 + c_1\frac{\tau}{\varepsilon} \|\dot{\alpha}_k(t)\|_{L^r} \right] \leq C \exp \left( \frac{C}{\varepsilon} \bar{\tau}_k(t) \right). \quad (3.23)$$

By the inequalities  $2A_i(A_i - A_{i-1}) \geq A_i^2 - A_{i-1}^2$  and  $D_i \leq C\frac{\tau}{\varepsilon} A_i$ , from (3.22) we get in particular that

$$A_i^2 - A_{i-1}^2 + B_i^2 \leq C_i^2 + C\frac{\tau}{\varepsilon} A_i^2,$$

for  $i = 2, \dots, k$ . We fix  $h \in \{2, \dots, k\}$  and we sum the inequality above for  $i = 2, \dots, h$ , deducing that

$$\varepsilon A_h^2 - \varepsilon A_1^2 + \sum_{i=2}^h B_i^2 \leq \varepsilon \sum_{i=2}^h C_i^2 + C \sum_{i=2}^h \tau A_i^2. \quad (3.24)$$

We claim that

$$\varepsilon A_1^2 \leq C \left[ \varepsilon C_1^2 + \tau A_1^2 + \frac{1}{\varepsilon} \right]. \quad (3.25)$$

Once (3.25) is proven, summing (3.24) and (3.25), by the initial assumption on  $w$  (2.8) we conclude that

$$\varepsilon A_h^2 \leq C \left[ \frac{1}{\varepsilon} + \sum_{i=1}^h \tau (\|\dot{w}_k^i\|_{W^{1,\bar{p}}}^2 + \|\dot{w}_k^{i-1}\|_{W^{1,\bar{p}}}^2) + \sum_{i=1}^h \tau A_i^2 \right] \leq C \left[ 1 + \frac{1}{\varepsilon} + \sum_{i=1}^h \tau A_i^2 \right] \quad (3.26)$$

for every  $h = 1, \dots, k$ . By a discrete Gronwall inequality on  $\varepsilon A_h^2$  we deduce that

$$\varepsilon A_h^2 \leq C \left( 1 + \frac{1}{\varepsilon} \right) \exp \left( C \frac{t_k^h}{\varepsilon} \right) \quad (3.27)$$

for every  $h = 1, \dots, k$ . Multiplying by  $\varepsilon$  and taking the square root, we get

$$\varepsilon A_h \leq C \exp \left( C \frac{t_k^h}{\varepsilon} \right) \quad (3.28)$$

and thus (3.23).

It remains to prove (3.25). Adding and subtracting  $\partial_\alpha \mathcal{E}(\alpha_0, u_0)$  to (3.10) evaluated at time  $t \in (0, t_k^1)$ , we deduce that

$$\langle \partial_\alpha \mathcal{E}(\alpha_k^1, u_k^1) - \partial_\alpha \mathcal{E}(\alpha_0, u_0), \dot{\alpha}_k^1 \rangle + \mathcal{R}(\dot{\alpha}_k^1, V_0) + \langle \partial_\alpha \mathcal{E}(\alpha_0, u_0), \dot{\alpha}_k^1 \rangle + \varepsilon \|\dot{\alpha}_k^1\|_{L^2}^2 = 0.$$

With computations similar to those previously done in (3.14)–(3.17) and using the assumption  $\partial_\alpha \mathcal{E}(\alpha_0, u_0) \in L^2(\Omega)$ , we infer that

$$\begin{aligned} \varepsilon \|\dot{\alpha}_k^1\|_{L^2}^2 + \tau \|\nabla \dot{\alpha}_k^1\|_{L^2}^2 &\leq C \tau \left[ \|\nabla u_k^1 + \nabla u_0\|_{L^2} \|\nabla \dot{u}_k^1\|_{L^p} \|\dot{\alpha}_k^1\|_{L^{q_1}} + \|\nabla u_k^1\|_{L^p}^2 \|\dot{\alpha}_k^1\|_{L^{q_1}}^2 \right] \\ &\quad + \|f\|_{L^\infty} \|\dot{\alpha}_k^1\|_{L^1} + \|\partial_\alpha \mathcal{E}(\alpha_0, u_0)\|_{L^2} \|\dot{\alpha}_k^1\|_{L^2} \\ &\leq C \tau \left[ \|\dot{w}_k^1\|_{W^{1,\bar{p}}}^2 + \|\dot{\alpha}_k^1\|_{L^r}^2 \right] + \frac{\varepsilon}{2} \|\dot{\alpha}_k^1\|_{L^2}^2 + \frac{C}{\varepsilon}. \end{aligned}$$

Using inequality (3.18) as above, it is not difficult to see that

$$\varepsilon \|\dot{\alpha}_k^1\|_{L^2}^2 + \tau \|\dot{\alpha}_k^1\|_{H^1}^2 \leq C \tau \left[ \|\dot{w}_k^1\|_{W^{1,\bar{p}}}^2 + \|\dot{\alpha}_k^1\|_{L^1}^2 + \frac{1}{\varepsilon} \right], \quad (3.29)$$

which in turn implies (3.25).

Proof of estimate (3.12). Inequalities (3.24) and (3.25) imply in particular that

$$\sum_{i=2}^h B_i^2 \leq \varepsilon \sum_{i=2}^h C_i^2 + C \sum_{i=2}^h \tau A_i^2 + C \left[ \varepsilon C_1^2 + \tau A_1^2 + \frac{1}{\varepsilon} \right]. \quad (3.30)$$

From (3.29) we deduce that

$$\varepsilon B_1^2 \leq C \left[ \varepsilon C_1^2 + \tau A_1^2 + \frac{1}{\varepsilon} \right] \quad (3.31)$$

Let us fix  $h \in \{1, \dots, k\}$ . Summing (3.30) and (3.31), by (2.8) we obtain that

$$\varepsilon \sum_{i=1}^h B_i^2 \leq C \left[ \frac{1}{\varepsilon} + \sum_{i=1}^h \tau (\|\dot{w}_k^i\|_{W^{1,\bar{p}}}^2 + \|\dot{w}_k^{i-1}\|_{W^{1,\bar{p}}}^2) + \sum_{i=1}^h \tau A_i^2 \right] \leq C \left[ 1 + \frac{1}{\varepsilon} + \sum_{i=1}^h \tau A_i^2 \right]$$

and thus, multiplying by  $\varepsilon$  and using (3.27),

$$\varepsilon \sum_{i=1}^h \tau \|\dot{\alpha}_k^i\|_{H^1}^2 \leq C \exp \left( C \frac{t_k^h}{\varepsilon} \right).$$

In the equality above we have integrated the exponential function in time and we have used the fact that  $\tau \ll \varepsilon$ .

This concludes the proof of (3.12).

Proof of estimate (3.13). By the discrete Gronwall estimate proved in [24, Lemma 4.1] we deduce that for every  $h = 2, \dots, k$

$$\begin{aligned} \left( \sum_{i=2}^h (1+\gamma)^{2(i-h)-1} B_i^2 \right)^{\frac{1}{2}} &\leq \left( (1+\gamma)^{-2h} A_1^2 + \sum_{i=2}^h (1+\gamma)^{2(i-h)-1} C_i^2 \right)^{\frac{1}{2}} + \sqrt{2} \sum_{i=2}^h (1+\gamma)^{i-k-1} D_i \\ &\leq \left[ 2(1+\gamma)^{-2h} A_1^2 + 2 \sum_{i=2}^h (1+\gamma)^{2(i-h)-1} C_i^2 + 4 \left( \sum_{i=2}^h (1+\gamma)^{i-k-1} D_i \right)^2 \right]^{\frac{1}{2}} \\ &\leq \sqrt{2} (1+\gamma)^{-h} A_1 + 2 \sum_{i=2}^h (1+\gamma)^{2(i-h)-1} C_i^2 + 1 + 2 \sum_{i=2}^h (1+\gamma)^{i-k-1} D_i \end{aligned} \quad (3.32)$$

Using the estimate

$$\gamma \sum_{i=2}^h (1+\gamma)^{2(i-h)-1} = \gamma (1+\gamma)^{-2h-1} \frac{(1+\gamma)^4 - (1+\gamma)^{2h+2}}{1 - (1+\gamma)^2} = \frac{1+\gamma}{2+\gamma} [1 - (1+\gamma)^{-2h+2}] \leq 1,$$

by the Cauchy-Schwarz inequality we estimate the left-hand side of (3.32) by

$$\gamma \sum_{i=2}^h (1+\gamma)^{2(i-h)-1} \|\dot{\alpha}_k^i\|_{H^1} = \sum_{i=2}^h (\gamma(1+\gamma)^{2(i-h)-1})^{\frac{1}{2}} (\gamma(1+\gamma)^{2(i-h)-1})^{\frac{1}{2}} \|\dot{\alpha}_k^i\|_{H^1} \leq C \left( \sum_{i=2}^h (1+\gamma)^{2(i-h)-1} B_i^2 \right)^{\frac{1}{2}},$$

for  $h = 2, \dots, k$ . Hence (3.32) reads

$$\begin{aligned} &\gamma \sum_{i=2}^h (1+\gamma)^{2(i-h)-1} \|\dot{\alpha}_k^i\|_{H^1} \\ &\leq C \left[ 1 + (1+\gamma)^{-h} A_1 + \gamma \sum_{i=2}^h (1+\gamma)^{2(i-h)-1} (\|\dot{w}_k^i\|_{W^{1,\bar{p}}}^2 + \|\dot{w}_k^{i-1}\|_{W^{1,\bar{p}}}^2) + \gamma \sum_{i=2}^h (1+\gamma)^{i-k-1} \|\dot{\alpha}_k^i\|_{L^1} \right], \end{aligned} \quad (3.33)$$

for  $h = 2, \dots, k$ . We multiply both sides of (3.33) by  $\tau$  and we sum over  $h = 2, \dots, k$ . Using the expression of the partial sums of the geometric series, it is possible to show that

$$\sum_{i=2}^k \tau \|\dot{\alpha}_k^i\|_{H^1} \leq C \left[ 1 + \varepsilon A_1 + \sum_{i=2}^k \tau (\|\dot{w}_k^i\|_{W^{1,\bar{p}}}^2 + \|\dot{w}_k^{i-1}\|_{W^{1,\bar{p}}}^2) + \sum_{i=2}^k \tau \|\dot{\alpha}_k^i\|_{L^1} \right]. \quad (3.34)$$

We refer to [24, Proposition 4.3] or [9, Proposition 3.8] for more details about the computations mentioned above.

Multiplying (3.29) by  $\tau$ , taking the square root and using the fact that  $\tau \ll \varepsilon$ , we infer that

$$\tau \|\dot{\alpha}_k^1\|_{H^1} \leq C \tau \left[ \|\dot{w}_k^1\|_{W^{1,\bar{p}}} + \|\dot{\alpha}_k^1\|_{L^1} + 1 \right].$$

Adding this last inequality to (3.34) we obtain that

$$\sum_{i=1}^k \tau \|\dot{\alpha}_k^i\|_{H^1} \leq C \left[ 1 + \varepsilon A_1 + \sum_{i=1}^k \tau (\|\dot{w}_k^i\|_{W^{1,\bar{p}}}^2 + \|\dot{w}_k^{i-1}\|_{W^{1,\bar{p}}}^2) + \sum_{i=1}^k \tau \|\dot{\alpha}_k^i\|_{L^1} \right].$$

To conclude the proof of (3.13), we observe that:  $\varepsilon A_1 \leq C$  by (3.28) evaluated for  $h = 1$ ; the second sum is bounded by a constant by the initial assumption on  $w$  (2.8); the third sum is actually a telescopic sum, namely

$$\sum_{i=1}^k \tau \|\dot{\alpha}_k^i\|_{L^1} = \int_{\Omega} (\alpha_0 - \alpha_k) dx \leq |\Omega|.$$

□

In order to obtain the energy dissipation balance for the evolution  $(\alpha_k, u_k)$ , in Proposition 3.7, we integrate in time the energy evaluated on these affine interpolations. We are allowed to do so because they are absolutely continuous (actually  $H^1$ ) in time. Since we also employ the Euler equation (3.10) of Lemma 3.4, that contains also the piecewise constant interpolations, we have to estimate the difference of the piecewise affine and constant interpolations. This is done in the following remark.

*Remark 3.6.* For every  $t \in [0, T]$

$$\|\alpha_k(t) - \bar{\alpha}_k(t)\|_{H^1} = \left\| \int_t^{\bar{t}_k(t)} \dot{\alpha}_k(s) \, ds \right\|_{H^1} \leq \int_t^{\bar{t}_k(t)} \|\dot{\alpha}_k(s)\|_{H^1} \, ds \leq \tau^{\frac{1}{2}} \|\alpha_k\|_{H^1(0, T; H^1(\Omega))}$$

and therefore, by (3.12),

$$\|\alpha_k - \bar{\alpha}_k\|_{L^\infty(0, T; H^1(\Omega))} \leq C_\varepsilon \tau^{\frac{1}{2}}. \quad (3.35a)$$

Similarly, we have

$$\|\alpha_k - \underline{\alpha}_k\|_{L^\infty(0, T; H^1(\Omega))} \leq C_\varepsilon \tau^{\frac{1}{2}}, \quad (3.35b)$$

$$\|u_k - \bar{u}_k\|_{L^\infty(0, T; W^{1, p}(\Omega))} \leq C_\varepsilon \tau^{\frac{1}{2}}, \quad \text{for } p \in [2, \tilde{p}), \quad (3.35c)$$

$$\|u_k - \underline{u}_k\|_{L^\infty(0, T; W^{1, p}(\Omega))} \leq C_\varepsilon \tau^{\frac{1}{2}}, \quad \text{for } p \in [2, \tilde{p}). \quad (3.35d)$$

**Discrete energy-dissipation balance.** Here we obtain the energy-dissipation balance, by employing the Euler condition (3.10), correcting with the piecewise affine interpolations in place of the piecewise constant ones.

**Proposition 3.7** (Discrete energy-dissipation balance).

$$\begin{aligned} \mathcal{E}(\alpha_k(T), u_k(T)) + \int_0^T \mathcal{R}(\dot{\alpha}_k(t); \underline{V}_k(t)) \, dt + \varepsilon \int_0^T \|\dot{\alpha}_k(t)\|_{L^2}^2 \, dt \\ = \mathcal{E}(\alpha_0, u_0) + \int_0^T \langle \mu(\bar{\alpha}_k(t)) \nabla \bar{u}_k(t), \nabla \dot{w}_k(t) \rangle_{L^2} \, dt + R_k, \end{aligned} \quad (3.36)$$

where  $R_k \rightarrow 0$  as  $k \rightarrow +\infty$ .

*Proof.* By (3.12)–(3.13), the piecewise affine interpolations  $\alpha_k(t)$  and  $u_k(t)$  are absolutely continuous in  $t$ . As a consequence,  $t \mapsto \mathcal{E}(\alpha_k(t), u_k(t))$  is absolutely continuous and

$$\begin{aligned} \frac{d}{dt} \left[ \mathcal{E}(\alpha_k(t), u_k(t)) \right] &= \langle \partial_\alpha \mathcal{E}(\alpha_k(t), u_k(t)), \dot{\alpha}_k(t) \rangle + \langle \partial_u \mathcal{E}(\alpha_k(t), u_k(t)), \dot{u}_k(t) \rangle \\ &= \langle \partial_\alpha \mathcal{E}(\bar{\alpha}_k(t), \bar{u}_k(t)), \dot{\alpha}_k(t) \rangle + \langle \partial_u \mathcal{E}(\bar{\alpha}_k(t), \bar{u}_k(t)), \dot{u}_k(t) \rangle + \eta_k(t) \end{aligned} \quad (3.37)$$

for a.e.  $t$ , where

$$\eta_k(t) := \langle \partial_\alpha \mathcal{E}(\alpha_k(t), u_k(t)) - \partial_\alpha \mathcal{E}(\bar{\alpha}_k(t), \bar{u}_k(t)), \dot{\alpha}_k(t) \rangle + \langle \partial_u \mathcal{E}(\alpha_k(t), u_k(t)) - \partial_u \mathcal{E}(\bar{\alpha}_k(t), \bar{u}_k(t)), \dot{u}_k(t) \rangle. \quad (3.38)$$

Using  $\dot{u}_k(t) - \dot{w}_k(t)$  as test function in (3.7), we deduce that

$$\langle \partial_u \mathcal{E}(\bar{\alpha}_k(t), \bar{u}_k(t)), \dot{u}_k(t) \rangle = \langle \partial_u \mathcal{E}(\bar{\alpha}_k(t), \bar{u}_k(t)), \dot{w}_k(t) \rangle = \langle \mu(\bar{\alpha}_k(t)) \nabla \bar{u}_k(t), \nabla \dot{w}_k(t) \rangle_{L^2}$$

Together with the Euler equation for  $\bar{\alpha}_k(t)$  (3.10) and (3.37), this gives

$$\frac{d}{dt} \left[ \mathcal{E}(\alpha_k(t), u_k(t)) \right] = -\varepsilon \|\dot{\alpha}_k(t)\|_{L^2}^2 - \mathcal{R}(\dot{\alpha}_k(t); \underline{V}_k(t)) + \langle \mu(\bar{\alpha}_k(t)) \nabla \bar{u}_k(t), \nabla \dot{w}_k(t) \rangle_{L^2} + \eta_k(t) \quad (3.39)$$

Integrating in time the previous equality, we obtain (3.36) with  $R_k := \int_0^T \eta_k(t) \, dt$ .



Let us show that  $R_k \rightarrow 0$ . By Hölder's Inequality, by (3.35a), by (3.6a), and by (3.13) we deduce that

$$\begin{aligned} \left| \int_0^T \int_{\Omega} (\mu'(\alpha_k(t)) - \mu'(\bar{\alpha}_k(t))) |\nabla u_k(t)|^2 \dot{\alpha}_k(t) \, dx \, dt \right| &\leq C \int_0^T \int_{\Omega} |\alpha_k(t) - \bar{\alpha}_k(t)| |\nabla u_k(t)|^2 |\dot{\alpha}_k(t)| \, dx \, dt \\ &\leq C \int_0^T \|\alpha_k(t) - \bar{\alpha}_k(t)\|_{H^1} \|u_k(t)\|_{W^{1,p}}^2 \|\dot{\alpha}_k(t)\|_{H^1} \, dt \leq C_\varepsilon \tau^{\frac{1}{2}} \end{aligned}$$

Furthermore by Hölder's Inequality, by (3.6a), by (3.35c), and by (3.13) we infer that

$$\begin{aligned} \left| \int_0^T \int_{\Omega} \mu'(\bar{\alpha}_k(t)) (|\nabla u_k(t)|^2 - |\nabla \bar{u}_k(t)|^2) \dot{\alpha}_k(t) \, dx \, dt \right| &\leq C \int_0^T \int_{\Omega} |\nabla u_k(t) + \nabla \bar{u}_k(t)| |\nabla u_k(t) - \nabla \bar{u}_k(t)| |\dot{\alpha}_k(t)| \, dx \, dt \\ &\leq C \int_0^T \|u_k(t) + \bar{u}_k(t)\|_{W^{1,p}} \|u_k(t) - \bar{u}_k(t)\|_{W^{1,p}} \|\dot{\alpha}_k(t)\|_{H^1} \, dt \leq C_\varepsilon \tau^{\frac{1}{2}}. \end{aligned}$$

Finally, by (3.35a) and (3.13) we get that

$$\left| \int_0^T \int_{\Omega} (\nabla \alpha_k(t) - \nabla \bar{\alpha}_k(t)) \cdot \nabla \dot{\alpha}_k(t) \, dx \, dt \right| \leq \int_0^T \|\alpha_k(t) - \bar{\alpha}_k(t)\|_{H^1} \|\dot{\alpha}_k(t)\|_{H^1} \, dt \leq C_\varepsilon \tau^{\frac{1}{2}}.$$

This shows that

$$\lim_{k \rightarrow +\infty} \int_0^T \langle \partial_\alpha \mathcal{E}(\alpha_k(t), u_k(t)) - \partial_\alpha \mathcal{E}(\bar{\alpha}_k(t), \bar{u}_k(t)), \dot{\alpha}_k(t) \rangle \, dt = 0.$$

With completely analogous computations it is not difficult to show that

$$\lim_{k \rightarrow +\infty} \int_0^T \langle \partial_u \mathcal{E}(\alpha_k(t), u_k(t)) - \partial_u \mathcal{E}(\bar{\alpha}_k(t), \bar{u}_k(t)), \dot{u}_k(t) \rangle \, dt = 0.$$

This concludes the proof.  $\square$

We observe that the energy balance (3.36) holds for any couple of times  $t_1 < t_2 \in [0, T]$ , as one can see arguing as in Proposition 3.7 and integrating (3.39) in the time interval  $(t_1, t_2)$ .

#### 4. EXISTENCE OF VISCOUS EVOLUTIONS

In this section we pass to the limit as  $k \rightarrow +\infty$  (i.e., as the time-step goes to zero). Notice that  $\varepsilon > 0$  is fixed in this section. The main result is the existence of *viscous evolutions*, defined as follows. Given  $\alpha_\varepsilon \in AC([0, T]; H^1(\Omega))$ ,  $u_\varepsilon \in AC([0, T]; H^1(\Omega))$  we define, as in (2.4),  $\zeta_\varepsilon := g(\alpha_\varepsilon) \nabla u_\varepsilon$  and, as in (2.5),

$$V_\varepsilon(t) := \int_0^t |\dot{\zeta}_\varepsilon(s)| \, ds, \quad (4.1)$$

as a Bochner integral in  $L^2(\Omega)$ . During the section we are in the constitutive assumptions of Section 2.

**Definition 4.1.** We say that a function  $(\alpha_\varepsilon, u_\varepsilon): [0, T] \rightarrow H^1(\Omega) \times W^{1,p}(\Omega)$  is an  $\varepsilon$ -*approximate viscous evolution* if  $\alpha_\varepsilon \in H^1(0, T; H^1(\Omega))$ ,  $u_\varepsilon \in H^1(0, T; W^{1,p}(\Omega))$  and the following conditions are satisfied:

(ev0) $_\varepsilon$  *irreversibility*:

$$[0, T] \ni t \mapsto \alpha_\varepsilon(t) \quad \text{is nonincreasing as a family of measurable functions on } \Omega,$$

that is  $\alpha_\varepsilon(t) \leq \alpha_\varepsilon(s)$  a.e. in  $\Omega$  for all  $s \leq t$ ;

(ev1) $_\varepsilon$  *equilibrium*: for every  $t \in [0, T]$ ,  $u_\varepsilon(t) \in H^1(\Omega)$  is a weak solution to the problem

$$\begin{cases} \operatorname{div}(\mu(\alpha_\varepsilon(t)) \nabla u_\varepsilon(t)) = 0 & \text{in } \Omega, \\ u_\varepsilon(t) = w(t) & \text{on } \partial_D \Omega. \end{cases} \quad (4.2)$$

(ev2) $_{\varepsilon}$  *Karush-Kuhn-Tucker inequality*: for a.e.  $t \in (0, T)$  and for every  $\beta \in H^1(\Omega)$  with  $\beta \leq 0$  a.e. in  $\Omega$  we have

$$\langle \partial_{\alpha} \mathcal{E}(\alpha_{\varepsilon}(t), u_{\varepsilon}(t)), \beta \rangle + \mathcal{R}(\beta; V_{\varepsilon}(t)) + \varepsilon \langle \dot{\alpha}_{\varepsilon}(t), \beta \rangle_{L^2} \geq 0. \quad (4.3)$$

(ev3) $_{\varepsilon}$  *energy balance*:

$$\mathcal{E}(\alpha_{\varepsilon}(T), u_{\varepsilon}(T)) + \int_0^T \mathcal{R}(\dot{\alpha}_{\varepsilon}(t); V_{\varepsilon}(t)) dt + \varepsilon \int_0^T \|\dot{\alpha}_{\varepsilon}(t)\|_{L^2}^2 dt = \mathcal{E}(\alpha_0, u_0) + \int_0^T \langle \mu(\alpha_{\varepsilon}(t)) \nabla u_{\varepsilon}(t), \nabla \dot{w}(t) \rangle_{L^2} dt.$$

All the section is devoted to the proof of the result below.

**Theorem 4.2.** *Let  $\tilde{p} > 2$  be given by Lemma 3.3. For every  $\varepsilon > 0$  and  $p < \tilde{p}$  there exists an  $\varepsilon$ -approximate viscous evolution  $(\alpha_{\varepsilon}, u_{\varepsilon})$  with  $(\alpha_{\varepsilon}(0), u_{\varepsilon}(0)) = (\alpha_0, u_0)$  and there is a constant  $C > 0$ , independent of  $\varepsilon$ , such that*

$$\int_0^T \|\dot{\alpha}_{\varepsilon}(s)\|_{H^1} ds + \int_0^T \|\dot{u}_{\varepsilon}(s)\|_{W^{1,p}} ds \leq C. \quad (4.4)$$

The strategy of the proof consists in showing first the existence of a weak form of  $\varepsilon$ -approximate viscous evolution. This satisfies the conditions (ev0) $_{\varepsilon}$ , (ev1) $_{\varepsilon}$ , and the (ev2) $_{\varepsilon}$ , (ev3) $_{\varepsilon}$  with a different expression of dissipation (Propositions 4.7, 4.8, and 4.10). Such a weak existence result allows us to improve, for fixed  $\varepsilon$ , the *a priori* convergences of the discrete-time evolutions (Proposition 4.11) and to express the dissipation in terms of  $V_{\varepsilon}(t)$ , the cumulation of  $\zeta_{\varepsilon}$  (cf. (4.1)), so recovering its form in Definition 4.1, by Lemma 4.4.

**Compactness.** We start by exploiting the *a priori* bounds found in Proposition 3.5 to deduce compactness of the discrete-time evolutions. By (3.12) we find a subsequence (which we do not relabel) such that

$$\alpha_k \rightharpoonup \alpha_{\varepsilon} \quad \text{weakly in } H^1(0, T; H^1(\Omega)), \quad (4.5)$$

$$u_k \rightharpoonup u_{\varepsilon} \quad \text{weakly in } H^1(0, T; W^{1,p}(\Omega)), \quad \text{for } p \in [2, \tilde{p}), \quad (4.6)$$

as  $k \rightarrow +\infty$ . (Actually, we also extract a subsequence independent of  $t$  such that the convergence in (4.20) below holds. We do not state this here for the sake of clarity in the presentation.) By the compact embeddings  $H^1(\Omega) \Subset L^q(\Omega)$  and  $W^{1,p}(\Omega) \Subset L^p(\Omega)$ , by the Aubin-Lions lemma [6], and by (3.35) we deduce that

$$\|\alpha_k - \alpha_{\varepsilon}\|_{C([0,T];L^q(\Omega))}, \|\bar{\alpha}_k - \alpha_{\varepsilon}\|_{L^{\infty}(0,T;L^q(\Omega))}, \|\underline{\alpha}_k - \alpha_{\varepsilon}\|_{L^{\infty}(0,T;L^q(\Omega))} \rightarrow 0, \quad \text{for } q \in [1, \infty), \quad (4.7a)$$

$$\|u_k - u_{\varepsilon}\|_{C([0,T];L^p(\Omega))}, \|\bar{u}_k - u_{\varepsilon}\|_{L^{\infty}(0,T;L^p(\Omega))}, \|u_k - u_{\varepsilon}\|_{L^{\infty}(0,T;L^p(\Omega))} \rightarrow 0, \quad \text{for } p \in [2, \tilde{p}). \quad (4.7b)$$

Moreover, from the inequality  $\|\alpha_k\|_{L^{\infty}(0,T;H^1(\Omega))} \leq C(1 + \|\alpha_k\|_{W^{1,1}(0,T;H^1(\Omega))})$  and by (3.6a) and (3.35a)–(3.35d) we deduce that for every  $t \in [0, T]$  we also have

$$\alpha_k(t), \bar{\alpha}_k(t), \underline{\alpha}_k(t) \rightharpoonup \alpha_{\varepsilon}(t) \quad \text{weakly in } H^1(\Omega), \quad (4.8)$$

$$u_k(t), \bar{u}_k(t), \underline{u}_k(t) \rightharpoonup u_{\varepsilon}(t) \quad \text{weakly in } W^{1,\tilde{p}}(\Omega). \quad (4.9)$$

In particular, for every  $s \leq t$  we have  $\alpha_{\varepsilon}(t) \leq \alpha_{\varepsilon}(s)$  a.e. in  $\Omega$ . Moreover, for every  $t \in [0, T]$  we have

$$\|u_{\varepsilon}(t)\|_{W^{1,\tilde{p}}} \leq \liminf_{k \rightarrow +\infty} \|u_k(t)\|_{W^{1,\tilde{p}}} \leq C. \quad (4.10)$$

In view of the convergences (4.5), (4.6), by (3.13) we get

$$\int_0^T \|\dot{\alpha}_{\varepsilon}(s)\|_{H^1} ds + \int_0^T \|\dot{u}_{\varepsilon}(s)\|_{W^{1,p}} ds \leq C, \quad (4.11)$$

where  $C$  is independent of  $\varepsilon$ , and then  $V_{\varepsilon}$  is well defined as in (4.1).

**Energy-dissipation balance and stability.** In this subsection we pass to the limit as  $k \rightarrow +\infty$  in the discrete energy-dissipation balance (3.36). We start by discussing the easiest terms in the energy-dissipation balance, namely the terms involving the energy, the viscous dissipation, and the work done by the boundary forces. The dissipation involving the fatigue term requires finer techniques and will be discussed below.

From the pointwise convergences (4.8)–(4.9) and the lower semicontinuity of the energy  $\mathcal{E}$  with respect to the weak convergence of  $\alpha$  in  $H^1(\Omega)$  and the weak convergence of  $u$  in  $W^{1,p}(\Omega)$  we deduce that

$$\mathcal{E}(\alpha_\varepsilon(T), u_\varepsilon(T)) \leq \liminf_{k \rightarrow +\infty} \mathcal{E}(\alpha_k(T), u_k(T)). \quad (4.12)$$

Moreover, since  $\dot{\alpha}_k \rightharpoonup \dot{\alpha}_\varepsilon$  weakly in  $L^2(0, T; L^2(\Omega))$ , we have that

$$\varepsilon \int_0^T \|\dot{\alpha}_\varepsilon(t)\|_{L^2}^2 dt \leq \liminf_{k \rightarrow +\infty} \left( \varepsilon \int_0^T \|\dot{\alpha}_k(t)\|_{L^2}^2 dt \right). \quad (4.13)$$

We claim that

$$\lim_{k \rightarrow +\infty} \int_0^T \langle \mu(\bar{\alpha}_k(t)) \nabla \bar{u}_k(t), \nabla \dot{w}_k(t) \rangle_{L^2} dt = \int_0^T \langle \mu(\alpha_\varepsilon(t)) \nabla u_\varepsilon(t), \nabla \dot{w}(t) \rangle_{L^2} dt. \quad (4.14)$$

To show the convergence above, first of all we notice that  $\mu(\bar{\alpha}_k(t)) \nabla \bar{u}_k(t) \rightharpoonup \mu(\alpha_\varepsilon(t)) \nabla u_\varepsilon(t)$  weakly in  $L^2(\Omega; \mathbb{R}^2)$  for every  $t \in [0, T]$  thanks to (4.8)–(4.9). In addition, (3.6a) and assumption (2.8) imply

$$|\langle \mu(\bar{\alpha}_k(t)) \nabla \bar{u}_k(t), \nabla \dot{w}_k(t) \rangle_{L^2}| \leq C \int_0^T \|\bar{u}_k(t)\|_{H^1} \|\dot{w}_k(t)\|_{H^1} dt \leq C.$$

Since  $\nabla \dot{w}_k(t) \rightarrow \nabla \dot{w}(t)$  strongly in  $L^2(\Omega; \mathbb{R}^2)$  for a.e.  $t \in (0, T)$ , by the Dominated Convergence Theorem the convergence in (4.14) holds true.

We consider now the limit of the dissipation involving the fatigue term. We start with the following lemma, which shows that the affine interpolation of the cumulation is close to the piecewise constant interpolation.

**Lemma 4.3.** *For every  $k \in \mathbb{N}$ ,  $\varepsilon > 0$  we have that*

$$\|f(V_k) - f(\underline{V}_k)\|_{L^2(0, T; L^2(\Omega))} \leq C\tau \left( \|\alpha_k\|_{H^1(0, T; L^2(\Omega))} + \|u_k\|_{H^1(0, T; H^1(\Omega))} \right) \leq C_\varepsilon \tau. \quad (4.15)$$

*Proof.* By (3.4) and (3.6a) we have

$$V_k(t) - \underline{V}_k(t) = \frac{t - \underline{t}_k(t)}{\tau} \left( [g(\alpha_k(t)) - g(\underline{\alpha}_k(t))] \nabla u_k(t) + g(\underline{\alpha}_k(t)) [\nabla u_k(t) - \nabla \underline{u}_k(t)] \right),$$

so that

$$\begin{aligned} |V_k(t) - \underline{V}_k(t)| &\leq \tau \left( \|g'\|_{L^\infty} |\dot{\alpha}_k(t)| |\nabla u_k(t)| + |g(\underline{\alpha}_k(t))| |\nabla \dot{u}_k(t)| \right) \\ &\leq C\tau (|\dot{\alpha}_k(t)| + |\nabla \dot{u}_k(t)|). \end{aligned}$$

Thus for any  $\beta \in L^2(0, T; L^2(\Omega))$  (recall that  $f$  is Lipschitz)

$$\begin{aligned} \int_0^T \int_\Omega |f(V_k(t)) - f(\underline{V}_k(t))| |\beta(t)| dx dt &\leq C\tau \int_0^T \int_\Omega (|\dot{\alpha}_k(t)| + |\nabla \dot{u}_k(t)|) |\beta(t)| dx dt \\ &\leq C\tau \int_0^T \left( \|\dot{\alpha}_k(t)\|_{L^2} + \|\dot{u}_k(t)\|_{H^1} \right) \|\beta(t)\|_{L^2} dt \\ &\leq C\tau \left( \|\alpha_k\|_{H^1(0, T; L^2(\Omega))} + \|u_k\|_{H^1(0, T; H^1(\Omega))} \right) \|\beta\|_{L^2(0, T; L^2(\Omega))}. \end{aligned}$$

Recalling (3.12), the estimate above gives (4.15). We notice that we arrive at the same conclusion also with  $\zeta$  defined by  $g(\alpha) |\nabla u|^\theta$ , for  $\theta \in [1, \tilde{p})$ , arguing similarly to what done to pass from (3.15) to (3.16) in Proposition 3.5.  $\square$

In the following lemma we show that a strong convergence of the discrete-time evolutions would guarantee the convergence of the dissipation term. We stress that the a priori bounds on  $u_k(t)$  found in Proposition 3.5 only guarantee the weak convergence (4.6). Therefore we are not allowed to apply Lemma 4.4 at the moment.

**Lemma 4.4.** *Assume that the following convergences for  $\alpha_k$  and  $u_k$  hold true:*

$$\alpha_k \rightarrow \alpha_\varepsilon \quad \text{strongly in } W^{1,1}(0, T; L^2(\Omega)), \quad (4.16a)$$

$$u_k \rightarrow u_\varepsilon \quad \text{strongly in } W^{1,1}(0, T; W^{1,p}(\Omega)), \quad \text{for } p \in [2, \tilde{p}). \quad (4.16b)$$

Then

$$f(V_k) \rightarrow f(V_\varepsilon) \quad \text{strongly in } L^2(0, T; L^2(\Omega)), \quad (4.17)$$

and

$$\lim_{k \rightarrow +\infty} \int_0^T \mathcal{R}(\dot{\alpha}_k(t); \underline{V}_k(t)) \, dt = \int_0^T \mathcal{R}(\dot{\alpha}_\varepsilon(t); V_\varepsilon(t)) \, dt, \quad (4.18)$$

where the cumulations  $V_k$  and  $V_\varepsilon$  are defined in (3.4) and (4.1), respectively.

*Proof.* For the proof it is convenient to introduce the function

$$dg(\beta, h) := \begin{cases} \frac{g(\beta + h) - g(\beta)}{h}, & \text{if } h \neq 0, \\ g'(\beta), & \text{if } h = 0, \end{cases}$$

for every  $\beta, h \in \mathbb{R}$ . Observe that  $g(\beta + h) = g(\beta) + h dg(\beta, h)$  and since  $g \in C^{1,1}(\mathbb{R})$

$$|dg(\beta, h) - g'(\beta)| \leq \|g''\|_{L^\infty} |h|.$$

Using the function  $dg$ , we can write for every  $s \in [0, T]$

$$\dot{\zeta}_k(s) = dg(\underline{\alpha}_k(s), \tau \dot{\alpha}_k(s)) \dot{\alpha}_k(s) \nabla \bar{u}_k(s) + g(\underline{\alpha}_k(s)) \nabla \dot{u}_k(s).$$

We now estimate  $V_k - V_\varepsilon$  by employing (3.5) and (4.1). For every  $t \in [0, T]$  we have

$$\begin{aligned} & \int_\Omega |V_k(t; x) - V_\varepsilon(t; x)| \, dx \leq \int_0^t \int_\Omega |\dot{\zeta}_k(s; x) - \dot{\zeta}_\varepsilon(s; x)| \, dx \, ds \\ & \leq \int_0^t \int_\Omega \left| dg(\underline{\alpha}_k(s), \tau \dot{\alpha}_k(s)) \dot{\alpha}_k(s) \nabla \bar{u}_k(s) + g(\underline{\alpha}_k(s)) \nabla \dot{u}_k(s) - g(\alpha_\varepsilon(s)) \dot{\alpha}_\varepsilon(s) \nabla u_\varepsilon(s) - g(\alpha_\varepsilon(s)) \nabla \dot{u}_\varepsilon(s) \right| \, dx \, ds \\ & \leq \int_0^t \int_\Omega \left[ \tau \|g''\|_{L^\infty} |\dot{\alpha}_k(s)|^2 |\nabla \bar{u}_k(s)| + \|g''\|_{L^\infty} |\underline{\alpha}_k(s) - \alpha_\varepsilon(s)| |\dot{\alpha}_k(s)| |\nabla \bar{u}_k(s)| \right. \\ & \quad + \|g'\|_{L^\infty} |\dot{\alpha}_k(s) - \dot{\alpha}_\varepsilon(s)| |\nabla \bar{u}_k(s)| + \|g'\|_{L^\infty} |\alpha_\varepsilon(s)| |\nabla \bar{u}_k(s) - \nabla u_\varepsilon(s)| \\ & \quad \left. + \|g'\|_{L^\infty} |\underline{\alpha}_k(s) - \alpha_\varepsilon(s)| |\nabla \dot{u}_k(s)| + \|g\|_{L^\infty} |\nabla \dot{u}_k(s) - \nabla \dot{u}_\varepsilon(s)| \right] \, dx \, ds \\ & \leq C (\|\underline{\alpha}_k - \alpha_\varepsilon\|_{L^\infty(0, T; L^q(\Omega))} \|\alpha_k\|_{W^{1,1}(0, T; L^q(\Omega))} + \|\alpha_k - \alpha_\varepsilon\|_{W^{1,1}(0, T; L^2(\Omega))}) \|\bar{u}_k\|_{L^\infty(0, T; W^{1,p}(\Omega))} \\ & \quad + C (\tau + \|u_k - u_\varepsilon\|_{L^\infty(0, T; W^{1,p}(\Omega))}) \|\alpha_\varepsilon\|_{W^{1,1}(0, T; L^q(\Omega))} \\ & \quad + C \|\underline{\alpha}_k - \alpha_\varepsilon\|_{L^\infty(0, T; L^q(\Omega))} \|u_k\|_{W^{1,1}(0, T; W^{1,p}(\Omega))} + C \|u_k - u_\varepsilon\|_{W^{1,1}(0, T; W^{1,p}(\Omega))}, \end{aligned}$$

for  $q \in (2, \infty)$  such that  $\frac{1}{q} + \frac{1}{p} < \frac{1}{2}$ .

Notice that we obtain the above inequality also if  $\zeta_\varepsilon = g(\alpha_\varepsilon)|\nabla u_\varepsilon|^\theta$ , with  $\theta \in [1, \bar{p})$ , up to consider  $q' > q$  with  $\frac{1}{q'} + \frac{\theta}{p} < \frac{1}{2}$  in the estimates of  $\alpha$ , since

$$\frac{d}{dt}|\nabla u|^\theta = \theta|\nabla u|^{\theta-2}\nabla u \cdot \nabla \dot{u}.$$

Let us now integrate in time the inequality obtained above for  $V_k - V_\varepsilon$ : using (3.6), (3.13), (3.35), (4.7), and (4.16) we deduce that

$$\|V_k - V_\varepsilon\|_{L^1(0,T;L^1(\Omega))} \rightarrow 0,$$

and then we get (4.17), since  $f$  is bounded.

Moreover, by weak convergence  $\dot{\alpha}_k \rightharpoonup \dot{\alpha}_\varepsilon$  in  $L^2(0,T;L^2(\Omega))$

$$\int_0^T \mathcal{R}(\dot{\alpha}_k(t); V_k(t)) \, dt = - \int_0^T \int_\Omega f(V_k(t)) \dot{\alpha}_k(t) \, dx \, dt \rightarrow - \int_0^T \int_\Omega f(V_\varepsilon(t)) \dot{\alpha}_\varepsilon(t) \, dx \, dt = \int_0^T \mathcal{R}(\dot{\alpha}_\varepsilon(t); V_\varepsilon(t)) \, dt$$

and, by (4.15),

$$\left| \int_0^T (\mathcal{R}(\dot{\alpha}_k(t); V_k(t)) - \mathcal{R}(\dot{\alpha}_k(t); \underline{V}_k(t))) \, dt \right| \leq \|f(V_k) - f(\underline{V}_k)\|_{L^2(0,T;L^2(\Omega))} \|\alpha_k\|_{H^1(0,T;L^2(\Omega))} \leq C_\varepsilon \tau \rightarrow 0,$$

as  $k \rightarrow +\infty$ . This concludes the proof.  $\square$

*Remark 4.5.* Combining (4.15) and (4.17) we obtain that if (4.16b) holds, then

$$f(\underline{V}_k) \rightarrow f(V_\varepsilon) \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \quad (4.19)$$

At the moment we do not have convergence (4.16b) at our disposal, and we cannot deduce that the convergence of the functions  $f(\underline{V}_k(t))$  to  $f(V_\varepsilon(t))$ . For this reason, in the following lemma we consider an additional variable  $\tilde{f}_\varepsilon(t)$  in the limit evolution, which later in the proof will turn out to be  $f(V_\varepsilon(t))$ .

**Lemma 4.6** (Compactness for the cumulated variable). *For every  $\varepsilon > 0$  there exist a nonincreasing function  $t \mapsto \tilde{f}_\varepsilon(t) \in L^\infty(\Omega)$  and a subsequence independent of  $t$  (which we do not relabel) such that*

$$f(\underline{V}_k(t)) \overset{*}{\rightharpoonup} \tilde{f}_\varepsilon(t) \quad \text{weakly* in } L^\infty(\Omega), \quad (4.20)$$

for every  $t \in [0, T]$ .

*Proof.* To prove the lemma we apply the generalized version of the classical Helly Theorem given in [18, Helly Theorem] in the space  $\mathcal{M}_b(\Omega)$ . For every  $t \in [0, T]$ , the sequence  $(f(\underline{V}_k(t)))_k$  is equibounded in  $L^\infty(\Omega)$ , and thus is relatively compact in  $\mathcal{M}_b(\Omega)$  with respect to the weak\* convergence. Moreover, the functions  $f(\underline{V}_k)$  have uniformly bounded variation in  $\mathcal{M}_b(\Omega)$ . Indeed, for  $s \leq t$  we have  $f(\underline{V}_k(t)) \leq f(\underline{V}_k(s))$  and thus, given a partition  $0 = s_0 < \dots < s_m = T$ , we get

$$\sum_{j=1}^m \int_\Omega |f(\underline{V}_k(s_j)) - f(\underline{V}_k(s_{j-1}))| \, dx = \int_\Omega f(\underline{V}_k(0)) - f(\underline{V}_k(T)) \, dx \leq \|f\|_{L^\infty}.$$

On the one hand, by [18, Helly Theorem] we deduce that there exists a subsequence independent of  $t$  (which we do not relabel) and a function  $t \mapsto \lambda_t \in \mathcal{M}_b(\Omega)$  such that

$$f(\underline{V}_k(t)) \mathcal{L}^2 \llcorner \Omega \overset{*}{\rightharpoonup} \lambda_t \quad \text{weakly* in } \mathcal{M}_b(\Omega). \quad (4.21)$$

On the other hand, for every  $t \in [0, T]$  there exists a function  $\tilde{f}_\varepsilon(t) \in L^\infty(\Omega)$  and a subsequence  $k_j(t)$  depending on  $t$  such that

$$f(\underline{V}_{k_j(t)}(t)) \overset{*}{\rightharpoonup} \tilde{f}_\varepsilon(t) \quad \text{weakly* in } L^\infty(\Omega). \quad (4.22)$$

By (4.21) and (4.22) we conclude that  $\lambda_t = \tilde{f}_\varepsilon(t)\mathcal{L}^2 \llcorner \Omega$  and the convergence in (4.22) holds on the whole subsequence  $k$  where (4.21) is satisfied. Notice that  $\tilde{f}_\varepsilon(t)$  is nonincreasing in  $t$ .  $\square$

The first step is to deduce the existence of an evolution where the fatigue term  $f(V_\varepsilon(t))$  is in fact replaced by the term  $\tilde{f}_\varepsilon(t)$ . We first prove one inequality in the energy-dissipation balance for the continuous-time evolutions. The opposite inequality will follow automatically from the differential conditions satisfied by  $\alpha_\varepsilon$ , see Proposition 4.10 below.

**Proposition 4.7** (Energy-dissipation balance in weak form: first inequality). *For every  $\varepsilon > 0$  we have*

$$\mathcal{E}(\alpha_\varepsilon(T), u_\varepsilon(T)) - \int_0^T \int_\Omega \tilde{f}_\varepsilon(t) \dot{\alpha}_\varepsilon(t) \, dx \, dt + \varepsilon \int_0^T \|\dot{\alpha}_\varepsilon(t)\|_{L^2}^2 \, dt \leq \mathcal{E}(\alpha_0, u_0) + \int_0^T \langle \mu(\alpha_\varepsilon(t)) \nabla u_\varepsilon(t), \nabla \dot{w}(t) \rangle_{L^2} \, dt. \quad (4.23)$$

*Proof.* In order to prove (4.23), we write the dissipation with the fatigue term as a supremum of finite sums which are continuous with respect to the convergence (4.20). Specifically

$$\int_0^T \mathcal{R}(\dot{\alpha}_k(t); \underline{V}_k(t)) \, dt = \sup_{0=s_0 < \dots < s_m=T} \left\{ \sum_{j=1}^m \int_\Omega f(\underline{V}_k(s_j)) (\alpha_k(s_{j-1}) - \alpha_k(s_j)) \, dx \right\}, \quad (4.24)$$

where the supremum is taken among all possible partitions  $0 = s_0 < \dots < s_m = T$ ,  $m \in \mathbb{N}$ , of the interval  $[0, T]$ . The supremum is in fact attained on the partition  $0 = t_k^0 < \dots < t_k^k = T$ . To check this, let us fix a partition  $0 = s_0 < \dots < s_m = T$  and let us prove that

$$\begin{aligned} \sum_{j=1}^m \int_\Omega f(\underline{V}_k(s_j)) (\alpha_k(s_{j-1}) - \alpha_k(s_j)) \, dx &\leq \sum_{i=1}^k \int_\Omega f(\underline{V}_k(t_k^i)) (\alpha_k(t_k^{i-1}) - \alpha_k(t_k^i)) \, dx \\ &= - \int_0^T \int_\Omega f(\underline{V}_k(t)) \dot{\alpha}_k(t) \, dx \, dt. \end{aligned} \quad (4.25)$$

Note that if we refine the partition  $0 = s_0 < \dots < s_m = T$  by including the nodes  $t_k^0, \dots, t_k^k$ , the dissipation increases, since the monotonicity of  $f(\underline{V}_k)$  and of  $\alpha_k$  yields the following triangular inequality:

$$\int_\Omega f(\underline{V}_k(r_3)) (\alpha_k(r_1) - \alpha_k(r_3)) \, dx \leq \int_\Omega f(\underline{V}_k(r_2)) (\alpha_k(r_1) - \alpha_k(r_2)) \, dx + \int_\Omega f(\underline{V}_k(r_3)) (\alpha_k(r_2) - \alpha_k(r_3)) \, dx$$

for  $0 \leq r_1 \leq r_2 \leq r_3 \leq T$ . Therefore we can assume without loss of generality that  $\{t_k^0, \dots, t_k^k\} \subset \{s_0, \dots, s_m\}$ . Let us now fix  $i \in \{1, \dots, k\}$  and  $1 \leq h_i < \ell_i \leq m$  such that  $t_k^{i-1} = s_{h_i} < \dots < s_{\ell_i} = t_k^i$ . Then the sum in the left-hand side of (4.25) can be rearranged as

$$\begin{aligned} \sum_{i=1}^k \sum_{j=h_i}^{\ell_i} \int_\Omega f(\underline{V}_k(s_j)) (\alpha_k(s_{j-1}) - \alpha_k(s_j)) \, dx &= \sum_{i=1}^k \sum_{j=h_i}^{\ell_i} \int_\Omega f(V_k^{i-1}) \frac{s_j - s_{j-1}}{\tau} (\alpha_k^{i-1} - \alpha_k^i) \, dx \\ &= \sum_{i=1}^k \int_\Omega f(V_k^{i-1}) \frac{t_k^i - t_k^{i-1}}{\tau} (\alpha_k^{i-1} - \alpha_k^i) \, dx = \sum_{i=1}^k \int_\Omega f(\underline{V}_k(t_k^i)) (\alpha_k(t_k^{i-1}) - \alpha_k(t_k^i)) \, dx. \end{aligned}$$

Now we pass to the limit in (4.24) as  $k \rightarrow +\infty$ . Let us fix a partition  $0 = s_0 < \dots < s_m = T$  and let us fix  $j \in \{0, \dots, m\}$ . By (4.8) we have in particular that  $\alpha_k(s_j) \rightarrow \alpha_\varepsilon(s_j)$  and  $\alpha_k(s_{j-1}) \rightarrow \alpha_\varepsilon(s_{j-1})$  strongly in  $L^1(\Omega)$  and therefore, by (4.20), we obtain that

$$\int_\Omega f(\underline{V}_k(s_j)) (\alpha_k(s_{j-1}) - \alpha_k(s_j)) \, dx \rightarrow \int_\Omega \tilde{f}_\varepsilon(s_j) (\alpha_\varepsilon(s_{j-1}) - \alpha_\varepsilon(s_j)) \, dx \quad (4.26)$$

as  $k \rightarrow +\infty$ .

On the other hand we have that

$$\sup_{0=s_0 < \dots < s_m=T} \left\{ \sum_{j=1}^m \int_{\Omega} \tilde{f}_{\varepsilon}(s_j) (\alpha_{\varepsilon}(s_{j-1}) - \alpha_{\varepsilon}(s_j)) \, dx \right\} = - \int_0^T \int_{\Omega} \tilde{f}_{\varepsilon}(t) \dot{\alpha}_{\varepsilon}(t) \, dx \, dt. \quad (4.27)$$

The equality above follows from a general lemma proved in [9, Lemma A.1] regarding the integral representation of weighted variations. To check the fulfillment of the assumptions required by [9, Lemma A.1] we remark that:

- $\alpha_{\varepsilon} \in AC([0, T]; L^2(\Omega))$ ;
- $\dot{\alpha}_{\varepsilon} \leq 0$  a.e. in  $\Omega$ ;
- $\tilde{f}_{\varepsilon}(t) \leq \tilde{f}_{\varepsilon}(s)$  a.e. in  $\Omega$  for  $s \leq t$ ;
- there exists a countable set  $E \subset [0, T]$  such that  $t \mapsto \tilde{f}_{\varepsilon}(t)$  is continuous for every  $t \in [0, T] \setminus E$  with respect to strong  $L^2$  topology (this follows from the monotonicity by [9, Lemma A.2]).

Applying [9, Lemma A.1] with  $X := L^2(\Omega)$  and  $F = L^2(\Omega)$ , we get (4.27).

By (4.24)–(4.27) we conclude that

$$- \int_0^T \int_{\Omega} \tilde{f}_{\varepsilon}(t) \dot{\alpha}_{\varepsilon}(t) \, dx \, dt \leq \liminf_{k \rightarrow +\infty} \int_0^T \mathcal{R}(\dot{\alpha}_k(t); \underline{V}_k(t)) \, dt. \quad (4.28)$$

We conclude the proof using the inequality above together with (4.12)–(4.14) and (3.36).  $\square$

**Proposition 4.8** (Stability in weak form). *Let  $\varepsilon > 0$ . For every  $t \in [0, T]$ ,  $u_{\varepsilon}(t)$  is a weak solution to the problem*

$$\begin{cases} \operatorname{div}(\mu(\alpha_{\varepsilon}(t)) \nabla u_{\varepsilon}(t)) = 0 & \text{in } \Omega, \\ u_{\varepsilon}(t) = w(t) & \text{on } \partial_D \Omega. \end{cases} \quad (4.29)$$

For a.e.  $t \in (0, T)$  and for every  $\beta \in H^1(\Omega)$  with  $\beta \leq 0$  a.e. in  $\Omega$  we have

$$\langle \partial_{\alpha} \mathcal{E}(\alpha_{\varepsilon}(t), u_{\varepsilon}(t)), \beta \rangle - \int_{\Omega} \tilde{f}_{\varepsilon}(t) \beta \, dx + \varepsilon \langle \dot{\alpha}_{\varepsilon}(t), \beta \rangle_{L^2} \geq 0. \quad (4.30)$$

*Proof.* To prove (4.29) it is sufficient to observe that from (3.7) we have that  $\bar{u}_k(t)$  is a weak solution to the problem

$$\begin{cases} \operatorname{div}(\mu(\bar{\alpha}_k(t)) \nabla \bar{u}_k(t)) = 0 & \text{in } \Omega, \\ \bar{u}_k(t) = w_k(t) & \text{on } \partial_D \Omega, \end{cases} \quad (4.31)$$

and pass (4.31) to the limit as  $k \rightarrow +\infty$  using (4.9) and (4.8).

Let us fix  $\beta \in H^1_-(\Omega)$ . Integrating (3.9) in time, we get

$$- \int_0^T \langle \partial_{\alpha} \mathcal{E}(\bar{\alpha}_k(t), \bar{u}_k(t)), \beta \rangle \, dt + \int_0^T \int_{\Omega} f(\underline{V}_k(t)) \beta \, dx \, dt - \varepsilon \int_0^T \langle \dot{\alpha}_k(t), \beta \rangle_{L^2} \, dt \leq 0. \quad (4.32)$$

First of all, we claim that for every  $t \in [0, T]$

$$- \langle \partial_{\alpha} \mathcal{E}(\alpha_{\varepsilon}(t), u_{\varepsilon}(t)), \beta \rangle \leq \liminf_{k \rightarrow +\infty} - \langle \partial_{\alpha} \mathcal{E}(\bar{\alpha}_k(t), \bar{u}_k(t)), \beta \rangle. \quad (4.33)$$

Indeed, since  $\mu'(\bar{\alpha}_k(t)) \beta \rightarrow \mu'(\alpha_{\varepsilon}(t)) \beta$  strongly in  $L^2(\Omega)$  and by (4.9), Ioffe Theorem [14, Theorem 3.23] yields

$$- \int_{\Omega} \mu'(\alpha_{\varepsilon}(t)) |\nabla u_{\varepsilon}(t)|^2 \beta \, dx \leq \liminf_{k \rightarrow +\infty} \left[ - \int_{\Omega} \mu'(\bar{\alpha}_k(t)) |\nabla \bar{u}_k(t)|^2 \beta \, dx \right]. \quad (4.34)$$

Furthermore, by (4.8)

$$\int_{\Omega} \nabla \bar{\alpha}_k(t) \cdot \nabla \beta \, dx \rightarrow \int_{\Omega} \nabla \alpha_{\varepsilon}(t) \cdot \nabla \beta \, dx, \quad (4.35)$$

for every  $t \in [0, T]$ . Summing (4.34)–(4.35) we obtain (4.33). Moreover, by convergence (4.20), we have

$$\int_{\Omega} f(\underline{V}_k(t))\beta \, dx \rightarrow \int_{\Omega} \tilde{f}_{\varepsilon}(t)\beta \, dx \quad (4.36)$$

for every  $t \in [0, T]$ .

Finally, (4.5) implies

$$\varepsilon \int_{t_1}^{t_2} \langle \dot{\alpha}_k(t), \beta \rangle_{L^2} \, dt \rightarrow \varepsilon \int_{t_1}^{t_2} \langle \dot{\alpha}_{\varepsilon}(t), \beta \rangle_{L^2} \, dt, \quad (4.37)$$

for every  $0 \leq t_1 \leq t_2 \leq T$ .

Collecting (4.33), (4.36), (4.37), and by (4.32), we infer that

$$- \int_{t_1}^{t_2} \langle \partial_{\alpha} \mathcal{E}(\alpha_{\varepsilon}(t), u_{\varepsilon}(t)), \beta \rangle \, dt + \int_{t_1}^{t_2} \int_{\Omega} \tilde{f}_{\varepsilon}(t)\beta \, dx \, dt - \varepsilon \int_{t_1}^{t_2} \langle \alpha_{\varepsilon}(t), \beta \rangle_{L^2} \, dt \leq 0$$

for every  $\beta \in H^1_{-}(\Omega)$  and  $0 \leq t_1 \leq t_2 \leq T$ . By the arbitrariness of  $t_1, t_2$ , a localisation argument gives (4.30).  $\square$

*Remark 4.9.* Using (4.29) we can improve the convergence in (4.7b), namely for every  $\varepsilon > 0$

$$\|u_k - u_{\varepsilon}\|_{C([0, T]; W^{1, p}(\Omega))}, \|\bar{u}_k - u_{\varepsilon}\|_{L^{\infty}(0, T; W^{1, p}(\Omega))}, \|\underline{u}_k - u_{\varepsilon}\|_{L^{\infty}(0, T; W^{1, p}(\Omega))} \rightarrow 0, \text{ for } p \in [2, \tilde{p}). \quad (4.38)$$

Indeed by (4.31) and (4.29) we deduce that the function  $v := u_{\varepsilon}(t) - \bar{u}_k(t) - w(t) + \bar{w}_k(t)$  is a weak solution to the problem

$$\begin{cases} \operatorname{div}(\mu(\alpha_{\varepsilon}(t))\nabla v) = \ell & \text{in } \Omega, \\ v = 0 & \text{on } \partial_D \Omega, \end{cases}$$

where  $\ell \in W^{-1, p}_{\partial_D \Omega}(\Omega)$  is defined by  $\ell := \operatorname{div}((\mu(\bar{\alpha}_k(t)) - \mu(\alpha_{\varepsilon}(t)))\nabla \bar{u}_k(t) + \operatorname{div}(\mu(\alpha_{\varepsilon}(t))(\nabla \bar{w}_k(t) - \nabla w(t)))$ . By

Remark 3.2, (3.6a), (4.7a), and (2.8) we deduce that

$$\begin{aligned} \|\bar{u}_k(t) - u_{\varepsilon}(t)\|_{W^{1, p}} &\leq C \left[ \|\bar{\alpha}_k(t) - \alpha_{\varepsilon}(t)\|_{L^q} \|\bar{u}_k(t)\|_{W^{1, \tilde{p}}} + \|\alpha_{\varepsilon}(t)\|_{L^q} \|\bar{w}_k(t) - w(t)\|_{W^{1, \tilde{p}}} \right] \\ &\leq C \left[ \|\bar{\alpha}_k(t) - \alpha_{\varepsilon}(t)\|_{L^q} + \|\bar{w}_k(t) - w(t)\|_{W^{1, \tilde{p}}} \right] \rightarrow 0 \end{aligned}$$

uniformly with respect to  $t$ , for a suitable  $q \in (2, \infty)$ . The convergence of  $u_k$  and  $\underline{u}_k$  follows from (3.35c)–(3.35d).

**Proposition 4.10** (Energy-dissipation balance in weak form). *For every  $\varepsilon > 0$  we have*

$$\mathcal{E}(\alpha_{\varepsilon}(T), u_{\varepsilon}(T)) - \int_0^T \int_{\Omega} \tilde{f}_{\varepsilon}(t)\dot{\alpha}_{\varepsilon}(t) \, dx \, dt + \varepsilon \int_0^T \|\dot{\alpha}_{\varepsilon}(t)\|_{L^2}^2 \, dt = \mathcal{E}(\alpha_0, u_0) + \int_0^T \langle \mu(\alpha_{\varepsilon}(t))\nabla u_{\varepsilon}(t), \nabla \dot{w}(t) \rangle_{L^2} \, dt.$$

*Proof.* One inequality has been proven in Proposition 4.7. To prove the opposite inequality, we observe that  $t \mapsto \mathcal{E}(\alpha_{\varepsilon}(t), u_{\varepsilon}(t))$  is absolutely continuous and

$$\begin{aligned} \frac{d}{dt} \left[ \mathcal{E}(\alpha_{\varepsilon}(t), u_{\varepsilon}(t)) \right] &= \langle \partial_{\alpha} \mathcal{E}(\alpha_{\varepsilon}(t), u_{\varepsilon}(t)), \dot{\alpha}_{\varepsilon}(t) \rangle + \langle \partial_u \mathcal{E}(\alpha_{\varepsilon}(t), u_{\varepsilon}(t)), \dot{u}_{\varepsilon}(t) \rangle \\ &\geq \int_{\Omega} \tilde{f}_{\varepsilon}(t)\dot{\alpha}_{\varepsilon}(t) \, dx - \varepsilon \|\dot{\alpha}_{\varepsilon}(t)\|_{L^2}^2 + \langle \mu(\alpha_{\varepsilon}(t))\nabla u_{\varepsilon}(t), \nabla \dot{w}(t) \rangle_{L^2} \end{aligned}$$

for a.e.  $t \in (0, T)$ , where in the last inequality we have used (4.30) and (4.29). Integrating the previous inequality in time, we complete the proof.  $\square$

The energy-dissipation balance obtained in Proposition 4.10 above allows us to get the desired strong convergence (4.16b).



**Proposition 4.11** (Strong convergence of discrete-time evolutions). *For every  $\varepsilon > 0$  we have*

$$\alpha_k \rightarrow \alpha_\varepsilon \quad \text{strongly in } W^{1,1}(0, T; L^q(\Omega)), \quad \text{for } q \in [1, \infty), \quad (4.39a)$$

$$u_k \rightarrow u_\varepsilon \quad \text{strongly in } W^{1,1}(0, T; W^{1,p}(\Omega)), \quad \text{for } p \in [2, \tilde{p}). \quad (4.39b)$$

*Proof.* From Proposition 3.7 and Proposition 4.10 and using the convergence of the work term (4.14), we deduce that

$$\begin{aligned} & \lim_{k \rightarrow +\infty} \left[ \mathcal{E}(\alpha_k(T), u_k(T)) + \int_0^T \mathcal{R}(\dot{\alpha}_k(t); \underline{V}_k(t)) \, dt + \varepsilon \int_0^T \|\dot{\alpha}_k(t)\|_{L^2}^2 \, dt \right] \\ &= \mathcal{E}(\alpha_\varepsilon(T), u_\varepsilon(T)) - \int_0^T \int_\Omega \tilde{f}_\varepsilon(t) \dot{\alpha}_\varepsilon(t) \, dx \, dt + \varepsilon \int_0^T \|\dot{\alpha}_\varepsilon(t)\|_{L^2}^2 \, dt. \end{aligned}$$

Notice that if  $(a_k)_k$  and  $(b_k)_k$  are two sequences such that  $a_k + b_k \rightarrow a + b$  and  $a \leq \liminf_k a_k$ ,  $b \leq \liminf_k b_k$ , then  $a_k \rightarrow a$  and  $b_k \rightarrow b$ . Therefore, by (4.12), (4.13), and (4.28) we obtain that

$$\lim_{k \rightarrow +\infty} \int_0^T \|\dot{\alpha}_k(t)\|_{L^2}^2 \, dt = \int_0^T \|\dot{\alpha}_\varepsilon(t)\|_{L^2}^2 \, dt.$$

As a consequence

$$\dot{\alpha}_k \rightarrow \dot{\alpha}_\varepsilon \quad \text{strongly in } L^2(0, T; L^2(\Omega)). \quad (4.40)$$

We want to deduce the strong convergence (4.39a) from (4.40). In order to do so, we shall control  $\|\dot{u}_\varepsilon(t) - \dot{u}_k(t)\|_{W^{1,p}}$  with  $\|\dot{\alpha}_\varepsilon(t) - \dot{\alpha}_k(t)\|_{L^q}$  for some  $q \in (2, \infty)$ , as we did in the proof of (3.6b). For this reason it is necessary to slightly improve the integrability in the target space in (4.40). More precisely, we claim that for every  $q \in [1, \infty)$

$$\dot{\alpha}_k \rightarrow \dot{\alpha}_\varepsilon \quad \text{strongly in } L^1(0, T; L^q(\Omega)). \quad (4.41)$$

Indeed, let us fix  $\theta \in (0, 1)$  and  $q > 2$  (the case  $q \leq 2$  being already covered by (4.40)) and let us define  $r > q$  in such a way that  $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{r}$ . Using the interpolation inequality between the spaces  $L^2(\Omega)$  and  $L^q(\Omega)$ , Hölder's Inequality, (3.13), (4.11), and (4.40) we obtain that

$$\begin{aligned} \int_0^T \|\dot{\alpha}_k(t) - \dot{\alpha}_\varepsilon(t)\|_{L^q} \, dt &\leq \int_0^T \|\dot{\alpha}_k(t) - \dot{\alpha}_\varepsilon(t)\|_{L^2}^\theta \|\dot{\alpha}_k(t) - \dot{\alpha}_\varepsilon(t)\|_{L^r}^{1-\theta} \, dt \\ &\leq \left( \int_0^T \|\dot{\alpha}_k(t) - \dot{\alpha}_\varepsilon(t)\|_{L^2} \, dt \right)^\theta \left( \int_0^T \|\dot{\alpha}_k(t) - \dot{\alpha}_\varepsilon(t)\|_{H^1} \, dt \right)^{1-\theta} \\ &\leq C \left( \int_0^T \|\dot{\alpha}_k(t) - \dot{\alpha}_\varepsilon(t)\|_{L^2} \, dt \right)^\theta \rightarrow 0 \end{aligned}$$

as  $k \rightarrow +\infty$ . This proves (4.41).

We are now ready to prove (4.39a). Differentiating (4.29) in time and by (3.8) we obtain that for a.e.  $t \in (0, T)$  the function  $v := \dot{u}_\varepsilon(t) - \dot{u}_k(t) - \dot{w}(t) + \dot{w}_k(t)$  is a weak solution to the problem

$$\begin{cases} \operatorname{div}(\mu(\alpha_\varepsilon(t)) \nabla v) = \ell & \text{in } \Omega, \\ v = 0 & \text{on } \partial_D \Omega, \end{cases}$$

where  $\ell \in W_{\partial_D \Omega}^{-1,p}(\Omega)$  is defined by

$$\begin{aligned} \ell := & -\operatorname{div} \left( \mu'(\alpha_\varepsilon(t)) \dot{\alpha}_\varepsilon(t) (\nabla u_\varepsilon(t) - \nabla \bar{u}_k(t)) \right) - \operatorname{div} \left( \mu(\alpha_\varepsilon(t)) (\nabla \dot{w}(t) - \nabla \dot{w}_k(t)) \right) \\ & + \operatorname{div} \left( (\mu(\underline{\alpha}_k(t)) - \mu(\alpha_\varepsilon(t))) \nabla \dot{u}_k(t) \right) + \operatorname{div} \left( \left( \frac{\mu(\underline{\alpha}_k(t) + \tau \dot{\alpha}_k(t)) - \mu(\underline{\alpha}_k(t))}{\tau} - \mu'(\alpha_\varepsilon(t)) \dot{\alpha}_\varepsilon(t) \right) \nabla \bar{u}_k(t) \right). \end{aligned}$$

Observe that for a.e.  $t \in \Omega$

$$\begin{aligned} & \left| \frac{\mu(\underline{\alpha}_k(t) + \tau \dot{\alpha}_k(t)) - \mu(\underline{\alpha}_k(t))}{\tau} - \mu'(\alpha_\varepsilon(t)) \dot{\alpha}_\varepsilon(t) \right| \\ & \leq \left| \frac{\mu(\underline{\alpha}_k(t) + \tau \dot{\alpha}_k(t)) - \mu(\underline{\alpha}_k(t))}{\tau} - \mu'(\underline{\alpha}_k(t)) \dot{\alpha}_k(t) \right| + \left| \mu'(\underline{\alpha}_k(t)) - \mu'(\alpha_\varepsilon(t)) \right| |\dot{\alpha}_k(t)| + \left| \mu'(\alpha_\varepsilon(t)) \right| |\dot{\alpha}_k(t) - \dot{\alpha}_\varepsilon(t)| \\ & \leq C \left[ \tau |\dot{\alpha}_k(t)| + |\underline{\alpha}_k(t) - \alpha_\varepsilon(t)| |\dot{\alpha}_k(t)| + |\dot{\alpha}_k(t) - \dot{\alpha}_\varepsilon(t)| \right] \end{aligned}$$

a.e. in  $\Omega$ . Therefore, by Remark 3.2, (3.6a), (3.6b), and (4.10) we get that

$$\begin{aligned} \|\dot{u}_k(t) - \dot{u}_\varepsilon(t)\|_{W^{1,p}} & \leq C \left[ \|\dot{\alpha}_\varepsilon(t)\|_{L^q} \|u_\varepsilon(t) - \bar{u}_k(t)\|_{W^{1,p_1}} + \|\dot{w}_k(t) - \dot{w}(t)\|_{W^{1,\tilde{p}}} \right. \\ & \quad + \|\underline{\alpha}_k(t) - \alpha_\varepsilon(t)\|_{L^q} \|\dot{u}_k(t)\|_{W^{1,p_1}} + \tau \|\dot{\alpha}_k(t)\|_{L^q} \|\bar{u}_k(t)\|_{W^{1,\tilde{p}}} \\ & \quad \left. + \|\underline{\alpha}_k(t) - \alpha_\varepsilon(t)\|_{L^q} \|\dot{\alpha}_k(t)\|_{L^r} \|\bar{u}_k(t)\|_{W^{1,p_1}} + \|\dot{\alpha}_k(t) - \dot{\alpha}_\varepsilon(t)\|_{L^q} \|\bar{u}_k(t)\|_{W^{1,\tilde{p}}} \right] \\ & \leq C \left[ \|\dot{\alpha}_\varepsilon(t)\|_{H^1} \|u_\varepsilon(t) - \bar{u}_k(t)\|_{W^{1,p_1}} + \|\dot{w}_k(t) - \dot{w}(t)\|_{W^{1,\tilde{p}}} \right. \\ & \quad \left. + \|\underline{\alpha}_k(t) - \alpha_\varepsilon(t)\|_{L^q} \|\dot{\alpha}_k(t)\|_{H^1} + \tau \|\dot{\alpha}_k(t)\|_{H^1} + \|\dot{\alpha}_k(t) - \dot{\alpha}_\varepsilon(t)\|_{L^q} \right], \end{aligned}$$

where  $q, r \in (2, \infty)$ , and  $p_1 \in (2, \tilde{p})$  are suitable exponents. Integrating in time the previous inequality and by Hölder's Inequality we obtain

$$\begin{aligned} \|\dot{u}_k - \dot{u}_\varepsilon\|_{L^1(0,T;W^{1,p}(\Omega))} & \leq C \left[ \|\alpha_\varepsilon\|_{W^{1,1}(0,T;H^1(\Omega))} \|u_\varepsilon - \bar{u}_k\|_{L^\infty(0,T;W^{1,p_1}(\Omega))} + \|w_k - w\|_{W^{1,1}(0,T;W^{1,\tilde{p}}(\Omega))} \right. \\ & \quad + \|\underline{\alpha}_k - \alpha_\varepsilon\|_{L^\infty(0,T;L^q(\Omega))} \|\alpha_k\|_{W^{1,1}(0,T;H^1(\Omega))} + \tau \|\dot{\alpha}_k\|_{W^{1,1}(0,T;H^1(\Omega))} \\ & \quad \left. + \|\dot{\alpha}_k - \dot{\alpha}_\varepsilon\|_{L^1(0,T;L^q(\Omega))} \right] \\ & \leq C \left[ \|u_\varepsilon - \bar{u}_k\|_{L^\infty(0,T;W^{1,p_1}(\Omega))} + \|w_k - w\|_{W^{1,1}(0,T;W^{1,\tilde{p}}(\Omega))} \right. \\ & \quad \left. + \|\underline{\alpha}_k - \alpha_\varepsilon\|_{L^\infty(0,T;L^q(\Omega))} + \tau + \|\dot{\alpha}_k - \dot{\alpha}_\varepsilon\|_{L^1(0,T;L^q(\Omega))} \right]. \end{aligned}$$

By (4.38), (2.8), (4.7a), and (4.41) we conclude that the right-hand side in the inequality above converges to zero as  $k \rightarrow +\infty$ .  $\square$

*Proof of Theorem 4.2.* For fixed  $\varepsilon > 0$ , Propositions 4.8 and 4.10 show that  $(\alpha_\varepsilon, u_\varepsilon)$ , obtained by (4.5), (4.6) as weak limit of a sequence of discrete-time evolutions  $(\alpha_k, u_k)$ , satisfy the conditions of Definition 4.1 in a weak sense. In fact,  $(\text{ev}0)_\varepsilon$ ,  $(\text{ev}1)_\varepsilon$  hold, while  $(\text{ev}2)_\varepsilon$ ,  $(\text{ev}3)_\varepsilon$  are satisfied with  $\tilde{f}_\varepsilon(t)$  in place of  $f(V_\varepsilon(t))$ , where  $\tilde{f}_\varepsilon(t)$  is such that (cf. (4.20))

$$f(\underline{V}_k(t)) \overset{*}{\rightharpoonup} \tilde{f}_\varepsilon(t) \quad \text{weakly* in } L^\infty(\Omega),$$

for every  $t \in [0, T]$ .

Actually we find, in Proposition 4.11, that *a posteriori* we have an enhanced convergence for the displacement evolutions that guarantees the strong convergence

$$f(\underline{V}_k) \rightarrow f(V_\varepsilon) \quad \text{strongly in } L^2(0, T; L^2(\Omega)),$$

by Lemma (4.4) and Remark 4.5. We conclude that for a.e.  $t \in (0, T)$

$$\tilde{f}_\varepsilon(t) = f(V_\varepsilon(t)),$$

so that  $(\text{ev}2)_\varepsilon$ ,  $(\text{ev}3)_\varepsilon$  are satisfied with  $f(V_\varepsilon(t))$  and  $(\alpha_\varepsilon, u_\varepsilon)$  is an  $\varepsilon$ -approximate viscous evolution. The estimate (4.4) follows immediately from (4.11).  $\square$

We conclude this section by a characterisation of the energy balance for  $\varepsilon$ -approximate viscous evolutions, that will be employed in the next section to pass to the limit as  $\varepsilon$  tends to 0. We first deduce the following lemma.

**Lemma 4.12.** *Let  $(\alpha_\varepsilon, u_\varepsilon) \in H^1(0, T; H^1(\Omega)) \times H^1(0, T; W^{1,p}(\Omega))$  satisfies  $(ev0)_\varepsilon$ ,  $(ev1)_\varepsilon$  of Definition 4.1. Then  $(ev3)_\varepsilon$  for  $(\alpha_\varepsilon, u_\varepsilon)$  is equivalent to:*

$(ev3')_\varepsilon$ : for a.e.  $t \in (0, T)$

$$\langle \partial_\alpha \mathcal{E}(\alpha_\varepsilon(t), u_\varepsilon(t)), \dot{\alpha}_\varepsilon(t) \rangle + \mathcal{R}(\dot{\alpha}_\varepsilon(t); V_\varepsilon(t)) + \varepsilon \|\dot{\alpha}_\varepsilon(t)\|_{L^2}^2 = 0. \quad (4.42)$$

*Proof.* Being  $\alpha_\varepsilon, u_\varepsilon$  absolutely continuous (in time) we get that  $t \mapsto \mathcal{E}(\alpha_\varepsilon(t), u_\varepsilon(t))$  is absolutely continuous and

$$\begin{aligned} \frac{d}{dt} \left[ \mathcal{E}(\alpha_\varepsilon(t), u_\varepsilon(t)) \right] &= \langle \partial_\alpha \mathcal{E}(\alpha_\varepsilon(t), u_\varepsilon(t)), \dot{\alpha}_\varepsilon(t) \rangle + \langle \partial_u \mathcal{E}(\alpha_\varepsilon(t), u_\varepsilon(t)), \dot{u}_\varepsilon(t) \rangle \\ &= \langle \partial_\alpha \mathcal{E}(\alpha_\varepsilon(t), u_\varepsilon(t)), \dot{\alpha}_\varepsilon(t) \rangle + \langle \mu(\alpha_\varepsilon(t)) \nabla u_\varepsilon(t), \nabla \dot{w}(t) \rangle_{L^2}, \end{aligned} \quad (4.43)$$

using  $(ev1)_\varepsilon$ . Differentiating in time  $(ev3)_\varepsilon$  gives then the equivalence between  $(ev3)_\varepsilon$  and  $(ev3')_\varepsilon$ .  $\square$

*Remark 4.13.* Arguing in a similar way (cf. also Proposition 4.10 and [11, Proposition 4.2]) it is not difficult to see that if  $(\alpha_\varepsilon, u_\varepsilon) \in H^1(0, T; H^1(\Omega) \times W^{1,p}(\Omega))$  satisfies  $(ev0)_\varepsilon$ ,  $(ev1)_\varepsilon$ ,  $(ev2)_\varepsilon$  of Definition 4.1 and

$(ev3'')_\varepsilon$ :

$$\mathcal{E}(\alpha_\varepsilon(T), u_\varepsilon(T)) + \int_0^T \mathcal{R}(\dot{\alpha}_\varepsilon(t); V_\varepsilon(t)) dt + \varepsilon \int_0^T \|\dot{\alpha}_\varepsilon(t)\|_{L^2}^2 dt \leq \mathcal{E}(\alpha_0, u_0) + \int_0^T \langle \mu(\alpha_\varepsilon(t)) \nabla u_\varepsilon(t), \nabla \dot{w}(t) \rangle_{L^2} dt,$$

then  $(\alpha_\varepsilon, u_\varepsilon)$  is an  $\varepsilon$ -approximate viscous evolution.

Let us introduce some notation in view of the characterisation of the energy balance for  $\varepsilon$ -approximate viscous evolutions.

For  $(\alpha, u) \in W^{1,1}(0, T; H^1(\Omega) \times H^1(\Omega))$  and  $\tilde{f} \in L^2(\Omega)$  (that we regard as an element of  $(H^1(\Omega))'$  with  $\langle \tilde{f}, \beta \rangle = \int_\Omega \tilde{f} \beta dx$ ) we define

$$\Phi(g) := \sup_{\beta \in F} \langle -g, \beta \rangle \quad \text{for every } g \in (H^1(\Omega))', \quad \Psi(\alpha, u, \tilde{f}) := \Phi(\partial_\alpha \mathcal{E}(\alpha, u) - \tilde{f}), \quad (4.44)$$

where

$$F := \{\beta \in H_-^1(\Omega) : \|\beta\|_{L^2} \leq 1\}$$

Employing Lemma 4.12 we obtain the following characterisation of the energy balance, which is invariant under time reparametrisation.

**Proposition 4.14.** *Let  $(\alpha_\varepsilon, u_\varepsilon)$  be an  $\varepsilon$ -approximate viscous evolution. Then with the notation above we have that*

$$\varepsilon \|\dot{\alpha}_\varepsilon(t)\|_{L^2} = \Psi(\alpha_\varepsilon(t), u_\varepsilon(t), f(V_\varepsilon(t))), \quad (4.45)$$

and one may recast the energy balance  $(ev3)_\varepsilon$  as

$$\begin{aligned} \mathcal{E}(\alpha_\varepsilon(T), u_\varepsilon(T)) + \int_0^T \mathcal{R}(\dot{\alpha}_\varepsilon(t); V_\varepsilon(t)) dt + \int_0^T \|\dot{\alpha}_\varepsilon(t)\|_{L^2} \Psi(\alpha_\varepsilon(t), u_\varepsilon(t), f(V_\varepsilon(t))) dt \\ = \mathcal{E}(\alpha_0, u_0) + \int_0^T \langle \mu(\alpha_\varepsilon(t)) \nabla u_\varepsilon(t), \nabla \dot{w}(t) \rangle_{L^2} dt. \end{aligned} \quad (4.46)$$

*Proof.* By  $(ev2)_\varepsilon$  we get that for every  $\beta \in H_-^1(\Omega)$

$$\varepsilon \langle \dot{\alpha}_\varepsilon(t), \beta \rangle \geq \langle -\partial_\alpha \mathcal{E}(\alpha_\varepsilon(t), u_\varepsilon(t)), \beta \rangle + \mathcal{R}(\beta; V_\varepsilon(t)).$$

On the other hand Lemma 4.12 implies that the equality above is attained for  $\bar{\beta} = \frac{\dot{\alpha}_\varepsilon(t)}{\|\dot{\alpha}_\varepsilon(t)\|_{L^2}}$  and this gives (4.45), since  $\bar{\beta}$  is in  $F$  and  $\mathcal{R}(\beta; V_\varepsilon(t)) = - \int_\Omega f(V_\varepsilon(t)) \beta dx$ . Then (4.46) follows immediately from the energy balance  $(ev3)_\varepsilon$ .  $\square$

*Remark 4.15.* Arguing in the same way of [11, Lemma 4.4] (see also [32, Lemma A.2]) we deduce that

$$\Phi(g) = d_2(g, G) \quad \text{for every } g \in (H^1(\Omega))',$$

where

$$G := \{h \in (H^1(\Omega))' : \langle h, \beta \rangle \geq 0 \text{ for every } \beta \in H_-^1(\Omega)\}, \quad d_2(g, G) := \min\{\|h\|_{L^2} : h \in L^2(\Omega), h + g \in G\}.$$

## 5. VANISHING VISCOSITY LIMIT

This section concerns the asymptotics of the viscous evolution, whose existence has been proven in Section 4, under the constitutive assumptions in Section 2, as the viscosity parameter  $\varepsilon$  vanishes. We use a rescaling technique, common to many other works (see e.g. [15, 24, 25, 11]). Let  $\{(\alpha_\varepsilon, u_\varepsilon)\}_{\varepsilon>0}$  be a family of  $\varepsilon$ -approximate viscous evolutions satisfying the uniform  $W^{1,1}$  bound in time (4.4), for a given  $p < \tilde{p}$ , where  $\tilde{p}$  is given by Lemma 3.3. The existence of these evolutions has been shown in Theorem 4.2. For  $\varepsilon > 0$  and  $t \in [0, T]$  we set

$$s_\varepsilon^\circ(t) := t + \int_0^t \|\dot{\alpha}_\varepsilon(s)\|_{H^1} ds + \int_0^t \|\dot{u}_\varepsilon(s)\|_{W^{1,p}} ds. \quad (5.1)$$

Then  $s_\varepsilon^\circ$  is absolutely continuous and

$$s_\varepsilon^\circ(t_2) - s_\varepsilon^\circ(t_1) \geq t_2 - t_1 \quad \text{for every } 0 \leq t_1 \leq t_2 \leq S_\varepsilon := s_\varepsilon^\circ(T),$$

in particular  $s_\varepsilon^\circ$  is strictly increasing and bijective on its domain. We denote by  $t_\varepsilon^\circ: [0, S_\varepsilon] \rightarrow [0, T]$  the inverse of  $s_\varepsilon^\circ$ . In view of (4.4), we have that  $T \leq S_\varepsilon < C$ , for  $C > 0$  independent of  $\varepsilon$ , and then, up to a subsequence,  $S_\varepsilon \rightarrow S$  as  $\varepsilon \rightarrow 0$ , with  $S \geq T$ . We define the rescaled evolution on  $[0, S_\varepsilon]$  by setting

$$\alpha_\varepsilon^\circ(s) := \alpha_\varepsilon(t_\varepsilon^\circ(s)), \quad u_\varepsilon^\circ(s) := u_\varepsilon(t_\varepsilon^\circ(s)), \quad \zeta_\varepsilon^\circ(s) := \zeta_\varepsilon(t_\varepsilon^\circ(s)), \quad V_\varepsilon^\circ(s) := V_\varepsilon(t_\varepsilon^\circ(s)). \quad (5.2)$$

Up to extending  $t_\varepsilon^\circ$  with  $t_\varepsilon^\circ(S_\varepsilon)$  in  $(S_\varepsilon, \bar{S}]$ , for  $\bar{S} := \sup_{\varepsilon>0} S_\varepsilon$  ( $\varepsilon$  small), we assume the rescaled functions above defined on the fixed time interval  $[0, S]$ . By a change of variable we have from (4.1) that

$$V_\varepsilon^\circ(s) = \int_0^s |\dot{\zeta}_\varepsilon^\circ(\sigma)| d\sigma \quad \text{a.e. in } \Omega, \text{ for every } s \in [0, S].$$

Since (5.1) gives that  $t_\varepsilon^\circ$  is nondecreasing and that

$$t_\varepsilon^\circ(s_2) - t_\varepsilon^\circ(s_1) + \|\alpha_\varepsilon^\circ(s_2) - \alpha_\varepsilon^\circ(s_1)\|_{H^1} + \|u_\varepsilon^\circ(s_2) - u_\varepsilon^\circ(s_1)\|_{W^{1,p}} \leq s_2 - s_1 \quad (5.3)$$

for every  $0 \leq s_1 \leq s_2 \leq S$ , we deduce (cf. also e.g. [15, 11, 25]) that, up to a (not relabeled) subsequence

$$(t_\varepsilon^\circ, \alpha_\varepsilon^\circ, u_\varepsilon^\circ) \xrightarrow{*} (t^\circ, \alpha^\circ, u^\circ) \quad \text{weakly}^* \text{ in } W^{1,\infty}(0, S; [0, T] \times H^1(\Omega) \times W^{1,p}(\Omega)), \quad (5.4)$$

for a suitable  $(t^\circ, \alpha^\circ, u^\circ)$  with

$$t^\circ(s) + \|\dot{\alpha}^\circ(s)\|_{H^1} + \|\dot{u}^\circ(s)\|_{W^{1,p}} \leq 1 \quad \text{for a.e. } s \in (0, S).$$

In view of the equicontinuity (with respect to  $\varepsilon$ ) of  $(\alpha_\varepsilon^\circ, u_\varepsilon^\circ)$ , it follows that for every  $s \in [0, S]$  and  $s_\varepsilon \rightarrow s$

$$\alpha_\varepsilon^\circ(s_\varepsilon) \rightharpoonup \alpha^\circ(s) \quad \text{weakly in } H^1(\Omega), \quad u_\varepsilon^\circ(s_\varepsilon) \rightharpoonup u^\circ(s) \quad \text{weakly in } W^{1,p}(\Omega). \quad (5.5)$$

Moreover, we define

$$w^\circ(s) := w(t^\circ(s)), \quad \text{for every } s \in [0, S]. \quad (5.6)$$

Similarly to the analogous situation in Section 4, the weak convergences above are not enough to guarantee pointwise convergence for the cumulations of the strains, even if the cumulation of  $\zeta^\circ$

$$V^\circ(s) = \int_0^s |\dot{\zeta}^\circ(\sigma)| d\sigma \quad \text{a.e. in } \Omega, \text{ for every } s \in [0, S]$$

is well defined as a Bochner integral in  $L^2(\Omega)$ . We may only say, passing through an Helly type selection principle as in Lemma 4.6 that there exists  $\tilde{f}^\circ: [0, S] \rightarrow L^\infty(\Omega)$ , increasing in time for a.e. fixed  $x \in \Omega$ , such that

$$f(V_\varepsilon^\circ(s)) \overset{*}{\rightharpoonup} \tilde{f}^\circ(s) \quad \text{weakly}^* \text{ in } L^\infty(\Omega), \text{ for every } s \in [0, S]. \quad (5.7)$$

However, differently from Section 4, in view of the loss of the viscous term in the limit evolution we are not able to improve the convergences (5.4) *a posteriori*, so to express  $\tilde{f}^\circ$  in terms of  $V^\circ$ , but we prove only an inequality, see Proposition 5.4.

We obtain then the following existence result for limit of rescaled  $\varepsilon$ -approximate viscous evolutions, that we call *rescaled quasistatic viscosity evolutions*, which is the main result of the paper.

**Theorem 5.1.** *With the notation above the function  $(t^\circ, \alpha^\circ, u^\circ) \in W^{1,\infty}(0, S; [0, T]) \times H^1(\Omega) \times W^{1,p}(\Omega)$ , defined as limit of rescaled  $\varepsilon$ -approximate viscous evolutions in (5.4), satisfies the following properties:*

(ev0) irreversibility:

$$[0, S] \ni s \mapsto \alpha^\circ(s) \quad \text{is nonincreasing as a family of measurable functions on } \Omega,$$

that is  $\alpha^\circ(t) \leq \alpha^\circ(s)$  a.e. in  $\Omega$  for all  $s \leq t$ ;

(ev1) equilibrium: for every  $s \in [0, S]$ ,  $u^\circ(s) \in H^1(\Omega)$  is a weak solution to the problem

$$\begin{cases} \operatorname{div}(\mu(\alpha^\circ(s))\nabla u^\circ(s)) = 0 & \text{in } \Omega, \\ u^\circ(s) = w^\circ(s) & \text{on } \partial_D \Omega. \end{cases} \quad (5.8)$$

(ev2) Karush-Kuhn-Tucker inequality : for a.e.  $s \in (0, S) \setminus U^\circ$  and for every  $\beta \in H^1(\Omega)$  with  $\beta \leq 0$  a.e. in  $\Omega$  we have

$$\langle \partial_\alpha \mathcal{E}(\alpha^\circ(s), u^\circ(s)) - \tilde{f}^\circ(s), \beta \rangle \geq 0, \quad (5.9)$$

where  $U^\circ := \{s \in [0, S] : t^\circ \text{ is constant in a neighbourhood of } s\}$ .

(ev3) energy balance:

$$\begin{aligned} \mathcal{E}(\alpha^\circ(S), u^\circ(S)) - \int_0^S \langle \tilde{f}^\circ(s), \dot{\alpha}^\circ(s) \rangle ds + \int_0^S \|\dot{\alpha}^\circ(s)\|_{L^2} \Psi(\alpha^\circ(s), u^\circ(s), \tilde{f}^\circ(s)) ds \\ = \mathcal{E}(\alpha_0, u_0) + \int_0^S \langle \mu(\alpha^\circ(s))\nabla u^\circ(s), \nabla \dot{w}^\circ(s) \rangle ds. \end{aligned}$$

Moreover, for every  $s \in [0, S]$  we have that

$$\mathcal{E}(\alpha^\circ(s), u^\circ(s)) = \lim_{\varepsilon \rightarrow 0} \mathcal{E}(\alpha_\varepsilon^\circ(s), u_\varepsilon^\circ(s)), \quad (5.10)$$

$$- \int_0^S \langle \tilde{f}^\circ(s), \dot{\alpha}^\circ(s) \rangle ds = - \lim_{\varepsilon \rightarrow 0} \int_0^S \langle f(V_\varepsilon^\circ(s)), \dot{\alpha}_\varepsilon^\circ(s) \rangle ds = \lim_{\varepsilon \rightarrow 0} \int_0^S \mathcal{R}(\dot{\alpha}_\varepsilon^\circ(s); V_\varepsilon^\circ(s)) ds,$$

and

$$\int_0^S \|\dot{\alpha}^\circ(s)\|_{L^2} \Psi(\alpha^\circ(s), u^\circ(s), \tilde{f}^\circ(s)) ds = \lim_{\varepsilon \rightarrow 0} \int_0^S \|\dot{\alpha}_\varepsilon^\circ(s)\|_{L^2} \Psi(\alpha_\varepsilon^\circ(s), u_\varepsilon^\circ(s), f(V_\varepsilon^\circ(s))) ds.$$

*Remark 5.2.* The Karush-Kuhn-Tucker inequality (ev2) is equivalent to  $\Psi(\alpha^\circ(s), u^\circ(s), \tilde{f}^\circ(s)) = 0$ , by the definition of  $\Psi$  (4.44), so that the term in  $\Psi$  in the energy balance gives a contribution only in the zones where the evolution is not stable. Notice also that  $\Psi(\alpha^\circ(s), u^\circ(s), \tilde{f}^\circ(s)) = d_2(\partial_\alpha \mathcal{E}(\alpha^\circ(s), u^\circ(s)) - \tilde{f}^\circ(s), G)$ , (cf. Remark 4.15) a sort of  $L^2$ -distance from the (first order) stability set  $G$ .

*Remark 5.3.* If  $(t^\circ, \alpha^\circ, u^\circ) \in W^{1,\infty}(0, S; [0, T] \times H^1(\Omega) \times W^{1,p}(\Omega))$  satisfies (ev0), (ev1), (ev2), then (ev3) is equivalent to say that for a.e.  $s \in (0, S)$

$$\langle \partial_\alpha \mathcal{E}(\alpha^\circ(s), u^\circ(s)) - \tilde{f}^\circ(s), \dot{\alpha}^\circ(s) \rangle + \|\dot{\alpha}^\circ(s)\|_{L^2} \Psi(\alpha^\circ(s), u^\circ(s), \tilde{f}^\circ(s)) = 0, \quad (5.11)$$

arguing as done for Lemma 4.12, by differentiation. Notice that, by definition of  $\Psi$  (4.44), we always have

$$\langle \partial_\alpha \mathcal{E}(\alpha^\circ(s), u^\circ(s)) - \tilde{f}^\circ(s), \beta \rangle + \|\beta\|_{L^2} \Psi(\alpha^\circ(s), u^\circ(s), \tilde{f}^\circ(s)) \geq 0, \quad (5.12)$$

for every  $\beta \in H^1(\Omega)$  with  $\beta \leq 0$  a.e. in  $\Omega$ .

By the convergence of the energies (5.10) we deduce the following relation between  $\tilde{f}^\circ$  and  $f(V^\circ)$ .

**Proposition 5.4.** *For every  $s \in [0, S]$*

$$\tilde{f}^\circ(s) \leq f(V^\circ) \quad \text{a.e. in } \Omega. \quad (5.13)$$

*Proof.* In this proof we use the notion of essential variation of a time-dependent family of functions, whose definition is given in Definition A.1 in the Appendix. Here we recall that, by Proposition A.4,

$$V_\varepsilon^\circ(s) = \text{ess Var}(\zeta_\varepsilon^\circ; 0, s) = \text{ess sup}_{0=s_0 < \dots < s_m=s} \left\{ \sum_{j=0}^m |\zeta_\varepsilon^\circ(s_j) - \zeta_\varepsilon^\circ(s_{j-1})| \right\}.$$

Hence, since  $f$  is nonincreasing, we have that for every partition  $0 \leq s_0 < \dots < s_m \leq s$

$$f(V_\varepsilon^\circ(s)) \leq f\left(\sum_{j=1}^m |\zeta_\varepsilon^\circ(s_j) - \zeta_\varepsilon^\circ(s_{j-1})|\right) \quad \text{a.e. in } \Omega. \quad (5.14)$$

Indeed

$$\sum_{j=1}^m |\zeta_\varepsilon^\circ(s_j) - \zeta_\varepsilon^\circ(s_{j-1})| \leq V_\varepsilon^\circ(s),$$

as functions on  $\Omega$ . By (5.5) and (5.10) we have that  $\nabla u_\varepsilon^\circ(s) \rightarrow \nabla u^\circ(s)$  in  $L^2(\Omega)$  for every  $s \in [0, S]$ , so that

$$\zeta_\varepsilon^\circ(s) \rightarrow \zeta^\circ(s) \quad \text{in } L^2(\Omega),$$

and then

$$\sum_{j=1}^m |\zeta_\varepsilon^\circ(s_j) - \zeta_\varepsilon^\circ(s_{j-1})| \rightarrow \sum_{j=1}^m |\zeta^\circ(s_j) - \zeta^\circ(s_{j-1})| \quad \text{in } L^2(\Omega) \quad (5.15)$$

as  $\varepsilon \rightarrow 0$  for every fixed partition. Testing (5.7) with characteristic functions of any Borel set  $B \subset \Omega$  and employing (5.14), (5.15), we can pass to  $\varepsilon \rightarrow 0$  and obtain

$$\int_B \tilde{f}^\circ(s) \, dx \leq \int_B f\left(\sum_{j=1}^m |\zeta^\circ(s_j) - \zeta^\circ(s_{j-1})|\right) \, dx,$$

that gives, since  $B \subset \Omega$  Borel is arbitrary,

$$\tilde{f}^\circ(s) \leq f\left(\sum_{j=1}^m |\zeta^\circ(s_j) - \zeta^\circ(s_{j-1})|\right).$$

By the arbitrariness of the partition, and since  $f$  is nonincreasing, this implies

$$\tilde{f}^\circ(s) \leq f(\text{essVar}(\zeta^\circ; 0, s)),$$

and (5.13) follows because  $V^\circ(s) = \text{essVar}(\zeta^\circ; 0, s)$  by Proposition A.4.  $\square$

*Remark 5.5.* By Proposition 5.4 we have that the Karush-Kuhn-Tucker inequality (ev2) holds also for  $f(V^\circ(s))$  in place of  $\tilde{f}^\circ(s)$ , that is for a.e.  $s \in (0, S) \setminus U^\circ$  and  $\beta \in H^1(\Omega)$ ,  $\beta \leq 0$ , we have

$$\langle \partial_\alpha \mathcal{E}(\alpha^\circ(s), u^\circ(s)) - f(V^\circ(s)), \beta \rangle \geq 0.$$

However we can guarantee only the inequality

$$\begin{aligned} \mathcal{E}(\alpha^\circ(S), u^\circ(S)) - \int_0^S \langle f(V^\circ(s)), \dot{\alpha}^\circ(s) \rangle \, ds + \int_0^S \|\dot{\alpha}^\circ(s)\|_{L^2} \Psi(\alpha^\circ(s), u^\circ(s), \tilde{f}^\circ(s)) \, ds \\ \geq \mathcal{E}(\alpha_0, u_0) + \int_0^S \langle \mu(\alpha^\circ(s)) \nabla u^\circ(s), \nabla \dot{w}^\circ(s) \rangle \, ds. \end{aligned}$$

in place of (ev3), if we consider  $f(V^\circ(s))$  instead of  $\tilde{f}^\circ$ .

*Proof of Theorem 5.1.* Since in general  $t^\circ$  is not invertible, we consider its left and right inverse, defined by

$$\begin{aligned} s_-^\circ(t) &:= \sup\{s \in [0, S] : t^\circ(s) < t\} \quad \text{for } t \in (0, T], s_-^\circ(0) := 0, \\ s_+^\circ(t) &:= \inf\{s \in [0, S] : t^\circ(s) > t\} \quad \text{for } t \in [0, T), s_+^\circ(T) := S. \end{aligned}$$

For every  $t \in [0, T]$  we have that  $t^\circ(s_-^\circ(t)) = t = t^\circ(s_+^\circ(t))$  and

$$s_-^\circ(t) \leq \liminf_{\varepsilon \rightarrow 0} s_\varepsilon^\circ(t) \leq \limsup_{\varepsilon \rightarrow 0} s_\varepsilon^\circ(t) \leq s_+^\circ(t), \quad (5.16)$$

while  $s_-^\circ(t^\circ(s)) \leq s \leq s_+^\circ(t^\circ(s))$  for every  $s \in [0, S]$ . The set

$$S^\circ := \{t \in [0, T] : s_-^\circ(t) < s_+^\circ(t)\} \quad (5.17)$$

is at most countable, and

$$U^\circ = \bigcup_{t \in S^\circ} (s_-^\circ(t), s_+^\circ(t)). \quad (5.18)$$

Arguing as done in Proposition 4.8, by (5.5), we pass (ev1) $_\varepsilon$  to the limit and obtain (ev1), while (ev0) is immediate from the pointwise convergence of  $\alpha_\varepsilon^\circ(s)$  to  $\alpha^\circ(s)$  for every  $s \in [0, S]$ .

**Proof of (ev2).** It is enough to show that  $A^\circ \subset U^\circ$ , where

$$A^\circ := \{s \in [0, S] : \Psi(\alpha^\circ(s), u^\circ(s), \tilde{f}^\circ(s)) > 0\}. \quad (5.19)$$

Arguing as in the proof Proposition 4.8 to obtain (4.33) and (4.36), we deduce that for every  $\beta \in H_-^1(\Omega)$  and every  $s \in [0, S]$

$$\langle -\partial_\alpha \mathcal{E}(\alpha^\circ(s), u^\circ(s)) + \tilde{f}^\circ(s), \beta \rangle \leq \liminf_{\varepsilon \rightarrow 0} \langle -\partial_\alpha \mathcal{E}(\alpha_\varepsilon^\circ(s), u_\varepsilon^\circ(s)) + f(V_\varepsilon^\circ(s)), \beta \rangle,$$

so that, passing to the supremum for  $\beta \in H_-^1(\Omega)$ ,

$$\Psi(\alpha^\circ(s), u^\circ(s), \tilde{f}^\circ(s)) \leq \liminf_{\varepsilon \rightarrow 0} \Psi(\alpha_\varepsilon^\circ(s), u_\varepsilon^\circ(s), f(V_\varepsilon^\circ(s))). \quad (5.20)$$

By (2.3) and the convergences (5.5) we have that the map  $s \mapsto \langle -\partial_\alpha \mathcal{E}(\alpha^\circ(s), u^\circ(s)), \beta \rangle$  is continuous for every  $\beta \in H_-^1(\Omega)$ . Also  $s \mapsto \langle \tilde{f}^\circ(s), \beta \rangle$  is continuous: indeed

$$|\langle f(V_\varepsilon^\circ(s_2)) - f(V_\varepsilon^\circ(s_1)), \beta \rangle| \leq \|f'\|_{L^\infty} \int_\Omega |\beta| \int_{s_1}^{s_2} |\dot{\zeta}_\varepsilon^\circ(\sigma)| \, d\sigma \, dx \leq \|f'\|_\infty \left( \int_{s_1}^{s_2} \|\dot{\zeta}_\varepsilon^\circ(\sigma)\|_{L^2} \, d\sigma \right) \|\beta\|_{L^2} \leq C(s_2 - s_1) \|\beta\|_{L^2},$$

since

$$\|\dot{\zeta}_\varepsilon^\circ(\sigma)\|_{L^2} = \|g'(\alpha_\varepsilon^\circ(\sigma)) \dot{\alpha}_\varepsilon^\circ(\sigma) \nabla u_\varepsilon^\circ(\sigma) + g(\alpha_\varepsilon^\circ(\sigma)) \nabla \dot{u}_\varepsilon^\circ(\sigma)\|_{L^2} \leq C(\|\dot{\alpha}_\varepsilon^\circ(\sigma)\|_{H^1} + \|\nabla \dot{u}_\varepsilon^\circ(\sigma)\|_{L^2}) \leq C,$$

and we pass to the limit as  $\varepsilon \rightarrow 0$  to get

$$|\langle \tilde{f}^\circ(s_2) - \tilde{f}^\circ(s_1), \beta \rangle| \leq C(s_2 - s_1) \|\beta\|_2.$$

Therefore  $s \mapsto \langle -\partial_\alpha \mathcal{E}(\alpha^\circ(s), u^\circ(s)) + \tilde{f}^\circ(s), \beta \rangle$  is continuous, and

$$s \mapsto \Psi(\alpha^\circ(s), u^\circ(s), \tilde{f}^\circ(s)) \quad \text{is lower semicontinuous.} \quad (5.21)$$

In particular,  $A^\circ$  is an open set. We now follow closely the argument in [11, Theorem 5.4, proof of (ev3) therein], to say that

$$\limsup_{\varepsilon \rightarrow 0} t_\varepsilon^\circ(s) > 0 \quad \text{for a.e. } s \in (0, S) \setminus D^\circ, \quad (5.22)$$

where  $D^\circ := \{s \in (0, S) : t^\circ(s) = 0\}$ .

Arguing by contradiction, there exists a measurable set  $B \subset (0, S) \setminus D^\circ$  with positive measure such that

$$\lim_{\varepsilon \rightarrow 0} t_\varepsilon^\circ(s) = 0 \quad \text{for every } s \in B,$$

$t_\varepsilon^\circ$  being nondecreasing. Since the functions  $t_\varepsilon^\circ$  are 1-Lipschitz, the Dominated Convergence Theorem implies that

$$\lim_{\varepsilon \rightarrow 0} \int_B t_\varepsilon^\circ(s) \, ds = 0.$$

On the other hand, from  $t_\varepsilon^\circ \rightharpoonup t^\circ$  weakly\* in  $W^{1,\infty}$  (see (5.5))

$$\lim_{\varepsilon \rightarrow 0} \int_B t_\varepsilon^\circ(s) \, ds = \int_B t^\circ(s) \, ds,$$

and this contradicts

$$\int_B t^\circ(s) \, ds > 0,$$

that follows from the definition of  $D^\circ$ .

By (5.20) and (4.45) evaluated in  $t = t_\varepsilon^\circ(s)$  (cf. (5.2)) we deduce

$$0 \leq \Psi(\alpha^\circ(s), u^\circ(s), \tilde{f}^\circ(s)) \leq \liminf_{\varepsilon \rightarrow 0} \Psi(\alpha_\varepsilon^\circ(s), u_\varepsilon^\circ(s), f(V_\varepsilon^\circ(s))) = \liminf_{\varepsilon \rightarrow 0} \varepsilon \|\dot{\alpha}_\varepsilon(t_\varepsilon^\circ(s))\|_{L^2} = \liminf_{\varepsilon \rightarrow 0} \varepsilon \frac{\|\dot{\alpha}_\varepsilon^\circ(s)\|_{L^2}}{t_\varepsilon^\circ(s)} = 0$$

for a.e.  $s \in (0, S) \setminus D^\circ$ . This implies that  $t^\circ(s) = 0$  for a.e.  $s \in A^\circ$ . Being  $A^\circ$  open by (5.21), every  $s \in A^\circ$  has an open neighborhood where  $t^\circ = 0$ ; then  $A^\circ \subset U^\circ$ , because  $t^\circ$  is Lipschitz.

**Proof of (ev3).** Looking at the version of the energy balance (4.46) proven in Proposition 4.14, this is invariant under time reparametrisation. Then, by the change of variables  $t = t_\varepsilon^\circ(s)$  (in the left hand side) we get

$$\begin{aligned} \mathcal{E}(\alpha_\varepsilon^\circ(S), u_\varepsilon^\circ(S)) + \int_0^S \mathcal{R}(\dot{\alpha}_\varepsilon^\circ(s); V_\varepsilon^\circ(s)) \, ds + \int_0^S \|\dot{\alpha}_\varepsilon^\circ(s)\|_{L^2} \Psi(\alpha_\varepsilon^\circ(s), u_\varepsilon^\circ(s), f(V_\varepsilon^\circ(s))) \, ds \\ = \mathcal{E}(\alpha_0, u_0) + \int_0^T \langle \mu(\alpha_\varepsilon(t)) \nabla u_\varepsilon(t), \nabla \dot{w}(t) \rangle_{L^2} \, dt. \end{aligned} \quad (5.23)$$

Arguing as done in Proposition 4.10 to deduce (4.28), we obtain

$$- \int_0^S \langle \tilde{f}^\circ(s), \dot{\alpha}^\circ(s) \rangle \, ds = \sup_{0=s_0 < \dots < s_m=S} \left\{ \sum_{j=1}^m \langle \tilde{f}^\circ(s_j), \alpha^\circ(s_{j-1}) - \alpha^\circ(s_j) \rangle \right\}$$

and then, since (5.5) and (5.7) give

$$\langle \tilde{f}^\circ(s_j), \alpha^\circ(s_{j-1}) - \alpha^\circ(s_j) \rangle = \lim_{\varepsilon \rightarrow 0} \langle f(V_\varepsilon^\circ(s_j)), \alpha_\varepsilon^\circ(s_{j-1}) - \alpha_\varepsilon^\circ(s_j) \rangle$$



for any  $s_{j-1}, s_j$ , we deduce

$$-\int_0^S \langle \tilde{f}^\circ(s), \dot{\alpha}^\circ(s) \rangle ds \leq \liminf_{\varepsilon \rightarrow 0} \int_0^S \mathcal{R}(\dot{\alpha}_\varepsilon^\circ(s); V_\varepsilon^\circ(s)) ds, \quad (5.24)$$

recalling (4.27). Moreover, we claim that

$$\int_{A^\circ} \|\dot{\alpha}^\circ(s)\|_{L^2} \Psi(\alpha^\circ(s), u^\circ(s), \tilde{f}^\circ(s)) ds \leq \liminf_{\varepsilon \rightarrow 0} \int_{A^\circ} \|\dot{\alpha}_\varepsilon^\circ(s)\|_{L^2} \Psi(\alpha_\varepsilon^\circ(s), u_\varepsilon^\circ(s), f(V_\varepsilon^\circ(s))) ds. \quad (5.25)$$

Indeed, for every compact set  $C \subset A^\circ$  and every continuous function  $\psi: C \rightarrow [0, +\infty)$  such that

$$\Psi(\alpha^\circ(s), u^\circ(s), \tilde{f}^\circ(s)) > \psi(s) \quad \text{for every } s \in C,$$

by the compactness of  $C$  and (5.20), for  $\varepsilon$  sufficiently small we get

$$\Psi(\alpha_\varepsilon^\circ(s), u_\varepsilon^\circ(s), f(V_\varepsilon^\circ(s))) > \psi(s) \quad \text{for every } s \in C.$$

Then, by approximating the semicontinuous function  $s \mapsto \Psi(\alpha^\circ(s), u^\circ(s), \tilde{f}^\circ(s))$  from below by continuous functions, in order to prove (5.25) it is sufficient to show

$$\int_C \|\dot{\alpha}^\circ(s)\|_{L^2} \psi(s) ds \leq \liminf_{\varepsilon \rightarrow 0} \int_C \|\dot{\alpha}_\varepsilon^\circ(s)\|_{L^2} \psi(s) ds$$

for every compact  $C \subset A^\circ$  and every continuous function  $\psi: C \rightarrow [0, +\infty)$ . This is done as in [11, Theorem 5.4] or [15, Lemma 6.4], using a localisation argument and the fact that for every  $\varphi \in C_c(\Omega)$  with  $\|\varphi\|_{L^2} = 1$  the functions  $s \mapsto \langle \varphi, \dot{\alpha}_\varepsilon^\circ(s) \rangle$  are equi-Lipschitz on  $[0, S]$  and converge to  $s \mapsto \langle \varphi, \dot{\alpha}^\circ(s) \rangle$  for every  $s$ .

By (5.24), (5.25), and the semicontinuity of the internal energy (cf. (4.12)) we obtain the lower semicontinuity of the left hand side of the energy balance (5.23).

Let us now study the limit with respect to  $\varepsilon$  of the right hand side of (5.23). Since for every  $t \in [0, T] \setminus S^\circ$  it holds that  $s_-^\circ(t) = \lim_{\varepsilon \rightarrow 0} s_\varepsilon^\circ(t)$  (see (5.16)), then

$$\alpha_\varepsilon(t) \rightharpoonup \alpha^\circ(s_-^\circ(t)) \quad \text{in } H^1(\Omega), \quad u_\varepsilon(t) \rightharpoonup u^\circ(s_-^\circ(t)) \quad \text{in } W^{1,p}(\Omega),$$

and

$$\int_0^T \langle \mu(\alpha^\circ(s_-^\circ(t))) \nabla u^\circ(s_-^\circ(t)), \nabla \dot{w}(t) \rangle_{L^2} dt = \lim_{\varepsilon \rightarrow 0} \int_0^T \langle \mu(\alpha_\varepsilon(t)) \nabla u_\varepsilon(t), \nabla \dot{w}(t) \rangle_{L^2} dt$$

by the Dominated Convergence Theorem. On the other hand, recalling (5.6),

$$\begin{aligned} \int_0^T \langle \mu(\alpha^\circ(s_-^\circ(t))) \nabla u^\circ(s_-^\circ(t)), \nabla \dot{w}(t) \rangle_{L^2} dt &= \int_0^S \langle \mu(\alpha^\circ(s_-^\circ(t^\circ(s)))) \nabla u^\circ(s_-^\circ(t^\circ(s))), \nabla \dot{w}(t^\circ(s)) \dot{t}^\circ(s) \rangle_{L^2} ds \\ &= \int_0^S \langle \mu(\alpha^\circ(s)) \nabla u^\circ(s), \nabla \dot{w}^\circ(s) \rangle_{L^2} ds, \end{aligned}$$

since  $\dot{t}^\circ(s) = 0$  for a.e.  $s \in U^\circ$  and  $s_-^\circ(t^\circ(s)) = s$  for a.e.  $s \in (0, S) \setminus U^\circ$ . Therefore the right hand side of (5.23) passes to the limit and we conclude the energy inequality

$$\begin{aligned} \mathcal{E}(\alpha^\circ(S), u^\circ(S)) - \int_0^S \langle \tilde{f}^\circ(s), \dot{\alpha}^\circ(s) \rangle ds + \int_0^S \|\dot{\alpha}^\circ(s)\|_{L^2} \Psi(\alpha^\circ(s), u^\circ(s), \tilde{f}^\circ(s)) ds \\ \leq \mathcal{E}(\alpha_0, u_0) + \int_0^S \langle \mu(\alpha^\circ(s)) \nabla u^\circ(s), \nabla \dot{w}^\circ(s) \rangle ds. \end{aligned}$$

To prove the converse inequality, we differentiate with respect to the time variable the energy (which is absolutely continuous, since  $\alpha^\circ$ ,  $u^\circ$  are Lipschitz). We obtain that for a.e.  $s \in (0, S)$

$$\begin{aligned} \frac{d}{ds} [\mathcal{E}(\alpha^\circ(s), u^\circ(s))] &= \langle \partial_\alpha \mathcal{E}(\alpha^\circ(s), u^\circ(s)), \dot{\alpha}^\circ(s) \rangle + \langle \partial_u \mathcal{E}(\alpha^\circ(s), u^\circ(s)), \dot{u}^\circ(s) \rangle \\ &= \langle \partial_\alpha \mathcal{E}(\alpha^\circ(s), u^\circ(s)), \dot{\alpha}^\circ(s) \rangle + \langle \mu(\alpha^\circ(s)) \nabla u^\circ(s), \nabla \dot{u}^\circ(s) \rangle_{L^2}, \\ &\geq \langle \tilde{f}^\circ(s), \dot{\alpha}^\circ(s) \rangle - \|\dot{\alpha}^\circ(s)\|_{L^2} \Psi(\alpha^\circ(s), u^\circ(s), \tilde{f}^\circ(s)) + \langle \mu(\alpha^\circ(s)) \nabla u^\circ(s), \nabla \dot{u}^\circ(s) \rangle_{L^2}, \end{aligned} \quad (5.26)$$

employing (ev1), evaluated in  $\beta = \dot{\alpha}^\circ(s)$  (which is in  $H^1(\Omega)$  for a.e.  $s$ ), in the second equality and (5.12) in the inequality above. We deduce the energy balance (ev3) by integrating (5.26) in  $(0, S)$ . As a byproduct, we also obtain that (5.24) and (5.25) hold true as limits as  $\varepsilon \rightarrow 0$ .  $\square$

We conclude by showing some properties of an evolution  $(t^\circ, \alpha^\circ, u^\circ)$ , obtained as limit of rescaled  $\varepsilon$ -approximate viscous evolution, in the spirit of e.g. [15, 24, 11]. We are in particular interested in its description in the time subset  $U^\circ \subset [0, S]$ , where it is not rate independent: if  $\alpha^\circ$  remains constant in  $(s_1, s_2) \subset U^\circ$ , then all the evolution is trivial in  $(s_1, s_2)$  (Remark 5.6); on the other hand, if  $\dot{\alpha}^\circ > 0$  in space in a time interval, up to a further time reparametrisation we have that the system is governed by an equation satisfied in the transition between the initial and final configurations: this equation (see (5.27) and (5.28) in Proposition 5.7) corresponds formally to consider 1 as viscosity parameter in (4.42) in Lemma 4.12, governing the  $\varepsilon$ -approximate viscous evolutions.

*Remark 5.6.* If  $\dot{\alpha}^\circ(s) = 0$  for every  $s \in (s_1, s_2) \subset U^\circ$ , then  $\dot{u}^\circ(s) = 0$  for every  $s \in (s_1, s_2) \subset U^\circ$ . Indeed, by definition of  $U^\circ$ , it follows that  $t^\circ(s) = t^\circ(s_1)$  and  $w^\circ(s) = w^\circ(s_1)$  for every  $s \in (s_1, s_2)$ , and then  $u^\circ(s) = u^\circ(s_1)$ , the unique solution of

$$\min_{u=w^\circ(s_1) \text{ on } \partial_D \Omega} \int_{\Omega} \mu(\alpha^\circ(s_1)) |\nabla u|^2 dx,$$

by (ev1) in Theorem 5.1.

**Proposition 5.7.** *Let  $(s_1, s_2) \subset A^\circ$  (with  $A^\circ$  defined in (5.19)) containing no subintervals where  $\dot{\alpha}^\circ(s) = 0$  in  $\Omega$  for a.e.  $s$ , and let for every  $s \in (s_1, s_2)$*

$$\varrho(s) := \int_{\frac{s_1+s_2}{2}}^s \frac{\|\dot{\alpha}^\circ(\sigma)\|_{L^2}}{\Psi(\alpha^\circ(\sigma), u^\circ(\sigma), \tilde{f}^\circ(\sigma))} d\sigma.$$

*Then  $\varrho$  is locally Lipschitz and strictly increasing, the function*

$$\alpha^\sharp(r) := \alpha^\circ(\varrho^{-1}(r)) \quad \text{for } r \in \varrho((s_1, s_2))$$

*has bounded variation and is continuous into  $H^1(\Omega)$ , and*

$$\|\dot{\alpha}^\circ(\varrho^{-1}(r))\|_{L^2}^2 \left[ \langle \partial_\alpha \mathcal{E}(\alpha^\sharp(r), u^\sharp(r)) - \tilde{f}^\sharp(r), \dot{\alpha}^\sharp(r) \rangle + \|\dot{\alpha}^\sharp(r)\|_{L^2}^2 \right] = 0, \quad (5.27)$$

*for every  $r \in \varrho^{-1}((s_1, s_2))$ , where  $u^\sharp(r) := u^\circ(\varrho^{-1}(r))$ ,  $\tilde{f}^\sharp(r) := \tilde{f}^\circ(\varrho^{-1}(r))$ .*

*If, moreover,  $\dot{\alpha}^\circ(s)$  is not 0 for every  $s \in (s_1, s_2)$  and  $\|\dot{\alpha}^\circ(s)\|_{L^2} > \delta_K$  for every  $K \Subset (s_1, s_2)$ , then  $\varrho$  is locally bi-Lipschitz,  $\alpha^\sharp$  is locally Lipschitz, and*

$$\langle \partial_\alpha \mathcal{E}(\alpha^\sharp(r), u^\sharp(r)) - \tilde{f}^\sharp(r), \dot{\alpha}^\sharp(r) \rangle + \|\dot{\alpha}^\sharp(r)\|_{L^2}^2 = 0. \quad (5.28)$$

*Proof.* By (5.19) and (5.21), for any  $K \Subset A^\circ$  we get that  $\Psi(\alpha^\circ(\sigma), u^\circ(\sigma), \tilde{f}^\circ(\sigma)) \geq \delta_K > 0$  for  $\sigma \in K$ . Then  $\varrho$  is locally Lipschitz on  $(s_1, s_2)$ , and it is strictly increasing by the assumptions that in no subintervals of  $(s_1, s_2)$

we have  $\dot{\alpha}^\circ(s) = 0$  in  $\Omega$ . This gives also  $\varrho^{-1}$  continuous and strictly increasing, so that  $\alpha^\sharp$  is continuous and with bounded variation, since  $\alpha^\circ$  is Lipschitz.

The change of variables  $s = \varrho^{-1}(r)$  in (5.11) gives

$$\langle \partial_\alpha \mathcal{E}(\alpha^\sharp(r), u^\sharp(r)) - \tilde{f}^\sharp(r), \dot{\alpha}^\circ(\varrho^{-1}(r)) \rangle + \Psi(\alpha^\sharp(r), u^\sharp(r), \tilde{f}^\sharp(r)) \|\dot{\alpha}^\circ(\varrho^{-1}(r))\|_{L^2} = 0,$$

that is

$$\|\dot{\alpha}^\circ(\varrho^{-1}(r))\|_{L^2} \langle \partial_\alpha \mathcal{E}(\alpha^\sharp(r), u^\sharp(r)) - \tilde{f}^\sharp(r), \dot{\alpha}^\circ(\varrho^{-1}(r)) \rangle + \Psi(\alpha^\sharp(r), u^\sharp(r), \tilde{f}^\sharp(r)) \|\dot{\alpha}^\circ(\varrho^{-1}(r))\|_{L^2}^2 = 0, \quad (5.29)$$

for every  $r \in \varrho^{-1}((s_1, s_2))$  and every  $\beta \in H^1_-(\Omega)$ . Now  $\alpha^\sharp$  is weakly differentiable in  $H^1(\Omega)$  at a.e.  $r \in \varrho^{-1}((s_1, s_2))$ , and we have the chain rule

$$\dot{\alpha}^\sharp(r) = \dot{\alpha}^\circ(\varrho^{-1}(r)) \frac{d}{dt} \varrho^{-1}(r) = \dot{\alpha}^\circ(\varrho^{-1}(r)) \frac{\Psi(\alpha^\sharp(r), u^\sharp(r), \tilde{f}^\sharp(r))}{\|\dot{\alpha}^\circ(\varrho^{-1}(r))\|_{L^2}} \quad \text{a.e. in } \Omega \quad (5.30)$$

for a.e.  $r$  such that  $\|\dot{\alpha}^\circ(\varrho^{-1}(r))\|_{L^2} > 0$ . Then (5.29) and (5.30) imply

$$\|\dot{\alpha}^\circ(\varrho^{-1}(r))\|_{L^2} \left[ \langle \partial_\alpha \mathcal{E}(\alpha^\sharp(r), u^\sharp(r)) - \tilde{f}^\sharp(r), \dot{\alpha}^\circ(\varrho^{-1}(r)) \rangle + \langle \dot{\alpha}^\sharp(r), \dot{\alpha}^\circ(\varrho^{-1}(r)) \rangle_{L^2} \right] = 0,$$

Recalling that  $\Psi(\alpha^\sharp(r), u^\sharp(r), \tilde{f}^\sharp(r)) > 0$  for every  $r \in \varrho^{-1}(s_1, s_2)$ , the two previous inequalities imply (5.27). At this stage, (5.28) follows easily since  $\|\dot{\alpha}^\circ(\varrho^{-1}(r))\|_{L^2}^2 > 0$  for every  $r \in \varrho^{-1}((s_1, s_2))$ .  $\square$

As usual in an analysis based on time rescaling, one could see that in the original, *faster*, time variable  $t \in [0, T]$ , the evolution is rate-independent outside an at most countable number of jump times, which is a subset of  $S^\circ$  introduced in (5.17). In order to describe the evolution of the system during these jump, one has to employ the description given by Remark 5.6 and Proposition 5.7. Here we do not perform directly this analysis, based on inverse rescaling in time, since it would be very similar to that in e.g. [16, Section 5] and [11, Proposition 6.7], to which we refer the interested reader.

#### A. AUXILIARY RESULTS

**The essential variation.** In this appendix  $X$  denotes a measure space. We do not label the measure on  $X$  and the notions of  $L^p$  space and of a.e.-equivalence refer to the measure on  $X$ . Moreover we fix  $n \geq 1$ .

We define here the notion of essential variation, namely the variation for a time-dependent family of measurable functions, in the sense of a.e. inequality.

**Definition A.1.** Let us consider a function  $t \mapsto \zeta(t)$ , with  $\zeta(t): X \rightarrow \mathbb{R}^n$ . Let  $0 \leq s \leq t \leq T$ . The *essential variation* of  $\zeta$  in the interval  $[s, t]$  is the function  $\text{ess Var}(\zeta; s, t): X \rightarrow [0, +\infty]$  defined by

$$\text{ess Var}(\zeta; s, t) := \text{ess sup}_{s=s_0 < \dots < s_m = T} \left\{ \sum_{j=0}^m |\zeta(s_j) - \zeta(s_{j-1})| \right\},$$

where the essential supremum is taken over all partitions  $0 = s_0 < \dots < s_m = t$ ,  $m \in \mathbb{N}$ .

*Remark A.2.* For every  $t_1 \leq t_2 \leq t_3$  we have

$$\text{ess Var}(\zeta; t_1, t_3) = \text{ess Var}(\zeta; t_1, t_2) + \text{ess Var}(\zeta; t_2, t_3) \quad \text{a.e. in } X.$$

For completeness, we recall here the definition of the essential supremum of a family of measurable functions.

**Definition A.3.** Let  $(v_i)_{i \in I}$  be a family of measurable functions from  $X$  to  $[-\infty, \infty]$ . Let  $\bar{v}: X \rightarrow [-\infty, \infty]$  be a measurable function such that

- (i)  $\bar{v} \geq v_i$  a.e. in  $X$ , for every  $i \in I$ ;
- (ii) if  $v: X \rightarrow [-\infty, \infty]$  is a measurable function such that  $v \geq v_i$  a.e. in  $X$ , for every  $i \in I$ , then  $v \geq \bar{v}$  a.e. in  $X$ .

The functions  $\bar{v}$  is called an *essential supremum* of the family  $(v_i)_{i \in I}$ . In fact, there exists a unique (up to a.e. equivalence) essential supremum  $\bar{v}$  of the family  $(v_i)_{i \in I}$ . We denote it by  $\operatorname{ess\,sup}_{i \in I} v_i := \bar{v}$ .

In the next proposition we provide an explicit formula for the essential variation of a function  $\zeta$  that is absolutely continuous in time. A quick survey about the notion and the main properties of the Bochner integral can be found in the appendix of [8]; for a more detailed treatment of the subject we refer to [19].

**Proposition A.4.** Let  $p \in [1, \infty)$ , and let  $\zeta \in AC([0, T]; L^p(X; \mathbb{R}^n))$ . Then

$$\operatorname{ess\,Var}(\zeta; 0, t)(x) = \int_0^t |\dot{\zeta}(r; x)| \, dr, \quad \text{for a.e. } x \in X,$$

where the integral in the right-hand side is a Bochner integral in  $L^p(X)$ .

*Proof.* We start by claiming that

$$\operatorname{ess\,Var}(\zeta; 0, \cdot) \in AC([0, T]; L^p(X)) \quad \text{and} \quad \frac{d}{dt} \operatorname{ess\,Var}(\zeta; 0, t) = |\dot{\zeta}(t)| \text{ in } L^p(X). \quad (\text{A.1})$$

To prove the claim, let us fix  $s \leq t$  and a partition  $s = s_0 < \dots < s_m = t$ . By the absolute continuity of  $\zeta$  we obtain that

$$\sum_{j=1}^m |\zeta(s_j) - \zeta(s_{j-1})| = \sum_{j=1}^m \left| \int_{s_{j-1}}^{s_j} \dot{\zeta}(r) \, dr \right| \leq \sum_{j=1}^m \int_{s_{j-1}}^{s_j} |\dot{\zeta}(r)| \, dr = \int_s^t |\dot{\zeta}(r)| \, dr \quad (\text{A.2})$$

a.e. in  $X$ , where the last integral is a Bochner integral in  $L^p(X)$ . Note that the second inequality in (A.2) can be proven, e.g., with an approximation argument via step functions. Taking the essential supremum in (A.2), by Remark A.2 we deduce that

$$\operatorname{ess\,Var}(\zeta; 0, t) - \operatorname{ess\,Var}(\zeta; 0, s) \leq \int_s^t |\dot{\zeta}(r)| \, dr \quad \text{a.e. in } X. \quad (\text{A.3})$$

Inequality (A.3) computed for  $s = 0$  yields, in particular, that  $\operatorname{ess\,Var}(\zeta; 0, t) \in L^p(X)$  for every  $t \in [0, T]$ . Moreover, it shows that  $\operatorname{ess\,Var}(\zeta; 0, \cdot) \in AC([0, T]; L^p(X))$ . By (A.3) and by Lebesgue's Differentiation Theorem for vector-valued functions [19, p. 217] we get that

$$\frac{d}{dt} \operatorname{ess\,Var}(\zeta; 0, t) = \lim_{s \rightarrow t^-} \frac{\operatorname{ess\,Var}(\zeta; 0, t) - \operatorname{ess\,Var}(\zeta; 0, s)}{t - s} \leq |\dot{\zeta}(t)|$$

if  $t$  is a differentiability point for  $\operatorname{ess\,Var}(\zeta; 0, \cdot)$  and it is a Lebesgue point for  $|\dot{\zeta}|$ , the limit being taken with respect to the  $L^p$ -norm.

On the other hand,  $s < t$  is a particular partition of the interval  $[s, t]$ , therefore

$$\frac{|\zeta(t) - \zeta(s)|}{t - s} \leq \frac{\operatorname{ess\,Var}(\zeta; 0, t) - \operatorname{ess\,Var}(\zeta; 0, s)}{t - s} \quad \text{a.e. in } X.$$

Taking the limit as  $s \rightarrow t^-$  with respect to the  $L^p$ -norm of both sides, we obtain

$$|\dot{\zeta}(t)| \leq \frac{d}{dt} \operatorname{ess\,Var}(\zeta; 0, t) \quad \text{a.e. in } X,$$

if  $t$  is a differentiability point for  $\operatorname{ess\,Var}(\zeta; 0, \cdot)$  and  $\zeta$ . This proves that  $\frac{d}{dt} \operatorname{ess\,Var}(\zeta; 0, t) = |\dot{\zeta}(t)|$ .

Finally, since  $\text{ess Var}(\zeta; 0, \cdot) \in AC([0, T]; L^p(X))$ , we conclude that

$$\text{ess Var}(\zeta; 0, t) = \int_0^t \frac{d}{dt} \text{ess Var}(\zeta; 0, r) \, dr = \int_0^t |\dot{\zeta}(t)| \, dr \quad \text{a.e. in } X.$$

□

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