GLOBALLY STABLE QUASISTATIC EVOLUTION FOR A COUPLED ELASTOPLASTIC–DAMAGE MODEL

VITO CRISMALE

Abstract. We show the existence of globally stable quasistatic evolutions for a rate-independent material model with elastoplasticity and incomplete damage, in small strain assumptions. The main feature of our model is that the scalar internal variable which describes the damage affects both the elastic tensor and the plastic yield surface. It is also possible to require that the history of plastic strain up to the current state influences the future evolution of damage.

Keywords: variational models, quasistatic evolution, energetic solutions, elastoplasticity, damage models, incomplete damage, softening.
2010 MSC: 74C05, 74R05, 74G65, 35Q74, 49J45.

Contents

1. Introduction

In this paper we study the problem of quasistatic evolution for a material model with elastoplasticity and incomplete damage, in small strain assumptions. The damage is described by a scalar internal variable, which affects both the elastic tensor and the plastic yield surface.

Models for elastic materials where the bulk energy depends on a scalar damage variable were considered for instance in [18, 30, 28, 4, 38, 16, 37, 21, 22] (without plasticity). In contrast, in the elastoplastic setting of e.g. [8, 9, 10, 11], and of the Perzyna model for viscoplasticity with damage in [33, Section 1.4 and Chapter 9], the plastic dissipation is function of a scalar internal variable, but the elastic tensor is constant.

The present formulation accounts for both these dependences and takes inspiration from [1], where a variational model for complete damage was proposed and certain closed-form solutions were given in dimension one. (See also [2] for a numerical analysis in dimension two.) In particular, the material exhibits a softening behavior: as damage increases, the stiffness decreases and the plastic yield surface shrinks.

We prove the existence of globally stable quasistatic evolutions following the so-called energetic approach for rate-independent processes (cf. e.g. [25] and references therein), as common in the study of plasticity, damage, as well as fracture (see for instance [7, 8], [4, 38, 37], and [13, 14, 12], respectively).

Preprint SISSA 34/2014/MATE.

We now briefly describe the problem, formulated in a reference configuration $\Omega \subset \mathbb{R}^n$. The evolution is driven by a time-dependent loading, limited for simplicity to a hard device, namely by a (sufficiently smooth) displacement $w(t)$ acting on the whole boundary $\partial\Omega$ of the domain during a time interval $[0,T]$. To introduce the model, we start from the classical formulation of perfect plasticity (cf. [7]), and introduce the dependence of the elastic tensor $\mathbb{C}$ and of the constraint set K on the damage variable α. Assuming isotropic damage, the values of α range in the interval $[0,1]$; here $\alpha(t,x) = 1$ stands for no damage and $\alpha(t,x) = 0$ for maximal damage in a neighbourhood of a point $x \in \Omega$ at time t.

The linearized strain Eu, defined as the symmetric part of the spatial gradient of the displacement u, is decomposed as the sum of the elastic strain e and the plastic strain p. The plastic strain p belongs to $\mathbb{M}^{n\times n}_D$, the space of trace free $n \times n$ symmetric matrices, as usual for materials which are insensitive to pressure. The effective Cauchy stress σ is determined by the relation $\mathbb{C}(\alpha)e$, and its deviatoric part σ_D lies in the constraint set $K(\alpha)$; when σ_D reaches the boundary of $K(\alpha)$ plasticity can evolve, according to the classical Prandtl-Reuss flow rule.

In a strong formulation we then have for every $t \in [0,T]$ and $x \in \Omega$ the following:

(sf1) additive decomposition: $Eu(t,x) = e(t,x) + p(t,x)$,
(sf2) constitutive equation: $\sigma(t,x) = \mathbb{C}(\alpha(t,x))e(t,x)$,
(sf3) equilibrium: div $\sigma(t,x) = 0$,
(sf4) stress constraint: $\sigma_D(t,x) \in K(\alpha(t,x))$,
(sf5) associative flow rule: $\dot{p}(t,x) \in N_{K(\alpha(t,x))}(\sigma_D(t,x))$,

where $N_{K(\alpha(t,x))}(\xi)$ is the normal cone to $K(\alpha(t,x)) \subset \mathbb{M}^{n\times n}_D$ at $\xi \in \mathbb{M}^{n\times n}_D$, and for $t \in [0,T]$ and $x \in \partial\Omega$

(sf6) prescribed boundary displacement: $u(t,x) = w(t,x)$ for $x \in \partial\Omega$.

As a result of damage, the material has a softening behavior: the function $e \mapsto \mathbb{C}(\alpha)e\colon e$ is nondecreasing in α for every e and equicoercive (indeed only incomplete damage is considered here), and

$$B_r(0) \subset K(0) \subset K(\alpha_1) \subset K(\alpha_2) \subset K(1) \subset B_R(0)\,,$$

for every $\alpha_1 \le \alpha_2$, with $0 < r < R$. Moreover we assume that $\alpha \mapsto \sup_{\sigma\in K(\alpha)} \sigma\colon \xi$ is continuous for every $\xi \in \mathbb{M}^{n\times n}_D$.

To present the equations governing the damage evolution, we introduce the mechanical energy $\mathcal{E}$. As in [29], this includes a continuous functional D and a regularizing gradient term, both depending only on α, and a quadratic form $\mathcal{Q} = \frac{1}{2}\int_\Omega \mathbb{C}(\alpha)e\colon e\,\mathrm{d}x$ of the elastic strain corresponding to the stored elastic energy. For a technical reason (see Lemma 2.3), it is not sufficient to take a regularizing term of the type $\|\nabla\alpha\|^2_{L^2}$ as done in [1] (where the setting is one dimensional). Thus, we choose $\|\nabla\alpha\|^\gamma_{L^\gamma}$ with $\gamma > n$, a regularization present also in [28] and more recently in [22], for example. Hence,

$$\mathcal{E}(\alpha,e) := \mathcal{Q}(\alpha,e) + D(\alpha) + \|\nabla\alpha\|^\gamma_{L^\gamma}\,.$$

We consider the damage as a unidirectional process whose evolution is governed by a threshold criterion of Kuhn Tucker-type (see e.g. [29], [30], [1], [2] and [33, Problem 9.2]). This gives in a strong formulation that for every t

(sf7) irreversibility: $\dot{\alpha}(t) \le 0$ in Ω,
(sf8) Kuhn Tucker condition: $\langle \partial_\alpha\mathcal{E}(\alpha(t),e(t)), \beta - \dot{\alpha}(t)\rangle \ge 0$ for every $\beta \in W^{1,\gamma}(\Omega)$, $\beta \le 0$,

where $\partial_\alpha\mathcal{E}$ is the Gâteaux derivative of $\mathcal{E}$ with respect to α.

Let us now introduce the plastic dissipation. The properties of $\alpha \mapsto K(\alpha)$ imply that the function $H\colon \mathbb{R}\times\mathbb{M}^{n\times n}_D \to \mathbb{R}^+ \cup \{0\}$ defined by

$$H(\alpha,\xi) := \sup_{\sigma\in K(\alpha)} \sigma\colon \xi$$

(i.e., $\xi \mapsto H(\alpha,\xi)$ is the support function of $K(\alpha)$), is convex and positively one-homogeneous in ξ, and continuous. In some cases, we require also that $\xi \mapsto H(\alpha_2,\xi) - H(\alpha_1,\xi)$ is convex for every $\alpha_1 \le \alpha_2$. As a particular case, we can choose a multiplicative definition for $K(\alpha)$ (see Remark 2.1).

In the energetic formulation of problems in linearized elastoplasticity without hardening it is natural to assume that p belongs to $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$, the space of $\mathbb{M}^{n\times n}_D$-valued Borel measures on $\overline{\Omega}$, since the plastic dissipation is considered one-homogeneous in p. Then, in accordance to the theory of convex functions of measures developed in [19], we define the plastic potential as

$$\mathcal{H}(\alpha,p) := \int_{\overline{\Omega}} H\Big(\alpha(x), \frac{\mathrm{d}p}{\mathrm{d}\mu}(x)\Big)\,\mathrm{d}\mu(x)\,,$$

where $\mu \in M_b(\overline{\Omega})^+$ is any measure such that $p \ll \mu$. In particular it is convex, positively one-homogeneous, and weakly* lower semicontinuous with respect to $p \in M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$, since $\alpha \in W^{1,\gamma}(\Omega) \subset C(\overline{\Omega})$.

Given two evolutions of damage and plastic strain $t \mapsto \alpha(t)$ and $t \mapsto p(t)$, the plastic dissipation is then defined as

$$\mathcal{V}_{\mathcal{H}}(\alpha,p;s,t) := \sup_{\mathcal{P}} \mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha,p;s,t), \quad \mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha,p;s,t) := \sum_{j=1}^{N} \mathcal{H}(\alpha(t_j), p(t_j)-p(t_{j-1}))\,,$$

where $\mathcal{P} = \{t_j\}_{j=0}^N$ is a partition of $[s,t]$, and the supremum is taken over the partitions. Notice that for each subinterval $[t_{i-1},t_i]$ one considers $\alpha(t_i)$ as a "weight" for this sort of variation. When p is sufficiently smooth in time and $t \mapsto \alpha(x,t)$ is nonincreasing, we can say (see the abstract result Lemma A.1 in the Appendix) that

$$\mathcal{V}_{\mathcal{H}}(\alpha,p;s,t) = \int_s^t \mathcal{H}(\alpha(\tau),\dot p(\tau))\,\mathrm{d}\tau\,. \tag{1.1}$$

To ease the reading, when $\alpha(t) = \overline{\alpha} \in C(\overline{\Omega};[0,1])$ for every t we use the symbol $\widehat{\mathcal{V}}_H$ instead of $\mathcal{V}_H$, so that

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\overline{\alpha},p;s,t) := \mathcal{V}_{\mathcal{H}}(\alpha,p;s,t)|_{\alpha(t)=\overline{\alpha}}\,.$$

If we consider the particular case when $K(\alpha) = V(\alpha)B(1)$, with $B(1)$ the unit ball of $\mathbb{M}^{n\times n}_D$, under regularity assumptions on p we get

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\overline{\alpha},p;0,t) = \int_\Omega V(\overline{\alpha}(x))\Big(\int_0^t |\dot p(s,x)|\,\mathrm{d}s\Big)\,\mathrm{d}x\,. \tag{1.2}$$

This term, which depends on the cumulated plastic strain $x \mapsto \int_0^t |\dot p(s,x)|\,\mathrm{d}s$, appears in the total mechanical energy in [1]. Therefore we define a more general total energy

$$\mathcal{E}_\lambda(\alpha,e;p,t) := \mathcal{E}(\alpha,e) + \lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha,p;0,t)\,,$$

with $\lambda \in [0,1]$ a parameter of the model. The Kuhn-Tucker condition for the damage becomes

(sf8)' $\langle \partial_\alpha \mathcal{E}_\lambda(\alpha(t),e(t);p,t), \beta - \dot\alpha(t)\rangle \ge 0$ for every $\beta \in W^{1,\gamma}(\Omega)$, $\beta \le 0$.

Notice that when λ is zero this reduces to (sf8). The presence of the term $\lambda\widehat{V}_{\mathcal{H}}$ in the energy is related to a fatigue phenomenon, see below in this introduction for a short discussion.

According to the general theory for energetic solutions, a quasistatic evolution corresponding to our choice of energy and dissipation is a function $t \mapsto (\alpha(t),u(t),e(t),p(t))$ characterized by the following conditions:

(qs0) *irreversibility* : for every $x \in \Omega$

$$t \in [0,T] \mapsto \alpha(t,x) \quad \text{is nonincreasing}\,;$$

(qs1) *global stability*: the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ has bounded variation, $(u(t),e(t),p(t)) \in A(w(t))$ for every $t \in [0,T]$, and

$$\mathcal{E}(\alpha(t),e(t)) + \lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha(t),p;0,t) \le \mathcal{E}(\beta,\eta) + \lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\beta,p;0,t) + \mathcal{H}(\beta,q-p(t))$$

for every $\beta \le \alpha(t)$ and $(v,\eta,q) \in A(w(t))$, where

$$A(w(t)) := \{(u,e,p) | Eu = e+p \text{ in } \Omega,\ p = (w(t)-u)\odot\nu\,\mathcal{H}^{n-1} \text{ on } \partial\Omega\}$$

is the set of admissible displacements with respect to the boundary datum $w(t)$;

(qs2) *energy balance*: for every $t \in [0,T]$

$$\mathcal{E}(\alpha(t),e(t)) + \lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha(t),p;0,t) + (1-\lambda)\mathcal{V}_{\mathcal{H}}(\alpha,p;0,t) = \mathcal{E}(\alpha(0),e(0)) + \int_0^t \langle \sigma(s), E\dot{w}(s)\rangle \,\mathrm{d}s\,,$$

where $\sigma(s) := \mathbb{C}(\alpha(s))e(s)$.

The last term in (qs2) is the work of the external loading due to the prescribed boundary condition. Notice that if the damage variable is constant in time we obtain for every λ an evolution for nonhomogeneous plastic materials, with yield surface continuous with respect to x (see [34]).

We point out here that the case of positive λ accounts for a fatigue phenomenon. Indeed, when one minimizes the energy with respect to β at a given time t, it is easier to damage portions of the material where the cumulated plastic strain is larger (recall (1.2) and the fact that V is nondecreasing), i.e. parts more affected by plastic evolution until t. Since the condition (qs1) requires that the current configuration is a minimum for the total energy plus the plastic dissipation among all the admissible states at the time t, we see that the damage process is affected by the history of the plastic strain up to t, and this is reflected also in (sf8'). Tuning between zero and one the parameter λ one can choose, possibly led by phenomenological considerations, how strong the interplay between damage growth and cumulated plastic strain is.

In Theorem 4.3 we prove an existence result for quasistatic evolutions. The proof is based on time discretization and on approximation by means of solutions to incremental minimum problems, following a method common in the study of quasistatic evolutions. As a technical note, we remark just that the monotonicity in time of α and the softening property of $\mathcal{H}$ allow us to prove that $\mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha,p;0,t)$ is indeed nondecreasing with respect to refinements of the partition $\mathcal{P}$ of $[0,t]$ (Lemma A.1). This is crucial in order to pass to the limit in the energy balance, see (4.19), and to recover (1.1) when p is more regular in time.

The global stability implies conditions (sf1)–(sf4) for every t and a.e. x, while the boundary datum is attained in a weak sense since a plastic slip can develop at the boundary. Assuming more regularity on the constitutive coefficients and on the evolution (which is strongly continuous except at most for countable many instants, see Proposition 5.1), one can derive (sf5), (sf7), (sf8'), cf. Propositions 5.4 and 5.6.

Let us conclude this presentation by some comments about the energetic formulation. It is known that the request of global stability may lead to unphysical jumps in time of the evolution, which may overtake energy barriers (see, for instance, [24, Ex. 6.1]); however, the description of the process is meaningful at least up to the first jump time. In order to overcome such a drawback, one may adopt the so-called vanishing viscosity technique (we refer the reader to e.g. [26, 27] for an abstract treatment and to [9], [10], [21], [22] for some applications): rate-independent evolutions are seen as limits of solutions to some rate-dependent systems containing a viscous dissipation that tends to zero. This viscous evolutions are regular in time, see in our context the Perzyna's law for viscoplasticity with damage considered in [33, Section 1.4 and Chapter 9], for instance. The forthcoming paper [6] adopts for the present problem a vanishing viscosity approach in the spirit of [21]. We remark that in [6] a different damage regularization and stronger regularity assumptions on the initial data and the constitutive coefficients are employed.

The structure of the paper is the following: in Section 2 we set the notation, describe the assumptions of the model, and introduce the energy and the dissipation terms with their main properties; Section 3 includes the results needed in order to solve the incremental problems and to assure convergence of the stability properties in the continuous time limit; Section 4 is devoted to prove the existence result; in Section 5 we show qualitative properties of the evolution. Finally, in the Appendix we analyse the particular variation used to define the plastic dissipation and show a property of increasing functions with values in L^p spaces, employed to prove more regularity for the evolution.

2. Preliminaries

Mathematical preliminaries. The Lebesgue measure on $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$ and the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. The space of bounded X-valued Radon measures on B is denoted by $M_b(B;X)$, for a locally compact subset B of $\mathbb{R}^n$ and a finite dimensional Hilbert space X. The indication of the space X is omitted when $X=\mathbb{R}$. The space $M_b(B;X)$ is endowed with the norm $\|\mu\|_1 := |\mu|(B)$, where $|\mu| \in M_b(B)$ is the variation of the measure μ, and it is identified with the dual of $C_0(B;X)$, the space of continuous functions $\varphi\colon B \to X$ such that $\{|\varphi| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$, by the Riesz Representation Theorem (see, e.g., [32, Theorem 6.19]). The weak* topology of $M_b(B;X)$ is defined by the duality.

The space $L^1(B;X)$ of X-valued $\mathcal{L}^n$-integrable functions is regarded as a subspace of $M_b(B;X)$, with the induced norm. The L^p norm, $1 \leq p \leq \infty$ is denoted by $\|\cdot\|_p$, while the brackets $\langle\cdot,\cdot\rangle$ denote the duality product between conjugate L^p spaces.

The space of *symmetric $n\times n$ matrices* is denoted by $\mathbb{M}^{n\times n}_{sym}$; it is endowed with the Euclidean scalar product $\xi\colon\eta := \sum_{ij}\xi_{ij}\eta_{ij}$ and with the corresponding Euclidean norm $|\xi| := (\xi\colon\xi)^{1/2}$. The symbol for the space of trace free matrices in $\mathbb{M}^{n\times n}_{sym}$ is $\mathbb{M}^{n\times n}_{D}$. For every $\xi \in \mathbb{M}^{n\times n}_{sym}$ the orthogonal projection of ξ on $\mathbb{R}I$ is $\frac{1}{n}\mathrm{tr}\,(\xi)I$. Therefore the orthogonal projection on $\mathbb{M}^{n\times n}_{D}$, called the *deviator* of ξ, is

$$\xi_D := \xi - \frac{1}{n}(\mathrm{tr}\,\xi)I\,.$$

The *symmetrized tensor product* $a \odot b$ of two vectors $a, b \in \mathbb{R}^n$ is the symmetric matrix with entries $(a_ib_j + a_jb_i)/2$. If X_1, X_2 are Banach spaces, $Lin(X_1;X_2)$ is the space of linear operators from X_1 into X_2, endowed with the usual operator norm.

For every $u \in L^1(U;\mathbb{R}^n)$, with U open in $\mathbb{R}^n$, let Eu be the $\mathbb{M}^{n\times n}_{sym}$-valued distribution on U whose components are defined by $E_{ij}u = \frac{1}{2}(D_ju_i + D_iu_j)$. The space $BD(U)$ of functions with *bounded deformation* is the space of all $u \in L^1(U;\mathbb{R}^n)$ such that $Eu \in M_b(U;\mathbb{M}^{n\times n}_{sym})$. It is easy to see that $BD(U)$ is a Banach space with the norm $\|u\|_1 + \|Eu\|_1$. It is possible to prove that $BD(U)$ is the dual of a normed space (see [36] and [23]), and this defines the weak* topology of $BD(U)$. A sequence u_k converges to u weakly* in $BD(U)$ if and only if $u_k \to u$ strongly in $L^1(U;\mathbb{R}^n)$ and $Eu_k \rightharpoonup Eu$ weakly* in $M_b(U;\mathbb{M}^{n\times n}_{sym})$. If U is a bounded open set with Lipschitz boundary, for every function $u \in BD(U)$ the trace of u on ∂U belongs to $L^1(\partial U;\mathbb{R}^n)$. It will always be denoted by the same symbol u. If u_k, $u \in BD(U)$, $u_k \to u$ strongly in $L^1(U;\mathbb{R}^n)$, and $\|Eu_k\|_1 \to \|Eu\|_1$, then $u_k \to u$ strongly in $L^1(\partial U;\mathbb{R}^n)$ (see [35, Chapter II, Theorem 3.1]). Moreover (see [35, Proposition 2.4 and Remark 2.5]), there exists a constant $C > 0$, depending on U, such that

$$\|u\|_{1,U} \leq C\,\|u\|_{1,\partial U} + C\,\|Eu\|_{1,U}\,, \tag{2.1}$$

$\|\cdot\|_{p,B}$ being the L^p norm of a function with domain a Borel set B. For the general properties of $BD(U)$ we refer to [35].

The reference configuration. Throughout the paper the *reference configuration* Ω is a *bounded connected open set* in $\mathbb{R}^n$, $n \geq 2$, with *Lipschitz boundary*. On $\partial\Omega$ we shall prescribe only Dirichlet boundary conditions, to simplify the presentation. This will be done by assigning a function $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, whose trace on $\partial\Omega$ (again denoted by w) is the prescribed boundary value. This choice is motivated by the fact that we do not want to impose "discontinuous" boundary data, so that, if the displacement develops sharp discontinuities, this is a result of energy minimization.

In our problem $u \in BD(\Omega)$ represents the *displacement* of an elasto-plastic body and Eu is the corresponding linearized *strain*. We now introduce the coupled elastoplastic damage model. As for modeling plasticity, we follow [7] and use the corrisponding notations.

The elastic and plastic strains. Given a displacement $u \in BD(\Omega)$ and a boundary datum $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, the *elastic* and *plastic strains* $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $p \in M_b(\overline{\Omega};\mathbb{M}^{n\times n}_{D})$ satisfy

the equations (weak kinematic compatibility conditions)

$$Eu = e + p \quad \text{in } \Omega\,, \tag{2.2a}$$

$$p = (w - u) \odot \nu\, \mathcal{H}^{n-1} \quad \text{on } \partial\Omega\,. \tag{2.2b}$$

Given $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, the *set of admissible displacements and strains* for the boundary datum w on $\partial\Omega$ is defined, with the same meaning and notation of [7], as

$$A(w) := \{(u,e,p) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D) |\, (2.2) \text{ hold}\}\,.$$

We shall also use *the space of admissible plastic strains*

$$\Pi(\Omega) := \{p \in M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D) |\, \exists\, (u,w,e) \in BD(\Omega) \times H^1(\mathbb{R}^n;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \text{ s.t. } (u,e,p) \in A(w)\}\,.$$

The damage variable and the associated dissipation. In addition to u, e, and p, we consider an internal variable $\alpha\colon \Omega \to \mathbb{R}$, which represents the damage state of the body. Actually this variable will take values in $[0,1]$ during the evolution. At a given point $x \in \Omega$, as $\alpha(x)$ decreases from 1 to 0, the material point x passes from a sound state to a fully damaged one.

For technical reasons we will introduce in the total energy a regularizing term $\|\nabla\alpha\|^\gamma_\gamma$, with $\gamma > n$, on the damage variable, cf. (2.7). In particular, whenever the enegy is finite the damage variable will be in $W^{1,\gamma}(\Omega) \subset C(\overline{\Omega})$. (Recall that this embedding is compact.) Therefore, in the following we define the other energy terms for $\alpha \in C(\overline{\Omega})$. Notice that the quantities depending on the damage variable are defined also for negative α, in order to consider variations with respect to α in the proof of Proposition 5.4.

We shall denote the *admissible damage states from a given* α_0 by

$$\mathcal{D}(\alpha_0) := \{\alpha \in C(\overline{\Omega}) |\, \alpha \le \alpha_0 \text{ in } \overline{\Omega}\}\,.$$

We assume that the damage process is irreversible, i.e., if α_0 is the current damage state, then the future damage states are in $\mathcal{D}(\alpha_0)$. Let us remark that

$$\mathcal{D}(\alpha_2) \subset \mathcal{D}(\alpha_1) \text{ for every } \alpha_2 \in \mathcal{D}(\alpha_1)\,. \tag{2.3}$$

As in [29], in the total energy we consider a term which accounts for the energy dissipated by the body during the damage process. Then we define for every $\alpha \in C(\Omega)$

$$D(\alpha) := \int_\Omega d(\alpha(x))\,\mathrm{d}x, \text{ with } d \in C(\mathbb{R};\mathbb{R}^+ \cup \{0\}),\ d(s) > d(0) \text{ for } s < 0\,. \tag{2.4}$$

The stored elastic energy. For every $(\alpha,e) \in C(\overline{\Omega}) \times L^2(\Omega,\mathbb{M}^{n\times n}_{sym})$, the *stored elastic energy* is given by

$$\mathcal{Q}(\alpha,e) := \frac{1}{2}\int_\Omega \mathbb{C}(\alpha(x))e(x)\colon e(x)dx = \frac{1}{2}\langle \mathbb{C}(\alpha)e, e\rangle_{L^2(\Omega;\mathbb{M}^{n\times n}_{sym})}\,.$$

We assume the following properties for the dependence of $\mathbb{C}$ on the damage variable:

$$\mathbb{C}\colon \mathbb{R} \to Lin(\mathbb{M}^{n\times n}_{sym};\mathbb{M}^{n\times n}_{sym}) \text{ is Lipschitz and } \mathbb{C}(\mathbb{R}^-) = \{\mathbb{C}(0)\}\,, \tag{2.5a}$$

$$\alpha \mapsto \mathbb{C}(\alpha)\xi\colon \xi \quad \text{is nondecreasing for every } \xi \in \mathbb{M}^{n\times n}_{sym}\,, \tag{2.5b}$$

$$\mathbb{C}(\alpha)\xi := \mathbb{C}_D(\alpha)\xi_D + \kappa(\alpha)(\operatorname{tr}\xi)I \text{ with } \mathbb{C}_D \in L^\infty(\mathbb{R};Sym(\mathbb{M}^{n\times n}_D;\mathbb{M}^{n\times n}_D)),\ \kappa \in L^\infty(\mathbb{R}), \tag{2.5c}$$

$$\gamma_1|\xi|^2 \le \mathbb{C}(\alpha)\xi\colon \xi \le \gamma_2|\xi|^2 \quad \text{for every } \alpha \in \mathbb{R}, \xi \in \mathbb{M}^{n\times n}_{sym}\,, \tag{2.5d}$$

where γ_1, γ_2 are positive constants independent of α and $Sym(\mathbb{M}^{n\times n}_D;\mathbb{M}^{n\times n}_D)$ is the set of symmetric endomorphisms on $\mathbb{M}^{n\times n}_D$. In particular, this implies

$$|\mathbb{C}(\alpha)\xi| \le 2\gamma_2|\xi|\,. \tag{2.6}$$

Assumption (2.5b) is reasonable since in applications the stiffness decreases as the material passes from the sound to the fully damaged state.

It is well known that for a given $\alpha \in C(\overline{\Omega})$ the function $e \mapsto \mathcal{Q}(\alpha,e)$ is weakly lower semicontinuous on $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$.

The total energy. We are now in a position to define the *total energy* of the body corresponding to a damage state α and an elastic strain e.

We set

$$\mathcal{E}(\alpha, e) := \mathcal{Q}(\alpha, e) + D(\alpha) + \|\nabla\alpha\|_\gamma^\gamma \,. \tag{2.7}$$

It is easy to see that $\mathcal{E}$ is lower semicontinuous with respect to the uniform convergence of α_k and the weak*-$L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ convergence of e_k, i.e.,

$$\mathcal{E}(\alpha, e) \le \liminf_{k\to\infty} \mathcal{E}(\alpha_k, e_k) \tag{2.8}$$

for every sequences α_k and e_k converging to α uniformly in $\overline{\Omega}$ and to e weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, respectively.

The constraint sets and their support functions. Let $(K(\alpha))_{\alpha\in\mathbb{R}}$ be a family of subsets of $\mathbb{M}^{n\times n}_D$ such that:

$$K(\alpha) \text{ is closed and convex, for every } \alpha \in \mathbb{R}\,, \tag{2.9a}$$

$$U \subset \mathbb{M}^{n\times n}_D \text{ open } \implies \{\alpha\in\mathbb{R} \,|\, K(\alpha)\cap U \neq \emptyset\} \text{ and } \{\alpha\in\mathbb{R}\,|\,K(\alpha)\subset U\} \text{ open} \tag{2.9b}$$

$$B_r(0) \subset K(0) \subset K(\alpha_1) \subset K(\alpha_2) \subset K(1) \subset B_R(0), \text{ for every } \alpha_1 \le \alpha_2\,, \tag{2.9c}$$

with $0 < r < R$. In particular we have that $K(\alpha) = K(0)$ for $\alpha \le 0$ and $K(\alpha) = K(1)$ for $\alpha \ge 1$. When (2.9b) holds we say that the multifunction $\alpha \mapsto K(\alpha)$ is continuous.

The sets above are called the constraint sets because we will see (see Corollary 5.3) that, during the evolution, $\sigma_D(t,x) \in K(\alpha(t,x))$ for every $t \in [0,T]$ and a.e. $x \in \Omega$, $\sigma := \mathbb{C}(\alpha)e$ being the *elastic stress*.

Let us consider the function $H\colon \mathbb{R}\times\mathbb{M}^{n\times n}_D \to \mathbb{R}^+\cup\{0\}$ defined by

$$H(\alpha,\xi) := \sup_{\sigma\in K(\alpha)} \sigma\colon\xi \text{ for every } \alpha\in\mathbb{R}\,,$$

namely $\xi \mapsto H(\alpha,\xi)$ is the support function of $K(\alpha)$. Arguing as in [34, Proposition 2.4], we can show that (2.9b) implies that

$$\alpha \mapsto H(\alpha,\xi) \text{ is continuous for every } \xi \in \mathbb{M}^{n\times n}_D\,. \tag{2.10}$$

Then we get, from (2.9), that the four conditions below are simultaneously satisfied:

$$H \text{ is continuous}\,, \tag{2.11a}$$

$$\alpha \mapsto H(\alpha,\xi) \text{ is nondecreasing for every } \xi \in \mathbb{M}^{n\times n}_D\,, \tag{2.11b}$$

$$\xi \mapsto H(\alpha,\xi) \text{ is convex and positively one-homogeneous for every } \alpha\in\mathbb{R}\,, \tag{2.11c}$$

$$r|\xi| \le H(\alpha,\xi) \le R|\xi| \text{ for every } \alpha\in\mathbb{R} \text{ and every } \xi\in\mathbb{M}^{n\times n}_D\,. \tag{2.11d}$$

Indeed, by [20, Theorem 5], we have that (2.9a) and (2.9c) are equivalent to (2.11b), (2.11c), and (2.11d). Since the functions $\xi \mapsto H(\alpha,\xi)$ are convex with respect to ξ for every α and locally equi-bounded with respect to α by (2.11d), condition (2.10) is equivalent to (2.11a).

In some of the results we will make the additional assumption that

$$\xi \mapsto H(\alpha_2,\xi) - H(\alpha_1,\xi) \text{ is convex, for every } \alpha_1 \le \alpha_2\,. \tag{2.12}$$

Remark 2.1. Let us consider a multiplicative setting for the constraint sets, i.e., let us define

$$K(\alpha) := V(\alpha)K(1)\,,$$

where $B_r(0) \subset K(1) \subset B_R(0)$, $K(1)$ is closed and convex, and

$$V\colon \mathbb{R}\to[m,M] \text{ is Lipschitz, nondecreasing, and constant in } (-\infty,0] \text{ and } [1,+\infty)$$

with r, R, m, M positive constants.

Then (2.9) and (2.12) hold.

The plastic potential. Basing on the theory of convex functions of measures developed in [19], we define the *plastic potential* $\mathcal{H}\colon C(\overline{\Omega})\times M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)\to\mathbb{R}$ by

$$\mathcal{H}(\alpha,p) := \int_{\overline{\Omega}} H\left(\alpha(x), \frac{\mathrm{d}p}{\mathrm{d}\mu}(x)\right)\mathrm{d}\mu(x)\,, \tag{2.13}$$

where $\mu \in M_b(\overline{\Omega})^+$ is any measure such that $p \ll \mu$ and $\frac{\mathrm{d}p}{\mathrm{d}\mu}$ denotes the Radon-Nikodym derivative of p with respect to μ; note that the homogeneity of H with respect to ξ implies that the integral does not depend on μ. When $\dot{p}$ is the rate of plastic strain and α the internal variable, $\mathcal{H}(\alpha, \dot{p})$ represents the rate of plastic dissipation. Moreover, applying [3, Proposition 2.37], we get that

$$p \mapsto \mathcal{H}(\alpha, p) \text{ is convex and positively one-homogeneous for every } \alpha \in C(\overline{\Omega})\,,$$

which implies

$$\mathcal{H}(\alpha, p_1 + p_2) \le \mathcal{H}(\alpha, p_1) + \mathcal{H}(\alpha, p_2) \tag{2.14}$$

for every $\alpha \in C(\overline{\Omega})$ and $p_1, p_2 \in M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$.

Remark 2.2. Since

$$\left|\frac{\mathrm{d}p}{\mathrm{d}|p|}(x)\right| = 1 \quad \text{for } |p|\text{-a.e. } x \in \overline{\Omega}\,, \tag{2.15}$$

from (2.11d) it follows immediately that for every $\alpha \in C(\overline{\Omega})$

$$r\|p\|_1 \le \mathcal{H}(\alpha, p) \le R\|p\|_1\,. \tag{2.16}$$

Moreover, by continuity of H there exists a modulus of continuity ω, namely an increasing function defined on $\mathbb{R}^+ \cup \{0\}$ which vanishes at 0, such that

$$|H(\alpha_1(x), \xi) - H(\alpha_2(x), \xi)| \le \omega(|\alpha_1(x) - \alpha_2(x)|)\,, \tag{2.17}$$

for every α_1, $\alpha_2 \in C(\overline{\Omega})$, $x \in \Omega$, and $\xi \in \mathbb{M}^{n\times n}_D$ with $|\xi| = 1$. Then, from (2.15) we obtain

$$|\mathcal{H}(\alpha_2, p) - \mathcal{H}(\alpha_1, p)| \le \omega(\|\alpha_1 - \alpha_2\|_\infty)\|p\|_1 \tag{2.18}$$

for every α_1, $\alpha_2 \in C(\overline{\Omega})$.

Lemma 2.3. *Let α_k and p_k be sequences in $C(\overline{\Omega})$ and $M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$ such that $\alpha_k \to \alpha$ uniformly and $p_k \rightharpoonup p$ weakly* in $M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$. Then*

$$\mathcal{H}(\alpha, p) \le \liminf_{k\to\infty} \mathcal{H}(\alpha_k, p_k)\,. \tag{2.19}$$

Proof. From (2.18) we obtain

$$\mathcal{H}(\alpha_k, p_k) \ge \mathcal{H}(\alpha, p_k) - \omega(\|\alpha_k - \alpha\|_\infty)\|p_k\|_1\,.$$

The lower semicontinuity result follows now from the weak* convergence of p_k and Reshetnyak's Lower Semicontinuity Theorem (see [31, Theorem 2]). □

Stress-strain duality. Let

$$\Sigma(\Omega) := \{\sigma \in L^2(\Omega; M^{n\times n}_{sym}) \,|\, \mathrm{div}\, \sigma \in L^n(\Omega; \mathbb{R}^n),\, \sigma_D \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D)\}\,.$$

If $\sigma \in \Sigma(\Omega)$, then $\sigma \in L^r(\Omega; \mathbb{M}^{n\times n}_{sym})$ for every $1 \le r < \infty$ by [17, Proposition 6.1]. For $\sigma \in \Sigma(\Omega)$ and $p \in \Pi(\Omega)$ such that $(u, e, p) \in A(w)$, we define, as in [17, Section 6], the distribution $[\sigma_D : p]$ on $\mathbb{R}^n$ by

$$\langle [\sigma_D : p], \varphi\rangle := -\int_\Omega \varphi\sigma\cdot(e - Ew)\,\mathrm{d}x - \int_\Omega \sigma\cdot[(u-w)\odot\nabla\varphi]\,\mathrm{d}x - \int_\Omega \varphi(\mathrm{div}\,\sigma)\cdot(u-w)\,\mathrm{d}x\,, \tag{2.20}$$

for every $\varphi \in C^\infty_c(\mathbb{R}^n)$. Notice that this is indeed a well defined distribution since $u \in L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n)$, being in $BD(\Omega)$. Moreover, in [17, Theorem 6.2 and Remark 6.3] it is proved that $[\sigma_D : p]$ is a bounded Radon measure such that

$$\|[\sigma_D : p]\|_1 \le \|\sigma_D\|_\infty \|p\|_1 \quad \text{in } \mathbb{R}^n\,.$$

In particular, we can consider its restriction, as a measure, to $\overline{\Omega}$, that we denote in the same way. We also define

$$\langle \sigma_D \,|\, p\rangle := [\sigma_D : p](\overline{\Omega})\,.$$

By (2.20), for $\varphi \in C^\infty_c(\mathbb{R}^n)$, $\varphi \equiv 1$ on $\overline{\Omega}$, we obtain the integration by parts formula

$$\langle \sigma_D \,|\, p\rangle = -\langle \sigma, e - Ew\rangle - \langle \mathrm{div}\,\sigma, u - w\rangle\,. \tag{2.21}$$

For a given $\alpha \in C(\overline{\Omega})$ let

$$\mathcal{K}_\alpha(\Omega) := \{\sigma \in L^2(\Omega; M^{n\times n}_{sym}) \,|\, \operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^n),\ \sigma_D(x) \in K(\alpha(x)) \text{ for a.e. } x \in \Omega\}\,.$$

Since the multifunction $\alpha \in [0,1] \mapsto K(\alpha)$ is continuous, that is (2.9b) holds, [17, Proposition 3.9] (which holds also if div σ is not identically 0) implies that for every $\sigma \in \mathcal{K}_\alpha(\Omega)$:

$$H\left(\alpha, \frac{\mathrm{d}p}{\mathrm{d}|p|}\right)|p| \geq [\sigma_D : p] \quad \text{as measures on } \overline{\Omega}, \tag{2.22}$$

and, arguing as in [34, Theorem 3.6 and Corollary 3.8], we deduce that for every $p \in \Pi(\Omega)$:

$$\mathcal{H}(\alpha, p) = \sup_{\sigma\in\mathcal{K}_\alpha(\Omega)} \langle \sigma_D \,|\, p\rangle\,. \tag{2.23}$$

The plastic dissipation. We introduce now a term which represents the plastic dissipation in a given time interval.

A function $p\colon [0,T] \to M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ will be regarded as a function defined on the time interval $[0,T]$ with values in the dual of the separable Banach space $C(\overline{\Omega};\mathbb{M}^{n\times n}_D)$, $\overline{\Omega} \subset \mathbb{R}^n$ being compact. For every $s,t \in [0,T]$ with $s \leq t$ the total variation of p on $[s,t]$ is defined by

$$\mathcal{V}(p;s,t) = \sup\left\{\sum_{j=1}^N \|p(t_j) - p(t_{j-1})\|_1 \,\middle|\, s = t_0 < t_1 < \cdots < t_N = t,\ N \in \mathbb{N}\right\}.$$

Let $\alpha\colon [0,T] \to C(\overline{\Omega})$. For every partition $\mathcal{P}$ of $[s,t]$, namely $\mathcal{P} := \{t_i\}_{0\leq i\leq N}$ with $s = t_0 < t_1 < \cdots < t_N = t$, we define

$$\mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha, p; s, t) := \sum_{i=1}^N \mathcal{H}(\alpha(t_i), p(t_i) - p(t_{i-1}))\,.$$

The *$\mathcal{H}$-variation of p with respect to α* on $[s,t]$, which will play the role of the plastic dissipation in the time interval $[s,t]$, is denoted by $\mathcal{V}_{\mathcal{H}}(\alpha,p;s,t)$ and is defined through

$$\begin{aligned}\mathcal{V}_{\mathcal{H}}(\alpha,p;s,t) &:= \sup\left\{\sum_{j=1}^N \mathcal{H}(\alpha(t_j), p(t_j) - p(t_{j-1})) \,\middle|\, s = t_0 < t_1 < \cdots < t_N = t,\ N \in \mathbb{N}\right\}\\ &= \sup\left\{\mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha,p;s,t) \,|\, \mathcal{P} \text{ partition of } [s,t]\right\}.\end{aligned} \tag{2.24}$$

Lemma A.1 in the Appendix states some properties of $\mathcal{V}_{\mathcal{H}}$ when the functions $t \mapsto \alpha(t,x)$ are nonincreasing for every $x \in \Omega$.

When $\alpha(t) = \overline{\alpha} \in C(\overline{\Omega};[0,1])$ for every t we use the symbols $\widehat{\mathcal{V}}_H$ and $\widehat{\mathcal{V}}^{\mathcal{P}}_{\mathcal{H}}$ instead of $\mathcal{V}_H$ and $\mathcal{V}^{\mathcal{P}}_{\mathcal{H}}$, so that

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\overline{\alpha},p;s,t) := \mathcal{V}_{\mathcal{H}}(\alpha,p;s,t)|_{\alpha(t)=\overline{\alpha}}\,, \quad \widehat{\mathcal{V}}^{\mathcal{P}}_{\mathcal{H}}(\overline{\alpha},p;s,t) := \mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha,p;s,t)|_{\alpha(t)=\overline{\alpha}}\,. \tag{2.25}$$

Remark 2.4. In the case of α constant in time we fit in the notion of $\mathcal{H}$-variation of p on $[s,t]$ introduced in [7, Appendix]. There the $\mathcal{G}$-variation of p on $[s,t]$ is defined, when p takes values in $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$, as

$$\begin{aligned}\mathcal{V}_{\mathcal{G}}(p;s,t) &:= \sup\left\{\sum_{i=1}^N \mathcal{G}(p(t_i) - p(t_{i-1}))\,\middle|\, s = t_0 < t_1 < \cdots < t_N = t,\ N \in \mathbb{N}\right\}\\ &= \sup\{\mathcal{V}^{\mathcal{P}}_{\mathcal{G}}(p;s,t) \,|\, \mathcal{P} \text{ partition of } [s,t]\}\,,\end{aligned}$$

where $\mathcal{G}(p) = \sup_{\varphi\in\mathcal{K}}\langle p,\varphi\rangle$ is the support function of a bounded closed convex set $\mathcal{K} \subset C(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ and $\mathcal{V}^{\mathcal{P}}_{\mathcal{G}}(p;s,t)$ is defined similarly to $\widehat{\mathcal{V}}^{\mathcal{P}}_{\mathcal{H}}(\overline{\alpha},p;s,t)$. By (2.23), it suffices to take $\mathcal{K} = \mathcal{K}_{\overline{\alpha}(\Omega)}$ in order to obtain $\mathcal{G} = \mathcal{H}(\overline{\alpha},\cdot)$ and

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\overline{\alpha},p;s,t) = \mathcal{V}_{\mathcal{G}}(p;s,t)\,.$$

The prescribed boundary displacement. In this paper the external loading will consist only in Dirichlet boundary conditions, for the sake of simplicity. However, similar results to those showed here hold also in the presence of external forces, under suitable regularity assumptions on $\partial\Omega$ and uniform safe load conditions, like the ones in [10, Section 2].

We assume that the prescribed boundary displacement w depends on time and satisfies the regularity assumption

$$w \in AC([0,T];H^1(\mathbb{R}^n;\mathbb{R}^n))\,, \tag{2.26}$$

so that the time derivative $t \mapsto \dot{w}(t)$ belongs to $L^1(0,T;H^1(\mathbb{R}^n;\mathbb{R}^n))$ and its strain $t \mapsto E\dot{w}(t)$ belongs to $L^1(0,T;L^2(\mathbb{R}^n;\mathbb{M}^{n\times n}_{sym}))$. For the main properties of absolutely continuous functions with values in reflexive Banach spaces we refer to [5, Appendix].

3. The minimization problem

In this section we study the minimization problem employed in the incremental formulation of the quasistatic evolution corresponding to a given parameter $\lambda \in [0,1]$. Therefore we deal with a problem of the type

$$\operatorname{argmin}\left\{\mathcal{E}_\lambda(\alpha,e;\overline{q},t)+\mathcal{H}(\alpha,p-\overline{p})\,|\,(\alpha,(u,e,p))\in\mathcal{D}(\overline{\alpha})\times A(w)\right\}, \tag{3.1}$$

where

$$\mathcal{E}_\lambda(\alpha,e;\overline{q},t) := \mathcal{E}(\alpha,e)+\lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha,\overline{q};0,t)\,. \tag{3.2}$$

The data are the current values $\overline{\alpha}\in W^{1,\gamma}(\Omega)$ and $\overline{p}\in\Pi(\Omega)$ of the damage variable and the plastic strain, and the updated value $w\in H^1(\mathbb{R}^n;\mathbb{R}^n)$ of the boundary displacement; if $\lambda>0$ we consider as an additional datum a function $\overline{q}\colon[0,t]\to M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ with bounded variation, which represents the evolution of the plastic strain up to the current time t. Solving this problem, we get the updated values α, u, e, and p of damage, displacement, elastic and plastic strain.

First we show the existence and the main properties of the solutions to (3.1). The second part of the section is devoted to prove a stability property with respect to variations of the data. Throughout this section, we suppose that (2.4), (2.5), (2.9), and (2.26) hold when $\lambda=0$; when $\lambda>0$ we will assume also (2.12).

Let us prove the existence of a solution to (3.1).

Theorem 3.1 (Existence of solutions to the incremental problem)**.** *Let $w\in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $\overline{\alpha}\in W^{1,\gamma}(\Omega)$, $\overline{p}\in\Pi(\Omega)$, and $\overline{q}\colon[0,t]\to M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ with bounded variation. Then* (3.1) *has a solution. Moreover, if $\overline{\alpha}\in W^{1,\gamma}(\Omega;[0,1])$, then for every $(\alpha,(u,e,p))$ solution of* (3.1) *we have that $\alpha\in W^{1,\gamma}(\Omega;[0,1])$.*

Proof. Let $(\alpha_k,(u_k,e_k,p_k))\in\mathcal{D}(\overline{\alpha})\times A(w)$ be a minimizing sequence for problem (3.1).

By (2.5d) and (2.16) the sequences α_k, e_k, and p_k are bounded in $W^{1,\gamma}(\Omega)$, $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$, respectively. Since $Eu_k=e_k+p_k$ in Ω, it follows that Eu_k is bounded in $M_b(\Omega;\mathbb{M}^{n\times n}_{sym})$. Since $(w-u_k)\odot\nu\,\mathcal{H}^{n-1}=p_k$ is bounded in $M_b(\partial\Omega;\mathbb{M}^{n\times n}_D)$, the traces of u_k are bounded in $L^1(\partial\Omega;\mathbb{R}^n)$. Therefore u_k is bounded in $BD(\Omega)$ by (2.1).

Up to extracting a subsequence, we may assume that $u_k\rightharpoonup u$ weakly* in $BD(\Omega)$, $e_k\rightharpoonup e$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $p_k\rightharpoonup p$ weakly* in $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$. By [7, Lemma 2.1], we have $(u,e,p)\in A(w)$.

The existence of solutions to (3.1) now follows from the lower semicontinuity of $\mathcal{E}$ (see (2.8)) and $\mathcal{H}$ (see (2.19)). Notice that if $\alpha\neq\alpha^+:=\alpha\vee 0$ then

$$\mathcal{E}_\lambda(\alpha^+,e;\overline{q},t)=\mathcal{E}(\alpha^+,e)+\lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha^+,\overline{q};0,t)<\mathcal{E}_\lambda(\alpha,e;\overline{q},t)=\mathcal{E}(\alpha,e)+\lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha,\overline{q};0,t)\,,$$

and this is enough to conclude that α takes values in $[0,1]$ if $\overline{\alpha}\in W^{1,\gamma}(\Omega;[0,1])$. □

The following lemma is not only useful in Lemma 3.3 below, but also in the proof of the stability for the approximate solutions in Theorem 4.3, when $\lambda=0$.

Lemma 3.2. *If $(\alpha,(u,e,p))$ solves* (3.1) *then*

$$\mathcal{E}_\lambda(\alpha,e;\overline{q},t)\le\mathcal{E}_\lambda(\tilde{\alpha},\tilde{e};\overline{q},t)+\mathcal{H}(\tilde{\alpha},\tilde{p}-p)\,, \tag{3.3}$$

for every $(\tilde{\alpha},(\tilde{u},\tilde{e},\tilde{p}))\in\mathcal{D}(\alpha)\times A(w)$.

Proof. Let $(\tilde\alpha,(\tilde u,\tilde e,\tilde p))\in\mathcal{D}(\alpha)\times A(w)$. Then, by (2.3), this quadruple belongs to $\mathcal{D}(\overline{\alpha})\times A(w)$ too. From our hypotesis, $\mathcal{E}_\lambda(\alpha,e;\overline{q},t)\le\mathcal{E}_\lambda(\tilde\alpha,\tilde e;\overline{q},t)+\mathcal{H}(\tilde\alpha,\tilde p-\overline{p})-\mathcal{H}(\alpha,p-\overline{p})$, and by (2.14) and (2.11b), $\mathcal{H}(\tilde\alpha,\tilde p-\overline{p})\le\mathcal{H}(\tilde\alpha,\tilde p-p)+\mathcal{H}(\tilde\alpha,p-\overline{p})\le\mathcal{H}(\tilde\alpha,\tilde p-p)+\mathcal{H}(\alpha,p-\overline{p})$. Thus we conclude. □

We now derive some differential conditions for a triple (u,e,p) such that $(\alpha,(u,e,p))$ solves (3.1), from a characterization of the solutions to (3.3).

Lemma 3.3. *Let $w\in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $\overline{q}\colon[0,t]\to M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$, $\alpha\in W^{1,\gamma}(\Omega)$, $(u,e,p)\in A(w)$ satisfy* (3.3)*, $\overline{q}$ having bounded variation. Then*

$$-\mathcal{H}(\alpha,q)\le\langle\mathbb{C}(\alpha)e,\eta\rangle\le\mathcal{H}(\alpha,-q)$$

for every $(v,\eta,q)\in A(0)$, and

$$\mathbb{C}(\alpha)e\in\mathcal{K}_\alpha(\Omega),\ \mathrm{div}\,(\mathbb{C}(\alpha)e)=0\quad\text{in }\Omega\,.$$

Proof. Let us assume fix $(v,\eta,q)\in A(0)$. Since for every $\varepsilon\in\mathbb{R}$

$$(\alpha,(u+\varepsilon v,e+\varepsilon\eta,p+\varepsilon q))\in\mathcal{D}(\alpha)\times A(w)\,,$$

we have

$$\mathcal{Q}(\alpha,e+\varepsilon\eta)+\mathcal{H}(\alpha,\varepsilon q)\ge\mathcal{Q}(\alpha,e)\ \text{ for every }\varepsilon\in\mathbb{R}\,.$$

The positive homogeneity of $\mathcal{H}$ implies

$$\mathcal{Q}(\alpha,e\pm\varepsilon\eta)+\varepsilon\mathcal{H}(\alpha,\pm q)\ge\mathcal{Q}(\alpha,e)\ \text{ for every }\varepsilon>0\,.$$

Dividing by ε and passing to the limit as $\varepsilon\to0$, we recover the former condition.

In order to get the latter one we can argue as in the first part of [7, Proposition 3.5], using the integration by parts formula (2.21). □

The following lemma shows, for pairs $(\alpha,(u,e,p))$ that satisfy (3.3), the Hölder dependence of u and e on α, p, and w.

Lemma 3.4. *For $i=1,2$ let $w_i\in H^1(\mathbb{R}^n,\mathbb{R}^n)$. Suppose that $(\alpha_i,(u_i,e_i,p_i))$ satisfies* (3.3) *with boundary datum $w=w_i$, and let*

$$\omega_{12}:=\|\alpha_2-\alpha_1\|_\infty+\|p_2-p_1\|_1^{1/2}+\|Ew_2-Ew_1\|_2\,.$$

Then

$$\|e_2-e_1\|_2\le C\,\omega_{12}\,,\tag{3.4}$$

where C is a positive constant depending on $\|e_1\|_2$, R, γ_1, γ_2, and Ω.

Proof. We modify the proof of [7, Theorem 3.8], considering that here $\mathbb{C}$ depends on α. Let

$$\begin{aligned}v&:=(u_2-w_2)-(u_1-w_1),\\ \eta&:=(e_2-Ew_2)-(e_1-Ew_1),\\ q&:=p_2-p_1\,.\end{aligned}$$

Since $(v,\eta,q)\in A(0)$, by Lemma 3.3 it follows that

$$\begin{aligned}-\mathcal{H}(\alpha_1,p_2-p_1)\le&\langle\mathbb{C}(\alpha_1)e_1,\eta\rangle,\\ &\langle\mathbb{C}(\alpha_2)e_2,\eta\rangle\le\mathcal{H}(\alpha_2,p_1-p_2)\,.\end{aligned}$$

Adding term by term and using (2.11d), we obtain

$$\langle\mathbb{C}(\alpha_2)(e_2-e_1),\eta)\rangle\le\langle[\mathbb{C}(\alpha_1)-\mathbb{C}(\alpha_2)]e_1,\eta\rangle+2R\|p_2-p_1\|_1\,.$$

Observe that above we have put an extra term $-\langle\mathbb{C}(\alpha_2)e_1,\eta\rangle$ on both sides. From the definition of η,

$$\begin{aligned}\langle\mathbb{C}(\alpha_2)(e_2-e_1),e_2-e_1)\rangle\le&\langle\mathbb{C}(\alpha_2)(e_2-e_1),Ew_2-Ew_1\rangle+\langle[\mathbb{C}(\alpha_1)-\mathbb{C}(\alpha_2)]e_1,e_2-e_1\rangle\\ &+\langle[\mathbb{C}(\alpha_1)-\mathbb{C}(\alpha_2)]e_1,Ew_1-Ew_2\rangle+2R\|p_2-p_1\|_1\,.\end{aligned}$$

By (2.5), this implies

$$2\gamma_1\|e_2-e_1\|_2^2 \le 2\gamma_2\|e_2-e_1\|_2\|Ew_2-Ew_1\|_2 \\ + \|e_1\|_2\|\alpha_2-\alpha_1\|_\infty(\|e_2-e_1\|_2+\|Ew_2-Ew_1\|_2)+2R\|p_2-p_1\|_1\,,$$

which yields (3.4) by the Cauchy inequality. □

Remark 3.5. We can also deduce the continuous dependence on α, p, and w of u, expressed (with the same notation as above) by

$$\|Eu_2-Eu_1\|_1 \le C\,(\omega_{12}+\|p_2-p_1\|_1)\,,$$
$$\|u_2-u_1\|_1 \le C\,(\omega_{12}+\|p_2-p_1\|_1+\|w_2-w_1\|_2)\,,$$

arguing as in the final part of [7, Theorem 3.8].

We now show some stability results for the solutions of problems of the type (3.1) with respect to the weak convergence of the data. To ease the reading we first consider, in Theorem 3.6, the case $\lambda=0$, and then we study, in Lemma 3.7, the additional term that appears when $\lambda>0$. The result for the case $\lambda>0$ (Theorem 3.8) follows from this lemma, arguing as in Theorem 3.6.

Theorem 3.6 (Stability, case $\lambda=0$)**.** *Let $w_k\in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $\alpha_k\in W^{1,\gamma}(\Omega)$, and $(u_k,e_k,p_k)\in A(w_k)$ for every k. Assume that $\alpha_k\rightharpoonup\alpha_\infty$ weakly in $W^{1,\gamma}(\Omega)$, $u_k\rightharpoonup u_\infty$ weakly* in $BD(\Omega)$, $e_k\rightharpoonup e_\infty$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p_k\rightharpoonup p_\infty$ weakly* in $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$, $w_k\rightharpoonup w_\infty$ weakly in $H^1(\mathbb{R}^n;\mathbb{R}^n)$. Then $(u_\infty,e_\infty,p_\infty)\in A(w_\infty)$. If, in addition,*

$$\mathcal{E}(\alpha_k,e_k)\le\mathcal{E}(\tilde\alpha_k,\tilde e_k)+\mathcal{H}(\tilde\alpha_k,\tilde p_k-p_k) \tag{3.6}$$

for every k and every $(\tilde\alpha_k,(\tilde u_k,\tilde e_k,\tilde p_k))\in\mathcal{D}(\alpha_k)\times A(w_k)$, then

$$\mathcal{E}(\alpha_\infty,e_\infty)\le\mathcal{E}(\alpha,e)+\mathcal{H}(\alpha,p-p_\infty) \tag{3.7}$$

for every $(\alpha,(u,e,p))\in\mathcal{D}(\alpha_\infty)\times A(w_\infty)$.

Proof. The fact that $(u_\infty,e_\infty,p_\infty)\in A(w_\infty)$ follows by [7, Lemma 2.1].

We fix $\alpha\in\mathcal{D}(\alpha_\infty)$ and $(u,e,p)\in A(w_\infty)$, and test (3.6) by

$$\tilde\alpha_k := \alpha\wedge\alpha_k\,,$$
$$\tilde u_k := u-u_\infty+u_k\,,$$
$$\tilde e_k := e-e_\infty+e_k\,,$$
$$\tilde p_k := p-p_\infty+p_k\,.$$

Then $\tilde\alpha_k\rightharpoonup\alpha$ and $\alpha\vee\alpha_k\rightharpoonup\alpha_\infty$ weakly in $W^{1,\gamma}(\Omega)$, $\tilde u_k\rightharpoonup u$ weakly* in $BD(\Omega)$, $\tilde e_k\rightharpoonup e$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $\tilde p_k\rightharpoonup p$ weakly* in $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$.

Since for every $\alpha\in W^{1,\gamma}(\Omega)$ and every $e_1,e_2\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ we have

$$\mathcal{Q}(\alpha,e_1)-\mathcal{Q}(\alpha,e_2)=\frac12\langle\mathbb{C}(\alpha)(e_1+e_2),e_1-e_2\rangle \tag{3.8}$$

and for every $\alpha,\beta\in W^{1,\gamma}(\Omega)$

$$\|\nabla(\alpha\vee\beta)\|_\gamma^\gamma+\|\nabla(\alpha\wedge\beta)\|_\gamma^\gamma=\|\nabla\alpha\|_\gamma^\gamma+\|\nabla\beta\|_\gamma^\gamma\,,$$

(3.6) can be rewritten, adding to both sides the term $-\mathcal{Q}(\tilde\alpha_k,e_k)$, as

$$\gamma_k :=\mathcal{Q}(\alpha_k,e_k)-\mathcal{Q}(\tilde\alpha_k,e_k)+D(\alpha_k)+\|\nabla(\alpha\vee\alpha_k)\|_\gamma^\gamma-\|\nabla\alpha\|_\gamma^\gamma \\ \le\frac12\langle\mathbb{C}(\tilde\alpha_k)(e-e_\infty+2e_k),e-e_\infty\rangle+D(\tilde\alpha_k)+\mathcal{H}(\tilde\alpha_k,p-p_\infty)=:\eta_k\,.$$

From (2.5a), for every $\alpha_1,\alpha_2\in C(\Omega)$ and $e\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$

$$|\mathcal{Q}(\alpha_1,e)-\mathcal{Q}(\alpha_2,e)|\le\mathrm{Lip}(\mathbb{C})\|\alpha_1-\alpha_2\|_\infty\|e\|_2^2\,.$$

Therefore,

$$\begin{aligned}\liminf_{k\to\infty} \mathcal{Q}(\alpha_k, e_k) - \mathcal{Q}(\tilde{\alpha}_k, e_k) &= \liminf_{k\to\infty} \mathcal{Q}(\alpha_\infty, e_k) - \mathcal{Q}(\alpha, e_k) \\ &= \liminf_{k\to\infty} \frac{1}{2}\langle [\mathbb{C}(\alpha_\infty) - \mathbb{C}(\alpha)]e_k, e_k\rangle\,.\end{aligned}$$

Since $\alpha \in \mathcal{D}(\alpha_\infty)$, by (2.5b) we have that $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \mapsto [\mathbb{C}(\alpha_\infty) - \mathbb{C}(\alpha)]e$: e is a positive semidefinite quadratic form. Hence, by lower semicontinuity,

$$\liminf_{k\to\infty} \gamma_k \geq \mathcal{Q}(\alpha_\infty, e_\infty) - \mathcal{Q}(\alpha, e_\infty) + D(\alpha_\infty) + \|\nabla(\alpha_\infty)\|^\gamma_\gamma - \|\nabla\alpha\|^\gamma_\gamma\,.$$

On the other hand,

$$\begin{aligned}\lim_{k\to\infty} \eta_k &= \frac{1}{2}\langle \mathbb{C}(\alpha)(e + e_\infty), e - e_\infty\rangle + D(\alpha) + \mathcal{H}(\alpha, p - p_\infty) \\ &= \mathcal{Q}(\alpha, e) - \mathcal{Q}(\alpha, e_\infty) + D(\alpha) + \mathcal{H}(\alpha, p - p_\infty)\,.\end{aligned}$$

This concludes the proof. □

From now on we treat the case $\lambda > 0$.

Lemma 3.7. *In addition to* (2.4), (2.5), (2.9), *and* (2.26), *let us assume also* (2.12). *Let* β_k *and* $\tilde{\beta}_k$ *be two sequences in* $C(\overline{\Omega})$ *such that* $\beta_k \to \beta_\infty$ *and* $\tilde{\beta}_k \to \beta$ *uniformly in* $\overline{\Omega}$, *and* $\tilde{\beta}_k \in \mathcal{D}(\beta_k)$ *for every* k. *Moreover let* q_k, q *be functions from* $[0,t]$ *into* $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ *such that* $q_k(s) \rightharpoonup q(s)$ *weakly* in* $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ *for every* $s \in [0,t]$. *Then*

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\beta_\infty, q; 0, t) - \widehat{\mathcal{V}}_{\mathcal{H}}(\beta, q; 0, t) \leq \liminf_{k\to\infty} [\widehat{\mathcal{V}}_{\mathcal{H}}(\beta_k, q_k; 0, t) - \widehat{\mathcal{V}}_{\mathcal{H}}(\tilde{\beta}_k, q_k; 0, t)]\,. \tag{3.9}$$

Proof. Let us consider the functionals $\widetilde{\mathcal{H}}_k$ and $\widetilde{\mathcal{H}}$ from $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ into $\mathbb{R}^+ \cup \{0\}$ defined, for every $p \in M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$, by

$$\begin{aligned}\widetilde{\mathcal{H}}(p) &:= \mathcal{H}(\beta_\infty, p) - \mathcal{H}(\beta, p)\,, \\ \widetilde{\mathcal{H}}_k(p) &:= \mathcal{H}(\beta_k, p) - \mathcal{H}(\tilde{\beta}_k, p)\,.\end{aligned}$$

By (2.12), $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}}_k$ are convex, positively one-homogeneous (and consequently subadditive), and weakly* lower semicontinuous, thanks to Reshetnyak's Lower Semicontinuity Theorem. We now show that

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\beta_\infty, q; 0, t) - \widehat{\mathcal{V}}_{\mathcal{H}}(\beta, q; 0, t) = \mathcal{V}_{\widetilde{\mathcal{H}}}(q; 0, t)\,, \tag{3.10}$$

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\beta_k, q; 0, t) - \widehat{\mathcal{V}}_{\mathcal{H}}(\tilde{\beta}_k, q; 0, t) = \mathcal{V}_{\widetilde{\mathcal{H}}_k}(q; 0, t)\,, \tag{3.11}$$

for every k. Indeed, let us fix $\varepsilon > 0$ and let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ be three partitions of $[0,t]$ such that

$$\begin{aligned}\widehat{\mathcal{V}}^{\mathcal{P}_1}_{\mathcal{H}}(\beta_\infty, q; 0, t) &> \widehat{\mathcal{V}}_{\mathcal{H}}(\beta_\infty, q; 0, t) - \varepsilon\,, \\ \widehat{\mathcal{V}}^{\mathcal{P}_2}_{\mathcal{H}}(\beta, q; 0, t) &> \widehat{\mathcal{V}}_{\mathcal{H}}(\beta, q; 0, t) - \frac{\varepsilon}{2}\,, \\ \mathcal{V}^{\mathcal{P}_3}_{\widetilde{\mathcal{H}}}(q; 0, t) &> \mathcal{V}_{\widetilde{\mathcal{H}}}(q; 0, t) - \frac{\varepsilon}{2}\,.\end{aligned}$$

It follows that

$$\mathcal{V}_{\widetilde{\mathcal{H}}}(q; 0, t) \geq \mathcal{V}^{\mathcal{P}_1}_{\widetilde{\mathcal{H}}}(q; 0, t) = \widehat{\mathcal{V}}^{\mathcal{P}_1}_{\mathcal{H}}(\beta_\infty, q; 0, t) - \widehat{\mathcal{V}}^{\mathcal{P}_1}_{\mathcal{H}}(\beta, q; 0, t) > \widehat{\mathcal{V}}_{\mathcal{H}}(\beta_\infty, q; 0, t) - \varepsilon - \widehat{\mathcal{V}}_{\mathcal{H}}(\beta, q; 0, t)\,.$$

On the other hand, we get

$$\begin{aligned}\widehat{\mathcal{V}}_{\mathcal{H}}(\beta_\infty, q; 0, t) - \widehat{\mathcal{V}}_{\mathcal{H}}(\beta, q; 0, t) &> \widehat{\mathcal{V}}^{\mathcal{P}_2\cup\mathcal{P}_3}_{\mathcal{H}}(\beta_\infty, q; 0, t) - \widehat{\mathcal{V}}^{\mathcal{P}_2}_{\mathcal{H}}(\beta, q; 0, t) - \frac{\varepsilon}{2} \\ &\geq \widehat{\mathcal{V}}^{\mathcal{P}_2\cup\mathcal{P}_3}_{\mathcal{H}}(\beta_\infty, q; 0, t) - \widehat{\mathcal{V}}^{\mathcal{P}_2\cup\mathcal{P}_3}_{\mathcal{H}}(\beta, q; 0, t) - \frac{\varepsilon}{2} \\ &= \mathcal{V}^{\mathcal{P}_2\cup\mathcal{P}_3}_{\widetilde{\mathcal{H}}}(q; 0, t) - \frac{\varepsilon}{2} > \mathcal{V}_{\widetilde{\mathcal{H}}}(q; 0, t) - \varepsilon\,,\end{aligned}$$

where the second inequality follows from Lemma A.1(1) and the last one comes from the subadditivity of $\widetilde{\mathcal{H}}$. This concludes the verification of (3.10). The proof of (3.11) is analogous.

Arguing as in Lemma 2.3 we have that

$$\widetilde{\mathcal{H}}(p) \le \liminf_{k\to\infty} \widetilde{\mathcal{H}}_k(p_k)\,,$$

for every $p_k \rightharpoonup p$ weakly* in $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$. Hence

$$\mathcal{V}_{\widetilde{\mathcal{H}}}(q;0,t) \le \liminf_{k\to\infty} \mathcal{V}_{\widetilde{\mathcal{H}}_k}(q_k;0,t)\,,$$

and we conclude by (3.10) and (3.11). □

Theorem 3.8 (Stability, case $\lambda > 0$)**.** *Besides* (2.4), (2.5), (2.9), *and* (2.26), *assume also* (2.12). *Let* $w_k \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $\alpha_k \in W^{1,\gamma}(\Omega)$, $(u_k,e_k,p_k) \in A(w_k)$, *and* q_k *be functions from* $[0,t]$ *into* $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ *of bounded variation, for every* k. *Suppose that* $\alpha_k \rightharpoonup \alpha_\infty$ *weakly in* $W^{1,\gamma}(\Omega)$, $u_k \rightharpoonup u_\infty$ *weakly** *in* $BD(\Omega)$, $e_k \rightharpoonup e_\infty$ *weakly in* $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p_k \rightharpoonup p_\infty$ *weakly** *in* $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$, $w_k \rightharpoonup w_\infty$ *weakly in* $H^1(\mathbb{R}^n;\mathbb{R}^n)$, *and* $q_k(s) \rightharpoonup q(s)$ *weakly** *in* $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ *for every* $s\in[0,t]$. *Then* $(u_\infty,e_\infty,p_\infty)\in A(w_\infty)$. *If, in addition,*

$$\mathcal{E}(\alpha_k,e_k)+\lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_k,q_k;0,t) \le \mathcal{E}(\tilde\alpha_k,\tilde e_k)+\lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\tilde\alpha_k,q_k;0,t)+\mathcal{H}(\tilde\alpha_k,\tilde p_k-p_k)$$

for every k *and every* $(\tilde\alpha_k,(\tilde u_k,\tilde e_k,\tilde p_k))\in\mathcal{D}(\alpha_k)\times A(w_k)$, *then*

$$\mathcal{E}(\alpha_\infty,e_\infty)+\lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_\infty,q;0,t) \le \mathcal{E}(\alpha,e)+\lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha,q;0,t)+\mathcal{H}(\alpha,p-p_\infty)\,,$$

for every $(\alpha,(u,e,p))\in\mathcal{D}(\alpha_\infty)\times A(w_\infty)$.

Proof. We can argue as in the proof of Theorem 3.6, choosing the same test functions, and adding to γ_k the term $\lambda(\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_k,q_k;0,t)-\widehat{\mathcal{V}}_{\mathcal{H}}(\tilde\alpha_k,q_k;0,t))$. The sequence of these terms is lower semicontinuous by Lemma 3.7 and this is enough to conclude. □

4. Quasistatic evolution

Fixed $\lambda\in[0,1]$, we now consider the problem of existence of globally stable quasistatic evolutions, where the time-dependent data are (only) Dirichlet boundary conditions $w \in AC([0,T];H^1(\mathbb{R}^n;\mathbb{R}^n))$. The functions α, u, e, p will be then functions from $[0,T]$ into the functional spaces $W^{1,\gamma}(\Omega;[0,1])$, $BD(\Omega)$, $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$, respectively.

The parameter λ accounts for the interplay between damage growth and cumulation of plastic strain. When $\lambda=1$ it is more convenient to damage material parts more affected by plastic evolution up to the current time. The physical meaning of λ will be explained in detail in Section 5, where we will study the properties of the evolutions. The case $\lambda=1$ corresponds to the model of [1] and [2], with a different gradient damage regularization.

The existence of quasistatic evolutions is shown in Theorem 4.3, the main result of the paper.

Definition 4.1. Let $\lambda\in[0,1]$. A *quasistatic evolution (corresponding to* λ*)* is a function $t\mapsto(\alpha(t),u(t),e(t),p(t))$ from $[0,T]$ into $W^{1,\gamma}(\Omega;[0,1])\times BD(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ that satisfies the following conditions:

(qs0) *irreversibility* : for every $x\in\Omega$

$$t\in[0,T]\mapsto\alpha(t,x) \quad \text{is nonincreasing;} \tag{4.1}$$

(qs1) *global stability*: the function $t\mapsto p(t)$ from $[0,T]$ into $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ has bounded variation, $(u(t),e(t),p(t))\in A(w(t))$ for every $t\in[0,T]$, and

$$\mathcal{E}(\alpha(t),e(t))+\lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha(t),p;0,t) \le \mathcal{E}(\beta,\eta)+\lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\beta,p;0,t)+\mathcal{H}(\beta,q-p(t)) \tag{4.2}$$

for every $(\beta,(v,\eta,q))\in\mathcal{D}(\alpha(t))\times A(w(t))$;

(qs2) *energy balance*: for every $t\in[0,T]$

$$\begin{aligned}\mathcal{E}(\alpha(t),e(t))+\lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha(t),p;0,t)+(1-\lambda)\mathcal{V}_{\mathcal{H}}(\alpha,p;0,t)\\ =\mathcal{E}(\alpha(0),e(0))+\int_0^t\langle\sigma(s),E\dot w(s)\rangle\,\mathrm{d}s\,,\end{aligned} \tag{4.3}$$

where $\sigma(s):=\mathbb{C}(\alpha(s))e(s)$ and $\mathcal{V}_{\mathcal{H}}$, $\widehat{\mathcal{V}}_{\mathcal{H}}$ are defined in (2.24) and (2.25), respectively.

Remark 4.2. The integral in (4.3) is well defined.

Indeed, from (4.2) it follows that $t \mapsto (\alpha(t), u(t), e(t), p(t))$ is a solution to the problem

$$\operatorname{argmin} \{\mathcal{E}_\lambda(\beta, \eta; p, t) + \mathcal{H}(\beta, q - p(t)) \,|\, (\beta, (v, \eta, q)) \in \mathcal{D}(\alpha(t)) \times A(w(t))\}\,,$$

for every $t \in [0, T]$, where $\mathcal{E}_\lambda$ is defined in (3.2). In view of Lemma 3.4, choosing $e_2 = e(t)$ for every $t \in [0, T]$ and $e_1 = e(0)$, we can observe that

$$\sup_{t\in[0,T]} \|e(t)\|_2 \le C\,, \tag{4.4}$$

where C is independent of time.

Let us now verify the measurability of $t \mapsto e(t)$. This follows from Lemma 3.4 if we show that $t \mapsto \alpha(t)$ is continuous for a.e. t with respect to the uniform convergence in $\overline{\Omega}$, since $t \mapsto p(t)$ is strongly continuous into $M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$ for a.e. t, having bounded variation. Now, by the irreversibility condition and the fact that for every $t \in [0, T]$ the function $\alpha(t)$ takes values in $[0, 1]$ we have, using Lemma A.2, that there exists a countable set $E \subset [0, T]$ such that α is continuous in every $t \in [0, T] \setminus E$ with respect to the L^p norm, with $1 \le p < \infty$. In other words, for every $t \in [0, T] \setminus E$

$$\alpha(s) \to \alpha(t) \text{ in } L^p(\Omega) \text{ as } s \to t\,.$$

From the stability condition, choosing $\beta \equiv 0$ and $(v, \eta, q) = (u(t), e(t), p(t))$, and using (4.4), it follows that

$$\sup_{t\in[0,T]} \|\nabla\alpha(t)\|^\gamma_\gamma < C$$

with C independent of $t \in [0, T]$. Then, by the Urysohn Property, α is continuous in every $t \in [0, T] \setminus E$ with respect to the weak convergence in $W^{1,\gamma}$, i.e., for every $t \in [0, T] \setminus E$

$$\alpha(s) \rightharpoonup \alpha(t) \text{ weakly in } W^{1,\gamma}(\Omega) \text{ as } s \to t\,.$$

The above convergence is uniform in $\overline{\Omega}$ by the compact Sobolev embedding.

Then e and σ belong to $L^\infty(0, T; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$. Finally, by (2.26), it follows that $\dot{w} \in L^1(0, T; H^1(\mathbb{R}^n; \mathbb{R}^n))$, and we conclude.

Theorem 4.3 (Existence of quasistatic evolutions)**.** *Let $\lambda \in [0, 1]$ and assume* (2.4)*,* (2.5)*,* (2.9)*, and* (2.26)*. If $\lambda > 0$ assume also* (2.12)*. Let $(\alpha_0, (u_0, e_0, p_0)) \in W^{1,\gamma}(\Omega; [0, 1]) \times A(w(0))$ satisfy the stability condition*

$$\mathcal{E}(\alpha_0, e_0) \le \mathcal{E}(\beta, \eta) + \mathcal{H}(\beta, q - p_0) \tag{4.5}$$

for every $(\beta, (v, \eta, q)) \in \mathcal{D}(\alpha_0) \times A(w(0))$. Then there exists a quasistatic evolution $t \mapsto (\alpha(t), u(t), e(t), p(t))$ corresponding to λ such that $\alpha(0) = \alpha_0$, $u(0) = u_0$, $e(0) = e_0$, $p(0) = p_0$.

Proof. The proof is based on discrete time approximation and is split into several steps.

Approximate solutions. Let us fix a sequence of subdivisions $(t^i_k)_{0\le i\le k}$ of the interval $[0, T]$, with

$$0 = t^0_k < t^1_k < \cdots < t^{k-1}_k < t^k_k = T\,,$$
$$\lim_{k\to\infty} \max_{1\le i\le k} (t^i_k - t^{i-1}_k) = 0\,.$$

For every k we define the approximate solutions α_k, u_k, e_k, and p_k by induction as follows. We set $(\alpha^0_k, (u^0_k, e^0_k, p^0_k)) := (\alpha_0, (u_0, e_0, p_0)) \in W^{1,\gamma}(\Omega; [0, 1]) \times A(w(0))$ and for $i = 1, \dots, k$ we define $(\alpha^i_k, (u^i_k, e^i_k, p^i_k))$ as a solution to the incremental problem

$$\operatorname{argmin} \{\mathcal{E}_\lambda(\beta, \eta; p_k, t^{i-1}_k) + \mathcal{H}(\beta, q - p^{i-1}_k) \,|\, (\beta, (v, \eta, q)) \in \mathcal{D}(\alpha^{i-1}_k) \times A(w^i_k)\}\,, \tag{4.6}$$

where $w^i_k := w(t^i_k)$ and, according to (3.2) and using Lemma A.1(2),

$$\mathcal{E}_\lambda(\beta, \eta; p_k, t^{i-1}_k) = \mathcal{E}(\beta, \eta) + \lambda \sum_{j=1}^{i-1} \mathcal{H}(\beta, p^j_k - p^{j-1}_k)\,,$$

with $p_k(t) := p_k^h$, h being the largest integer such that $t_k^h \le t$. The existence of a solution to this problem and the fact that $\alpha_k^i \in W^{1,\gamma}(\Omega;[0,1])$ for every $k \in \mathbb{N}$ and $0 \le i \le k$ follow from Theorem 3.1.

For every $t \in [0,T]$ we define the piecewise constant interpolations

$$\alpha_k(t) := \alpha_k^i,\ \ u_k(t) := u_k^i,\ \ e_k(t) := e_k^i,\ \ \sigma_k(t) := \mathbb{C}(\alpha_k^i)e_k^i,\ \ w_k(t) := w_k^i\,, \tag{4.7}$$

where i is the largest integer such that $t_k^i \le t$. By definition $t \mapsto \alpha_k(t)$ is nonincreasing, $\alpha_k(t) \in W^{1,\gamma}(\Omega;[0,1])$ and $(u_k(t), e_k(t), p_k(t)) \in A(w_k(t))$ for every $t \in [0,T]$. By Lemma A.1(2) it follows that

$$\mathcal{E}_\lambda(\alpha_k^i, e_k^i; p_k, t_k^i) = \mathcal{E}_\lambda(\alpha_k^i, e_k^i; p_k, t_k^{i-1}) + \lambda\mathcal{H}(\alpha_k^i, p_k^i - p_k^{i-1})\,. \tag{4.8}$$

Then (4.6) implies that

$$\begin{aligned}\mathcal{E}_\lambda(\alpha_k^i, e_k^i; p_k, t_k^i) + (1-\lambda)\mathcal{H}(\alpha_k^i, p_k^i - p_k^{i-1}) &= \mathcal{E}_\lambda(\alpha_k^i, e_k^i; p_k, t_k^{i-1}) + \mathcal{H}(\alpha_k^i, p_k^i - p_k^{i-1})\\ &\le \mathcal{E}_\lambda(\beta, \eta; p_k, t_k^{i-1}) + \mathcal{H}(\beta, q - p_k^{i-1})\end{aligned} \tag{4.9}$$

for every k, $1 \le i \le k$, and $(\beta,(v,\eta,q)) \in \mathcal{D}(\alpha_k^i) \times A(w_k^i)$. Since

$$\begin{aligned}\mathcal{H}(\beta, q - p_k^{i-1}) &\le \mathcal{H}(\beta, p_k^i - p_k^{i-1}) + \mathcal{H}(\beta, q - p_k^i)\\ &\le \lambda\mathcal{H}(\beta, p_k^i - p_k^{i-1}) + (1-\lambda)\mathcal{H}(\alpha_k^i, p_k^i - p_k^{i-1}) + \mathcal{H}(\beta, q - p_k^i)\,,\end{aligned}$$

from (4.9) we get the discrete formulation of global stability

$$\mathcal{E}_\lambda(\alpha_k(t), e_k(t); p_k, t) \le \mathcal{E}_\lambda(\beta, \eta; p_k, t) + \mathcal{H}(\beta, q - p_k(t)) \tag{4.10}$$

for every k, $t \in [0,T]$, and $(\beta,(v,\eta,q)) \in \mathcal{D}(\alpha_k(t)) \times A(w_k(t))$. Notice that if $\lambda = 0$ the equation (4.10) follows directly from Lemma 3.2.

The discrete energy inequality. We now derive an energy estimate for the solutions of the incremental problems. Let us fix $i \in \{1,\dots,k\}$ and for a given integer h with $1 \le h \le i$ let $v := u_k^{h-1} - w_k^{h-1} + w_k^h$ and $\eta := e_k^{h-1} - Ew_k^{h-1} + Ew_k^h$.

Since $(\alpha_k^{h-1}, (v, \eta, p_k^{h-1})) \in \mathcal{D}(\alpha_k^{h-1}) \times A(w_k^h)$, by the minimality condition (4.6) we obtain

$$\begin{aligned}&\mathcal{E}_\lambda(\alpha_k^h, e_k^h; p_k, t_k^{h-1}) + \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) \le \mathcal{E}_\lambda(\alpha_k^{h-1}, e_k^{h-1}; p_k, t_k^{h-1})\\ &\quad + \mathcal{Q}(\alpha_k^{h-1}, Ew_k^h - Ew_k^{h-1}) + \langle \mathbb{C}(\alpha_k^{h-1})e_k^{h-1}, Ew_k^h - Ew_k^{h-1}\rangle\,,\end{aligned} \tag{4.11}$$

where we have used the identity

$$\mathcal{Q}(\alpha, e_1 + e_2) = \mathcal{Q}(\alpha, e_1) + \mathcal{Q}(\alpha, e_2) + \langle \mathbb{C}(\alpha)e_1, e_2\rangle$$

for every $\alpha \in W^{1,\gamma}(\Omega)$ and e_1, $e_2 \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. From the absolute continuity of w with respect to t we obtain

$$w_k^h - w_k^{h-1} = \int_{t_k^{h-1}}^{t_k^h} \dot w(t)\,\mathrm{d}t\,,$$

where we use a Bochner integral of a function with values in $H^1(\mathbb{R}^n;\mathbb{R}^n)$. This implies that

$$Ew_k^h - Ew_k^{h-1} = \int_{t_k^{h-1}}^{t_k^h} E\dot w(t)\,\mathrm{d}t\,, \tag{4.12}$$

where the integral is again in the sense of Bochner and the target space is $L^2(\mathbb{R}^n;\mathbb{M}^{n\times n}_{sym})$. By (2.5d) and (4.12) we get

$$\mathcal{Q}(\alpha_k^{h-1}, Ew_k^h - Ew_k^{h-1}) \le \gamma_2\Big(\int_{t_k^{h-1}}^{t_k^h} \|E\dot w(t)\|_2\,\mathrm{d}t\Big)^2\,.$$

From (4.8), (4.11), and (4.12) it follows that

$$\begin{aligned}\mathcal{E}_\lambda(\alpha_k^h, e_k^h; p_k, t_k^h) + (1-\lambda)\mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) &\le \mathcal{E}_\lambda(\alpha_k^{h-1}, e_k^{h-1}; p_k, t_k^{h-1})\\ &+ \int_{t_k^{h-1}}^{t_k^h} \langle \mathbb{C}(\alpha_k^{h-1})e_k^{h-1}, E\dot w(t)\rangle\,\mathrm{d}t + \omega_k \int_{t_k^{h-1}}^{t_k^h} \|E\dot w(t)\|_2\,\mathrm{d}t\,,\end{aligned} \tag{4.13}$$

where

$$\omega_k := \gamma_2 \max_{1\le h\le k} \int_{t_k^{h-1}}^{t_k^h} \|E\dot w(t)\|_2 \,\mathrm{d}t \to 0$$

by the absolute continuity of the integral. Iterating now inequality (4.13) for $1 \le h \le i$, we have

$$\mathcal{E}_\lambda(\alpha_k^i, e_k^i; p_k, t_k^i) + (1-\lambda)\sum_{h=1}^{i} \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) \le \mathcal{E}(\alpha_0, e_0) + \int_0^{t_k^i} \langle \sigma_k(s), E\dot w(s)\rangle \,\mathrm{d}s + \delta_k\,, \quad (4.14)$$

with $\delta_k := \omega_k \int_0^T \|E\dot w(t)\|_2 \,\mathrm{d}t \to 0$.

A priori estimates. Using the hypoteses (2.5d) and (2.11d) in the left-hand side, as well as (2.6) and the fact that the function $t \mapsto \|E\dot w(t)\|_2$ is integrable on $[0,T]$ in the right-hand side, we find

$$\gamma_1\|e_k(t)\|_2^2 + D(\alpha_k(t)) + \|\nabla\alpha_k(t)\|_\gamma^\gamma + r(1-\lambda)\sum_{h=1}^{i}\|p_k^h - p_k^{h-1}\|_1$$
$$\le \mathcal{E}(\alpha_0, e_0) + 2\gamma_2 \sup_{t\in[0,T]}\|e_k(t)\|_2 \int_0^T \|E\dot w(s)\|_2 \,\mathrm{d}s + \delta_k$$

for every k and $t \in [t_k^1, T]$, where i is the largest integer such that $t_k^i \le t$.

Thus, by the Cauchy inequality,

$$\sup_{t\in[0,T]} \|e_k(t)\|_2 \le C\,. \quad (4.15)$$

Henceforth, C denotes a suitable constant depending only on γ_1, γ_2, r, and on the functions α_0, e_0, and w. We immediately deduce that

$$\sup_{t\in[0,T]} \|\nabla\alpha_k(t)\|_\gamma^\gamma \le C\,, \quad (4.16)$$

and, from the fact that $t \mapsto p_k(t)$ is constant on the intervals $[t_k^{h-1}, t_k^h[$, that

$$\mathcal{V}(p_k; 0, T) = \sum_{i=1}^{k} \|p_k^i - p_k^{i-1}\|_1 \le C\,. \quad (4.17)$$

Passage to the limit. Since the functions α_k are nonincreasing in time and take values in $[0,1]$, by virtue of (4.16) we can apply the generalized version of the classical Helly Theorem given in [15, Helly Theorem] to conclude that there exist a subsequence, still denoted α_k, and a function $\alpha\colon [0,T] \to W^{1,\gamma}(\Omega;[0,1])$ nonincreasing in time such that $\alpha_k(t) \to \alpha(t)$ strongly in $L^1(\Omega)$ for every $t \in [0,T]$. By (4.16) and the Urysohn Property we have weak convergence in $W^{1,\gamma}(\Omega)$ and thus uniform convergence in $\overline{\Omega}$. In particular (qs0) holds.

In the same way, using now (4.17) and [7, Lemma 7.2], we can assume that there exists $p\colon [0,T] \to M_b(\overline{\Omega}; \mathbb{M}_D^{n\times n})$ with bounded variation on $[0,T]$ such that $p_k(t) \rightharpoonup p(t)$ weakly* in $M_b(\overline{\Omega}; \mathbb{M}_D^{n\times n})$, for every $t \in [0,T]$.

Following the same argument used in the proof of Theorem 3.1, by (4.15) and (4.17) we can deduce that

$$\sup_{t\in[0,T]} \|u_k(t)\|_{BD(\Omega)} \le C\,. \quad (4.18)$$

Let us fix $t \in [0,T]$. From (4.15) and (4.18) it follows that there exist an increasing sequence k_j (possibly depending on t) and two functions $\tilde u \in BD(\Omega)$ and $\tilde e \in L^2(\Omega; \mathbb{M}_{sym}^{n\times n})$ such that $u_{k_j}(t) \rightharpoonup \tilde u$ weakly* in $BD(\Omega)$ and $e_{k_j}(t) \rightharpoonup \tilde e$ weakly in $L^2(\Omega; \mathbb{M}_{sym}^{n\times n})$. By (4.10) we can apply Theorem 3.8 (or Theorem 3.6 if $\lambda = 0$) and find that $(\alpha(t), (\tilde u, \tilde e, p(t)))$ is a solution to the problem

$$\operatorname{argmin}\{\mathcal{E}_\lambda(\beta, \eta; p, t) + \mathcal{H}(\beta, q - p(t)) \,|\, (\beta, (v, \eta, q)) \in \mathcal{D}(\alpha(t)) \times A(w(t))\}\,.$$

In particular $(\tilde u, \tilde e)$ minimizes the functional $(v, \eta) \mapsto \mathcal{Q}(\alpha(t), \eta)$, which is strictly convex in η, on the convex set $K := \{(v, \eta) :\ (v, \eta, p(t)) \in A(w(t))\}$. Then $(\tilde u, \tilde e)$ is uniquely determined,

using also Korn's inequality; defining $(u(t),e(t)) := (\tilde{u},\tilde{e})$, we have that $u_k(t) \rightharpoonup u(t)$ in $BD(\Omega)$ and $e_k(t) \rightharpoonup e(t)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. Therefore (qs1) holds.

To prove that $t \mapsto (\alpha(t),u(t),e(t),p(t))$ is a quasistatic evolution it remains to show the energy balance (qs2).

Energy balance. We consider now the asymptotics of the discrete energy inequality (4.14). Later we will show that also the equality holds in the limit.

Since p_k is piecewise constant and continuous from the right, α_k is nonincreasing, and (2.11b) holds, by Lemma A.1(2) we have

$$\mathcal{V}_{\mathcal{H}}(\alpha_k,p_k;0,t) = \sum_{h=1}^{i} \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1})\,, \tag{4.19}$$

where i is the largest integer such that $t_k^i \le t$. From the lower semicontinuity of $\mathcal{H}$ (Lemma 2.3) and the definition of plastic dissipation (2.24) it follows that

$$\mathcal{V}_{\mathcal{H}}(\alpha,p;0,t) \le \liminf_{k\to\infty} \mathcal{V}_{\mathcal{H}}(\alpha_k,p_k;0,t), \text{ and } \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha(t),p;0,t) \le \liminf_{k\to\infty} \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_k(t),p_k;0,t)\,. \tag{4.20}$$

Moreover, since $\alpha_k(t) \to \alpha(t)$ uniformly in $\overline{\Omega}$ and $e_k(t) \rightharpoonup e(t)$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ for every $t\in[0,T]$, by (2.5), (2.26), (4.15), and the Dominated Convergence Theorem we get

$$\int_0^t \langle \sigma(s), E\dot{w}(s)\rangle \,\mathrm{d}s = \lim_{k\to\infty} \int_0^{t_k^i} \langle \sigma_k(s), E\dot{w}(s)\rangle \,\mathrm{d}s\,, \tag{4.21}$$

where $\sigma(s) := \mathbb{C}(\alpha(s))e(s)$ for every $s\in[0,T]$.

Collecting (4.19)–(4.21), from (4.14) and the lower semicontinuity of the remaining terms the inequality

$$\mathcal{E}_\lambda(\alpha(t),e(t);p,t) + (1-\lambda)\mathcal{V}_{\mathcal{H}}(\alpha,p;0,t) \le \mathcal{E}(\alpha(0),e(0)) + \int_0^t \langle \sigma(s), E\dot{w}(s)\rangle \,\mathrm{d}s$$

follows, for every $t\in[0,T]$.

Conversely, let us fix $t\in[0,T]$ and let $(s_k^i)_{0\le i\le k}$ be a sequence of subdivisions of the interval $[0,t]$ satisfying

$$0 = s_k^0 < s_k^1 < \cdots < s_k^{k-1} < s_k^k = t\,,$$
$$\lim_{k\to\infty} \max_{1\le i\le k} (s_k^i - s_k^{i-1}) = 0\,.$$

For every $i=1,\dots,k$ let $v := u(s_k^i) - w(s_k^i) + w(s_k^{i-1})$ and $\eta := e(s_k^i) - Ew(s_k^i) + Ew(s_k^{i-1})$. Since $(\alpha(s_k^i),(v,\eta,p(s_k^i))) \in \mathcal{D}(\alpha(s_k^{i-1})) \times A(w(s_k^{i-1}))$, by the global stability (4.2) we have

$$\begin{aligned}\mathcal{E}_\lambda(\alpha(s_k^{i-1}),e(s_k^{i-1});p,s_k^{i-1}) \le{}& \mathcal{E}_\lambda(\alpha(s_k^i),e(s_k^i);p,s_k^{i-1}) + \mathcal{Q}(\alpha(s_k^i),Ew(s_k^{i-1}) - Ew(s_k^i))\\ &- \langle \sigma(s_k^i), Ew(s_k^i) - Ew(s_k^{i-1})\rangle + \mathcal{H}(\alpha(s_k^i),p(s_k^i) - p(s_k^{i-1}))\,.\end{aligned} \tag{4.22}$$

By definition of $\widehat{\mathcal{V}}_{\mathcal{H}}$ it follows that

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha(s_k^i),p;0,s_k^{i-1}) + \mathcal{H}(\alpha(s_k^i),p(s_k^i) - p(s_k^{i-1})) \le \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha(s_k^i),p;0,s_k^i)\,,$$

and then, recalling the definition of $\mathcal{E}_\lambda$, (4.22) implies that

$$\begin{aligned}\mathcal{E}_\lambda(\alpha(s_k^{i-1}),e(s_k^{i-1});p,s_k^{i-1}) \le{}& \mathcal{E}_\lambda(\alpha(s_k^i),e(s_k^i);p,s_k^i) + \mathcal{Q}(\alpha(s_k^i),Ew(s_k^{i-1}) - Ew(s_k^i))\\ &- \langle \sigma(s_k^i), Ew(s_k^i) - Ew(s_k^{i-1})\rangle + (1-\lambda)\mathcal{H}(\alpha(s_k^i),p(s_k^i) - p(s_k^{i-1}))\,.\end{aligned}$$

Now, following the same argument used in (4.13), we find that there exists a sequence $\omega_k \to 0^+$ such that

$$\begin{aligned}\mathcal{E}_\lambda(\alpha(s_k^{i-1}),\, e(s_k^{i-1});p,s_k^{i-1}) \le{}& \mathcal{E}_\lambda(\alpha(s_k^i),e(s_k^i);p,s_k^i) + (1-\lambda)\mathcal{H}(\alpha(s_k^i),p(s_k^i) - p(s_k^{i-1}))\\ &- \int_{s_k^{i-1}}^{s_k^i} \langle \sigma(s_k^i), E\dot{w}(t)\rangle dt + \omega_k \int_{s_k^{i-1}}^{s_k^i} \|E\dot{w}(t)\|_2 \,\mathrm{d}t\,.\end{aligned}$$

On $[0,t]$ we define the piecewise constant function $\overline{\sigma}_k(s) := \sigma(s_k^i)$, where i is the smallest index such that $s \le s_k^i$.

Since $\sum_i \mathcal{H}(\alpha(s_k^i), p(s_k^i) - p(s_k^{i-1})) \le \mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t)$, iterating the last inequality for $1 \le i \le k$ we obtain

$$\mathcal{E}(\alpha(0), e(0)) \le \mathcal{E}_\lambda(\alpha(t), e(t); p, t) + (1-\lambda)\mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t) - \int_0^t \langle \overline{\sigma}_k(s), E\dot{w}(s)\rangle ds + \delta_k \,,$$

where $\delta_k := \omega_k \int_0^T \|E\dot{w}(s)\|_2 \,\mathrm{d}s$. By Remark 4.2 the set of discontinuity points of $s \mapsto \alpha(s)$ and $s \mapsto e(s)$ is at most countable, and $\|\alpha(s)\|_\infty$, $\|e(s)\|_2$ are uniformly bounded in s. Therefore $\overline{\sigma}_k(s) \to \sigma(s)$ in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ for a.e. s, so that

$$\int_0^t \langle \sigma(s), E\dot{w}(s)\rangle \,\mathrm{d}s = \lim_{k\to\infty} \int_0^t \langle \overline{\sigma}_k(s), E\dot{w}(s)\rangle \,\mathrm{d}s \,,$$

by Dominated Convergence Theorem. This concludes the proof. □

5. Qualitative properties of quasistatic evolutions

In this section we show some qualitative properties of quasistatic evolutions, whose existence is proved in Theorem 4.3.

First, in Proposition 5.1, we deduce that $t \mapsto u(t)$, $t \mapsto e(t)$, and $t \mapsto p(t)$ are continuous, with respect to the norms of their spaces, at the continuity points for $t \mapsto \alpha(t)$ with respect to the uniform convergence in $\overline{\Omega}$. Then the time discontinuities of the quasistatic evolutions are at most countable, by Remark 4.2. This regularity in time of α also permits to say that $\mathcal{H}(\alpha(\bar{t}), \dot{p}(\bar{t}))$ represents the rate of plastic dissipation at $\bar{t}$, and then to understand the physical meaning of the term in λ in (qs1) (cf. Remark 5.2).

In Corollary 5.3 we derive from (qs1) Euler conditions with respect to the variation of u, e, and p, corresponding to equilibrium and stress constraint properties. In the last part of the section we assume suitable regularity properties on $\mathbb{C}$, D and $\mathcal{H}$, and absolute continuity of the evolutions. In Proposition 5.4 is shown an Euler condition for α and the differential counterpart of the energy balance (qs2): together with the irreversibility, these are Kuhn Tucker conditions (see e.g. [30] for this terminology) governing the evolution of the damage variable α. Moreover, it is deduced the Hill's maximum plastic work principle that, if p is regular enough, implies the Prandtl-Reuss flow rule with damage.

Throughout this section, we suppose that (2.4), (2.5), (2.9), and (2.26) hold when $\lambda = 0$; when $\lambda > 0$ we will assume also (2.12).

Except for countable many instants, every quasistatic evolution is continuous in time, as shown in the following result.

Proposition 5.1. *Every quasistatic evolution $t \mapsto (\alpha(t), u(t), e(t), p(t))$ is strongly continuous from $[0,T]$ into $C(\overline{\Omega}; [0,1]) \times BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$ except for a countable subset of $[0,T]$, which is the set of discontinuity points of α with respect to the uniform convergence in $\overline{\Omega}$.*

Proof. From the energy balance condition (qs2), written for a time interval $[s,t]$, we deduce

$$\begin{aligned} &Q(\alpha(t), e(t)) - Q(\alpha(s), e(s)) + \mathcal{H}(\alpha(t), p(t) - p(s)) \\ &\le \int_s^t \langle \sigma(\tau), E\dot{w}(\tau)\rangle \,\mathrm{d}\tau + D(\alpha(s)) - D(\alpha(t)) + \|\nabla(\alpha(s))\|_\gamma^\gamma - \|\nabla(\alpha(t))\|_\gamma^\gamma \,, \end{aligned}$$

using also (1) of Lemma A.1 both for $(1-\lambda)\mathcal{V}_{\mathcal{H}}(\alpha, p; s, t)$ and for $\lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha(t), p; s, t)$.

Notice now that

$$D(\alpha(s)) - D(\alpha(t)) + \|\nabla(\alpha(s))\|_\gamma^\gamma - \|\nabla(\alpha(t))\|_\gamma^\gamma \le 0 \,.$$

Indeed, if the term above were strictly positive, from (2.11b) and (2.5b) we would have

$$\mathcal{E}(\alpha(t), e(s)) + \lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha(t), p; 0, t) < \mathcal{E}(\alpha(s), e(s)) + \lambda\widehat{\mathcal{V}}_{\mathcal{H}}(\alpha(s), p; 0, t) \,,$$

which contradicts (qs1) since $(\alpha(t), (u(s), e(s), p(s))) \in \mathcal{D}(\alpha(s)) \times A(w(s))$.

Then

$$Q(\alpha(t),e(t))-Q(\alpha(s),e(s))+\mathcal{H}(\alpha(t),p(t)-p(s))\le\int_s^t\langle\sigma(\tau),E\dot w(\tau)\rangle\,\mathrm{d}\tau \tag{5.1}$$

Now, by Lemma 3.3 it follows that

$$-\langle\sigma(s),e(t)-e(s)-(Ew(t)-Ew(s))\rangle\le\mathcal{H}(\alpha(s),p(t)-p(s))\,, \tag{5.2}$$

because $(u(t)-u(s)-(w(t)-w(s)),e(t)-e(s)-(Ew(t)-Ew(s)),p(t)-p(s))\in A(0)$. Summing (5.1) and (5.2) we get

$$\begin{aligned}Q(\alpha(s),e(t)-e(s))\le{}&\frac12\langle\big[\mathbb{C}(\alpha(s))-\mathbb{C}(\alpha(t))\big]e(t),e(t)\rangle-\langle\sigma(s),Ew(t)-Ew(s)\rangle\\&+\int_s^t\langle\sigma(\tau),E\dot w(\tau)\rangle\,\mathrm{d}\tau+\mathcal{H}(\alpha(s),p(t)-p(s))-\mathcal{H}(\alpha(t),p(t)-p(s))\end{aligned}$$

which implies

$$\|e(t)-e(s)\|_2^2\le C\Big(\|\alpha(t)-\alpha(s)\|_\infty+\omega(\|\alpha(t)-\alpha(s)\|_\infty)+\|Ew(t)-Ew(s)\|_2\Big)\,, \tag{5.3}$$

where ω was introduced in (2.17) and C depends on $\mathrm{Lip}(\mathbb{C})$, γ_1, γ_2, and $\sup_t\|e(t)\|_2$ (recall that, from (qs2), the variation of p is bounded by such a C).

By (5.1), (2.16), and (5.3), we obtain

$$\|p(t)-p(s)\|_2^2\le\widetilde C\Big(\|\alpha(t)-\alpha(s)\|_\infty+\omega(\|\alpha(t)-\alpha(s)\|_\infty)+\|Ew(t)-Ew(s)\|_2\Big)\,,$$

$\widetilde C$ depending on C, r, and $\sup_t\|Ew(t)\|_2$. An analogous estimate holds for u, arguing as in [7, Theorem 3.8]. Then we conclude by Remark 4.2, where it is stated that the discontinuity points of $t\mapsto\alpha(t)$ with respect to the uniform convergence in $\overline\Omega$ are countable many. □

In order to establish the differential formulation of the energy balance the following remark turns to be useful. Moreover it allows us to explain the role of λ in the model.

Remark 5.2. If in addition $p\in AC([0,T];M_b(\overline\Omega;\mathbb{M}^{n\times n}_D))$ then

$$\mathcal{V}_{\mathcal{H}}(\alpha,p;0,t)=\int_0^t\mathcal{H}(\alpha(s),\dot p(s))\,\mathrm{d}s \tag{5.4}$$

for every $t\in[0,T]$.

Indeed this follows from Lemma A.1(4) in the Appendix, since $\alpha\colon[0,T]\to\mathbb{C}(\overline\Omega;[0,1])$ has at most countable many discontinuity points.

In the light of (5.4), we point out that the term in $\lambda\widehat{\mathcal{V}}_{\mathcal{H}}$ in (qs1) makes it easier to damage, at a given instant t, a part of the material more affected by plastic evolution until t: indeed, if $p\in AC([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_D))$ and $\beta\in C(\overline\Omega;[0,1])$, we get that

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\beta,p;0,t)=\int_\Omega\int_0^t H(\beta(x),\dot p(s,x))\,\mathrm{d}s\,\mathrm{d}x\,.$$

To fix the ideas, let us consider the simplest case of a multiplicative setting (see Remark 2.1) where $K(1)=B(1)$, the unit ball of $\mathbb{M}^{n\times n}_D$. Here the above formula reads as

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\beta,p;0,t)=\int_\Omega V(\beta(x))\Big(\int_0^t|\dot p(s,x)|\,\mathrm{d}s\Big)\,\mathrm{d}x\,.$$

By the monotonicity property of V, in order to minimize $\widehat{\mathcal{V}}_{\mathcal{H}}(\beta,p;0,t)$ in (qs1) it is convenient to take β smaller when the *cumulated plastic strain* $\int_0^t|\dot p(s,\cdot)|\,\mathrm{d}s$ is greater. Therefore the parameter λ is related to a fatigue phenomenon; when λ increases the cumulated plastic strain affects more seriously the damage growth.

The stability condition (qs1) and Lemma 3.3 imply the following result, which states Euler conditions with respect to variations of u, e, and p: (5.5a) is the equilibrium condition, while (5.5b) gives a constraint for the elastic stress.

Corollary 5.3. *Let $t \in [0,T] \mapsto (\alpha(t), u(t), e(t), p(t))$ be a quasistatic evolution corresponding to $\lambda \in [0,1]$. Then we have that for every $t \in [0,T]$:*

$$\operatorname{div}(\sigma(t)) = 0 \text{ in } \Omega\,, \tag{5.5a}$$

$$\sigma(t) \in \mathcal{K}_{\alpha(t)}(\Omega)\,. \tag{5.5b}$$

Let us now assume the multiplicative setting of Remark 2.1, C^1 regularity for $\mathbb{C}$, D, V, and absolute continuity for the quasistatic evolution. Then we can obtain a differential condition also for the damage variable α and a differential formulation of the energy balance.

Proposition 5.4. *Besides the assumptions* (2.4), (2.5), *and* (2.9), *let us assume that*

$$d \in C^1(\mathbb{R})\,, \tag{5.6a}$$

$$\mathbb{C} \in C^1(\mathbb{R}; Lin(\mathbb{M}^{n\times n}_{sym}; \mathbb{M}^{n\times n}_{sym}))\,, \tag{5.6b}$$

$$K(\alpha) = V(\alpha)K(1), \text{ with } K(1) \text{ closed and convex, } B_r(0) \subset K(1) \subset B_R(0),\ V \in C^1(\mathbb{R})\,. \tag{5.6c}$$

Let $t \in [0,T] \mapsto (\alpha(t), u(t), e(t), p(t))$ be a quasistatic evolution corresponding to $\lambda \in [0,1]$ absolutely continuous into $W^{1,\gamma}(\Omega;[0,1]) \times BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$. Then for every t the functional $\beta \mapsto \widehat{\mathcal{V}}_{\mathcal{H}}(\beta, p; 0, t)$ belongs to $C^1(C(\overline{\Omega}))$ and $W^{1,\gamma}(\Omega) \ni \beta \mapsto \mathcal{E}_\lambda(\beta, e(t); p, t)$ is differentiable at $\alpha(t)$ with Gâteaux derivative in the direction $\beta \in W^{1,\gamma}(\Omega)$

$$\begin{aligned}\langle \partial_\alpha \mathcal{E}_\lambda(\alpha(t), e(t); p, t), \beta\rangle &= \frac{1}{2}\langle \mathbb{C}'(\alpha(t))\beta e(t), e(t)\rangle + \langle D'(\alpha(t)), \beta\rangle \\ &\quad + \gamma \int_\Omega |\nabla\alpha(t)|^{\gamma-2}\nabla\alpha(t)\cdot\nabla\beta\,\mathrm{d}x + \lambda\left\langle \partial_\alpha \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha(t), p; 0, t), \beta\right\rangle\,.\end{aligned} \tag{5.7}$$

Moreover the following hold:

$$\langle \partial_\alpha \mathcal{E}_\lambda(\alpha(t), e(t); p, t), \beta\rangle \geq 0 \tag{5.8}$$

for every $t \in [0,T]$ and $\beta \in W^{1,\gamma}(\Omega)$, $\beta \leq 0$ in Ω,

$$\langle \partial_\alpha \mathcal{E}_\lambda(\alpha(t), e(t); p, t), \dot\alpha(t)\rangle = 0\,, \tag{5.9}$$

and

$$\mathcal{H}(\alpha(t), \dot p(t)) = \langle (\sigma(t))_D \,|\, \dot p(t)\rangle\,, \tag{5.10}$$

for a.e. $t \in (0,T)$, with $\sigma(t) := \mathbb{C}(\alpha(t))e(t)$.

Proof. By Dominated Convergence Theorem and (5.6) it follows that $\beta \mapsto \widehat{\mathcal{V}}_{\mathcal{H}}(\beta, p; 0, t) \in C^1(C(\overline{\Omega}))$ and that $W^{1,\gamma}(\Omega) \ni \beta \mapsto \mathcal{E}_\lambda(\beta, e(t); p, t)$ is differentiable at $\alpha(t)$ with Gâteaux derivative given by (5.7).

Let $t \in [0,T]$ and $\beta \in W^{1,\gamma}(\Omega)$, with $\beta \leq 0$ in Ω. Using $(\alpha(t) + h\beta, (u(t), e(t), p(t)))$ as a test pair in (qs1) for every $h > 0$, we get

$$\frac{\mathcal{E}_\lambda(\alpha(t) + h\beta, e(t); p, t) - \mathcal{E}_\lambda(\alpha(t), e(t); p, t)}{h} \geq 0\,,$$

and taking the limit as $h \to 0$ we deduce (5.8).

By [7, Lemma 5.5] we have that for a.e. $t \in (0,T)$

$$(\dot u(t), \dot e(t), \dot p(t)) \in A(\dot w(t))\,. \tag{5.11}$$

Thus, by (5.5a), (5.11), and the integration by parts formula (2.21) we get

$$\langle (\sigma(t))_D \,|\, \dot p(t)\rangle = \langle \sigma(t), E\dot w(t) - \dot e(t)\rangle \tag{5.12}$$

and by (qs2), recalling (5.4), it follows that for a.e. $t \in (0,T)$

$$\langle \sigma(t), \dot e(t)\rangle + \mathcal{H}(\alpha(t), \dot p(t)) + \langle \partial_\alpha \mathcal{E}_\lambda(\alpha(t), e(t); p, t), \dot\alpha(t)\rangle = \langle \sigma(t), E\dot w(t)\rangle\,. \tag{5.13}$$

From (5.12) and (5.13) we obtain that

$$\mathcal{H}(\alpha(t), \dot p(t)) - \langle (\sigma(t))_D \,|\, \dot p(t)\rangle + \langle \partial_\alpha \mathcal{E}_\lambda(\alpha(t), e(t); p, t), \dot\alpha(t)\rangle = 0 \tag{5.14}$$

for a.e. $t \in (0,T)$. Since by (5.5b) and (2.22) it follows that

$$\mathcal{H}(\alpha(t), \dot{p}(t)) - \langle (\sigma(t))_D \,|\, \dot{p}(t) \rangle \geq 0\,,$$

we conclude (5.9) and (5.10) by (5.14) and (5.8). □

We can now use the maximal dissipation property (5.10) (also called Hill's maximum plastic work principle) to show the validity of the elastoplastic flow rule $\mathcal{L}^n$–a.e. on the support $\{|\dot{p}(t)| > 0\}$ of the measure $\dot{p}(t)$. The following remark is useful to prove Proposition 5.6.

Remark 5.5. From (5.5b), (2.22), and (5.10) we deduce that for a.e. $t \in (0,T)$

$$H\left(\alpha(t), \frac{\mathrm{d}\dot{p}(t)}{\mathrm{d}|\dot{p}(t)|}\right)|\dot{p}(t)| = [\sigma_D(t) : \dot{p}(t)] \quad \text{as measures on } \overline{\Omega}, \tag{5.15}$$

where the measure denoted by square brackets is defined in (2.20).

Proposition 5.6 (Flow rule)**.** *In the hypoteses of Proposition 5.4, for a.e. $t \in (0,T)$*

$$\frac{\mathrm{d}\dot{p}(t)}{\mathrm{d}|\dot{p}(t)|}(x) \in N_{K(\alpha(t,x))}(\sigma_D(t,x)) \quad \textit{for } \mathcal{L}^n\textit{–a.e. } x \in \{|\dot{p}(t)| > 0\}\,, \tag{5.16}$$

where $\sigma_D(t,x)$ denotes the value of $\sigma_D(t)$ at the point x and $N_{K(\alpha(t,x))}(\sigma_D(t,x))$ is the normal cone to the closed convex set $K(\alpha(t,x))$ at $\sigma_D(t,x)$. In particular, if $\dot{p}(t) \in L^1(\Omega)$ for a.e. $t \in (0,T)$, we have that

$$\dot{p}(t,x) \in N_{K(\alpha(t,x))}(\sigma_D(t,x)) \quad \textit{for } \mathcal{L}^n\textit{–a.e. } x\,. \tag{5.17}$$

Proof. It is enough to argue as in the proof of [17, Theorem 3.13]. □

A. Auxiliary results

In this Appendix we analyse the particular variation used to define the plastic dissipation and show a property of monotone functions with values in L^p spaces.

A.1. **A "weighted" variation.** Let X be a Banach space, F a set, and $\mathcal{H}\colon F \times X \to \mathbb{R}^+ \cup \{0\}$. Given $\alpha\colon [0,T] \to F$, $p\colon [0,T] \to X$, $a,b \in [0,T]$ with $a < b$, and $\mathcal{P} := \{t_i\}_{0 \leq i \leq N}$ with $a = t_0 < t_1 < \cdots < t_N = b$, we define

$$\mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha, p; a, b) := \sum_{i=1}^{N} \mathcal{H}(\alpha(t_i), p(t_i) - p(t_{i-1}))\,.$$

and the *$\mathcal{H}$-variation of p with respect to α on $[a,b]$* as

$$\begin{aligned}\mathcal{V}_{\mathcal{H}}(\alpha, p; a, b) &:= \sup\Big\{ \sum_{i=1}^{N} \mathcal{H}(\alpha(t_i), p(t_i) - p(t_{i-1})) \,|\, a = t_0 < t_1 < \cdots < t_N = b,\, N \in \mathbb{N}\Big\} \\ &= \sup\Big\{ \mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha, p; a, b) \,|\, \mathcal{P} \text{ partition of } [a,b]\Big\}\,.\end{aligned} \tag{A.1}$$

When $\alpha(t) = \overline{\alpha} \in F$ for every t we use the symbols $\widehat{\mathcal{V}}_H$ and $\widehat{\mathcal{V}}^{\mathcal{P}}_{\mathcal{H}}$ instead of $\mathcal{V}_H$ and $\mathcal{V}^{\mathcal{P}}_{\mathcal{H}}$, so that

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\overline{\alpha}, p; a, b) := \mathcal{V}_{\mathcal{H}}(\alpha, p; a, b)|_{\alpha(t) = \overline{\alpha}}\,, \quad \widehat{\mathcal{V}}^{\mathcal{P}}_{\mathcal{H}}(\overline{\alpha}, p; a, b) := \mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha, p; a, b)|_{\alpha(t) = \overline{\alpha}}\,.$$

Let us assume that

$$\mathcal{H}(\alpha(t_2), f) \leq \mathcal{H}(\alpha(t_1), f), \text{ for every } \quad 0 \leq t_1 \leq t_2 \leq T,\, f \in X\,, \tag{A.2a}$$

$$\mathcal{H}(\beta, 0) = 0, \text{ for every } \quad \beta \in F\,, \tag{A.2b}$$

$$\mathcal{H}(\beta, f_1 + f_2) \leq \mathcal{H}(\beta, f_1) + \mathcal{H}(\beta, f_2), \quad \text{for every } \beta \in F,\, f_1, f_2 \in X\,. \tag{A.2c}$$

Lemma A.1. *With the notations and assumptions above, it follows that:*

(1) *If $\mathcal{P}_1, \mathcal{P}_2$ are partitions of $[a,b]$, with $\mathcal{P}_1 \subset \mathcal{P}_2$, then*

$$\mathcal{V}^{\mathcal{P}_1}_{\mathcal{H}}(\alpha, p; a, b) \leq \mathcal{V}^{\mathcal{P}_2}_{\mathcal{H}}(\alpha, p; a, b)\,.$$

(2) *For every* $p\colon [a,b]\to X$ *piecewise constant and continuous from the right, with discontinuities at the points* $s_1,\dots,s_N$ *with* $a<s_1<s_2<\dots<s_N\le b$,

$$\mathcal{V}_{\mathcal{H}}(\alpha,p;a,b)=\sum_{i=1}^{N}\mathcal{H}(\alpha(s_i),p(s_i)-p(s_{i-1}))\,,$$

where $s_0:=a$.

(3) *For every* $a\le t_1<t_2<t_3\le b$,

$$\mathcal{V}_{\mathcal{H}}(\alpha,p;t_1,t_3)=\mathcal{V}_{\mathcal{H}}(\alpha,p;t_1,t_2)+\mathcal{V}_{\mathcal{H}}(\alpha,p;t_2,t_3)\,.$$

(4) *Assume in addition that* F *is a measurable topological space,* X *is the dual of a separable Banach space* Y, $p\in AC([a,b];X)$, $\alpha\colon[a,b]\to F$ *is continuous for a.e.* $t\in[a,b]$, *and*

$$\mathcal{H}(\beta,tf)=t\mathcal{H}(\beta,f)\ \text{for every } \beta\in F,\ f\in X,\ \text{and } t>0\,, \tag{A.3a}$$

$$f\mapsto\mathcal{H}(\beta,f)\ \text{is weakly}^*\ \text{lower semicontinuous in } X \text{ for every } \beta\in F\,, \tag{A.3b}$$

$$\mathcal{H}(\beta_k,f)\to\mathcal{H}(\beta,f)\quad \text{for every } \beta_k\to\beta \text{ in } F \text{ and } f\in X\,. \tag{A.3c}$$

Then $t\mapsto\mathcal{H}(\alpha(t),\dot p(t))$ *is measurable and*

$$\mathcal{V}_{\mathcal{H}}(\alpha,p;a,b)=\int_a^b\mathcal{H}(\alpha(t),\dot p(t))\,\mathrm{d}t\,. \tag{A.4}$$

Proof. (1) It is enough to see that for every $a\le t_1\le t_2\le t_3\le b$

$$\mathcal{H}(\alpha(t_3),p(t_3)-p(t_1))\le\mathcal{H}(\alpha(t_3),p(t_3)-p(t_2))+\mathcal{H}(\alpha(t_2),p(t_2)-p(t_1))\,.$$

This is true because, by (A.2c), $\mathcal{H}(\alpha(t_3),p(t_3)-p(t_1))\le\mathcal{H}(\alpha(t_3),p(t_3)-p(t_2))+\mathcal{H}(\alpha(t_3),p(t_2)-p(t_1))$; apply then (A.2a) to the second term in the right-hand side.

(2) Observe firstly that given a partition $\mathcal{P}:=\{t_i\}_{0\le i\le M}$ of $[a,b]$ it is possible to choose a set of indices $1\le i_1<i_2<\dots<i_k\le N$ such that

$$\mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha,p;a,b)\le\sum_{j=1}^{k}\mathcal{H}(\alpha(s_{i_j}),p(s_{i_j})-p(s_{i_j-1}))\,. \tag{A.5}$$

In fact, if $s_i\le t_j<t_{j+1}<s_{i+1}$, then

$$\mathcal{H}(\alpha(t_{j+1}),p(t_{j+1})-p(t_j))=\mathcal{H}(\alpha(t_{j+1}),p(s_i)-p(s_i))=0\,,$$

while if $s_i\le t_j<s_{i+1}<\dots<s_{i+l}\le t_{j+1}<s_{i+l+1}$ it follows that

$$\mathcal{H}(\alpha(t_{j+1}),p(t_{j+1})-p(t_j))=\mathcal{H}(\alpha(t_{j+1}),p(s_{i+l})-p(s_i))\le\mathcal{H}(\alpha(s_{i+l}),p(s_{i+l})-p(s_i))\,,$$

by (A.2b) and (A.2a). From (1) and (A.5), for every $\mathcal{P}$ partition of $[a,b]$ the inequalities

$$\mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha,p;a,b)\le\sum_{i=1}^{N}\mathcal{H}(\alpha(s_i),p(s_i)-p(s_{i-1}))\le\mathcal{V}_{\mathcal{H}}(\alpha,p;a,b)$$

hold. The conclusion follows by taking the supremum over the partitions of $[a,b]$.

(3) It is always true that $\mathcal{V}_{\mathcal{H}}(\alpha,p;t_1,t_3)\ge\mathcal{V}_{\mathcal{H}}(\alpha,p;t_1,t_2)+\mathcal{V}_{\mathcal{H}}(\alpha,p;t_2,t_3)$ because for every partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[t_1,t_2]$ and $[t_2,t_3]$, $\mathcal{P}:=\mathcal{P}_1\cup\mathcal{P}_2$ is a partition of $[t_1,t_3]$. On the other hand, for every $\mathcal{P}$ partition of $[t_1,t_3]$, $\tilde{\mathcal{P}}:=\mathcal{P}\cup\{t_2\}$ is the union of two partitions of $[t_1,t_2]$ and $[t_2,t_3]$ respectively; since, by (1),

$$\mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha,p;a,b)\le\mathcal{V}^{\tilde{\mathcal{P}}}_{\mathcal{H}}(\alpha,p;a,b)\,,$$

the latter inequality holds.

(4) From (A.2c), (A.3a), and (A.3b), we have that for every $\beta\in F$ the function $f\mapsto\mathcal{H}(\beta,f)$ is weakly* lower semicontinuous, convex and positively one-homogeneous. Then, by [20, Theorem 5], for every $\beta\in F$ there exists a bounded closed convex set $\mathcal{K}_\beta\subset Y$ such that

$$\mathcal{H}(\beta,f)=\sup_{y\in\mathcal{K}_\beta}\langle y,f\rangle\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and Y. Being Y separable, we get $\mathcal{H}(\beta,f)=\sup_{y\in\mathcal{K}^0_\beta}\langle y,f\rangle$, where $\mathcal{K}^0_\beta$ is a countable dense subset of $\mathcal{K}_\beta$.

Since $p \in AC([a,b];X)$, the weak*-limit

$$\dot{p}(t) := w^*\text{-}\lim_{s\to t} \frac{p(s)-p(t)}{s-t}$$

exists for a.e. $t \in [a,b]$, and then the function $t \mapsto \langle y, \dot{p}(t)\rangle$ is measurable for every $y \in Y$. Therefore $t \mapsto \mathcal{H}(\beta, \dot{p}(t))$ is measurable for every $\beta \in F$. Moreover, from [7, Theorem 7.1],

$$\widehat{\mathcal{V}}_{\mathcal{H}}(\beta, p; t_1, t_2) = \int_{t_1}^{t_2} \mathcal{H}(\beta, \dot{p}(t))\,\mathrm{d}t\,, \tag{A.6}$$

for every $a \le t_1 < t_2 \le b$ and every $\beta \in F$.

Let us fix $\varepsilon > 0$. There exist points $t_0, \dots, t_N$, with $a = t_0 < t_1 < t_2 < \cdots < t_N \le b$, such that

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t) - \varepsilon \le \sum_{i=1}^{N} \mathcal{H}(\alpha(t_i), p(t_i) - p(t_{i-1}))\,. \tag{A.7}$$

For every $k \in \mathbb{N}$ we consider the set $\left(a + i\frac{b-a}{k}\right)_{i=0}^{k} \cup (t_j)_{j=1}^{N} =: s_0^k < s_1^k < \cdots < s_{M(k)}^K$, with $s_0^k = a$, and we define α_k as

$$\alpha_k(t) := \alpha(s_{j+1}) \quad \text{when } t \in (s_j, s_{j+1}]$$

and $\alpha_k(a) = \alpha(a)$. In other words α_k is the left-continuous piecewise constant interpolation of α with nodes $(s_j)_j$. By construction

$$\alpha_k(t_j) = \alpha(t_j) \quad \text{for every } j \in \{1, \dots, N\} \tag{A.8}$$

and by (A.2a) and (A.3c) we get that for every $f \in X$

$$\mathcal{H}(\alpha_k(s), f) \le \mathcal{H}(\alpha_{k+1}(s), f) \le \mathcal{H}(\alpha(s), f) \tag{A.9}$$

for every $s \in [a,b]$, and

$$\mathcal{H}(\alpha_k(s), f) \to \mathcal{H}(\alpha(s), f) \tag{A.10}$$

for every s continuity point of α.

Since the functions α_k are piecewise constant, from the point (3) and (A.6) we have that

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(\alpha_k, p; a, b) &= \sum_{j=1}^{M(k)} \mathcal{V}_{\mathcal{H}}(\alpha_k, p; s_{j-1}^k, s_j^k) = \sum_{j=1}^{M(k)} \widehat{\mathcal{V}}_{\mathcal{H}}(\alpha_k(s_j^k), p; s_{j-1}^k, s_j^k) \\ &= \sum_{j=1}^{M(k)} \int_{s_{j-1}}^{s_j} \mathcal{H}(\alpha_k(s_j^k), \dot{p}(t))\,\mathrm{d}t = \int_a^b \mathcal{H}(\alpha_k(t), \dot{p}(t))\,\mathrm{d}t\,. \end{aligned} \tag{A.11}$$

Moreover the fact that α is continuous for a.e. $t \in [a,b]$ and (A.10) imply that

$$t \mapsto \mathcal{H}(\alpha(t), \dot{p}(t)) \text{ is measurable}\,,$$

as well as

$$\int_a^b \mathcal{H}(\alpha(t), \dot{p}(t))\,\mathrm{d}t = \lim_{k\to\infty} \int_a^b \mathcal{H}(\alpha_k(t), \dot{p}(t))\,\mathrm{d}t\,, \tag{A.12}$$

using the Monotone Convergence Theorem.

By (A.7), (A.8), and (A.9) we obtain

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; a, b) - \varepsilon \le \sum_{i=1}^{N} \mathcal{H}(\alpha_k(t_i), p(t_i) - p(t_{i-1})) \le \mathcal{V}_{\mathcal{H}}(\alpha_k, p; a, b) \le \mathcal{V}_{\mathcal{H}}(\alpha, p; a, b)\,,$$

and using (A.11) and (A.12) we can pass to the limit as $k \to \infty$ and get

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; a, b) - \varepsilon \le \int_a^b \mathcal{H}(\alpha(t), \dot{p}(t))\,\mathrm{d}t \le \mathcal{V}_{\mathcal{H}}(\alpha, p; a, b)\,.$$

We therefore conclude since ε is arbitrary.

□

A.2. **A remark about monotone functions from time into L^p spaces.**

Lemma A.2. *Let (X,μ) a measure space with $\mu(X)<\infty$, and $\alpha\colon [0,T]\to L^\infty(X,\mu)$ such that $\|\alpha(t)\|_\infty\le M$ for every $t\in[0,T]$ and*

$$\alpha(t_2)\le\alpha(t_1)\,\mu\text{-a.e. in } X \text{ for every } t_1\le t_2\,. \tag{A.13}$$

Then there exists a countable set $E\subset[0,T]$ such that for every $1\le p<\infty$ the function α is continuous in every $t\in[0,T]\setminus E$ with respect to the $L^p(X,\mu)$ norm.

Proof. For every $s\in(0,T]$ and $t\in[0,T)$ we define

$$\alpha^-(s):=\inf_{n\in\mathbb{N}}\alpha(t_n^-),\quad \alpha^+(t):=\sup_{n\in\mathbb{N}}\alpha(t_n^+)\,,$$

where $t_n^-<s$ and $t<t_n^+$ are sequences in $[0,T]$ convergent to s and t, and

$$\alpha^-(0):=\alpha(0),\quad \alpha^+(T):=\alpha(T)\,.$$

By (A.13) these definitions are well posed. Indeed, let for instance $t<s_n^+$ be a sequence that converges to t, and $\tilde\alpha(t^+):=\sup_{n\in\mathbb{N}}\alpha(s_n^+)$. For every $m\in\mathbb{N}$, there exists n_m such that $t<s_n^+\le t_m^+$ for every $n\ge n_m$: therefore $\tilde\alpha(t^+)\ge\alpha(s_n^+)\ge\alpha(t_m^+)$ for every m, and $\tilde\alpha(t^+)\ge\alpha(t^+)$, taking the supremum over m. The opposite inequality follows by interchanging the two sequences. Moreover for every $t\in[0,T]$

$$\alpha(t_n^+)\to\alpha^+(t),\quad \alpha(t_n^-)\to\alpha^-(t)\quad \text{strongly in } L^p(X,\mu)\,, \tag{A.14}$$

by Monotone Convergence Theorem and (A.13) again, and

$$\alpha^-(t)\ge\alpha(t)\ge\alpha^+(t)\,,$$

for every $t\in[0,T]$. Let us consider now the function

$$g(t):=\int_X\big(\alpha^-(t)-\alpha^+(t)\big)\,\mathrm{d}\mu\,.$$

It takes values in $\mathbb{R}^+\cup\{0\}$ and for every $t_1<\dots<t_k\in E:=\{t\in[0,T]|\,g(t)>0\}$ we get, using in particular (A.13), that

$$\sum_{i=1}^k g(t_i)\le\int_X\big(\alpha^-(t_1)-\alpha^+(t_k)\big)\,\mathrm{d}\mu\le 2M\mu(X)\,.$$

By a standard argument, we deduce that E is a countable set. By definition of E, $\alpha^+(t)=\alpha^-(t)=\alpha(t)$ μ-a.e. for every $t\in[0,T]\setminus E$ and we conclude by (A.14). □

Acknowledgments. The author wishes to thank Gianni Dal Maso and Giuliano Lazzaroni for having proposed the problem and for many helpful discussions. This work is part of the Project "Calculus of Variations" 2010-11, supported by the Italian Ministry of Education, University, and Research.

References

[1] R. Alessi, J.-J. Marigo, and S. Vidoli, *Gradient damage models coupled with plasticity and nucleation of cohesive cracks*, Arch. Ration. Mech. Anal., 214 (2014), pp. 575–615.

[2] ———, *Gradient damage models coupled with plasticity: Variational formulation and main properties*, Mechanics of Materials, 80 (2015), pp. 351–367.

[3] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[4] G. Bouchitté, A. Mielke, and T. Roubíček, *A complete-damage problem at small strains*, Z. Angew. Math. Phys., 60 (2009), pp. 205–236.

[5] H. Brezis, *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam-London; American Elsevier, New York, 1973.

[6] V. Crismale and G. Lazzaroni, *Viscous approximation of evolutions for a coupled elastoplastic-damage model*. Preprint SISSA 05/2015/MATE, 2015.

[7] G. Dal Maso, A. DeSimone, and M. G. Mora, *Quasistatic evolution problems for linearly elastic-perfectly plastic materials*, Arch. Ration. Mech. Anal., 180 (2006), pp. 237–291.

[8] G. Dal Maso, A. DeSimone, M. G. Mora, and M. Morini, *Globally stable quasistatic evolution in plasticity with softening*, Netw. Heterog. Media, 3 (2008), pp. 567–614.

[9] ———, *A vanishing viscosity approach to quasistatic evolution in plasticity with softening*, Arch. Ration. Mech. Anal., 189 (2008), pp. 469–544.
[10] G. Dal Maso, A. DeSimone, and F. Solombrino, *Quasistatic evolution for Cam-Clay plasticity: a weak formulation via viscoplastic regularization and time rescaling*, Calc. Var. Partial Differ. Equ., 40 (2011), pp. 125–181.
[11] ———, *Quasistatic evolution for Cam-Clay plasticity: properties of the viscosity solution*, Calc. Var. Partial Differential Equations, 44 (2012), pp. 495–541.
[12] G. Dal Maso and G. Lazzaroni, *Quasistatic crack growth in finite elasticity with non-interpenetration*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), pp. 257–290.
[13] G. Dal Maso and R. Toader, *A model for the quasi-static growth of brittle fractures: existence and approximation results*, Arch. Ration. Mech. Anal., 162 (2002), pp. 101–135.
[14] ———, *Quasistatic crack growth in elasto-plastic materials: the two-dimensional case*, Arch. Ration. Mech. Anal., 196 (2010), pp. 867–906.
[15] M. Duchoň and P. Maličký, *A Helly theorem for functions with values in metric spaces*, Tatra Mt. Math. Publ., 44 (2009), pp. 159–168.
[16] A. Fiaschi, D. Knees, and U. Stefanelli, *Young-measure quasi-static damage evolution*, Arch. Ration. Mech. Anal., 203 (2012), pp. 415–453.
[17] G. A. Francfort and A. Giacomini, *Small-strain heterogeneous elastoplasticity revisited*, Comm. Pure Appl. Math., 65 (2012), pp. 1185–1241.
[18] M. Frémond and B. Nedjar, *Damage, gradient of damage and principle of virtual power*, Internat. J. Solids Structures, 33 (1996), pp. 1083–1103.
[19] C. Goffman and J. Serrin, *Sublinear functions of measures and variational integrals*, Duke Math. J., 31 (1964), pp. 159–178.
[20] L. Hörmander, *Sur la fonction d'appui des ensembles convexes dans un espace localement convexe*, Ark. Math., 3 (1954), pp. 181–186.
[21] D. Knees, R. Rossi, and C. Zanini, *A vanishing viscosity approach to a rate-independent damage model*, Math. Models Methods Appl. Sci., 23 (2013), pp. 565–616.
[22] ———, *A quasilinear differential inclusion for viscous and rate-independent damage systems in non-smooth domains*, Preprint 2013.
[23] H. Matthies, G. Strang, and E. Christiansen, *The saddle point of a differential program*, in Energy Methods in Finite Element Analysis, Z. O. Glowinski R., Rodin E., ed., Wiley, New York, 1979, pp. 309–318.
[24] A. Mielke, *Energetic formulation of multiplicative elasto-plasticity using dissipation distances*, Contin. Mech. Thermodyn., 15 (2003), pp. 351–382.
[25] ———, *Evolution of rate-independent systems*, in Evolutionary equations. Vol. II, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2005, pp. 461–559.
[26] A. Mielke, R. Rossi, and G. Savaré, *BV solutions and viscosity approximations of rate-independent systems*, ESAIM Control Optim. Calc. Var., 18 (2012), pp. 36–80.
[27] ———, *Balanced viscosity (BV) solutions to infinite-dimensional rate-independent systems*. to appear on J. Eur. Math. Soc., 2015.
[28] A. Mielke and T. Roubíček, *Rate-independent damage processes in nonlinear elasticity*, Math. Models Methods Appl. Sci., 16 (2006), pp. 177–209.
[29] K. Pham and J.-J. Marigo, *Approche variationnelle de l'endommagement: II. Les modèles à gradient*, Comptes Rendus Mécanique, 338(4) (2010), pp. 199–206.
[30] ———, *From the onset of damage to rupture: construction of responses with damage localization for a general class of gradient damage models*, Contin. Mech. Thermodyn., 25 (2013), pp. 147–171.
[31] Y. G. Reshetnyak, *Weak convergence of completely additive vector functions on a set*, Siberian Math. J., 9 (1968), pp. 1039–1045.
[32] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1966.
[33] M. Sofonea, W. Han, and M. Shillor, *Analysis and approximation of contact problems with adhesion or damage*, vol. 276 of Pure and Applied Mathematics (Boca Raton), Chapman & Hall/CRC, Boca Raton, FL, 2006.
[34] F. Solombrino, *Quasistatic evolution problems for nonhomogeneous elastic plastic materials*, J. Convex Anal., 16 (2009), pp. 89–119.
[35] R. Temam, *Mathematical problems in plasticity*, Gauthier-Villars, Paris, 1985. Translation of Problèmes mathématiques en plasticité. Gauthier-Villars, Paris, 1983.
[36] R. Temam and G. Strang, *Duality and relaxation in the variational problem of plasticity*, J. Mécanique, 19 (1980), pp. 493–527.
[37] M. Thomas, *Quasistatic damage evolution with spatial BV-regularization*, Discrete Contin. Dyn. Syst. Ser. S, 6 (2013), pp. 235–255.
[38] M. Thomas and A. Mielke, *Damage of nonlinearly elastic materials at small strain—existence and regularity results*, ZAMM Z. Angew. Math. Mech., 90 (2010), pp. 88–112.

SISSA, Via Bonomea 265, 34136 Trieste, Italy
E-mail address: vito.crismale@sissa.it