DENSITY IN *SBD* AND APPROXIMATION OF FRACTURE ENERGIES

VITO CRISMALE

Abstract. We prove three density theorems, in the strong BD topology, for the three subspaces of SBD functions: SBD; SBD^p_∞, where the absolutely continuous part of the symmetric gradient is in L^p, with $p>1$; SBD^p, whose functions are in SBD^p_∞ and the jump set has finite $\mathcal{H}^{n-1}$-measure. We compare them with existing results, discussing related approximation of fracture energies.

Keywords: special bounded deformation functions, strong approximation, Γ-convergence, free discontinuity problems, cohesive fracture

MSC 2010: 49Q20, 26A45, 49J45, 74R99, 35Q74.

1. Main results and comments

Special Bounded Deformation (SBD) functions have been introduced by Ambrosio, Coscia, and Dal Maso [2], as the Bounded Deformation (BD) functions whose symmetric distributional gradient $\mathrm{E}u = \frac{\mathrm{D}u + \mathrm{D}u^T}{2}$ has no Cantor part. Given $\Omega \subset \mathbb{R}^n$ open bounded, $u\colon \Omega \to \mathbb{R}^n$ is in $SBD(\Omega)$ if it is in L^1 and the bounded Radon measure $\mathrm{E}u$ has the form

$$\mathrm{E}u = e(u)\mathcal{L}^n + ([u] \odot \nu_u)(x)\mathcal{H}^{n-1} \llcorner J_u\,,$$

where $e(u)$ is the density of $\mathrm{E}u$ with respect to $\mathcal{L}^n$, the *jump set* J_u is the set of points x at which u has two different approximate limits $u^+(x)$, $u^-(x)$ with respect to a suitable direction $\nu_u(x)$, and $[u](x) := u^+(x) - u^-(x)$ is the *jump* ($\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ are the n-dimensional Lebesgue and the $(n{-}1)$-dimensional Hausdorff measures, $\odot$ the symmetric tensor product).

For $p>1$, consider also the subspaces

$$SBD^p(\Omega) := \{u \in SBD(\Omega)\colon e(u) \in L^p(\Omega;\mathbb{M}^{n\times n}_{sym})\,,\ \mathcal{H}^{n-1}(J_u) < \infty\}$$

and

$$SBD^p_\infty(\Omega) := \{u \in SBD(\Omega)\colon e(u) \in L^p(\Omega;\mathbb{M}^{n\times n}_{sym})\}\,.$$

We prove the following density results for these spaces, through functions in

$$\begin{aligned}\mathcal{U}(\Omega;\mathbb{R}^n):=\{v\in SBV(\Omega;\mathbb{R}^n)\cap L^\infty(\Omega;\mathbb{R}^n)\colon & J_v \text{ closed and included in a}\\ & \text{finite union of closed connected pieces of } C^1 \text{ hypersurfaces},\\ & v\in C^\infty(\overline{\Omega}\setminus J_v;\mathbb{R}^n)\cap W^{m,\infty}(\Omega\setminus J_v;\mathbb{R}^n) \text{ for all } m\in\mathbb{N}\}\end{aligned}$$

assuming Ω Lipschitz (or, more in general, that the trace of u be well defined and integrable on $\partial\Omega$). Notice that the properties on the jump sets are attained up to $\mathcal{H}^{n-1}$-negligible sets, that is *essentially* attained.

Theorem 1.1. *Let $u\in SBD^p(\Omega)$. Then there exist $u_k\in\mathcal{U}(\Omega;\mathbb{R}^n)$ such that*

$$\lim_{k\to\infty}\Big(\|u_k-u\|_{BD(\Omega)}+\|e(u_k)-e(u)\|_{L^p(\Omega;\mathbb{M}^{n\times n}_{sym})}+\mathcal{H}^{n-1}(J_{u_k}\triangle J_u)\Big)=0\,.$$

Moreover, (if $p\in\big[1,\frac{n}{n-1}\big]$ this is trivial) there are Borel sets $E_k\subset\Omega$ such that

$$\lim_{k\to\infty}\mathcal{L}^n(E_k)=\lim_{k\to\infty}\int\limits_{\Omega\setminus E_k}|u_k-u|^p\,\mathrm{d}x=0\,.$$

Theorem 1.2. *Let $u\in SBD(\Omega)$. Then there exist $u_k\in\mathcal{U}(\Omega;\mathbb{R}^n)$ such that J_{u_k} is (essentially) a finite union of pairwise disjoint C^1 compact hypersurfaces strictly contained in Ω and*

$$\lim_{k\to\infty}\big(\|u_k-u\|_{BD(\Omega)}+\mathcal{H}^{n-1}(J_{u_k}\setminus J_u)\big)=0\,.$$

Theorem 1.3. *Let $u\in SBD^p_\infty(\Omega)$, with $p>1$. Then there exist $u_k\in\mathcal{U}(\Omega;\mathbb{R}^n)$ such that*

$$\lim_{k\to\infty}\Big(\|u_k-u\|_{BD(\Omega)}+\|e(u_k)-e(u)\|_{L^p(\Omega;\mathbb{M}^{n\times n}_{sym})}\Big)=0\,.$$

We compare below these theorems with existing density results of two types: those for subspaces of SBV (the space of BV functions whose distributional gradient has no Cantor part, see e.g. [3]); those for SBD and for $GSBD$, the space of *Generalised-SBD* functions.

APPROXIMATIONS FOR SBV

The first approximation in BV-norm for $SBV^p\cap L^\infty$ functions is due to Braides and Chiadò-Piat [7]: the approximating functions u_k are C^1 outside some closed countably rectifiable sets R_k (in the sense of [24, 3.2.14]) and $J_{u_k}\subset R_k$, with no information on the shape of J_{u_k}.

De Philippis, Fusco, and Pratelli [23] have recently proven three approximations for SBV^p, SBV, SBV^p_∞, in BV-norm, through functions in

$$\mathcal{V}(\Omega;\mathbb{R}):=\{v\in SBV(\Omega)\colon J_v\Subset\Omega \text{ closed } C^1 \text{ manifold}, v\in C^\infty(\Omega\setminus J_v)\}\,,$$

under weak regularity assumptions on Ω similar to those in theorems above. These read as follows (∇u denotes the density of the absolutely continuous part of the distributional gradient $\mathrm{D}u$ with respect to $\mathcal{L}^n$):

Theorem 1.4 ([23], Theorems A, B, C)**.** *The following holds:*

- *If* $u \in SBV^p(\Omega)$*, there exist* $u_k \in \mathcal{V}(\Omega;\mathbb{R})$ *such that*

$$\lim_{k\to\infty} \Big(\|u_k - u\|_{BV(\Omega)} + \|\nabla u_k - \nabla u\|_{L^p(\Omega;\mathbb{R}^n)} + \mathcal{H}^{n-1}(J_{u_k} \triangle J_u)\Big) = 0\,;$$

- *If* $u \in SBV(\Omega)$*, there exist* $u_k \in \mathcal{V}(\Omega;\mathbb{R})$ *such that*

$$\lim_{k\to\infty} \big(\|u_k - u\|_{BV(\Omega)} + \mathcal{H}^{n-1}(J_{u_k} \setminus J_u)\big) = 0\,;$$

- *If* $u \in SBV^p_\infty(\Omega)$*, there exist* $u_k \in \mathcal{V}(\Omega;\mathbb{R})$ *such that*

$$\lim_{k\to\infty} \Big(\|u_k - u\|_{BV(\Omega)} + \|\nabla u_k - \nabla u\|_{L^p(\Omega;\mathbb{R}^n)}\Big) = 0\,.$$

We observe that the "distances" of u_k from u are analogous to those in our results (with BV, ∇u in place of BD, $e(u)$), while the main difference lies in the classes $\mathcal{U}$ and $\mathcal{V}$ (in SBV one may consider also vector valued functions, arguing componentwise). The functions in $\mathcal{U}$ are regular up to both the boundary and the jump set, while in $\mathcal{V}$ only in the interior. On the other hand the jump set of functions in $\mathcal{U}$ is less regular, since the C^1 hypersurfaces could overlap. However, this regularity could be improved by an argument in [23] (see Lemma 5.2 and Part B in proof of Theorem C therein) or by the capacitary argument in [19, Corollary 3.11]. In Theorem 1.2 we are able to separate the manifolds one from each other, obtaining a complete generalisation of the corresponding SBV result.

As the analogous for SBV, Theorems 1.1 and 1.2 are sharp, since they strongly approximate all the relevant quantities in the definition of SBD^p and SBD and also the measure of $J_{u_k} \setminus J_u$, while in Theorem 1.3 we do not control $\mathcal{H}^{n-1}(J_{u_k} \setminus J_u)$.

A further SBV-approximation result has been proven by Cortesani and Toader [20]. The approximating functions are in the class

$$\begin{aligned}\mathcal{W}(\Omega;\mathbb{R}) := \{u \in SBV(\Omega)\colon\ & J_u \text{ the intersection of } \Omega \text{ with a finite union of}\\ & (n{-}1)\text{-dimensional closed simplexes}, u \in W^{m,\infty}(\Omega \setminus J_u) \text{ for all } m\}.\end{aligned}$$

Theorem 1.5 ([20], Theorem 3.1)**.** *Let* $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$*. There exist* $u_k \in \mathcal{W}(\Omega;\mathbb{R})$ *such that*

$$\lim_{k\to\infty} \Big(\|u_k - u\|_{L^1(\Omega)} + \|\nabla u_k - \nabla u\|_{L^p(\Omega;\mathbb{R}^n)} + \mathcal{H}^{n-1}(J_{u_k} \triangle J_u)\Big) = 0\,,$$

$$\lim_{k\to\infty} \int\limits_{J_{u_k}} \phi(x, u_k^+, u_k^-, \nu_{u_k})\,\mathrm{d}\mathcal{H}^{n-1} = \int\limits_{J_u} \phi(x, u^+, u^-, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\,,$$

for every ϕ *strictly positive, continuous, and* BV*-elliptic (see e.g.* [1] *or* [20, equation (2.4)] *for the notion of* BV*-ellipticity).*

The approximation is not in BV-norm, since the geometry of the jump set changes, and u is required to be in L^∞. This result could be however used in combination with the previous theorems, that do not assume any integrability on u and give approximants $u_k \in L^\infty$.

Approximations for SBD and $GSBD$

The space SBD has been introduced to represent displacements in elastic materials with fractures. The *elastic strain* corresponds to $e(u)$, the *crack* to J_u. The first density result in SBD is the following, due to Chambolle [9, 10].

Theorem 1.6 ([10], Theorem 1)**.** *Let* $u \in SBD^2(\Omega) \cap L^2(\Omega;\mathbb{R}^n)$. *There exist* $u_k \in \mathcal{U}(\Omega;\mathbb{R}^n)$ *such that*

$$\lim_{k\to\infty} \Big(\|u_k - u\|_{L^2(\Omega;\mathbb{R}^n)} + \|e(u_k) - e(u)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{sym})} + \mathcal{H}^{n-1}(J_{u_k} \triangle J_u) \Big) = 0\,.$$

The main improvement by Theorem 1.1 is that we approximate also the jump part of Eu. Moreover, we do not assume any *a priori* integrability on u, and we consider SBD^p with any $p > 1$, not necessarily $p = 2$.

Density theorem are in general very useful to prove Γ-convergence approximations of energies, through more regular ones. Theorem 1.6 has been developed to study the *brittle fracture* Griffith energy [29, 26]

$$\int\limits_\Omega \mathbb{C}e(u)\colon e(u)\,\mathrm{d}x + \mathcal{H}^{n-1}(J_u)\,, \tag{G}$$

sum of the elastic bulk energy ($\mathbb{C}$ being the Cauchy stress tensor) and the surface energy dissipated in the crack. Theorem 1.1 permits to approximate also energies depending on the jump amplitude $[u]$, such as

$$\int\limits_\Omega \mathbb{C}e(u)\colon e(u)\,\mathrm{d}x + \mathcal{H}^{n-1}(J_u) + \int\limits_{J_u} \big|[u] \odot \nu_u\big|\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{C}$$

considered by Focardi and Iurlano in [25] (see also [8]).

Actually, SBD^2 is the right ambient space for (G) only for displacements in L^∞ (see [5]). The proper space is in fact $GSBD^2$, introduced by Dal Maso [22] requiring only the SBD^2 slicing properties to hold, and not even $u \in L^1$. With Antonin Chambolle, we recently proved a sharp approximation result for $GSBD^p$ in [13], removing the simplifying assumption of dimension 2 in [27] and of L^p-integrability in [30, 17]. This permits to prove the following $GSBD$ counterpart of $GSBV$ Ambrosio-Tortorelli approximation [4], widely used in Fracture Mechanics for numerical simulations (see e.g. [6]). Moreover, we have a suitable convergence of minimisers to a minimiser for the Dirichlet problem, whose existence has been shown in [14] (in [28] in 2d). To simplify the notation, we assume the boundary datum on all $\partial\Omega$ and Ω star-shaped. We denote by $\mathrm{tr}_{\partial\Omega}$ the trace on $\partial\Omega$.

Theorem 1.7 ([13, 14])**.** *Let* $u_0 \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, ε_k, $\eta_k > 0$ *with* $\varepsilon_k \to 0$, $\frac{\eta_k}{\varepsilon_k} \to 0$ *as* $k \to \infty$, $H^1_{u_0}(\Omega;\mathbb{R}^n) := u_0 + H^1_0(\Omega;\mathbb{R}^n)$, *and* $V^1_k := \{v \in 1 + H^1_0(\Omega)\colon \eta_k \leq v \leq 1\}$. *Then*

$$D^2_k(u,v) := \begin{cases} \displaystyle\int\limits_\Omega \Big(v\,\mathbb{C}e(u)\colon e(u) + \frac{(1-v)^2}{4\varepsilon_k} + \varepsilon_k|\nabla v|^2\Big)\,\mathrm{d}x & \textit{in } H^1_{u_0}(\Omega;\mathbb{R}^n)\times V^1_k\,,\\ +\infty & \textit{otherwise,}\end{cases}$$

Γ*-converge with respect to the topology of convergence in measure for* u *and* v *to*

$$D^2(u,v) := \begin{cases} \displaystyle\int_\Omega \mathbb{C}e(u)\colon e(u)\,\mathrm{d}x + \mathcal{H}^{n-1}\big(J_u \cup \{\mathrm{tr}_{\partial\Omega}u \neq \mathrm{tr}_{\partial\Omega}u_0\}\big) & \text{in } GSBD^2(\Omega)\times\{v=1\},\\ +\infty & \text{otherwise.}\end{cases}$$

Moreover, if $(u_k, v_k) \in H^1_{u_0}(\Omega;\mathbb{R}^n)\times V^1_k$ *are minimisers of* D^2_k*, then, for a subsequence* (u_h, v_h)*,* v_h *converges to 1 in* $L^1(\Omega)$*, the set* $A := \{x \in \Omega\colon |u_h(x)| \to \infty\}$ *has finite perimeter, and there exists* $u \in GSBD(\Omega)$ *minimiser of* D^2 *with* $u = 0$ *in* A*, such that* $\partial^* A \subset J_u$*,* $u_h \to u$ $\mathcal{L}^n$*-a.e. in* $\Omega \setminus A$*,*

$$\int_\Omega \mathbb{C}e(u)\colon e(u)\,\mathrm{d}x = \lim_{h\to\infty}\int_\Omega v_h\,\mathbb{C}e(u_h)\colon e(u_h)\,\mathrm{d}x\,, \tag{1.2a}$$

$$\mathcal{H}^{n-1}(J_u) = \lim_{h\to\infty}\int_\Omega \Big(\frac{(1-v_h)^2}{4\varepsilon_h} + \varepsilon_h|\nabla v_h|^2\Big)\,\mathrm{d}x\,. \tag{1.2b}$$

Conversely, the energy space for (C) is SBD^2. In [25] (C) is obtained, assuming an *a priori* L^∞ bound on displacements, by a phase-field approximation, with the difference that now v is in $\widehat{V}^1_k := \{v \in 1 + H^1_0(\Omega)\colon \varepsilon_k \le v \le 1\}$. We remove any assumption on u, obtaning the following result (with the notation of Theorem 1.7).

Theorem 1.8 ([21, 16])**.** *The functionals*

$$\widehat{D}^2_k(u,v) := \begin{cases} \displaystyle\int_\Omega \Big(v\,\mathbb{C}e(u)\colon e(u) + \frac{(1-v)^2}{4\varepsilon_k} + \varepsilon_k|\nabla v|^2\Big)\,\mathrm{d}x & \text{in } H^1_{u_0}(\Omega;\mathbb{R}^n)\times\widehat{V}^1_k\,,\\ +\infty & \text{otherwise,}\end{cases}$$

Γ*-converge in the strong* $L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega)$ *topology to* $\widehat{D}^2(u,v)$*, defined as*

$$\int_\Omega \mathbb{C}e(u)\colon e(u)\,\mathrm{d}x + \mathcal{H}^{n-1}\big(J_u \cup \{\mathrm{tr}_{\partial\Omega}u \neq \mathrm{tr}_{\partial\Omega}u_0\}\big) + \int_{J_u} \big|[u]\odot\nu_u\big|\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial\Omega}\big|\mathrm{tr}_{\partial\Omega}(u-u_0)\odot\nu_\Omega\big|\,\mathrm{d}\mathcal{H}^{n-1}$$

if $u \in SBD^2(\Omega)$*,* $v = 1$ *a.e., and* $+\infty$ *otherwise. Moreover, there is convergence of minima and minimisers, up to a subsequence.*

The two theorems above hold also for bulk energy with p-growth in $e(u)$, thanks to our density results. In [16] we study a phase-field approximation “intermediate” between Theorems 1.7 and 1.8.

2. Strategy of the proof

We notice first that, since $\mathrm{tr}_{\partial\Omega}u$ is integrable on $\partial\Omega$, for a bounded $\widetilde{\Omega} \supset \Omega$ the extension of u with 0 in $\widetilde{\Omega}\setminus\Omega$ is in $SBD(\widetilde{\Omega})$, and for any $\varepsilon > 0$ there exists $\widetilde{\Gamma}_\varepsilon \subset J_u$

with $\mathcal{H}^{n-1}(\widetilde{\Gamma}_\varepsilon) < \infty$ and (we argue for the extended u, not relabeled)

$$\int_{J_u \setminus \widetilde{\Gamma}_\varepsilon} |[u]| \, \mathrm{d}\mathcal{H}^{n-1} < \varepsilon \,. \tag{2.1}$$

By a covering argument we find a C^1 set $\widehat{\Gamma}$ (depending on ε) with $\mathcal{H}^{n-1}(\widetilde{\Gamma}_\varepsilon \setminus \widehat{\Gamma}) < \varepsilon$ and pairwise disjoint cubes $Q_1, \dots, Q_N$ with center $x_j \in \widetilde{\Gamma}_\varepsilon$ and sidelength ϱ_j, $j = 1, \dots, N$, for which $\widehat{\Gamma} \subset \bigcup_j Q_j$,

$$\mathcal{H}^{n-1}\big((\widetilde{\Gamma}_\varepsilon \triangle \Gamma_j) \cap \overline{Q_j}\big) < \varepsilon (2\varrho_j)^{n-1} < \frac{\varepsilon}{1-\varepsilon} \mathcal{H}^{n-1}(\widetilde{\Gamma}_\varepsilon \cap \overline{Q_j}) \,, \tag{2.2}$$

being $\Gamma_j := \widehat{\Gamma} \cap Q_j$, and Γ_j is a C^1 graph in direction $\nu_u(x_j)$ with Lipschitz constant less than ε. Then J_u is almost a diameter for each Q_j (recall $\widetilde{\Gamma}_\varepsilon \subset J_u$, with $\widetilde{\Gamma}_\varepsilon = J_u$ if $\mathcal{H}^{n-1}(J_u) < \infty$), and $\widetilde{\Omega}$ is partitioned, up to a $\mathcal{L}^n$-negligible set, by the family of subdomains given by the two (open) halves of each Q_j, and $\widetilde{\Omega} \setminus \bigcup_j \overline{Q}_j$. In every subdomain, the jump energy $\int_{J_u} |[u]| \, \mathrm{d}x$ is small, as well as the measure of the jump set if $\mathcal{H}^{n-1}(J_u) < \infty$.

The guiding idea, in the spirit of [9], is to construct in each of these subdomains a rough approximation u_k, in the following sense: u_k converge in L^1 to u; the trace of $u_k - u$ vanishes in k on each Γ_j; in a small neighbourhood the L^p norm of $e(u_k)$ is controlled by that of $e(u)$, up to a factor $1 + o_{k\to\infty}(1)$, while $\int_{J_{u_k}} (1 + |[u_k]|) \, \mathrm{d}\mathcal{H}^{n-1}$ is less than $C \int_{J_u} (1 + |[u]|) \, \mathrm{d}\mathcal{H}^{n-1}$, for $C > 0$ independent of k.

At this stage, we first join the two rough approximation for the two halves of each Q_j, and then we glue all the resulting functions with the rough approximation in the complement of the cubes, avoiding jumps on each ∂Q_j. Since the traces on Γ_j are well approximated, and Γ_j is almost covered by J_u, we do not increase (as $\varepsilon, k^{-1} \to 0$) on each almost-diameter both the measure of the jump set and the jump energy. Conversely, we increased the jump energy outside $\widehat{\Gamma}$ by the factor C, but there this energy is less than ε. If $\mathcal{H}^{n-1}(J_u)$ is finite, also $\mathcal{H}^{n-1}(J_{u_k} \setminus \widehat{\Gamma}) < C\,\varepsilon$.

In order to avoid jumps on each ∂Q_j, [9, 30, 17] use a partition of the unity to glue the pieces. In such a case, due to the Leibniz rule $e(\varphi u) = \varphi\, e(u) + \nabla\varphi \odot u$, to control the L^p norm of $e(u)$ one needs that the approximants converge in L^p to u, and then that $u \in L^p$. This issue is overcome in [13], by developing a procedure for the rough approximation similar in each subdomain, that permits to glue simply by characteristic functions, still avoiding (almost all) jumps on each ∂Q_j.

Another point where [9, 30, 17] use partitions of the unity is to extend the original function a little bit outside each subdomain, to construct then the rough approximation. This extension is done in [13] by taking the same function u a litte bit outside each Q_j, and applying an argument derived by Nitsche [31] to extend along the direction $\nu_u(x_j)$ on the two sides with respect to Γ_j: since Γ_j is almost flat, we find an hyperplane at distance less than ε and we extend in the domain reflected with respect to the hyperplane, without creating jump and keeping the energy controlled. Notice that we cannot simply reflect the function since we would loose the control on $e(u)$.

The construction of [21] is inspired by the one in [13], that is crucial to avoid any *a priori* integrability assumption, but improves it both in the rough approximation and in the extension procedure, to control the resulting jump energy.

As for the rough approximation, we use a different method for each result. For Theorem 1.2 it is enough to take a convolution with $\varphi_k(x) := k^n\varphi(kx)$, for $\varphi \in C_c^\infty(B(0,1))$ radially symmetric. Indeed, for any subdomain U

$$\int_U |e(u*\varphi_k)|\,\mathrm{d}x \leq |\mathrm{E}u|(U+B(0,k^{-1}))\,,$$

so $\|e(u*\varphi_k)\|_{L^1} \leq \|e(u)\|_{L^1} + |\mathrm{E}^j u|(U+B(0,k^{-1})$, but we know that the jump energy is small in each subdomain. Conversely, when $e(u)$ is accounted with a power $p>1$ and $\mathrm{E}^j u$ linearly, we have to separate the two contributions, so we cannot use only convolution.

In fact, for the other results, we partition any subdomain in cubes q of sidelength k^{-1} and we distinguish the "bad" cubes where the relative jump is large either in measure for Theorem 1.1, that is

$$\mathcal{H}^{n-1}(J_u\cap 4q) > \theta k^{-(n-1)}\,,$$

for a small parameter θ, or in energy for Theorem 1.3, that is

$$|\mathrm{E}^j u|(4q) > k^{-n}\,.$$

In the good cubes we take convolution with φ_k. In the first case the energy is controlled by a technical argument based on the Korn-Poincaré-type estimate in [11] by Chambolle, Conti, and Francfort (used also in [12, 18, 15]), which gives

$$\|e(\tilde{u}*\varphi_k) - e(u)*\varphi_k\|^p_{L^p(q)} \leq C\Big(\frac{\mathcal{H}^{n-1}(J_u\cap 4q)}{k^{-(n-1)}}\Big)^r \|e(u)\|^p_{L^p(4q)} \leq \theta^r\|e(u)\|^p_{L^p(4q)}\,,$$

for $\tilde{u}$ a modification of u in a small exceptional set, and r depending only on p and n. In the other case we use the easy estimate (cf. [21, Lemma 5.1])

$$\|e(u*\varphi_k) - e(u)*\varphi_k\|^p_{L^p(q)} \leq \|\varphi\|^p_{L^p}\, k^{n(p-1)}|\mathrm{E}^j u|^p(2q) \leq C\,|\mathrm{E}^j u|(2q)\,.$$

In the bad cubes we define in both cases u_k as the affine function a_q obtained by the classical Korn-Poincaré inequality in BD, such that $e(a_q)=0$ and

$$\|u-a_q\|_{L^1(2q)} \leq C\,k^{-1}|\mathrm{E}u|(2q)\,. \tag{2.3}$$

Of course the u_k jump on the boundary of bad cubes, but we estimate the jump energy with $C|\mathrm{E}u|(\bigcup_{q\text{ bad}} 2q)$. Notice that the number of bad cubes is less than $\varepsilon\,\theta^{-1}k^{n-1}$ in the first case and than $\varepsilon\,k^n$ in the second case, by (2.2) and (2.2), since we are in a subdomain. Thus $\mathcal{L}^n(\bigcup_{q\text{ bad}} 2q)$ vanishes as $\varepsilon \to 0$ (for $\varepsilon \ll \theta$).

A difference with respect to [13] is that therein we put in the bad cubes u_k equal to 0. This seems good since it does not create jump between two neighbouring bad cubes, but instead it gives no control on the amplitude of the jump between good and bad cubes. By adding the energy contribution for any small cube, we obtain the energy rough estimate, while it is not hard to guarantee convergence of the functions and of the traces.

We remark that since we have used the same approximation procedure in each subdomain, we do not create jump on ∂Q_j, except in the zone when we extend, modifying the original function. A crucial difference with respect to [13] is related to this zone: indeed, if we consider an hyperplane at distance of order ε from Γ_j, then we create a jump for u_k at the intersection between ∂Q_j and a neighbourhood of the diameter of thickness ε. Since we consider convolution at scale $k \ll \varepsilon$, we are not able to control $[u_k]$ therein, even if we could control the measure of the union of all these jump sets by $C\,\varepsilon\mathcal{H}^{n-1}(J_u)$, as in [13]. For this reason, we have to keep the reflected zone of height Ck^{-1}, so comparable to the size of the small cubes and of the convolution kernels. Thus we divide the two halves of each Q_j in parallelepipeds whose basis is a $(n{-}1)$-dimensional cube of sidelength $(\eta_\varepsilon k)^{-1}$, and extend separately. This introduces also jumps at the common boundary of two adjacent parallelepides, but we choose $\eta_\varepsilon \geq \varepsilon$ in such a way that $\lim_{\varepsilon\to 0} \eta_\varepsilon = 0$ and both the jump energy and the measure of these jumps vanish as $\varepsilon \to 0$.

Acknowledgements. I have been supported by a public grant as part of the *Investissement d'avenir* project, reference ANR-11-LABX-0056-LMH, LabEx LMH, and acknowledge the financial support from the Laboratory Ypatia and the CMAP. I am currently founded by the Marie Skłodowska-Curie Standard European Fellowship No. 793018. I am grateful to Antonin Chambolle for his generous and fruitful advices.

References

[1] L. Ambrosio, *Existence theory for a new class of variational problems*, Arch. Rational Mech. Anal., 111 (1990), pp. 291–322.

[2] L. Ambrosio, A. Coscia, and G. Dal Maso, *Fine properties of functions with bounded deformation*, Arch. Ration. Mech. Anal., 139 (1997), pp. 201–238.

[3] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[4] L. Ambrosio and V. M. Tortorelli, *Approximation of functionals depending on jumps by elliptic functionals via* Γ*-convergence*, Comm. Pure Appl. Math., 43 (1990), pp. 999–1036.

[5] G. Bellettini, A. Coscia, and G. Dal Maso, *Compactness and lower semicontinuity properties in* $\mathrm{SBD}(\Omega)$, Math. Z., 228 (1998), pp. 337–351.

[6] B. Bourdin, *Numerical implementation of the variational formulation for quasistatic brittle fracture*, Interfaces Free Bound., 9 (2007), pp. 411–430.

[7] A. Braides and V. Chiadò Piat, *Integral representation results for functionals defined on* $\mathrm{SBV}(\Omega;\mathbf{R}^m)$, J. Math. Pures Appl. (9), 75 (1996), pp. 595–626.

[8] M. Caroccia and N. Van Goethem, *Damage-driven fracture with low-order potentials: asymptotic behavior and applications*, 2018, Preprint arXiv:1712.08556.

[9] A. Chambolle, *An approximation result for special functions with bounded deformation*, J. Math. Pures Appl. (9), 83 (2004), pp. 929–954.

[10] ———, *Addendum to: "An approximation result for special functions with bounded deformation" [J. Math. Pures Appl. (9)* **83** *(2004), no. 7, 929–954; mr2074682]*, J. Math. Pures Appl. (9), 84 (2005), pp. 137–145.

[11] A. Chambolle, S. Conti, and G. A. Francfort, *Korn-Poincaré inequalities for functions with a small jump set*, Indiana Univ. Math. J., 65 (2016), pp. 1373–1399.

[12] ———, *Approximation of a brittle fracture energy with a constraint of non-interpenetration*, Arch. Ration. Mech. Anal., 228 (2018), pp. 867–889.

[13] A. Chambolle and V. Crismale, *A density result in $GSBD^p$ with applications to the approximation of brittle fracture energies*, In press on Arch. Ration. Mech. Anal. (2018), https://doi.org/10.1007/s00205-018-01344-7.

[14] ———, *Compactness and lower semicontinuity in $GSBD$*, Preprint arXiv:1802.03302 (2018), accepted for publication on J. Eur. Math. Soc. (JEMS).

[15] ———, *Existence of strong solutions to the Dirichlet problem for Griffith energy*, Preprint arXiv:1811.07147 (2018).

[16] ———, *Phase-field approximation of a class of cohesive fracture energies with an activation threshold*, Preprint arXiv:1812.05301 (2018).

[17] S. Conti, M. Focardi, and F. Iurlano, *Approximation of fracture energies with p-growth via piecewise affine finite elements*, Accepted for publication on ESAIM Control Optim. Calc. Var. DOI 10.1051/cocv/2018021.

[18] ———, *Existence of strong minimizers for the Griffith static fracture model in dimension two*, In press on Ann. Inst. H. Poincaré Anal. Non Linéaire.

[19] G. Cortesani, *Strong approximation of $GSBV$ functions by piecewise smooth functions*, Ann. Univ. Ferrara Sez. VII (N.S.), 43 (1997), pp. 27–49 (1998).

[20] G. Cortesani and R. Toader, *A density result in SBV with respect to non-isotropic energies*, Nonlinear Anal., 38 (1999), pp. 585–604.

[21] V. Crismale, *On the approximation of SBD functions and some applications*, Preprint arXiv:1806.03076 (2018).

[22] G. Dal Maso, *Generalised functions of bounded deformation*, J. Eur. Math. Soc. (JEMS), 15 (2013), pp. 1943–1997.

[23] G. de Philippis, N. Fusco, and A. Pratelli, *On the approximation of SBV functions*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 28 (2017), pp. 369–413.

[24] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.

[25] M. Focardi and F. Iurlano, *Asymptotic analysis of Ambrosio-Tortorelli energies in linearized elasticity*, SIAM J. Math. Anal., 46 (2014), pp. 2936–2955.

[26] G. A. Francfort and J.-J. Marigo, *Revisiting brittle fracture as an energy minimization problem*, J. Mech. Phys. Solids, 46 (1998), pp. 1319–1342.

[27] M. Friedrich, *A Piecewise Korn Inequality in SBD and Applications to Embedding and Density Results*, SIAM J. Math. Anal., 50 (2018), pp. 3842–3918.

[28] M. Friedrich and F. Solombrino, *Quasistatic crack growth in 2d-linearized elasticity*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 35 (2018), pp. 27–64.

[29] A. A. Griffith, *The phenomena of rupture and flow in solids*, Philos. Trans. Roy. Soc. London Ser. A, 221 (1920), pp. 163–198.

[30] F. Iurlano, *A density result for GSBD and its application to the approximation of brittle fracture energies*, Calc. Var. Partial Differential Equations, 51 (2014), pp. 315–342.

[31] J. A. Nitsche, *On Korn's second inequality*, RAIRO Anal. Numér., 15 (1981), pp. 237–248.

CMAP, École Polytechnique, UMR CNRS 7641, 91128 Palaiseau Cedex, France
E-mail address, Vito Crismale: `vito.crismale@polytechnique.edu`