Connectivity in waves and vibrations: one-to-six, one-to-all, all-to-all and random connections

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Abstract
This paper develops a theory of propagation based on connectivity templates. Connectivity describes how the elastic connections distribute. A visual counterpart is the structure of the stiffness matrix. D'Alembert equation refers to classical elasticity based on closest neighbors connectivity and is characterized by propagation of waves, which can be classified as *one-to-six*, since each particle of the scheme is connected only with two other particles for each direction. However, very different connectivity schemes can be introduced, e.g. a *one-to-all connectivity* scheme, in which one particle can be connected with a cluster of particles, or *all-to-all* where each particle is connected with any other. Moreover, connections are not instantaneous: the information flows is delayed due to the connection length. Waves exhibit unbelievable behaviour changing the system connectivity. Nondissipative structures shows damping. Energy can propagate backwards in respect to wave direction. Waves can stop or localize at some points. Negative mass effect can emerge. These effects will be discussed in the present paper.

1 Introduction

This paper has its focus in synthesizing new results in the field of vibrations and waves the authors and the group of structural dynamics of Sapienza introduced in some recent works [1-37]. The dominant concept is that of topological connectivity and its effect on wave propagation, vibration characteristics, damping, modal behavior.

The connectivity in a vibrating system (or more in general, in a dynamical system) is the attribute that indicates the topology of the force exchange among the degrees of freedom of the system under investigation. Thinking to linear systems, interactions can be reduced to displacement proportional force, and the concept of connectivity becomes, for discrete systems, a characteristic of the stiffness matrix. For continuous systems, a general connectivity scheme is represented by integral-differential equations of motion and the connective topology is characterized by the properties of the integral kernel.

Conventional vibrating systems, especially continuous elastic structures, are characterized by closest neighbors interactions: each mass interacts only with its adjacent masses. This connectivity paradigm, denominated here *short-range* interaction, is the basis of the traditional theory of local elasticity. This way to build the connectivity texture in a three-dimensional space and in a regular cubic lattice, includes a six-connection topology: each particle in the lattice is elastically connected to 6 other particles, namely the preceding and the following particles along the three axes x, y, z, respectively. It appears clearly a more arbitrary topological connectivity can be introduced, deeply different from the standard local elasticity, since many others connective schemes can be proposed. Broadly speaking, any other connective template that is more general than the paradigm of the six-connection topology, implies to connect particles in the lattice that not adjacent, but located at distant positions. The reason to indicate these connective schemes as *long-range* [3-5].

This paper makes the point on some fundamental differences that can be expected when passing from a purely *short-range* connectivity to a *long-range* type in vibration characteristics. Correspondently, we highlight some structural differences between the dynamics of *local-elasticity* and *nonlocal-elasticity*. However, our previous investigations show that, in the family of long-range interactions, the emergence of a wide class of different behaviors is observed [3-5, 8, 10, 16-18]. Note, in fact, the *long-range* interaction textures can be deeply different, including many different topologies of connections [3-5].

In this paper we analyze three main types of *long-range* connectivity to compare to *short-range* or closest neighbors: all-all full-range, , all-all limited range , all-all randomly and sparsely connected, one-all. The general mathematical framework is given, but the focus is to outline only the general scenario, and any mathematical detail can be found in the references.

The particles of the scheme behave as individuals of a population, and our investigation represents an attempt to classify the collective behavior of these individuals depending on their freedom to communicate. Implications of this point of view are in vibration properties of the populations as a collective behavior that can affect wave propagation, synchronization, decoherence, damping properties, mode shapes, modal density and localization. These implications can be of practical use in the analysis of microscopical vibrations of lattices (thermal baths) in design of new metamaterials, design of new vibration absorbing devices and waves at interface between two different media [1, 6, 7, 10, 12-14, 16-20, 24-27, 30].

2 Connectivity topology scenarios and properties

In this paper we analyze four main types of *long-range* connectivity to compare to *short-range* or closest neighbors: one-all, all-all full-range, all-all randomly and sparsely connected, all-all limited range.

A set of individuals in a population of masses are connected in a circle (the representation is useful graphically, the system has not to be mechanically connected necessarily in a closed chain), called *ring*. Figure 1 represents the typical connections in a *short-range* interaction fashion, where individuals can communicate, i.e. exchange forces, only with those individuals that are adjacent (red connecting line represent the *short-range* interactions). This is the typical connectivity that mechanically represent the world of *local-elasticity*. The behaviour of such a connected system is well known. One of the main characteristic is the chance of having travelling disturbances we call waves. Energy travels across the ring and two typical speeds can be introduced: phase and group. The spectral properties of theses kind of systems are associated to eigenvectors the shape of which is typically an oscillating space function that involves the whole ring. The characteristic correlation expressed by the waves is the motion of the particles is strongly correlated at a given distance, depending on the propagation velocity (phase or group, depending on the characteristic we desire to correlate).

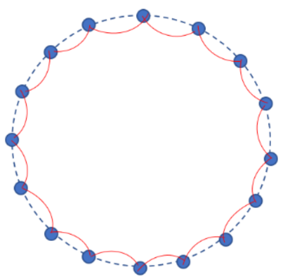

Figure 1. Connectivity based on neighbors interaction, short-range classical case

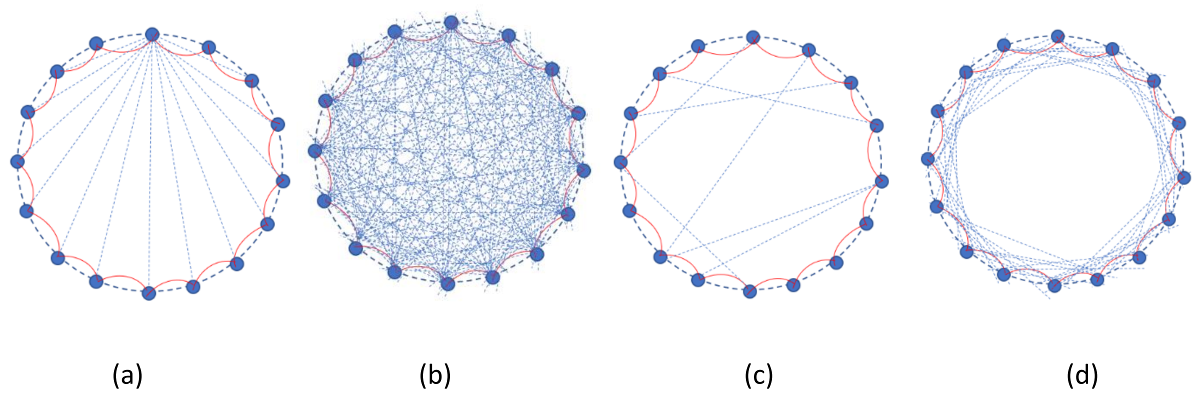

Figure 2.

Different kind of connectivity: (a) one-all, (b) all-all, (c) all-all random sparse, (d) all-all limited range

However, very different topology of connections can be introduced and each characterized by the common fact the interaction is *long-range* in the sense that, along the ring, the connectivity is not only between neighbors, but among distant individuals along the ring. In particular, we would like to describe four basic connectivity templates, that emerge form a previous experience in detailing the dynamics of such systems [3-8, 11, 17, 18, 25, 26, 29] that enter the domain of the propagation in *nonlocal elasticity*.

The one-all scheme is the case in which only a single mass, we call *leader*, generally larger with respect to the other in the ring, communicates with all the individuals of the ring. The short-range connections are general of smaller intensity with respect to the long-range, or they can be even completely absent. This kind of system has been investigated for a long time by the authors and exhibits completely new characteristics with respect to the short-range properties [7, 8, 11, 17-19, 25, 26, 29, 31, 32, 36, 37].

We observed, using mathematical models, simulations and experiments the following:

(i) Waves propagate along the ring, and this happens even in the absence of the short-range forces(!); the propagation speed depends on the distribution law of the intensity of the connections of the scheme.

(ii) For some particular distribution of the interaction forces, one-all connectivity produces a phenomenon of wave stopping: the energy transported along the ring when exciting the connected mass, can stop and the excited mass does not receive any echo from the ring; this means the group velocity along the ring exhibits frequencies at which it vanishes.

(iii) Modes of the ring are localized, and the leader location is never the place at which the modes localize;

(iv) The energy initially stored in the leader tends to be very fast transferred to the ring and it is possible to modulate the intensity of the connections in a way this process is irreversible.

(v) The modes of the ring tend to group at a special frequency, where they are accumulated producing a singularity in the modal density.

The all-all scheme full-range is realized in such systems in which any possible connection between any arbitrariy pair of individuals in the ring is promoted. A recent analysis carried on in [3-5], shows the following remarkable results:

(i) All-all full-range produces a phenomenon of wave stopping along the ring;

(ii) Moreover, at some frequencies, an anomalous propagation phenomenon appears: superluminal propagation of waves along the ring can be observed, meaning the waves transported along the ring itself reach an infinite group velocity.

(iii) Modes of the ring are localized.

(iv) Singularity in the modal density are observed at those frequencies at which the wave stopping is produced.

For the case (d), all-all limited range, the same phenomena, under a qualitative point of view, are identical to those observed in the all-all full-range.

The case (c), in which a small number of connections are present, randomly involving pair of individuals along the ring, i.e. the connections are randomly sparse, exhibits additional effects [41], that are based on the small world theory, originally proposed in social sciences [38-40]:

(i) The number of long-range connections to be introduced to deeply depart from the short-range characteristic propagation is very small. Even a few percent of activated long-range connections permit to observe the following phenomena:

(ii) Strong synchronization of the individual motion, meaning the motion of a large group of masses along the ring move in-phase, and no phase-delay is observed as it happens in the propagation of conventional waves.

(iii) Very high speed of propagation of the disturbance along the ring, with amplification of the group velocity of the system, in part analogous the case observed for limited range and full-range long-distance interaction.

3 *Long-range* effects general modelling

The mathematical model to which we can reduce all the examined cases has a common root into integral-differential equations. Differential equations, both in space and time, is the typical ground on which the local-elasticity operates. A theory of long-range connections includes in general integral convolution terms, analogously in space and time. All the previous cases can be included in the same general model, but with different definition of the kernel of the convolutional part. This kind of model are certainly claimed in the new generation of metamaterials, where the connections can be built up by using, for example, adding manufacturing techniques. The field of vibration absorbers based on new concepts is also part of the possibilities of these methods. The investigation of thermodynamic systems, as the thermal bath used in physics to model Brownian motion for the meso-scale analysis is presented in [6, 14, 16, 20]. In the next section, as an example of more conventional system, generally investigated by coupled differential equations, is considered: the case of a fluid-loaded plate. We show that, even these kind of coupled problems, can be reduced to an integral-differential model governed by topological connectivity features.

The general structure of the cases (a), (b), (c), (d) is kept by the general integral-differential equation:

$$L_x^{(n)}\{w(x,t)\} + m'(x)\frac{\partial^2 w(x,t)}{\partial\, t^2} + K(x,t) ** w(x,t) = 0$$

*wh*ere double convolution indicates, in general, space and time are involved. Differential terms involve $L_x^{(n)}$, a *n-th* order differential operator with respect to space *x*, and an inertia term second-order in time $\frac{\partial^2 w(x,t)}{\partial\, t^2}$. The nature of the kernel *K* is the key to describe the kind of connections. If the connections are simple elastic elements, then the convolution is only space-based and the time *t* does not play any role and the term can be reduced to:

$$K(x,t) ** w(x,t) = \int_{-\infty}^{-\infty} K(x,\xi)\, w(x-\xi,t) d\xi$$

However, if the channel of communication between the pair of individuals of the ring implies delayed information transport itself (or also waves mechanisms), then the double convolution $K(x,t) ** w(x,t)$ brings both space and time effects:

$$K(x,t) ** w(x,t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{-\infty} K(x,\xi,t,\tau)\, w(x-\xi, t-\tau) d\xi\, d\tau$$

The general physical interpretation of the double convolution term is simple: it brings at the place *x* all the information that is coming from other locations of the ring that are distant $x - \xi$ in space and of which the flight time to reach *x* from the place $x - \xi$ is $\tau = x - \xi / c(x - \xi)$.

Most of the propagation properties that are involved in the previous schemes can be determined in the frequency and wavenumber counterpart of this equation:

$$P^{(n)}(k) - m'\omega^2 + K(k,\omega) = 0$$

where P is a polynomial of order *n* in the wavenumber *k* and $K(k,\omega)$ is in general a complicated expression depending on both the wavenumber and the frequency.

This is the basis to investigate the wave dispersion relationship, that because of the highly unconventional term $K(k,\omega)$, produces very new and characteristic effects that are related to propagation in the field of nonlocal elasticity.

4 *An example: interface waves as a long-range connectivity problem*

Propagation of waves at the interface between two different media is an example that can be approached with the connectivity analysis presented in section 2. The case of a fluid-loaded flexural elastic plate is considered. The most frequent modelling of this system is made by coupling the differential equation of the plate (simplified in a beam for a two dimensional problem in the *x,y* plane), governed by its displacement *w(x.t)* and the differential equation of the compressible fluid, governed by the potential *φ(x,y,t)*:

$$EI \frac{\partial^4 w(x,t)}{\partial\, x^4} + \rho A \frac{\partial^2 w(x,t)}{\partial\, t^2} = p, \;\; \nabla^2 \varphi(x,y,t) - \frac{1}{c^2} \frac{\partial^2}{\partial\, t^2} \varphi(x,y,t) = 0$$

where the coupling condition, relating the potential and the pressure, is:

$$p(x,y,t) = -\rho \frac{\partial}{\partial t} \varphi(x,y,t)$$

Therefore, the two partial differential equations can be solved with the help of the coupling condition, that is the conventional way to formulate the problem.

Let us try to bring the problem on the ground of the connectivity explained in section 2. We can consider that the sections of the beam are the nodes of the network, connected by *short-length* interactions represented by the differential term $EI \frac{\partial^4 w(x,t)}{\partial\, x^4}$, but there is a further channel of communication among them represented by the waves that propagate in the compressible fluid. Therefore, the effect of the displacement signal of the beam connects distant sections, providing the *long-range* channel among the nodes of the network.

The last condition implies that along the plate surface, one can write:

$$p(x,y,t) = -\rho \frac{\partial}{\partial t} \varphi(x,y,t) \quad \rightarrow \quad \frac{\partial}{\partial x} p(x,y,t) = -\rho \frac{\partial}{\partial t} \frac{\partial}{\partial x} \varphi(x,y,t) = -\rho \frac{\partial}{\partial t} v(x,y,t)$$

i.e.:

$$\frac{\partial}{\partial y} p(x,0,t) = -\rho \frac{\partial^2}{\partial t^2} w(x,t)$$

$$\nabla^2 \varphi(x,y,t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varphi(x,y,t) = 0 \quad \rightarrow \quad \nabla^2 p(x,y,t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p(x,y,t) = 0$$

Transforming into the Fourier domain the beam equation and the last two pressure equations (space and time):

$$EI{k_x}^4 W - \rho A \omega^2 W = P, \quad jk_y P = \rho \omega^2 W, \quad {k_x}^2 + {k_y}^2 = \frac{\omega^2}{c^2}$$

and after substitution:

$$EI{k_x}^4 W - \rho A \omega^2 W = \pm j \omega W G$$

$$G\left(\frac{ck_x}{\omega}\right) = \frac{\rho c}{\sqrt{1 - \left(\frac{ck_x}{\omega}\right)^2}}, \quad \frac{\omega}{c} H_0^{(1)}\left(\left|\frac{\omega}{c} x\right|\right) = {F_{k_x}}^{-1}\left\{G\left(\frac{ck_x}{\omega}\right)\right\}$$

and finally:

$$EI \frac{\partial^4 w(x,t)}{\partial x^4} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} = \pm \frac{1}{x} h_0^{(1)}\left(\frac{ct}{x}\right) ** w(x,t)$$

$$\text{where: } h_0^{(1)}(t) = {F_\omega}^{-1}\left\{H_0^{(1)}(\omega)\right\}$$

revealing the convolution long-range nature of the phenomenon described in the previous section.

Concluding Remarks

This paper highlights some effects that are disclosed in a series of researches, some very recent, developed by the authors. The point considered here is related to characteristic response that a population of oscillators, connected through communication channels of given topology that bring force information, exhibits as a collective behavior. The problem is based on consideration of long-range interaction forces, opposed to the short-range, characteristic of the classical local-elasticity. Relevant effects, in this field of non-local elasticity propagation, emerge as synchronization, wave stopping, superluminal group velocity, mode localization and singularity in the modal density.

Different scenarios are considered, and the mathematical framework is based on the theory of integral-differential equations. It put the basis for the analysis of dispersion relationship of unconventional form that is responsible of the investigated effects.

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