STANDARD MONOMIAL THEORY FOR WONDERFUL VARIETIES

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Abstract. A general setting for a standard monomial theory on a multiset is introduced and applied to the Cox ring of a wonderful variety. This gives a degeneration result of the Cox ring to a multicone over a partial flag variety. Further, we deduce that the Cox ring has rational singularities.

1. Introduction

The first appearance of the idea of a standard monomial theory may be traced back to Hodge's study of Grassmannians in [20], [21]. Then Doubilet, Rota and Stein found a similar theory for the coordinate ring of the space of matrices in [18]. This was reproved and generalized to the space of symmetric and antisymmetric matrices by De Concini and Procesi in [15].

A systematic program for the development of a standard monomial theory for quotients of reductive groups by parabolic subgroups was then started by Seshadri in [31] where the case of minuscule parabolics is considered. Further, in [24] Seshadri and Lakshmibai noticed that the above mentioned results could be obtained as specializations of their general theory.

This program was finally completed by Littelmann. Indeed, in [25], he found a combinatorial character formula for representations of symmetrizable Kac-Moody groups introducing the language of L-S paths. Moreover, he used L-S paths to construct a standard monomial theory for Schubert varieties of symmetrizable Kac-Moody groups in [27]. This theory has been developed in the context of LS algebras over posets with bonds in [9], [10] and [11].

We want now to briefly recall what a standard monomial theory is, the reader may see [12] for further details about this general setting. Let $\mathbb{A}$ be a finite subset of an algebra A and suppose we are given a transitive antisymmetric binary relation $\longleftarrow$ on $\mathbb{A}$. We define a formal monomial $\mathsf{a}_1\mathsf{a}_2\cdots\mathsf{a}_N$ of elements of $\mathbb{A}$ as standard if $\mathsf{a}_1 \longleftarrow \mathsf{a}_2 \longleftarrow \cdots \longleftarrow \mathsf{a}_N$. If the set of standard monomials is a basis of the algebra A as a vector space then we say that $(\mathbb{A}, \longleftarrow)$ is a standard monomial theory for A. Suppose, further, we have an order $\leqslant$ on the monomials of elements of $\mathbb{A}$. By the previous assumption, we may write any non-standard monomial m' as a linear combination of standard monomials. If in such an expression only standard monomials m with $\mathsf{m}'\leqslant\mathsf{m}$ appear, then we say that we have a straightening relation for m'. If we have a straightening relation for each non-standard monomial, then we say that $(\mathbb{A}, \longleftarrow, \leqslant)$ is a standard monomial theory with straightening relations.

Given a simply connected semisimple algebraic group G over an algebraically closed field $\Bbbk$ of characteristic 0, a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$, let $\Lambda^+ \subset \Lambda$ be the monoid of dominant weights and the lattice of weights, respectively. For a dominant weight λ, let V_λ be the irreducible G–module of highest weight λ. Let $B \subset P \subset G$ be a parabolic subgroup of G contained in the stabilizer

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of the line generated by a highest weight vector in V_λ. Moreover, we denote by $\mathbb{B}_\lambda$ the set of L-S paths of shape λ (see Section 3 for details).

Littelmann's construction provides a basis $\mathbb{A}_\lambda = \{\mathsf{p}_\pi \,|\, \pi \in \mathbb{B}_\lambda\}$, indexed by L-S paths, for the module $\Gamma(G/P, \mathcal{L}_\lambda) \simeq V_\lambda^*$, where $\mathcal{L}_\lambda$ is the line bundle over G/P associated with λ. The ring of sections $A_\lambda = \bigoplus_{n\geqslant 0} \Gamma(G/P, \mathcal{L}_{n\lambda})$ is generated in degree one and it is the coordinate ring of the cone over the closed embedding $G/P \hookrightarrow \mathbb{P}(V_\lambda)$ induced by $\mathcal{L}_\lambda$. On the basis $\mathbb{A}_\lambda$ one may define a relation $\longleftarrow$ and an order $\leqslant$ such that $(\mathbb{A}_\lambda, \longleftarrow, \leqslant)$ is a standard monomial theory with straightening relations for A_λ.

In [13], the second and fourth named authors adapted Littelmann's basis to the Cox ring (see below) of complete symmetric varieties; this class of varieties has been introduced by De Concini and Procesi in [16]. As a consequence, they proved the degeneration of the Cox ring to the coordinate ring of a suitable multicone over a flag variety. This degeneration allowed a new proof of the rational singularity property for the Cox ring of complete symmetric varieties.

The purpose of the present paper is a further extension of these results to the Cox ring of wonderful varieties. As a first step, we take the opportunity to introduce a general setting for a standard monomial theory on a multiset modelled on the above recalled one. This setting may be briefly summarized as follows, see Section 2 below for details.

Let $\mathbb{A} \doteq \mathbb{A}_1 \sqcup \mathbb{A}_2 \sqcup \cdots \sqcup \mathbb{A}_n$ be the union of disjoint finite subsets of an algebra A. Suppose we have a binary relation $\longleftarrow$ on $\mathbb{A}$ such that $\longleftarrow$ restricted to $\mathbb{A}_i$ is transitive and antisymmetric for all $i = 1, 2, \ldots, n$ and, further, suppose we have bijective maps $\phi_{i,j}$, called swaps, from the set of comparable pairs $\mathsf{a} \longleftarrow \mathsf{b}$ of $\mathbb{A}_i \times \mathbb{A}_j$ to the set of comparable pairs $\mathsf{b}' \longleftarrow \mathsf{a}'$ of $\mathbb{A}_j \times \mathbb{A}_i$ satisfying $\phi_{i,i} = \mathrm{Id}$ and $\phi_{i,j}\phi_{j,i} = \mathrm{Id}$. We define a formal monomial $\mathsf{a}_1\mathsf{a}_2\cdots\mathsf{a}_N$ as weakly standard if $\mathsf{a}_1 \longleftarrow \mathsf{a}_2 \longleftarrow \cdots \longleftarrow \mathsf{a}_N$, and we say it is standard if all monomials obtained by repeatedly swapping adjacent pairs in all possible ways, via the $\phi_{i,j}$'s, are weakly standard. If the set of standard monomials is a basis for A as a vector space, we say that $(\mathbb{A}, \longleftarrow, \phi_{i,j})$ is a standard monomial theory on the multiset $\mathbb{A}$ for the algebra A. As above we introduce also the notions of order for monomials and straightening relations for non-standard monomials.

We prove that the kernel of the natural map from the symmetric algebra over $\mathbb{A}$ to A is generated by the straightening relations of minimally non-standard monomials, that is by the straightening relations of those non-standard monomials which are not a product of non-standard monomials of smaller degree. In particular, we show that if any weakly standard monomial is standard then such kernel is generated in degree two.

We also show how, given a valuation map for monomials that is compatible with the order $\leqslant$, one may construct a flat degeneration of A.

As a motivating example for this setting one may see the standard monomial theory on a multiset for the multicone over a flag variety constructed by the second named author in [11]. This is described in details in Section 3. In the same Section the multicone associated to fundamental weights for SL and a multicone for $\mathsf{SL}_2 \times \mathsf{SL}_2$ are studied in great detail.

Now we recall which is the type of varieties we are interested in. A G–variety X is wonderful of rank r if it satisfies the following conditions:

– X is smooth and projective;
– X possesses an open orbit whose complement is a union of r smooth prime divisors, called the boundary divisors, with non-empty transversal intersections;

– any orbit closure in X equals the intersection of the prime divisors which contain it.

Examples of wonderful varieties are the flag varieties, which are the wonderful varieties of rank zero, and the complete symmetric varieties. Wonderful varieties are instances of spherical varieties ([28]): normal G–varieties with a dense orbit under the action of a Borel subgroup of G. See [5] for a general introduction to wonderful varieties.

If X is a wonderful G–variety, then the Picard group $\mathrm{Pic}(X)$ is freely generated by the classes of the B–stable prime divisors of X which are not G–stable (see, for example, Section 2.2 in [7]). These divisors are called the colors of X. Since X contains an open B–orbit, the colors form a finite set Δ, so that $\mathrm{Pic}(X)$ is a free lattice of finite rank. Given $D \in \mathbb{Z}\Delta$, we denote by $\mathcal{L}_D$ the corresponding line bundle.

The direct sum

$$C(X) \quad \doteq \bigoplus_{\mathcal{L}\in \mathrm{Pic}(X)} \Gamma(X, \mathcal{L}),$$

has a ring structure (see Section 4 below) and it is called the Cox ring of X.

Denote by $\sigma_1, \ldots, \sigma_r$ the boundary divisors of X, and let s_i be a section of $\mathcal{O}(\sigma_i)$ defining σ_i, for $i = 1, \ldots, r$. As an algebra $C(X)$ is generated by the sections of the line bundles $\mathcal{L}_D = \mathcal{O}(D)$ with $D \in \Delta$ together with the sections $s_1, \ldots, s_r$.

By definition, X contains a unique closed G–orbit $Y \simeq G/P$ for a suitable parabolic subgroup P, and given $D \in \mathbb{N}\Delta$ we denote by λ_D the highest weight of the dual of the simple G–module $\Gamma(Y, \mathcal{L}_D\big|_Y)$, so that $\mathcal{L}_D\big|_Y \simeq \mathcal{L}_{\lambda_D}$ corresponds to the equivariant line bundle on G/P associated to the dominant weight λ_D. By taking into account the decomposition of $\Gamma(X, \mathcal{L}_D)$ as a G–module (see Proposition 4.1), we lift Littelmann's basis of $\Gamma(Y, \mathcal{L}_{\lambda_D})$ to X, and we take as algebra generators for $C(X)$ this set of lifts together with the sections $s_1, \ldots, s_r$.

Consider the coordinate ring $C(Y) = \oplus_{E\in\mathbb{N}\Delta}\Gamma(Y, \mathcal{L}_E\big|_Y)$ of the multicone over the flag variety Y associated to the dominant weights λ_D, with $D \in \Delta$. In Section 4 we construct a standard monomial theory on a multiset for $C(X)$ by extending, in a natural way, that of $C(Y)$. Further an explicit example of our construction is given.

As a consequence of our standard monomial theory, we obtain a flat deformation which degenerates $C(X)$ to the product $\mathbb{k}[s_1, \ldots, s_r] \otimes C(Y)$. Since multicones over flag varieties have rational singularities by [22] and, since the property of having rational singularities is stable under deformation by [19], it follows that the Cox ring $C(X)$ has rational singularities as well. From this it follows at once that, given $D \in \mathbb{Z}\Delta$, also the ring $C_D(X) = \oplus_{n\geqslant 0}\Gamma(X, \mathcal{L}_{nD})$ has rational singularities. Both $C(X)$ and $C_D(X)$, for any $D \in \mathbb{Z}\Delta$, can be seen as coordinate rings of affine spherical varieties, see Section 3.1 in [7], and the fact that affine spherical varieties have rational singularities was already known, see [30] and [1].

2. Standard monomial theory on a multiset

In this section, as a first step, we introduce the notion of a standard monomial theory on a multiset. This requires some technical machinery which we express in a very abstract setting. In the next section we see the application to the multicone over a flag variety and in Section 4 that to the Cox ring of a wonderful variety.

For further details the reader may see the various referenced papers as suggested below. In particular, the standard monomial theory we are going to introduce is modelled on the general definition of a standard monomial theory given in [14]; see also [10] and [11] where such kind of standard monomial theory is developed in the context of posets with bonds.

We begin with a field $\mathbb{k}$ and a commutative $\mathbb{k}$–algebra A. Let $\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_n$ be disjoint finite subsets of A and let $\mathbb{A} \doteq \mathbb{A}_1 \sqcup \mathbb{A}_2 \sqcup \cdots \sqcup \mathbb{A}_n$, we call $\mathbb{A}$ a *multiset*. By *formal monomial* we mean a monomial in the elements of $\mathbb{A}$ in the free associative algebra generated by $\mathbb{A}$. Define the *shape* of $\mathsf{a} \in \mathbb{A}_i$ as the index i and extend the notion of shape to formal monomials $\mathsf{m} \doteq \mathsf{a}_1\mathsf{a}_2\cdots\mathsf{a}_N$ of elements of $\mathbb{A}$ by declaring that the shape of m is $(i_1, i_2, \ldots, i_N)$ if a_h has shape i_h for $h = 1, 2, \ldots, N$.

Suppose we have a binary relation $\longleftarrow$ on $\mathbb{A}$ that is antisymmetric and transitive when restricted to $\mathbb{A}_i$ for all i. We say that a formal monomial $\mathsf{a}_1\mathsf{a}_2\cdots\mathsf{a}_N$ of elements of $\mathbb{A}$ is *weakly standard* if $\mathsf{a}_1 \longleftarrow \mathsf{a}_2 \longleftarrow \cdots \longleftarrow \mathsf{a}_N$. Given i, j, let $\phi_{i,j}$ be a map from the set of weakly standard formal monomials of shape (i, j) to the set of weakly standard formal monomials of shape (j, i). We assume that these maps verify $\phi_{i,i} = \mathrm{Id}$ and $\phi_{i,j}\phi_{j,i} = \mathrm{Id}$, and we call them *swap maps*.

Now let $\mathsf{m} = \mathsf{a}_1\mathsf{a}_2\cdots\mathsf{a}_N$ be a weakly standard formal monomial. Since any pair $\mathsf{a}_j\mathsf{a}_{j+1}$ is weakly standard, we may swap it and obtain a new monomial

$$\mathsf{m}' \doteq \mathsf{a}_1\cdots\mathsf{a}_{j-1}\mathsf{a}'_j\mathsf{a}'_{j+1}\mathsf{a}_{j+2}\cdots\mathsf{a}_N,$$

where $\mathsf{a}'_j\mathsf{a}'_{j+1}$ is the swap of $\mathsf{a}_j\mathsf{a}_{j+1}$. If also m' is weakly standard, then we may apply another swap, etc. If the shape of m is a non-decreasing sequence and if all monomials obtained from m by swaps are weakly standard, then we say that m is a *standard formal monomial* (notice that the number of swaps of m is surely finite since $\mathbb{A}$ is a finite set).

We say that a (commutative) monomial in the symmetric algebra $\mathsf{S}(\mathbb{A})$ is weakly standard, respectively, standard if it is the image of a weakly standard, respectively, standard formal monomial of elements of $\mathbb{A}$ via the natural map from formal monomials to monomials in $\mathsf{S}(\mathbb{A})$.

We say that the above datum $(\mathbb{A}, \phi_{i,j}, \longleftarrow)$ is a *standard monomial theory on the multiset* $\mathbb{A}$ for A if: the images of the standard monomials of $\mathsf{S}(\mathbb{A})$ in A via the natural map $\mathsf{S}(\mathbb{A}) \longrightarrow A$ are all distinct and, moreover, the set of these standard monomials is a basis of A as a $\mathbb{k}$–vector space.

A standard monomial theory usually has another feature, the straightening relations. We express them by introducing an order $\leqslant$ on (commutative) monomials in $\mathbb{A}$ as elements of A with the following properties:

i) if m, m', m'' are monomials in $\mathbb{A}$ and if $\mathsf{m}' \leqslant \mathsf{m}''$, then $\mathsf{m}\mathsf{m}' \leqslant \mathsf{m}\mathsf{m}''$,

ii) for every monomial m the set of monomials m' such that $\mathsf{m} \leqslant \mathsf{m}'$ is a finite set.

Since the standard monomials are a $\mathbb{k}$–basis of A, for every non-standard monomial m' of elements of $\mathbb{A}$ we have a relation

$$\mathsf{m}' = \sum_{\mathsf{m}} a_{\mathsf{m}}\mathsf{m}$$

in A, expressing m' as a linear combination of the standard monomials m. If we have $\mathsf{m}' \leqslant \mathsf{m}$ whenever $a_{\mathsf{m}} \neq 0$ we say that the above relation is a *straightening relation* for m'. If, further, we have straightening relations for all non-standard monomials then we say that $(\mathbb{A}, \phi_{i,j}, \longleftarrow, \leqslant)$ is a standard monomial theory on a multiset *with straightening relations* for A.

A formal non-standard monomial $\mathsf{m} = \mathsf{a}_1\mathsf{a}_2\cdots\mathsf{a}_N$ is *minimally* non-standard if for all proper subsequences $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant N$ of indexes, the monomial $\mathsf{a}_{i_1}\mathsf{a}_{i_2}\cdots\mathsf{a}_{i_k}$ is a formal standard monomial. We say that a monomial in $\mathsf{S}(\mathsf{A})$ is minimally non-standard if it is the image of a minimally non-standard formal monomial.

Notice that the straightening relations generate the kernel $\mathcal{R}$ of the map $\mathsf{S}(\mathbb{A}) \longrightarrow A$, i.e. the ideal of relations in the generators $\mathbb{A}$ for A. But fewer relations can suffice as we see in the following theorem.

Theorem 2.1. *If* $(\mathbb{A}, \phi_{i,j}, \longleftarrow, \leqslant)$ *is a standard monomial theory with straightening relations for* A*, then* $\mathcal{R}$ *is generated by the straightening relations of the minimally non-standard monomials.*

Proof. Let $\mathcal{I}$ be the ideal of $\mathsf{S}(\mathbb{A})$ generated by the straightening relations for minimally non-standard m. Since by definition $\mathcal{I} \subseteq \mathcal{R}$ we have a surjective map $\mathsf{S}(\mathbb{A})/\mathcal{I} \longrightarrow \mathsf{S}(\mathbb{A})/\mathcal{R}$. Moreover the set of standard monomials are a $\Bbbk$–basis for $\mathsf{S}(\mathbb{A})/\mathcal{R} \simeq A$, so if we show that the images of standard monomials generate $\mathsf{S}(\mathbb{A})/\mathcal{I}$ as a vector space we have $\mathcal{I} = \mathcal{R}$.

In order to prove that in $\mathsf{S}(\mathbb{A})/\mathcal{I}$ any monomial m is a $\Bbbk$–linear combination of standard monomials we use induction on the order $\leqslant$. In particular, if m is $\leqslant$–maximal then it is standard by the order requirement in the straightening relations.

So now suppose that m is non-standard and let m_1, m_2 be monomials such that $\mathsf{m} = \mathsf{m}_1\mathsf{m}_2$ with m_1 minimally non-standard. In $\mathsf{S}(\mathbb{A})/\mathcal{I}$ we have $\mathsf{m}_1 = \sum_{\mathsf{n}} a_{\mathsf{n}}\mathsf{n}$ where the sum runs over the standard monomials n with $\mathsf{m}_1 < \mathsf{n}$. Hence

$$\mathsf{m} = \mathsf{m}_1\mathsf{m}_2 \equiv \sum a_{\mathsf{n}}\mathsf{n}\mathsf{m}_2 \pmod{\mathcal{I}}$$

and $\mathsf{m} < \mathsf{n}\mathsf{m}_2$ for all n. Using the inductive hypothesis on $\leqslant$, all $\mathsf{n}\mathsf{m}_2$'s are sums of standard monomials, hence also m is sum of standard monomials in $\mathsf{S}(\mathbb{A})/\mathcal{I}$. □

As a corollary we have the following result.

Corollary 2.2. *If all weakly standard monomials are standard and the ideal* $\mathcal{R}$ *of relations is homogeneous for the total degree of monomials, then* $\mathcal{R}$ *is generated in degree two.*

Proof. First of all we show that for a minimally non-standard monomial $\mathsf{m} = \mathsf{a}_1\mathsf{a}_2\cdots\mathsf{a}_N$ we have $N = 2$. Indeed suppose $N \geqslant 3$, then $\mathsf{a}_i\mathsf{a}_{i+1}$ is standard for all $i = 1, 2, \ldots, N-1$. So $\mathsf{a}_1 \longleftarrow \mathsf{a}_2 \longleftarrow \cdots \longleftarrow \mathsf{a}_N$ and m is weakly standard, hence it is standard.

We know that $\mathcal{R}$ is generated by the straightening relations of the minimally non-standard monomials by the previous theorem. We have just seen that minimally non-standard monomials have degree 2, so $\mathcal{R}$ is generated by the straightening relations of non-standard monomials of total degree 2. Let m be such a monomial, let

$$R \doteq \mathsf{m} - \sum_{\mathsf{n}} a_{\mathsf{n}}\mathsf{n}$$

be its straightening relations and finally let

$$R_2 \doteq \mathsf{m} - \sum_{\mathsf{n}} a_{\mathsf{n}}\mathsf{n}$$

where the sum is over all standard monomials n of total degree 2. The ideal $\mathcal{R}$ is homogeneous, hence $R_2 \in \mathcal{R}$; but then $R' \doteq R_2 - R \in \mathcal{R}$ and R' is a sum of standard monomials. This is clearly possible only if $R' = 0$ since the standard monomials are linearly independent in $A \simeq \mathsf{S}(\mathbb{A})/\mathcal{R}$. This shows that $R = R_2$ is an homogeneous relation of total degree 2 and completes the proof. □

In the last part of this section we see how a degeneration for A may be constructed using the straightening relations (see also [10] for further details about this kind of degeneration in the language of LS algebras). Suppose we have a *valuation*: a map $\delta : \mathbb{A} \longrightarrow \mathbb{N}$ such that, when extended to monomials by $\delta(\mathsf{m}\mathsf{n}) = \delta(\mathsf{m}) + \delta(\mathsf{n})$ for all m, n, we have $\delta(\mathsf{m}) \leqslant \delta(\mathsf{n})$ if $\mathsf{m} \leqslant \mathsf{n}$.

For an integer n, let K_n be the ideal of A generated by those monomials m such that $\delta(\mathsf{m}) \geqslant n$ and consider the Rees algebra

$$\mathcal{A} \doteq \cdots \oplus At^2 \oplus At \oplus A \oplus K_1t^{-1} \oplus K_2t^{-2} \oplus \cdots$$

as a subalgebra of $\mathbb{k}[t,t^{-1}]\otimes A$. This algebra is a torsion-free $\mathbb{k}[t]$–module, hence it is a flat $\mathbb{k}[t]$–algebra. For $a\in\mathbb{k}$ let $\mathcal{A}_a \doteq \mathcal{A}/(t-a)$ be the fiber over a. Notice that we have an action of $\mathbb{k}^*$ on $\mathcal{A}$ given by $\lambda\cdot t=\lambda t$, for all $\lambda\in\mathbb{k}^*$; hence isomorphisms $\mathcal{A}_a \longrightarrow \mathcal{A}_{\lambda^{-1}a}$ between the fibers. In particular all *generic fibers*, i.e. $\mathcal{A}_a$ with $a\neq 0$, are isomorphic to $\mathcal{A}_1\simeq A$. On the other hand the *special fiber* $\mathcal{A}_0=\mathcal{A}/(t)$ is isomorphic to the associated graded algebra

$$A/K_1\oplus K_1/K_2\oplus K_2/K_3\oplus\cdots$$

We may now state our deformation result.

Theorem 2.3. *Let $(A,\varphi_{i,j},\longleftarrow,\leqslant)$ be a standard monomial theory with straightening relations for the ring A and let δ be a valuation as above. Then there exists a flat $\mathbb{k}^*$–equivariant degeneration of A to $\mathcal{A}_0$ whose all generic fibers are isomorphic to A while the special fiber $\mathcal{A}_0$ is isomorphic to the quotient of the symmetric algebra $\mathsf{S}(\mathbb{A})$ by the ideal generated by the relations*

$$\mathsf{m}' \quad - \sum_{\delta(\mathsf{m})=\delta(\mathsf{m}')} a_{\mathsf{m}}\mathsf{m}$$

where m' is a minimally non-standard monomial and $\mathsf{m}'-\sum_{\mathsf{m}} a_{\mathsf{m}}\mathsf{m}$ is its straightening relation.

Proof. We have only to prove the last part about $\mathcal{A}_0$. Consider the symmetric algebra $\mathsf{T}\doteq\mathsf{S}(\mathbb{A},t)$ with indeterminates the set of generators $\mathbb{A}$ and the parameter t. Let B be the quotient of T by the ideal generated by the modified straightening relations

$$\mathsf{m}'-\sum_{\mathsf{m}} a_{\mathsf{m}}\mathsf{m}t^{\delta(\mathsf{m})-\delta(\mathsf{m}')}$$

for all m' minimally non-standard. We may define a map $\mathsf{T}\longrightarrow\mathbb{k}[t,t^{-1}]\otimes A$ by $\mathbb{A}\ni\mathsf{a}\longmapsto \mathsf{a}t^{-\delta(\mathsf{a})}$ and by $t\longmapsto t$. It is clear that this map is well defined also on B. Its image is $\mathcal{A}$ by definition of this last algebra. Moreover it is an injective map since $\mathcal{A}$ has a standard monomial theory on a multiset defined in terms of that of A with any monomial m replaced by $\mathsf{m}t^{-\delta(\mathsf{m})}$.

It is now clear that $\mathcal{A}_0\simeq B\big|_{t=0}$ is as claimed in the statement of the theorem. □

3. Standard monomial theory for multicones over flag varieties

In this section we apply the abstract construction of the previous section to the multicone over a flag variety; this is the motivating example for the above general setting of a standard monomial theory on a multiset.

Let G be a simply connected semisimple algebraic group over an algebraically closed field $\mathbb{k}$ of characteristic 0 and let $T\subset B\subset G$ be a maximal torus and a Borel subgroup of G, respectively. Denote by W the Weyl group and by $\Lambda\supset\Lambda^+$ the lattice of integral weights and the monoid of dominant weights associated to the choice of T and B. For a dominant weight λ denote by $W_\lambda\subseteq W$ its stabilizer and by $W^\lambda\subseteq W$ the set of minimal length representatives of the cosets W/W_λ; denote, moreover, by $\leqslant$ the Bruhat order on W and on W^λ.

Now let $P\supseteq B$ be a parabolic subgroup of G stabilizing the line generated by a highest weight vector v_λ in the irreducible G–module V_λ of highest weight λ. We have a natural map $G/P\ni gP\longmapsto[g\cdot v_\lambda]\in\mathbb{P}(V_\lambda)$ from the flag variety G/P to the projective space over V_λ. We use this map to define the line bundle $\mathcal{L}_\lambda$ on G/P as the pull-back of $\mathcal{O}(1)$ on $\mathbb{P}(V_\lambda)$; we denote the space of its sections by $\Gamma(G/P,\mathcal{L}_\lambda)$. Notice that, as G–modules, we have $\Gamma(G/P,\mathcal{L}_\lambda)\simeq V_\lambda^*$, the dual of V_λ. In the sequel we denote by λ^* the unique dominant weight such that $V_{\lambda^*}\simeq V_\lambda^*$ as G–modules.

In [25] Littelmann associated to a fixed piece-wise linear path $\pi : [0,1]_{\mathbb{Q}} \longrightarrow \Lambda\otimes\mathbb{Q}$ completely contained in the dominant Weyl chamber and ending in $\lambda \doteq \pi(1) \in \Lambda^+$, a set $\mathbb{B}_\pi$ of piece-wise linear paths. The set $\mathbb{B}_\pi$ gives the character of the irreducible module V_λ:

$$\operatorname{char} V_\lambda = \sum_{\eta\in\mathbb{B}_\pi} e^{\eta(1)}$$

In particular, if we start with the path $\pi_\lambda : t \longmapsto t\lambda$ we obtain the set $\mathbb{B}_\lambda$ of *L-S paths* of *shape* λ; they may be combinatorially described in the following way.

Given a pair $\tau < s_\beta\tau$ of adjacent elements in W^λ, where s_β is the reflection with respect to the positive root β, we define the positive integer $f_\lambda(\tau, s_\beta\tau) \doteq \langle\tau(\lambda), \check\beta\rangle$. Further, we extend f_λ to comparable pairs $\sigma < \tau$ in W^λ by choosing a chain $\sigma = \tau_1 < \tau_2 < \cdots < \tau_u = \tau$ of adjacent elements in W^λ and defining $f_\lambda(\sigma,\tau) \doteq \gcd\{f_\lambda(\tau_1,\tau_2),\ldots,f_\lambda(\tau_{u-1},\tau_u)\}$; indeed such gcd is independent of the chain used to compute it (see [17]).

A pair $\eta \doteq (\tau_1,\tau_2,\ldots,\tau_r; a_0,a_1,\ldots,a_r)$, where $\tau_1 < \tau_2 < \cdots < \tau_r$ is a sequence of comparable elements of W^λ and $0 = a_0 < a_1 < \cdots < a_r = 1$ are rational numbers, is an L-S path if the integral condition $a_i f_\lambda(\tau_i,\tau_{i+1}) \in \mathbb{N}$ holds for all $i = 1,2,\ldots,r-1$. The pair η is identified with the path

$$\pi_{(a_r-a_{r-1})\tau_r(\lambda)} * \pi_{(a_{r-1}-a_{r-2})\tau_{r-1}(\lambda)} * \cdots * \pi_{(a_1-a_0)\tau_1(\lambda)}$$

where we denote concatenation of paths by $*$. The set $\operatorname{supp}\eta \doteq \{\tau_1,\tau_2,\ldots,\tau_r\}$ is called the *support* of the path η.

Let $\mathcal{W}$ be the set of words in the alphabet W and denote by N_λ the least common multiple of the image of f_λ; we define the *word* $w(\eta)$ of the L-S path $\eta = (\tau_1,\tau_2,\ldots,\tau_r; a_0,a_1,\ldots,a_r)$ as $w(\eta) \doteq \tau_1^{N_\lambda(a_1-a_0)}\cdots\tau_r^{N_\lambda(a_r-a_{r-1})}$; this will be needed in the sequel to define an order on monomials.

The set $\mathbb{B}_\lambda$ not only describes the character of the irreducible G–module V_λ, but also, in [27], Littelmann associates a section p_π in $\Gamma(G/P,\mathcal{L}_\lambda)$ to an L-S path $\pi \in \mathbb{B}_\lambda$. The set $\mathbb{A}_\lambda \doteq \{\mathsf{p}_\pi \mid \pi \in \mathbb{B}_\lambda\}$ of these sections, of shape λ, may be used to construct a standard monomial theory as follows. For more details about the combinatorics of L-S paths and their application to the geometry of Schubert varieties one may see [9].

Given two dominant weights λ,μ we lift the Bruhat order on W^λ and W^μ to $W^\lambda \sqcup W^\mu$ by defining $W^\lambda \ni \sigma \leqslant \tau \in W^\mu$ if there exist $\sigma',\tau' \in W$ such that $\sigma' W_\lambda = \sigma W_\lambda$, $\tau' W_\mu = \tau W_\mu$ and $\sigma' \leqslant \tau'$ with respect to the Bruhat order of W. For details we refer to [11]. Notice that this lift is still the Bruhat order if λ and μ have the same stabilizer in W. We use this order to define a relation $\underset{\lambda,\mu}{\longleftarrow}$ on pairs in $\mathbb{B}_\lambda \times \mathbb{B}_\mu$ as follows: we set

$$\pi \underset{\lambda,\mu}{\longleftarrow} \eta \quad \text{if } \pi \in \mathbb{B}_\lambda,\ \eta \in \mathbb{B}_\mu, \text{ and } \max\operatorname{supp}\pi \leqslant \min\operatorname{supp}\eta$$

Notice that if $\lambda = \mu$, then $\underset{\lambda,\lambda}{\longleftarrow}$ is a transitive and antisymmetric relation.

Recall that the set of pairs $(\pi,\eta) \in \mathbb{B}_\lambda \times \mathbb{B}_\mu$ such that $\pi \underset{\lambda,\mu}{\longleftarrow} \eta$ is in natural bijection with the basis $\mathbb{B}_{\pi_\lambda * \pi_\mu}$ as proved by Littelmann in Theorem 10.1 in [26]. Further the two bases $\mathbb{B}_{\pi_\lambda*\pi_\mu}$ and $\mathbb{B}_{\pi_\mu*\pi_\lambda}$ of the module $V_{\lambda+\mu}$ are in bijection by a unique isomorphism of crystal graphs as follows at once by Theorem 6.3 in [25]. So we have the diagram

$$\begin{array}{ccc} \{\pi * \eta \mid \pi \underset{\lambda,\mu}{\longleftarrow} \eta\} & \longrightarrow & \mathbb{B}_{\pi_\lambda*\pi_\mu} \\ & & \downarrow \\ \{\eta' * \pi' \mid \eta' \underset{\mu,\lambda}{\longleftarrow} \pi'\} & \longleftarrow & \mathbb{B}_{\pi_\mu*\pi_\lambda} \end{array}$$

and, for $\mathbb{B}_\lambda \ni \pi \underset{\lambda,\mu}{\longleftarrow} \eta \in \mathbb{B}_\mu$, we define $\phi_{\lambda,\mu}(\mathsf{p}_\pi \mathsf{p}_\eta) \doteq \mathsf{p}_{\eta'}\mathsf{p}_{\pi'}$ if (η', π') corresponds to (π, η) under the composition of the above three bijections.

Finally let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be dominant weights stabilized by the parabolic subgroup P. As seen before for a pair of dominant weights, we define an order $\leqslant$ on $W^{\lambda_1} \sqcup W^{\lambda_2} \sqcup \cdots \sqcup W^{\lambda_n}$ by: $W^{\lambda_i} \ni \sigma \leqslant \tau \in W^{\lambda_j}$ if $i \leqslant j$ and there exist σ', $\tau' \in W$ such that $\sigma' W_{\lambda_i} = \sigma W_{\lambda_i}$, $\tau' W_{\lambda_j} = \tau W_{\lambda_j}$ and $\sigma' \leqslant \tau'$ with respect to the Bruhat order on W. Further we refine this order to a total order $\leqslant_t$ such that $W^{\lambda_i} \ni \sigma \leqslant_t \tau \in W^{\lambda_j}$ only for $i \leqslant j$.

Let $\leqslant_{t,\mathrm{lex}}$ be the lexicographic order on the set of words $\mathcal{W}$ defined as follows: $\tau_1 \cdots \tau_u \leqslant_{t,\mathrm{lex}} \sigma_1 \cdots \sigma_v$ if there exists i such that $\tau_j = \sigma_j$ for all $j = 1, 2, \ldots, i$ and either $i = v$ or $i < u, v$ and $\tau_{i+1} <_t \sigma_{i+1}$.

For an L-S path section p_π in $\mathbb{A} \doteq \mathbb{A}_{\lambda_1} \sqcup \mathbb{A}_{\lambda_2} \sqcup \cdots \sqcup \mathbb{A}_{\lambda_n}$ let $\lambda(\mathsf{p}_\pi) = \lambda_i$ if λ_i is the shape of π and extend the shape to monomials as $\lambda(\mathsf{p}_{\pi_1}\mathsf{p}_{\pi_2}\cdots\mathsf{p}_{\pi_u}) = \lambda(\pi_1) + \lambda(\pi_2) + \cdots + \lambda(\pi_u)$.

On formal monomials in $\mathbb{A}$ we define the following order:

$$\mathsf{p}_{\eta_1}\mathsf{p}_{\eta_2}\cdots\mathsf{p}_{\eta_u} \lessdot \mathsf{p}_{\epsilon_1}\mathsf{p}_{\epsilon_2}\cdots\mathsf{p}_{\epsilon_v}$$

if

- the shapes of the two monomials are equal, and
- $w(\eta_1)w(\eta_2)\cdots w(\eta_u) \leqslant_{t,\mathrm{lex}} w(\epsilon_1)w(\epsilon_2)\cdots w(\epsilon_v)$.

Notice that this order, for our purpose, is equivalent to the one used in Section 7 of [23] (see Proposition 32 in [9] and Proposition 2.1 in [11]) since the relations we are going to see are homogeneous with respect to the shape. Further this order verifies the conditions in Section 2 for an order on monomials.

Finally notice that we may define a relation $\longleftarrow$ on $\mathbb{A}$ by declaring $\mathsf{p}_\pi \longleftarrow \mathsf{p}_\eta$ if $\pi \in \mathbb{B}_{\lambda_i}$, $\eta \in \mathbb{B}_{\lambda_j}$ and $\pi \underset{\lambda_i,\lambda_j}{\longleftarrow} \eta$.

Now consider the $\Bbbk$–algebra

$$A(\lambda_1, \lambda_2, \ldots, \lambda_n) \doteq \bigoplus \Gamma(G/P, \mathcal{L}_{m_1\lambda_1 + m_2\lambda_2 + \cdots + m_n\lambda_n})$$

where the sum runs over all n–tuples of non-negative integers $m_1, m_2, \ldots, m_n$. This algebra is the coordinate ring of the multicone over the partial flag variety G/P mapped diagonally in $\mathbb{P}(V_{\lambda_1}) \times \cdots \times \mathbb{P}(V_{\lambda_n})$.

Now we need a result about the sections p_π in order to state our main theorem about standard monomial theory.

Proposition 3.1 (See Proposition 7.3 in [23])**.** *If $\pi_1, \pi_2 \ldots, \pi_N$ are L-S paths in $\mathbb{B}_{\lambda_1} \sqcup \mathbb{B}_{\lambda_2} \sqcup \cdots \mathbb{B}_{\lambda_n}$, then $\mathsf{p}_{\pi_1}\mathsf{p}_{\pi_2}\cdots\mathsf{p}_{\pi_N} = \sum a_{\eta_1,\eta_2,\ldots,\eta_N}\mathsf{p}_{\eta_1}\mathsf{p}_{\eta_2}\cdots\mathsf{p}_{\eta_N}$, where $\mathsf{p}_{\eta_1}\mathsf{p}_{\eta_2}\cdots\mathsf{p}_{\eta_N}$ is standard and $a_{\eta_1,\eta_2,\ldots,\eta_N} \neq 0$ only if $\mathsf{p}_{\pi_1}\mathsf{p}_{\pi_2}\cdots\mathsf{p}_{\pi_N} \lessdot \mathsf{p}_{\eta_1}\mathsf{p}_{\eta_2}\cdots\mathsf{p}_{\eta_N}$.*

In [23], this proposition is stated and proved only for $N = 2$ and $n = 1$ (i.e. for products of two sections of the same shape). However the proof there may be verbatim generalized.

We finally have all we need to state the main result of standard monomial theory for the multicone.

Theorem 3.2 (Proposition 4.1 in [11])**.** *The set of generators $\mathbb{A}$, the swap maps $\phi_{\lambda_i,\lambda_j}$ and the relation $\longleftarrow$ define a standard monomial theory for the algebra $A(\lambda_1, \lambda_2, \ldots, \lambda_n)$ on the multiset $\mathbb{A}$. With respect to the order $\lessdot$, any non-standard monomial in the p_π's has a straightening relation.*

We stress that for this standard monomial theory we cannot apply Corollary 2.2 since the notion of weakly standard and standard do not coincide. Indeed, in general, there exist minimally non-standard monomials of degree 3 (see Example 3.1 below). However, as proved in Proposition 2 in [22], the ideal of relations is still generated in degree two. In particular, it is generated by the straightening relations for the non-standard monomials of degree 2.

We point out that in the proof of Proposition 4.1 in [11] there is a slight inaccuracy. Only the proof that the relations of degree 2 are straightening relations is correct as given there; indeed, in [11], Proposition 7.3 in [23] is used while one needs its generalization in Proposition 3.1.

3.1. **Example of multicones for type A.** We see an example of a standard monomial theory on a multiset. We develop first some combinatorics about rows and tableaux and then we apply these to the multicones over partial flag varieties for $\mathsf{SL}_{\ell+1}$. More details and all proofs about the combinatorics may be found in [8], while one may see [11] about the application to the multicones. For the particular type of multicones we are going to discuss, our standard monomial theory is completely explicit.

We fix a positive integer ℓ and denote by $\mathsf{T}(k)$ the set of increasing sequences $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant \ell + 1$ of integers; we call such a sequence a *row* while k is its *shape*.

We define a (partial) order $\longleftarrow$ on the set of rows in the following way: if $R = i_1 i_2 \cdots i_k \in \mathsf{T}(k)$ and $S = j_1 j_2 \cdots j_h \in \mathsf{T}(h)$ then $R \longleftarrow S$ if either (i) $k \geqslant h$ and $i_1 \leqslant j_1$, $i_2 \leqslant j_2$, …, $i_h \leqslant j_h$ or (ii) $k \leqslant h$ and $i_1 \leqslant j_{h-k+1}$, $i_2 \leqslant j_{h-k+2}$, …, $i_k \leqslant j_h$. This order may simply be described as follows. Align the two rows R, S to the left if R has shape greater than or equal to that of S or to the right otherwise, then compare the numbers in the columns: if these numbers are non-decreasing then $R \longleftarrow S$. For example $135 \longleftarrow 14$ while $45 \not\longleftarrow 135$.

Suppose we are given two rows R, S, with $R \longleftarrow S$, of shapes $k \leqslant h$, respectively. One can prove that the set of subrows S' of S of shape k such that $R \longleftarrow S'$ has a minimum S^0 for the order $\longleftarrow$. In the same way, there exists a maximum R^0 for $\longleftarrow$ in the set of rows R', of shape h, containing R and such that $R' \longleftarrow S$. Further $R^0 \longleftarrow S^0$ and $R^0 \cup S^0 = R \cup S$ counting entries with multiplicities.

Analogously, if the two rows R, S have shapes $k \geqslant h$, respectively, we define R^0 as the $\longleftarrow$–maximal subrow of R, of shape h, which is $\longleftarrow$–less or equal to S, and S^0 as the minimum of the rows containing S, of shape k, and $\longleftarrow$–greater or equal to R. Also in this case we have $R^0 \longleftarrow S^0$ and $R^0 \cup S^0 = R \cup S$ with multiplicities.

So we have defined a *swap map* $\phi_{k,h}$ for pairs of comparable rows by defining: $\phi_{k,h}(R, S) = (R^0, S^0)$. We have $\phi_{k,k} = \mathrm{Id}$ and $\phi_{h,k}\phi_{k,h} = \mathrm{Id}$. For example $\phi_{2,4} : (25, 1346) \longmapsto (1245, 36)$.

A sequence $T = R_1, R_2, \ldots, R_N$ of rows is called a *(skew) tableau*; we think to its rows as aligned by the above recipe, each one with respect to the following one. The *shape* of T is the sequence $(k_1, k_2, \ldots, k_N)$ of the shapes of its rows and we denote by $\mathsf{T}(k_1, k_2, \ldots, k_N)$ the set of all tableaux of a given shape. The tableau T is *weakly standard* if $R_1 \longleftarrow R_2 \longleftarrow \cdots \longleftarrow R_N$, i.e. if the numbers in its columns are non-decreasing. For example $24, 134, 2$ is a weakly standard tableau.

Suppose in the above tableau T we have $R_i \longleftarrow R_{i+1}$, we may then define a new tableau $\tau_i(T) \doteq R_1, \ldots, R_{i-1}, R_i^0, R_{i+1}^0, R_{i+2}, \ldots, R_N$ where $(R_i^0, R_{i+1}^0) = \phi_{k_i,k_{i+1}}(R_i, R_{i+1})$; the tableau $\tau_i(T)$ has shape $(k_1, \ldots, k_{i-1}, k_{i+1}, k_i, k_{i+2}, \ldots, k_N)$. In particular $\tau_i(T)$ is defined if T is weakly standard.

Now suppose that T is weakly standard and that also $\tau_i(T)$ is weakly standard, then we may define $\tau_j(\tau_i(T))$ by swapping two other rows. If all tableaux that we obtain by applying the τ_i's to T are weakly standard, then we say that T is a *standard tableau.* For example $24, 134, 3$ is a standard tableau while $24, 134, 2$ is *not* a standard tableau, indeed if we swap its first two rows we have $124, 34, 2$ which is not weakly standard.

We denote by $\mathsf{ST}(k_1, k_2, \ldots, k_N)$ the set of standard tableaux of shape $(k_1, k_2, \ldots, k_N)$. Further let

$$\mathsf{ST}\{k_1, k_2, \ldots, k_N\} \doteq \bigcup \mathsf{ST}(k_{\tau(1)}, k_{\tau(2)}, \ldots, k_{\tau(N)})$$

where τ runs over all permutations in the symmetric group S_N. The maps τ_i's are defined on $\mathsf{ST}\{k_1, k_2, \ldots, k_N\}$ and they give an action of S_N on this set.

Let us fix for the sequel a shape $\mathsf{k} = (\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_n)$, called the *reference shape*. We say that a shape $(k_1, k_2, \ldots, k_N)$ is *adapted* to the reference shape k if: (i) for all $1 \leqslant i \leqslant N$ there exists j_i such that $k_i = \bar{k}_{j_i}$ and (ii) $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_N$. In the same way, we say that a tableau has *adapted shape* if its shape is adapted to k.

Given two tableaux $T = R_1, R_2, \ldots, R_N$ and $T' = R'_1, R'_2, \ldots, R'_N$ with adapted shapes we define $T \leqslant T'$ if:

- T and T' have the same shape, and
- either $T = T'$ or there exists an index j such that
  - i) $R_1 = R'_1$, $R_2 = R'_2, \ldots, R_j = R'_j$, $R_{j+1} \neq R'_{j+1}$ and
  - ii) $R_{j+1} \longleftarrow R'_{j+1}$.

Now we see how the combinatorial data seen above is linked to the standard monomial theory. Let $G = \mathsf{SL}_{\ell+1}(\mathbb{C})$, B its Borel subgroup of upper triangular matrices, let $\omega_1, \omega_2, \ldots, \omega_\ell$ be the fundamental weights numbered as in [3] and let $P \supseteq B$ be a parabolic subgroup stabilizing the fundamental weights $\omega_{\bar{k}_i}$ for $i = 1, 2, \ldots, n$, where $\mathsf{k} = (\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_n)$ is the above fixed reference shape.

We want to describe a standard monomial theory for the $\mathbb{C}$–algebra

$$A \doteq \bigoplus \Gamma(G/P, \mathcal{L}_{m_1\omega_{\bar{k}_1} + m_2\omega_{\bar{k}_2} + \cdots + m_n\omega_{\bar{k}_n}})$$

where the sum runs over all n–tuples of non-negative integers $m_1, m_2, \ldots, m_n$. This is the same algebra previously studied in this section once we choose $\lambda_1 = \omega_{\bar{k}_1}$, $\lambda_2 = \omega_{\bar{k}_2}, \ldots, \lambda_n = \omega_{\bar{k}_n}$.

The set of rows $\mathsf{T}(k)$ is in bijection with the L-S paths of shape ω_k, so we have a map $\mathsf{T}(k) \ni R \longmapsto \mathsf{p}_R \in \Gamma(G/P, \mathcal{L}_{\omega_k})$. These sections p_R's are nothing else but the classical Plücker coordinates for the Grassmannian of k–dimensional subspaces in $\mathbb{C}^{\ell+1}$ pulled back to G/P.

The order $\longleftarrow$ for rows is the same order defined in the general part in this section by lifting the Bruhat order. The swap maps $\phi_{h,k}$ correspond to the general swap maps (for L-S path sections) and may be defined on sections by: $\phi_{\omega_k,\omega_k}(\mathsf{p}_R, \mathsf{p}_S) = (\mathsf{p}_{R^0}, \mathsf{p}_{S^0})$ if $\mathsf{T}(k) \ni R \longleftarrow S \in \mathsf{T}(h)$ and $\phi_{k,h}(R, S) = (R^0, S^0)$.

Notice that if the reference shape is decreasing then a tableau is standard if and only if it is weakly standard. This is clear since the set of weakly standard tableaux of decreasing shape are a particular instance of a path model. So, in order to check that a weakly standard tableau T is standard one may use the swap maps and make its shape decreasing, obtaining a new tableau T', and then check that T' is (weakly) standard, i.e. check whether the entries of T' are not decreasing in the columns. Further, if T' is standard, then it is uniquely determined by T since the swap maps give an action of the symmetric group. Of course all of this is true also for increasing reference shapes.

If $\mathsf{p}_{R_1}\mathsf{p}_{R_2} \cdots \mathsf{p}_{R_N}$ is a (commutative) monomial in A, we may always assume that the shape of the tableau $R_1, R_2, \ldots, R_N$ is adapted to k; so we compare monomials in the p_R's via the order $\leqslant$ on the corresponding adapted tableaux. This order corresponds to the order defined via the lexicographic order $\leqslant_{t,\mathrm{lex}}$ on words associated to L-S paths in our situation.

So the standard monomial theory for A may be seen in terms of rows and tableaux. But also its straightening relations are quite explicit. In the sequel we give a set of generators for the ideal $\mathcal{R}$ of relations among the generators p_R's, i.e. $\mathcal{R}$ is the kernel of the natural map from the polynomial algebra

$$\mathsf{S}[\mathsf{p}_R \,|\, R \text{ a row of shape in } \mathsf{k}]$$

to A.

We need a slight generalization of these generators: let $R = i_1 i_2 \cdots i_k$ be any sequence of integers in $\{1, 2, \ldots, \ell+1\}$, we define $[R]$ either as 0 if the entries of R are not distinct, or as $(-1)^\sigma \mathsf{p}_{R'}$ if σ is the unique permutation of $1, 2, \ldots, k$ such that $R' = i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(k)}$ is a row.

Now suppose that (k, h), with $k > h$, is a shape adapted to the reference shape and suppose we have two rows

$$\begin{array}{lcllllllllll} R & = & i_1 & i_2 & \ldots & i_{t-1} & u_{t+1} & u_{t+2} & \cdots & \cdots & \cdots & u_{k+1} \\ S & = & u_1 & u_2 & \ldots & u_{t-1} & u_t & j_1 & j_2 & \cdots & j_{h-t} & \end{array}$$

with shapes k, h, respectively, such that $i_1 < u_1$, $i_2 < u_2, \ldots, i_{t-1} < u_{t-1}$ but $u_{t+1} > u_t$ so that $T \doteq R, S$ is not a standard tableau; we say that t is the *index of violation* of standardness in T. Then the polynomial

$$\sum (-1)^\sigma [i_1, \ldots, i_{t-1}, u_{\sigma(t+1)}, \ldots, u_{\sigma(k+1)}][u_{\sigma(1)}, \ldots, u_{\sigma(t)}, j_1, \ldots, j_{h-k}]$$

where the sum runs over a set of representatives for the quotient $\mathsf{S}_{k+1}/\mathsf{S}_t \times \mathsf{S}_{k+1-t}$, is in $\mathcal{R}$. Such a relation is called a *shuffling relation.* The case of decreasing adapted shape $k < h$ results in similar shuffling relations.

Notice that a shuffling relation may not be a straightening relation; indeed other non-standard tableaux besides T (corresponding to σ = the identity permutation) may appear. But any other non-standard tableau appearing in this relation has index of violation greater than t. So we may use a finite number of shuffling relations and reach eventually a straightening relation for T.

Finally, since standard and weakly standard coincide for tableaux with two rows, and since, by [22], we know that $\mathcal{R}$ is generated in degree 2, we conclude that the shuffling relations generate $\mathcal{R}$.

Let us see an example with $\ell = 3$ and reference shape $\mathsf{k} = (2, 3, 1)$. The tableau $T \doteq 24, 134, 2$ is weakly standard but not standard (as already seen above). So $[24][134][2] = \mathsf{p}_{24}\mathsf{p}_{134}\mathsf{p}_2$ is a linear combination of sections associated to standard tableaux.

We have the following shuffling relations (in particular they are also straightening relations):

$$[234][14] \quad - \quad [134][24] \quad + \quad [124][34] \quad = \quad 0$$

$$[34][2] \quad - \quad [24][3] \quad + \quad [23][4] \quad = \quad 0$$

If we multiply the first one by [2], use the second and move T to the left hand side, we have

$$T \quad = \quad [14][234][2] \quad + \quad [24][124][3] \quad - \quad [23][124][4]$$

As one can easily check, the three tableaux in the right hand side are all standard; hence we have obtained the straightening relation for T.

Now let $\mathcal{R}_0$ be the ideal of $\mathsf{S}[\mathsf{p}_R]$ generated by $\mathsf{p}_{R_1}\mathsf{p}_{R_2}\cdots\mathsf{p}_{R_N}$ for all non-standard tableaux $R_1, R_2, \ldots, R_N$. The quotient $A_0 = \mathsf{S}[\mathsf{p}_R]/\mathcal{R}_0$ is called the *discrete algebra* for the multicone with reference shape k. Notice that it is possible to define a certain valuation and, using Theorem 2.3, degenerate A to A_0. In particular, in our example above with $\ell = 3$ and $\mathsf{k} = (2, 3, 1)$, the ideal $\mathcal{R}_0$ is no more generated in degree 2 since the tableau T, for example, is weakly standard but not standard. The same is true for any non-decreasing or non-increasing reference shape.

3.2. **A multicone for type $\mathsf{A}_1 \times \mathsf{A}_1$.** Now we see a very simple example of multicone which will be used in the next Section. Let $G \doteq \mathsf{SL}_2(\mathbb{C}) \times \mathsf{SL}_2(\mathbb{C})$, a group of type $\mathsf{A}_1 \times \mathsf{A}_1$, let ω, ω' be the two fundamental weights and take $\lambda_1 \doteq \omega$, $\lambda_2 = \omega+\omega'$ and $\lambda_3 = \omega'$ (we are using the symbols λ_1, λ_2, λ_3 with the same meaning as in the main part of this Section). The multicone for this example is $\mathsf{SL}_2(\mathbb{C})/B \times \mathsf{SL}_2(\mathbb{C})/B$, which is clearly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We will describe the combinatorics and the straightening relations using tableaux with rows made of two boxes with some boxes filled with the integers 1 and 2.

In particular here is the correspondence between rows and L-S paths for $\mathbb{B}_{\lambda_i}$:

$$\begin{array}{cccc}
\begin{array}{|c|c|}\hline 1 & \ \\ \hline\end{array} & \longmapsto & \pi_{\omega} & \in \mathbb{B}_{\lambda_1} \\
\begin{array}{|c|c|}\hline 2 & \ \\ \hline\end{array} & \longmapsto & \pi_{-\omega} & \in \mathbb{B}_{\lambda_1} \\
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} & \longmapsto & \pi_{\omega+\omega'} & \in \mathbb{B}_{\lambda_2} \\
\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} & \longmapsto & \pi_{-\omega+\omega'} & \in \mathbb{B}_{\lambda_2} \\
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} & \longmapsto & \pi_{\omega-\omega'} & \in \mathbb{B}_{\lambda_2} \\
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array} & \longmapsto & \pi_{-\omega-\omega'} & \in \mathbb{B}_{\lambda_2} \\
\begin{array}{|c|c|}\hline \ & 1 \\ \hline\end{array} & \longmapsto & \pi_{\omega'} & \in \mathbb{B}_{\lambda_3} \\
\begin{array}{|c|c|}\hline \ & 2 \\ \hline\end{array} & \longmapsto & \pi_{-\omega'} & \in \mathbb{B}_{\lambda_3}
\end{array}$$

In the sequel we write $\boxed{i}$ where i may be 1, 2 or nothing and we denote pairs of rows and tableaux as stacked rows of two boxes. Notice that by $\begin{array}{|c|c|}\hline i & j \\ \hline\end{array}$ we mean one of the rows in the above list, so that the row $\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}$ is not allowed in our context.

We define the following relation on boxes:

$$\boxed{i} \leqslant \boxed{j}$$

for all pairs (i, j) with i, j equal to 1,2 or nothing, but the pair $i = 2$ and $j = 1$. The relation $\longleftarrow$, defined by lifting the Bruhat order, corresponds to the following relation via the above bijection from rows to paths:

$$\begin{array}{|c|c|}\hline i & j \\ \hline\end{array} \longleftarrow \begin{array}{|c|c|}\hline h & k \\ \hline\end{array} \qquad \text{if and only if} \qquad \boxed{i} \leqslant \boxed{h} \ \text{ and } \ \boxed{j} \leqslant \boxed{k}$$

Hence a tableau

$$T = \begin{array}{|c|c|}\hline i_1 & j_1 \\ \hline i_2 & j_2 \\ \hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline i_N & j_N \\ \hline\end{array}$$

is weakly standard if and only if: in each column, consecutive integers do not decrease.

The crystal graph isomorphisms are very easy to compute and, denoting by i, j, k integers in $\{1, 2\}$ (so that $\boxed{i}$, $\boxed{j}$ and $\boxed{k}$ are not empty boxes), the resulting swap maps are:

$$\begin{array}{|c|c|}\hline i & j \\ \hline k & \ \\ \hline\end{array} \longleftrightarrow \begin{array}{|c|c|}\hline i & \ \\ \hline k & j \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline \ & i \\ \hline j & k \\ \hline\end{array} \longleftrightarrow \begin{array}{|c|c|}\hline j & i \\ \hline \ & k \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline \ & i \\ \hline j & \ \\ \hline\end{array} \longleftrightarrow \begin{array}{|c|c|}\hline j & \ \\ \hline \ & i \\ \hline\end{array}$$

The swap maps may be summarized in words as: vertically exchange the empty boxes with the filled ones. Hence a tableau T as above is standard if in each column the integer entries, read out by skipping the empty boxes, are non-decreasing. In particular, if in a tableau the empty boxes are only $j_1, j_2, \ldots, j_h$ and $i_k, i_{k+1}, \ldots, i_N$ for certain h, k, then it is standard if and only if it is weakly standard.

The coordinate ring of the multicone is

$$A \doteq \bigoplus_{n_1,n_2,n_3} \Gamma(\mathsf{SL}_2(\mathbb{C})/B \times \mathsf{SL}_2(\mathbb{C})/B, \mathcal{L}_{(n_1+n_2)\omega+(n_2+n_3)\omega'})$$

and we have a surjective map from the polynomial algebra $\mathsf{S}(\begin{array}{|c|c|}\hline i & j \\ \hline\end{array})$ with indeterminates indexed by the rows $\begin{array}{|c|c|}\hline i & j \\ \hline\end{array}$, to A whose kernel $\mathcal{R}$ is generated by the polynomials

$$\begin{array}{|c|c|}\hline 2 & \\ \hline 1 & i \\ \hline\end{array} - \begin{array}{|c|c|}\hline 1 & \\ \hline 2 & i \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & 2 \\ \hline\end{array} - \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline i & 2 \\ \hline & 1 \\ \hline\end{array} - \begin{array}{|c|c|}\hline i & 1 \\ \hline & 2 \\ \hline\end{array}$$

where $i \in \{1, 2\}$.

4. Standard monomial theory for the Cox ring of a wonderful variety

Let X be a wonderful G–variety with (unique) closed G–orbit Y, and let $P \supseteq B$ be the parabolic subgroup such that $Y \simeq G/P$. By [28], X is *spherical*, i.e. it possesses an open B–orbit, say $B \cdot x_0 \subset X$. Since $B \cdot x_0$ is affine, $G \cdot x_0 \smallsetminus B \cdot x_0$ is a union of finitely many B–stable divisors and we denote by Δ the set of their closures in X:

$$\Delta \doteq \{D \subset X \,:\, D \text{ is a } B\text{–stable prime divisor}, \, D \cap G \cdot x_0 \neq \varnothing\}.$$

The elements of Δ are called the *colors* of X.

Denote by B^- the opposite Borel subgroup of B and let $y_0 \in Y$ be the unique B^-–fixed point of X. The normal space $\mathrm{T}_{y_0}X/\mathrm{T}_{y_0}Y$ of Y in X at y_0 is a multiplicity-free T–module. The elements of the set

$$\Sigma \doteq \{T\text{–weights of } \mathrm{T}_{y_0}X/\mathrm{T}_{y_0}Y\}$$

are called the *spherical roots* of X. If $\sigma \in \Sigma$, there exists a unique G–stable divisor X_σ of X such that the weight of T on $\mathrm{T}_{y_0}X/\mathrm{T}_{y_0}X_\sigma$ is σ. This gives a natural correspondence between the set Σ and the irreducible boundary divisors of X.

Recall that every line bundle on X and on Y has a unique G–linearization. As a group, $\mathrm{Pic}(X)$ is freely generated by the equivalence classes of line bundles $\mathcal{L}_D \doteq \mathcal{O}(D)$, for $D \in \Delta$ (see [6, Proposition 2.2]). For all $E \in \mathbb{Z}\Delta$, the associated line bundle $\mathcal{L}_E \doteq \mathcal{O}(E)$ is globally generated, respectively ample, if and only if E is a non-negative, respectively positive, combination of colors. Notice that $\mathbb{Z}\Sigma$ is a sublattice of $\mathbb{Z}\Delta$.

The restriction of line bundles to the closed orbit induces a map $\lambda : \mathrm{Pic}(X) \longrightarrow \Lambda$; given $E \in \mathbb{Z}\Delta$ we set $\lambda_E \doteq \lambda(\mathcal{L}_E)$ in such a way that $\Gamma(Y, \mathcal{L}_E\big|_Y) \simeq V^*_{\lambda_E}$ and, moreover, we set $V_E \doteq V^*_{\lambda_E}$ for short. (Hence $\mathcal{L}_E\big|_Y \simeq \mathcal{L}_{\lambda_E}$ where this last line bundle is defined in the previous section.) Moreover, in particular, $\Gamma(X, \mathcal{L}_E)$ contains a copy of V_E and, since X is spherical the decomposition of the G–module $\Gamma(X, \mathcal{L}_E)$ is multiplicity-free.

If $\gamma \doteq \sum a_\sigma \sigma \in \mathbb{N}\Sigma$, we denote by $s^\gamma \in \Gamma(X, \mathcal{L}_{X^\gamma})$ a section whose divisor is equal to $X_\gamma \doteq \sum a_\sigma X_\sigma$; notice that this section is G–invariant. If $E, F \in \mathbb{Z}\Delta$ are such that $F - E \in \mathbb{N}\Sigma$, then we write $E \leqslant_\Sigma F$. If $E \in \mathbb{N}\Delta$, $F \in \mathbb{Z}\Delta$ and $E \leqslant_\Sigma F$

the multiplication by s^{F-E} induces a G–equivariant map from the sections of $\mathcal{L}_E$ to the sections of $\mathcal{L}_F$, in particular we have $s^{F-E}V_E \subseteq \Gamma(X, \mathcal{L}_F)$. Moreover

Proposition 4.1 ([6, Proposition 2.4]). *Let $F \in \mathbb{Z}\Delta$, then*

$$\Gamma(X, \mathcal{L}_F) \quad = \bigoplus_{E\in\mathbb{N}\Delta\,:\,E\leqslant_\Sigma F} s^{F-E}V_E.$$

Since $\mathrm{Pic}(X) \simeq \mathbb{Z}\Delta$ is a free lattice, the space

$$C(X) \doteq \bigoplus_{D\in\mathbb{Z}\Delta} \Gamma(X, \mathcal{L}_D)$$

is a ring; in analogy with the toric case $C(X)$ is called the *Cox ring* of X. The ring $C(X)$ was studied in [13] and [12] in the case of a wonderful symmetric variety (where it is called respectively the *ring of sections* of X and the *coordinate ring* of X), and in [7] in the case of a wonderful variety (where it is called the *total coordinate ring* of X).

Since X is irreducible, by Proposition 4.1, $C(X)$ is generated as a $\mathbb{k}$–algebra by the sections s^σ, for $\sigma \in \Sigma$, and by the modules $V_D \subseteq \Gamma(X, \mathcal{L}_D)$ for $D \in \Delta$. It follows that $C(X)$ is a quotient of the symmetric algebra

$$\mathsf{S}(X) \doteq \mathbb{k}[s_1, \ldots, s_r] \otimes \mathsf{S}\Big(\bigoplus_{D\in\Delta} V_D\Big)$$

where we fix an ordering $\Sigma = \{\sigma_1, \ldots, \sigma_r\}$ and we set $s_i \doteq s^{\sigma_i}$ for short. Further, notice that the quotient of $C(X)$ by the ideal generated by the sections $s_1, \ldots, s_r$ is isomorphic to the coordinate ring of a multicone over the flag variety $Y \simeq G/P$, that is

$$C(Y) \doteq A(\lambda_{D_1}, \lambda_{D_2}, \ldots, \lambda_{D_q}) = \bigoplus_{D\in\mathbb{N}\Delta} \Gamma(Y, \mathcal{L}_D\big|_Y) \simeq \bigoplus_{D\in\mathbb{N}\Delta} V_D.$$

where $\Delta = \{D_1, \ldots, D_q\}$ is any fixed ordering of Δ. Therefore we have surjective maps

$$\mathsf{S}(X) \longrightarrow C(X) \longrightarrow C(Y),$$

The rings $\mathsf{S}(X)$, $C(X)$ and $C(Y)$ all have natural $\mathbb{Z}\Delta$–gradings, and the previous maps are morphisms of $\mathbb{Z}\Delta$–graded G–algebras.

By Theorem 3.2 we have a standard monomial theory with straightening relations for $C(Y)$. Our aim is to extend it to a standard monomial theory for the Cox ring $C(X)$, and deduce a degeneration result for such a ring. A description of the ideal $\mathcal{I}_X$ defining $C(X)$ as a quotient of $\mathsf{S}(X)$ is given in terms of straightening relations of our standard monomial theory.

Given $D \in \Delta$ we denote by $\mathbb{B}_D \doteq \mathbb{B}_{\lambda_D}$ the set of L-S paths of shape λ_D for short. Given $\pi \in \mathbb{B}_D$, let $\mathsf{x}_\pi \in V_D \subset \Gamma(X, \mathcal{L}_D)$ be the unique section such that $\mathsf{x}_\pi\big|_Y = \mathsf{p}_\pi$ and let

$$\mathbb{A}_D \doteq \{\mathsf{x}_\pi \,:\, \pi \in \mathbb{B}_D\} \subset V_D.$$

Then $\mathbb{A}_D$ is a basis of $V_D \subseteq \Gamma(X, \mathcal{L}_D)$. Further, define

$$\mathbb{A}_\Sigma \doteq \{s_i \,:\, i = 1, \ldots, r\}, \quad \mathbb{A}_\Delta \doteq \bigsqcup_{D\in\Delta} \mathbb{A}_D \quad \text{and } \mathbb{A}_X \doteq \mathbb{A}_\Sigma \sqcup \mathbb{A}_\Delta.$$

In particular, $\mathsf{S}(X)$ is the symmetric algebra in the indeterminates $\mathbb{A}_X$. The multiset of our standard monomial theory is $\mathbb{A}_X = \mathbb{A}_\Sigma \sqcup \bigsqcup_{D\in\Delta} \mathbb{A}_D$ so that if $\mathsf{x}_\pi \in \mathbb{A}_\Delta$, its *shape* is the unique $D \in \Delta$ such that $\mathsf{x}_\pi \in \mathbb{A}_D$ and the shape of s_i is Σ. Recall that in the definition of a standard monomial theory, in order to define a standard monomial theory we have to fix an order on the possible shapes. Here we have already fixed a total order on Δ and we extend it by declaring $\Sigma < D$ for all $D \in \Delta$.

Let $\mathbb{B}_\Delta \doteq \bigsqcup_{D\in\Delta}\mathbb{B}_D$ and set

$$\mathbb{A}_Y \doteq \{\mathsf{p}_\pi : \pi \in \mathbb{B}_\Delta\},$$

which is naturally identified with the subset $\mathbb{A}_\Delta \subset \mathbb{A}_X$ via the bijection $\mathsf{x}_\pi \longmapsto \mathsf{p}_\pi = \mathsf{x}_\pi\big|_Y$.

By Theorem 3.2 we have a standard monomial theory for $C(Y)$. We denote by $\mathsf{M}(Y) \subset \mathsf{S}(\mathbb{A}_Y) \simeq \mathsf{S}(\mathbb{A}_\Delta)$ the set of monomials in the coordinates $\mathbb{A}_Y$, and by $\mathsf{SM}(Y) \subset \mathsf{M}(Y)$ the subset of standard monomials. In particular, using the bijections $\mathbb{A}_D \simeq \mathbb{A}_{\lambda_D}$, for all $D, D' \in \Delta$ we have swap maps $\phi_{D,D'}$, a relation $\longleftarrow$ on $\mathbb{A}_\Delta$ and an order $\leqslant$ on $\mathsf{M}(Y)$ as defined in the previous section.

First we extend the relation $\longleftarrow$ to $\mathbb{A}_X$ by declaring $s_1 \longleftarrow s_2 \longleftarrow \cdots \longleftarrow s_r$ and $s_i \longleftarrow \mathsf{x}_\pi$, $\mathsf{x}_\pi \longleftarrow s_i$ for all $i = 1, 2, \ldots, r$ and all $\pi \in \mathbb{B}_\Delta$. Next we extend the swap maps by

$$\phi_{\Sigma,D}(s_i, \mathsf{x}_\pi) \doteq (\mathsf{x}_\pi, s_i) \qquad\qquad \phi_{D,\Sigma}(\mathsf{x}_\pi, s_i) \doteq (s_i, \mathsf{x}_\pi)$$

for all $i = 1, 2, \ldots, r$ and all $\pi \in \mathbb{B}_\Delta$.

Let $\mathsf{m} \doteq s^\gamma \mathsf{x}_{\pi_1}\cdots\mathsf{x}_{\pi_N}$ be a monomial; then $\nu(\mathsf{m}) \doteq \gamma$ is called the *vanishing* of m . Further if $D_i \in \Delta$ is the shape of $\mathsf{x}_{\pi_i} \in \mathbb{A}_\Delta$, we define the *Picard weight* of m as

$$\gamma + \sum_{i=1}^N D_i,$$

namely the degree of m with respect to the $\mathbb{Z}\Delta$–grading.

Finally, we extend the order $\leqslant$ to monomials in $\mathbb{A}_X$: if $\mathsf{m}_1, \mathsf{m}_2$ are two monomials in $\mathsf{S}(\mathbb{A}_\Delta) \simeq \mathsf{S}(\mathbb{A}_Y)$ and if γ_1, $\gamma_2 \in \mathbb{N}\Delta$, then we set $s^{\gamma_1}\mathsf{m}_1 \leqslant s^{\gamma_2}\mathsf{m}_2$ if:

- the Picard weights of $s^{\gamma_1}\mathsf{m}_1$ and of $s^{\gamma_2}\mathsf{m}_2$ are equal, and
- either $\gamma_1 <_\Sigma \gamma_2$ or $\gamma_1 = \gamma_2$ and $\mathsf{m}_1\big|_Y \leqslant \mathsf{m}_2\big|_Y$ (with respect to the order $\leqslant$ defined in Section 3).

We denote by $\mathsf{M}(X) \subset \mathsf{S}(X)$ the set of the monomials in the indeterminates $\mathbb{A}_X$, endowed with the order $\leqslant$.

The inclusion $\mathbb{A}_Y \hookrightarrow \mathbb{A}_X$ defines a shape-preserving bijection between $\mathsf{M}(Y)$ and the subset of $\mathsf{M}(X)$ of the monomials m such that $\nu(\mathsf{m}) = 0$. Given $\mathsf{n} \in \mathsf{M}(Y)$ we denote by $\widetilde{\mathsf{n}} \in \mathsf{M}(X)$ the corresponding monomial, so we have $\widetilde{\mathsf{n}}\big|_Y = \mathsf{n}$ and, in particular, $\widetilde{\mathsf{p}}_\pi = \mathsf{x}_\pi$. Conversely, given a monomial $\mathsf{m} \in \mathsf{M}(X)$ we may define a monomial $\overline{\mathsf{m}} \in \mathsf{M}(X)$ with $\nu(\overline{\mathsf{m}}) = 0$ by setting $\overline{\mathsf{m}} \doteq s^{-\nu(\mathsf{m})}\mathsf{m}$. Notice that $\mathsf{m} \in \mathsf{M}(X)$ is a standard monomial if and only if $\overline{\mathsf{m}} \in \mathsf{M}(X)$ is a standard monomial, if and only if $\overline{\mathsf{m}}\big|_Y \in \mathsf{M}(Y)$ is a standard monomial. We denote by $\mathsf{SM}(X) \subset \mathsf{M}(X)$ the set of standard monomials in $\mathbb{A}_X$, and if $E \in \mathbb{Z}\Delta$ we denote by $\mathsf{SM}_E(X) \subset \mathsf{M}_E(X)$ the set of standard monomials and the set of all monomials of Picard weight E, respectively.

Following [13, Theorem 3], we are now able to construct a standard monomial theory for the Cox ring $C(X)$ of a wonderful variety X.

Theorem 4.2. i) *Given $E \in \mathbb{Z}\Delta$, the images of the standard monomials of Picard weight E form a basis of $\Gamma(X, \mathcal{L}_E)$.*

ii) *Given a non-standard monomial m' the equality*

$$\mathsf{m}' \quad = \sum_{\mathsf{m}\in\mathsf{SM}(X)} a_\mathsf{m}\mathsf{m}$$

guaranteed by i) *is a straightening relation in $C(X)$; that is, we have $\mathsf{m}' \leqslant \mathsf{m}$ whenever $a_\mathsf{m} \neq 0$. Moreover,*

$$\overline{\mathsf{m}}' \quad = \sum_{\mathsf{m}\in\mathsf{SM}(X),\nu(\mathsf{m})=\nu(\mathsf{m}')} a_\mathsf{m}\overline{\mathsf{m}}$$

is a straightening relation in $C(Y)$.

iii) *The defining ideal $\mathcal{I}_X \subset \mathsf{S}(X)$ is generated by the straightening relations for the non-standard monomials of degree two.*

In particular, the set of generators $\mathbb{A}_X$ together with the above defined swap maps, relation and order define a standard monomial theory on the multiset $\mathbb{A}_X$ with straightening relations for the Cox ring $C(X)$.

Proof. We prove the first two statements together. Let $\pi_1, \ldots, \pi_N \in \mathbb{B}_\Delta$ be such that $\mathsf{x}_{\pi_1} \cdots \mathsf{x}_{\pi_N}$ is not standard. In $A(Y)$, by Theorem 3.2, we have a straightening relation

$$\mathsf{p}_{\pi_1} \cdots \mathsf{p}_{\pi_N} = \sum_{\mathsf{n} \in \mathsf{SM}(Y)} a_{\mathsf{n}} \mathsf{n},$$

where $\mathsf{p}_{\pi_1} \cdots \mathsf{p}_{\pi_N} \leqslant \mathsf{n}$ for all n such that $a_{\mathsf{n}} \neq 0$.

Since $X \smallsetminus G \cdot x_0$ is a normal crossing divisor with smooth irreducible components, a section in $C(X)$ vanishes on the closed orbit Y if and only if it is in the ideal generated by the sections $s_1, \ldots, s_r$. By construction the difference $\mathsf{x}_{\pi_1} \cdots \mathsf{x}_{\pi_N} - \sum_{\mathsf{n}} a_{\mathsf{n}} \widetilde{\mathsf{n}}$ is homogeneous w.r.t. the $\mathbb{Z}\Delta$–grading, and it vanishes on Y. Hence we have

$$\mathsf{x}_{\pi_1} \cdots \mathsf{x}_{\pi_N} = \sum_{\mathsf{n} \in \mathsf{SM}(Y)} a_{\mathsf{n}} \widetilde{\mathsf{n}} + \sum_{\mathsf{m} \in \mathsf{M}_E(X) \,:\, \nu(\mathsf{m}) \neq 0} a_{\mathsf{m}} \mathsf{m},$$

where $\mathsf{x}_{\pi_1} \cdots \mathsf{x}_{\pi_N} \leqslant \widetilde{\mathsf{n}}$ for all $\mathsf{n} \in \mathsf{SM}(Y)$ with $a_{\mathsf{n}} \neq 0$.

Proceeding inductively on the partial order $\leqslant_\Sigma$, the previous equality implies that in $C(X)$ the image of every monomial $\mathsf{m}' \in \mathsf{M}_E(X)$ may be written as the image of a sum of standard monomials $\mathsf{m} \in \mathsf{SM}_E(X)$ with $\mathsf{m}' \leqslant \mathsf{m}$.

Therefore the image of the standard monomials of $\mathsf{SM}_E(X)$ in $C(X)$ is a set of generators for $\Gamma(X, \mathcal{L}_E)$ as a vector space. On the other hand, by Theorem 3.2, the images of the standard monomials $\mathsf{n} \in \mathsf{SM}(Y)$ form a basis for $C(Y)$; hence for all $F \in \mathbb{N}\Delta$ the images of the standard monomials $\mathsf{n} \in \mathsf{SM}(Y)$ of Picard weight F form a basis for the graded component $C(Y)_F = V_F$. So using Proposition 4.1, we have

$$\begin{aligned} \dim \Gamma(X, \mathcal{L}_E) &= \textstyle\sum_{F \in \mathbb{N}\Delta \,:\, F \leqslant_\Sigma E} \dim V_F \\ &= \textstyle\sum_{F \in \mathbb{N}\Delta \,:\, F \leqslant_\Sigma E} |\mathsf{SM}_F(Y)| \\ &= |\mathsf{SM}_E(X)|, \end{aligned}$$

and this finishes the proof of i) and ii).

Now, in order to prove iii), let J be the ideal of $\mathsf{S}(X)$ generated by the straightening relations for the non-standard monomials of degree two. Clearly $J \subseteq \mathcal{I}_X$ and we want to show that these two ideals are equal.

The quotient $\mathsf{S}(X)/\langle J, s_1, \ldots, s_r \rangle$ is isomorphic to $C(Y)$ since the relations for this last ring are generated by the quadratic straightening relations; indeed it is generated in degree 2 by [22, Proposition 2]. So, if m' is a non-standard monomial $\mathsf{m}' + \langle s_1, \ldots, s_r \rangle$ is a sum of standard monomials modulo J. Hence in $\mathsf{S}(X)/J$ the monomial m' is a sum of standard monomials m with $\nu(\mathsf{m}) = 0$ plus $s_1 y_1 + \cdots + s_r y_r$ for some homogeneous elements $y_1, \ldots, y_r$ whose Picard weights are $<_\Sigma$ of that of m'.

Proceeding again by induction on the Picard weight of a non-standard monomial we see that any straightening relation is an element of J. So $J = \mathcal{I}_X$ and the last statement of the Theorem is proved.

□

Remark 4.3. The results stated in the above Theorem overlap with Proposition 3.3.1 in [7]. In that Proposition a description of the quadratic relations of the ring $C(X)$ is given. However in [7] the standard monomial structure is not considered explicitly. Notice that even if the relations of the ring $C(X)$ are generated in degree two, the relations of degree two are not enough to construct a standard monomial theory as we have already noticed in the final part of Example 3.1.

The standard monomial theory constructed in the previous theorem is compatible with the G–orbit closures in X in the following sense. Recall that the subsets $I \subseteq \Sigma$ parametrize the G–orbits in X; that is, for every $x \in X$ there is a unique $I \subseteq \Sigma$ such that $\overline{G \cdot x} = \bigcap_{\sigma \in \Sigma \smallsetminus I} X^\sigma \doteq X_I$.

The G–stable subvariety X_I is again a wonderful variety; its set of spherical roots coincides with I. Given $I \subseteq \Sigma$, we say that a standard monomial $\mathsf{m} \in \mathsf{SM}(X)$ is *I–standard* if $\nu(\mathsf{m}) \in \langle \Sigma \smallsetminus I \rangle_{\mathbb{Z}}$. We denote by $\mathsf{SM}^I(X)$ the set of I–standard monomials, and by $\mathsf{SM}^I_E(X)$ the set of the I–standard monomials of Picard weight $E \in \mathbb{Z}\Delta$.

Corollary 4.4. *Given $E \in \mathbb{Z}\Delta$, the images of the I–standard monomials $\mathsf{SM}^I_E(X)$ are a basis for $\Gamma(X_I, \mathcal{L}_E\big|_{X_I})$.*

Proof. Let $J \doteq \Sigma \smallsetminus I$. Then the restriction of sections $\rho : \Gamma(X, \mathcal{L}) \longrightarrow \Gamma(X_I, \mathcal{L}_E\big|_{X_I})$ is a surjective map, and we have

$$\ker \rho \quad = \bigoplus_{F \in \mathbb{N}\Delta \,:\, F \leqslant_J E} s^{E-F} V_F,$$

where we write $F \leqslant_J E$ if and only if $E - F \in \mathbb{N}J$. It follows that the images of the J–standard monomials of Picard weight E give a basis for $\ker \rho$, whereas the restrictions of the images of the I–standard monomials of Picard weight E give a basis for $\Gamma(X_I, \mathcal{L}_E\big|_{X_I})$. □

When X is the wonderful compactification of a semisimple adjoint group regarded as a homogeneous $G \times G$–variety, the above constructed standard monomial theory is even compatible with the $B \times B$–orbit closures (see [2]).

4.1. **An example of the Cox ring for a wonderful variety.** We illustrate our theory in a simple example. We will make use of the results and conventions introduced in Section 3.2. Let $V = \mathbb{C}^2$ and define

$$X = \{([\varphi], [A], [v]) \in \mathbb{P}(V^*) \times \mathbb{P}(\mathrm{End}(V)) \times \mathbb{P}(V) \,:\, \varphi(Av) = 0\}.$$

X is a wonderful variety for the group $G = \mathsf{SL}(V) \times \mathsf{SL}(V)$ acting by $(g, h)(\varphi, A, v) = (g \cdot \varphi, gAh^{-1}, h \cdot v)$. We choose the maximal torus given by the diagonal matrices and the Borel subgroup given by the upper triangular matrices. We denote by α the simple root of the first factor of G and by α' the simple root of the second factor.

The Picard group is generated by the pull-backs of the three line bundles $\mathcal{O}_{\mathbb{P}(V^*)}(1)$, $\mathcal{O}_{\mathbb{P}(\mathrm{End}(V))}(1)$ and $\mathcal{O}_{\mathbb{P}(V)}(1)$ and we denote by D_1, D_2, D_3 the associated colors, respectively. We have two spherical roots

$$\sigma_1 = \alpha = D_1 + D_2 - D_3 \text{ and } \sigma_2 = \alpha' = D_2 + D_3 - D_1.$$

The closed orbit Y in this case is $\mathbb{P}^1 \times \mathbb{P}^1$ and the ring $C(Y)$ is the one studied in Section 3.2.

The ring $C(X)$ is generated by the eight generators x_π that we denote with rows of length two as in Section 3.2, together with s_1 and s_2. The standard monomials are the monomials of the form $s_1^a s_2^b \tilde{\mathsf{m}}$, where $\tilde{\mathsf{m}}$ is a standard monomial for the ring $C(Y)$. In particular we have the same minimally non-standard monomials as in the ring $C(Y)$, and the straightening relations are given by

$$\begin{array}{|c|c|}\hline 2 & \\ \hline 1 & i \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & \\ \hline 2 & i \\ \hline\end{array} + s_1 \begin{array}{|c|c|}\hline & i \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & 2 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array} + s_1 s_2$$

$$\begin{array}{|c|c|}\hline i & 2 \\ \hline & 1 \\ \hline\end{array} = \begin{array}{|c|c|}\hline i & 1 \\ \hline & 2 \\ \hline\end{array} + s_2 \begin{array}{|c|c|}\hline i & \\ \hline\end{array}$$

where $i \in \{1, 2\}$.

5. Degeneration and rational singularities

Any straightening relation involves monomials with higher power of the sections $s_1, \ldots, s_r$. This allows us to degenerate $\operatorname{Spec} C(X)$ to the product of the affine space $\Bbbk^r$ with a multicone over the flag variety $G/P \simeq Y$. Let us see the details for such a degeneration.

Corollary 5.1. *There exists a flat $G \times \Bbbk^*$–equivariant degeneration $\mathcal{C}$ of $C(X)$ to the ring $\Bbbk[s_1, \ldots, s_r] \otimes C(Y)$; further all generic fibers of $\mathcal{C}$ are isomorphic to $C(X)$.*

Proof. We define a map $\delta : \mathbb{A}_X \longrightarrow \mathbb{N}$ by $\delta(\mathsf{x}_\pi) = 0$ for all $\pi \in \mathbb{B}_\Delta$ and $\delta(s_i) = 1$ for all $i = 1, 2, \ldots, r$. This map is a valuation for the standard monomial theory of $C(X)$ by Theorem 4.2. Hence we may apply Theorem 2.3. The special fiber is isomorphic to the ring in the statement of the theorem again by Theorem 4.2.

Moreover, for this valuation map the Rees algebra is

$$\mathcal{C} = \cdots \oplus C(X)t^2 \oplus C(X)t \oplus C(X) \oplus Kt^{-1} \oplus K^2t^{-1} \oplus \cdots$$

with K the ideal of $C(X)$ generated by the sections $s_1, s_2, \ldots, s_r$. So, being K generated by G–invariants, the action of G on $C(X)$ induces an action on $\mathcal{C}$ by letting G act trivially on t. In particular G acts on each fiber and it is clear that this G–action commutes with the isomorphisms $\mathcal{C}_a \longrightarrow \mathcal{C}_{\lambda^{-1}a}$ for any $a \in \Bbbk$ and $\lambda \in \Bbbk^*$. So the deformation is also G–equivariant. □

We now apply this degeneration result to the study of the singularities of the algebra $C(X)$. A variety X is said to have *rational singularities* if there exists a resolution of singularities $\pi : Y \longrightarrow X$ of X such that $R^i\pi_*\mathcal{O}_Y = 0$ for $i > 0$ and $\pi_*\mathcal{O}_Y = \mathcal{O}_X$. If such a property holds for a resolution then it holds for all resolutions. Finally a ring A is said to have rational singularities if $\operatorname{Spec} A$ has rational singularities.

We have the following properties:

(a) a multicone over a flag variety has rational singularities (see [22], Theorem 2)
(b) if X is an affine G–variety with rational singularities and G is a reductive group then $X/\!\!/G$ has rational singularities (see [4])
(c) if $(\mathcal{X}, X) \longrightarrow (S, s_0)$ is a flat deformation of a variety with rational singularities X then there exists a neighbourhood U of s_0 such that for $s \in U$ the fiber over s has also rational singularities (see [19], Théorème 4).

Given $D \in \mathbb{Z}\Delta$ consider the subalgebra of $C(X)$ defined as follows

$$C_D(X) \doteq \bigoplus_{n \geqslant 0} \Gamma(X, \mathcal{L}_{nD}).$$

This is the projective coordinate ring of a spherical variety, namely the image of X in the projective space $\mathbb{P}(\Gamma(X, \mathcal{L}_D)^*)$. It is known that these rings have rational singularities (see [30], or also [1, Remark 2.5] for another proof which is closer to the constructions of this paper).

Proposition 5.2. *Let X be a wonderful variety and let $D \in \mathbb{Z}\Delta$. Then $C(X)$ and $C_D(X)$ have rational singularities.*

Proof. By Corollary 5.1 we have that $C(X)$ is a deformation of a multicone over a flag variety, which has rational singularities by (a), hence $C(X)$ has rational singularities as well by (c).

In order to show the second claim, let $\widetilde{D} \in \mathbb{Z}\Delta$ be such that $\mathbb{Q}D \cap \mathbb{Z}\Delta = \mathbb{Z}\widetilde{D}$. Then the inclusions $\mathbb{Z}D \subset \mathbb{Z}\widetilde{D} \subset \mathbb{Z}\Delta$ define a torus $S \doteq \mathrm{Hom}(\mathbb{Z}\Delta/\mathbb{Z}\widetilde{D}, \mathbb{C}^*)$ and a finite group $\Gamma \doteq \mathrm{Hom}(\mathbb{Z}\widetilde{D}/\mathbb{Z}D, \mathbb{C}^*)$. Moreover, we have natural actions of S on $C(X)$ and of Γ on $C(X)^S$, and $C_D(X) = (C(X)^S)^\Gamma$. Therefore, by (b) it follows that $C_D(X)$ has rational singularities as well. □

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