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An invariant region for the collisional dynamics of two bodies on Keplerian orbits

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Abstract We study the dynamics of two bodies moving on elliptic Keplerian orbits around a fixed center of attraction and interacting only by means of elastic or inelastic collisions. We show that there exists a bounded invariant region: for suitable values of the total energy and the total angular momentum (explicitly computable) the orbits of the bodies remain elliptic, whatever are the number and the details of the collisions.

The invariant region exists also in the case of two bodies interacting by short range potential.

Keywords Planetary systems, Planetary rings, Elastic collisions, Inelastic collisions

Mathematics Subject Classification (2000) MSC 70F15, MSC 37N05

1 Introduction

The interest in the collisional dynamics in a planetary system goes back to Poincaré. In particular, in [12] he studies the planetary three-body problem, with one body of large mass (the Sun) and two bodies of small mass (the planets). He indicates how to find periodic solutions (of *deuxième espèce*), as perturbations of periodic collisional solutions he can get if the mass of the planets are infinitely small and the distance between them becomes infinitely small. In this approximation, the two bodies are on Keplerian ellipses until the "choc" (i.e. the interaction), which moves the bodies on two other Keplerian ellipses. During the interaction, only the total energy and the total momentum are conserved, and the choc acts as an elastic collision. It should be noted that the collision can move the bodies also on hyperbolic orbits but Poincaré is only interested in elliptic case.

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In this work, we prove that, for this collisional dynamics, there exists an invariant bounded region of positive measure in the phase space. More precisely, we consider two bodies moving on elliptic Keplerian orbits around a center (the “Sun” in the sequel), interacting only by means of collisions. A collision changes the orbital parameters of the bodies, and a sequence of collisions can move one of the bodies out of the system, on parabolic or hyperbolic orbits. We show that for suitable values of the total energy and the total angular momentum (easily computable), the bodies remain on elliptic colliding orbits. Moreover, we extend this result to the case of two point particles interacting by means a bounded short range potential.

We are neglecting the gravitational interaction between the bodies and the influence on the Sun. These approximations can be only justified for very light particles and for a few revolution times (the time between two consecutive collisions can be very long with respect to the revolution period of the particles). Considering this, our result has the following interpretation for real systems: the particles can not leave the system due to the collisions, unless other perturbations change enough the orbital parameters.

Our result seems not known in the literature, despite its simplicity and despite the great interest in research on planetary systems. We run in the result while we were studying numerical models for the dynamics of inelastic particles of planetary rings. It is known (see [13], [1], and [2]) that the inelasticity of the collisions is sufficient to guarantee the persistence of the rings. In contrast, in the case of elastic collisions, almost all the particles leave the system on hyperbolic orbits. Here we prove that in the case of only two particles, the inelasticity is not needed to avoid that the orbits become hyperbolic or parabolic; in this sense, two colliding particles are a stable subsystem of a ring. This observation and its consequences can be interesting for the study of various models of planetary rings. In particular, in some models it is assumed that the collisions are elastic for small relative velocities (see [5] for experimental result on the inelasticity of ice balls).

From the mathematical point of view, it can be interesting to study how the particles moves on the invariant region. Preliminary two-dimensional numerical simulations, in which the impact parameter is randomly chosen, show that the two orbits are approximately tangent, for most of the time. The numerical study of this behaviour in the more realistic three-dimensional case is difficult: it is necessary to find an efficient way to determine the values of the anomalies which correspond to collisional configurations. Some useful suggestion and some technical insight can be obtained analyzing the solutions to the problem of finding the critical points of the distance from two confocal elliptic orbits in $\mathbb{R}^3$ (see [4]).

The paper is organized as follows. In section 2, we establish the mathematical notation and the exact nature of the problem. In section 3, we analyze a simplified model considering a two-dimensional *dynamics of the orbits* instead of the dynamics of the two bodies. We consider more general cases in section 4 (two bodies in $\mathbb{R}^2$) and in section 5, in which we analyze the case of two bodies in $\mathbb{R}^3$; here we also analyze the case of point particles interacting by means a short range potential.

2 The problem

The system we analyze consists in two spherical bodies, of mass m_1 and m_2 and radii R_1 and R_2 respectively, which are attracted by a fixed sun and interact by means of elastic or inelastic collisions. We indicate with $\mathbf{x}_i$ and $\mathbf{v}_i$ the position of the center and the velocity of the body $i = 1,2$, with $M = m_1 + m_2$ the total mass and with $\mu_i = m_i/M$ the fraction of the total mass carried by the body i. We choose the units of measure in such that the gravitational potential energy of the body i is m_i/r_i, where $r_i = |\mathbf{x}_i|$ is the distance from the Sun.

In the case of hard spheres, inelastic collisions can be modeled supposing that only a part of the normal impulse is transferred, while the tangential one is conserved. Let us denote with $\mathbf{n} = (\mathbf{x}_1 - \mathbf{x}_2)/(R_1 + R_2)$ the direction of the relative position at the moment of the impact, and with $\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2$ the relative velocity. The particles are in the incoming configuration iff $\mathbf{n} \cdot \mathbf{w} < 0$. The relative velocity after the collision is

$$\mathbf{w}' = (I - \mathbf{n} \times \mathbf{n})\ \mathbf{w} - (1 - 2\varepsilon)(\mathbf{n} \times \mathbf{n})\ \mathbf{w} \tag{1}$$

where I is the identity matrix, $(\mathbf{x} \times \mathbf{y})_{ij} = x_i y_j$ is the tensor product, $\mathbf{n} \times \mathbf{n}$ is the projector on the direction of $\mathbf{n}$, $I - \mathbf{n} \times \mathbf{n}$ is the projector on the orthogonal plane to $\mathbf{n}$, and finally $\varepsilon \in [0, 0.5]$ is the parameter of inelasticity, which is 0 in the case of the elastic collision.

The outgoing velocities $\mathbf{v}_1'$, $\mathbf{v}_2'$ can be obtained from $\mathbf{w}'$ using the conservation of the velocity of the center of mass

$$\mathbf{v}' = \mu_1 \mathbf{v}_1' + \mu_2 \mathbf{v}_2' = \mu_1 \mathbf{v}_1 + \mu_2 \mathbf{v}_2 = \mathbf{v}$$

(where $\mu_i = m_i/(m_1 + m_2)$). If the collision is inelastic, the normal component of $\mathbf{w}$ is reduced in modulus:

$$\mathbf{w}' \cdot \mathbf{n} = -(1 - 2\varepsilon)\mathbf{w} \cdot \mathbf{n}, \quad |\mathbf{w}' \cdot \mathbf{n}| \le |\mathbf{w} \cdot \mathbf{n}|, \tag{2}$$

where $(1 - 2\varepsilon) \in [0, 1)$ is the *coefficient of restitution* which is 1 in the elastic case. The kinetic energy $T = m_1|\mathbf{v}_1|^2/2 + m_2|\mathbf{v}_2|^2/2$ decreases and becomes

$$T' = \frac{1}{2}\left(m_1|\mathbf{v}_1'|^2 + m_2|\mathbf{v}_2'|^2\right) = T - 2M\mu_1\mu_2\varepsilon(1 - \varepsilon)(\mathbf{w} \cdot \mathbf{n})^2.$$

We remark that in some model it is assumed that also the tangential component $(I - \mathbf{n} \times \mathbf{n})\ \mathbf{w}$ is reduced (see [2] and [7]). Moreover, the restitution coefficient can depend on the relative velocity, as shown in a huge number of theoretical and experimental studies (see e.g. [3], [6], and references therein). In all these models the kinetic energy decreases.

Let us describe what it can be happen to the orbits after a collision of the two bodies. Before the collision, both energies are negative:

$$\frac{m_i}{2}\mathbf{v}_i^2 - \frac{m_i}{|\mathbf{x}_i|} < 0, \quad i \in 1,2.$$

These conditions are equivalent to

$$|\mathbf{v} + \mu_2 \mathbf{w}|^2 < 2/|\mathbf{x}_1|, \quad |\mathbf{v} - \mu_1 \mathbf{w}|^2 < 2/|\mathbf{x}_2|, \tag{3}$$

where $|\mathbf{x}_1 - \mathbf{x}_2| = R_1 + R_2$. It would be easy to find an invariant region, if the bodies could collide only in $\mathbf{x}_1$, $\mathbf{x}_2$: if

$$|\mathbf{w}| < \min\Big((\sqrt{2/|\mathbf{x}_1|} - |\mathbf{v}|)/\mu_2, (\sqrt{2|/\mathbf{x}_2|} - |\mathbf{v}|)/\mu_1\Big) \tag{4}$$

the inequalities (3) are satisfied, and the orbits remain elliptic after the collision, because $|\mathbf{w}'| \le |\mathbf{w}|$, as follows from eq.s (1), (2). The hypothesis (4) is not sufficient to avoid that one of the particles leaves the system, because the next collision can take place at other points with very different values of $\mathbf{w}$, $\mathbf{v}$, $\mathbf{x}_1$ and $\mathbf{x}_2$ More in general, a sequence of collisions on different points can end with a particle which leaves the system on a hyperbolic orbit.

Let us remark that, if we consider sticky point particles, i.e. $R_1 = R_2 = 0$ and $\mathbf{w}' = \mathbf{0}$ (see, for istance, [8]), the only hypotesis required is that the two body orbits are elliptic before the collision, as follows from (3) with $\mathbf{x}_1 = \mathbf{x}_2$. Note that the sticky condition is more restrictive than the condition of inelasticity (1) with $\varepsilon = 1/2$; in the latter case, one of the body can still escape on an hyperbolic orbit after a few collisions. In this paper, we consider only elastic or inelastic collision, as in (1), with $\varepsilon \in [0, 1/2]$.

In the next section we show how to control the condition of ellipticity, regardless the collision history. Let us first note that, for bodies with vanishing radii, the possible points of collision are at most two, which is the maximum number of intersection of two non identical co-focal Keplerian orbits. As noted by Poincaré in [12], if two orbits have two points of intersections the points and the Sun are on a line or the two orbits are in a plane. Then, we start our analysis. in section 3, with the two dimensional case, also simplifying the dynamics considering two point particles.

3 The invariant region for two point-particles in $\mathbb{R}^2$

In this section we consider a simplified bi-dimensional mathematical model. We suppose that the bodies are point particles moving on co-focal Keplerian orbits in a plane. In order to allow the particles collide, we have to assume that the orbits intersect (note that this can happen at most in two points). Although this condition is satisfied, the particles can not collide because the cross section is zero for dimensionless bodies. In order to avoids this problem, we considered the *dynamics of the orbits*: we choose as variables the parameters of two orbits o_1 and o_2, and we evolve the system with the following procedure: we choose one of the points of intersection, we consider in that point two fictitious particles on the two orbits, we choose an impact parameter $\mathbf{n}$ and we consider the resulting orbits o_1' and o_2' after the collision of the two particles. We will show, in Theorem 1, that for sufficiently low value of the total energy, the new orbits o_1' and o_2' are ellipses and intersect, for any choice between the two possible collision points and for any choice of the impact parameter $\mathbf{n}$. The key points of the proof are the conservation of the total momentum, the decrease of the energy, and the fact that the condition of intersection of the orbits is a feature preserved by the dynamics.

For the orbit o_i of the particle i, ω_i is the angle between the x axis and the position of the periapsis (the point of the orbit of the minimal distances from the Sun); ϑ_i is the true anomaly, i.e. the angle in the orbital plane between the particle and the periapsis of the orbit; $E_i = \mathbf{v}_i^2/2 - 1/r_i$ is the specific energy (i.e. the energy for unit of mass); $L_i = r_i^2 \dot{\vartheta}_i$ is the specific angular momentum; $e_i = \sqrt{1+2E_iL_i^2}$ is the eccentricity. The position of the particle i in orbits is given by

$$\mathbf{x}_i = \frac{L_i^2}{1+e_i\cos\vartheta_i}\begin{pmatrix}\cos(\vartheta_i+\omega_i)\\ \sin(\vartheta_i+\omega_i)\end{pmatrix} \tag{5}$$

All the quantities E_i and L_i are conserved during the Keplerian motion but only the combinations $m_1E_1+m_2E_2$ (the total energy) and $m_1L_1+m_2L_2$ (the total angular momentum) are conserved in the elastic collisions, moreover $m_1E_1+m_2E_2$ decreases if the collisions are inelastic. We fix the initial values of the specific energy and the specific angular momentum of the whole system (remember that $\mu_i = m_i/(m_1+m_2)$):

$$\begin{aligned} L &= \mu_1L_1+\mu_2L_2 \\ E &= \mu_1E_1+\mu_2E_2 \end{aligned} \tag{6}$$

We can assume $L \geq 0$ without loss of generality, and we consider $E < 0$, which is the case of a couple of elliptic orbits. We rewrite the orbital parameters in terms of the differences of the energy and angular momentum:

$$\begin{aligned} \delta E &= E_1 - E_2 \\ \delta L &= L_1 - L_2 \end{aligned} \quad \text{from which} \quad \begin{aligned} E_1 &= E+\mu_2\delta E \\ E_2 &= E-\mu_1\delta E \end{aligned} \quad \text{and} \quad \begin{aligned} L_1 &= L+\mu_2\delta L \\ L_2 &= L-\mu_1\delta L. \end{aligned} \tag{7}$$

From these quantities we can obtain the shapes (i.e. the eccentricities) and the dimensions of the orbits. Theirs positions in the framework are specified by the angles ω_i, but we are only interested in the relative position of the two orbits, which is given by $\delta\omega = \omega_1 - \omega_2$.

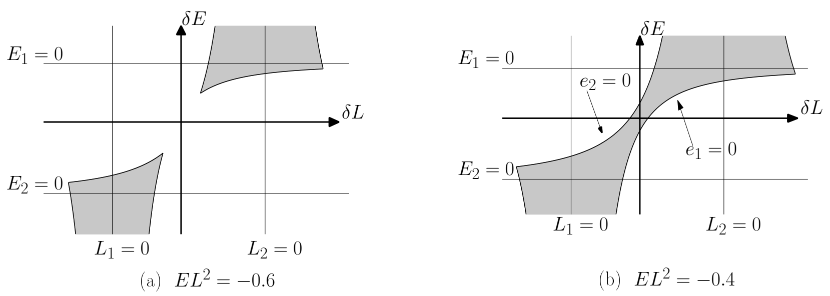


Fig. 1 $\mu_1 = 0.45$: the region of the admissible values for $EL^2 = -0.6$ and $EL^2 = -0.4$.

Not all the values of δE and δL correspond to couple of orbits: namely the energy E_i and the angular momentum L_i must satisfy the condition $0 \leq e_i^2 = 1+2E_iL_i^2$, i.e.

$$\frac{1}{\mu_2}\left(-E-\frac{1}{2L_1^2}\right) \leq \delta E \leq \frac{1}{\mu_1}\left(E+\frac{1}{2L_2^2}\right) \tag{8}$$

These inequalities define, in the space $\delta L, \delta E$, the region of admissibility

$$A = \{(\delta L, \delta E)|\,\text{eq.s (8) hold}\},$$

which we show in fig. 1. The boundary of A corresponds to $e_1 = 0$, i.e. $\delta E = -\frac{1}{\mu_2}\left(E + \frac{1}{2L_1^2}\right)$, and $e_2 = 0$, i.e. $\delta E = \frac{1}{\mu_1}\left(E + \frac{1}{2L_2^2}\right)$. As follows from easy calculations, the topology of the set A depends on the value of EL^2. If $EL^2 < -1/2$, as in fig. 1 (a), the region is not connected (this condition is equivalent to the non existence of the 'mean orbit', i.e. the orbit of energy E and angular momentum L, whose eccentricity is $\sqrt{1+2EL^2}$). If $-1/2 \le EL^2 < 0$, as in fig. 1 (b), the region is connected.

If $\delta E \in (E/\mu_1, -E/\mu_2)$ the orbits are both elliptic, while if $\delta E > -E/\mu_2$ (i.e. $E_1 > 0$) the first orbit is hyperbolic and if $\delta E < E/\mu_1$ (i.e. $E_2 > 0$) the second orbit is hyperbolic. If $\delta L \in (-L/\mu_1, L/\mu_2)$ both the particles move counterclockwise, while if $\delta L = L/\mu_2$ (i.e. $L_1 = 0$) or $\delta L = -L/\mu_1$ (i.e. $L_2 = 0$) one of the orbits degenerates.

With these notations, we can state the first theorem. We fix $\mu_1 \le \mu_2$ without loss of generality, and we consider the case $E < 0$.

Theorem 1 *Let be*

$$\sigma = \sigma(\mu_1, \mu_2) = -\frac{(1-\xi^2)(\mu_1^2 + \mu_2^2\xi)^2}{2\mu_2\xi^2} \tag{9}$$

where

$$\xi = \left(\sqrt{(\mu_1/\mu_2)^4 + 8(\mu_1/\mu_2)^2} - (\mu_1/\mu_2)^2\right)/4. \tag{10}$$

If initially

$$EL^2 < \sigma \tag{11}$$

the orbits remain elliptic for all times.

Moreover, $|L_1|$ *and* $|L_2|$ *are bounded and* $e_i \le c_i < 1$ *for suitable constants* c_1, c_2.

Remarks.

i. As we will see in the proof, if the orbits intersect, the value of EL^2 is bounded from below: $EL^2 \ge -1/2$.
ii. According to the spatial scale invariance of the problem, the behavior of the system depends only on the product EL^2 of the two invariant quantities E and L.
iii. In the proof, we study the condition of intersection in terms of the orbital parameters; for a similar analysis see [8].
iv. If M is the mass of the Sun, G the Newton constant and $k = GM$ the standard gravitational parameter of the system, the major semi-axes of the orbit i is $L_i^2/(k(1-e_i))$, where the eccentricity is $e_i = \sqrt{1+2L_i^2E_i/k^2}$. The condition (11) must be rewritten in term of EL^2/k^2.
v. The condition (11) is equivalent to $e < \sqrt{1-2|\sigma|}$, where $e = \sqrt{1+2EL^2}$ is the eccentricty of the 'mean orbit', i.e. the orbit of energy E and angular momentum L. In term of the ratio of the semi-axes $a = L^2/(1-e^2)$ and $b = L^2/\sqrt{1-e^2}$ of the mean orbit, the condition becomes $b/a < \sqrt{2|\sigma|}$.

vi. The invariant region is large, from two point of view: it contains couples of orbits which can be very different, and the orbits live in a huge region on the configuration space. For instance, in the case $\mu_1 = \mu_2 = 0.5$, the critical value is $EL^2 = -27/64$, and, for this value, if one of the particles has the same orbit of the Earth, the other particle can intersect the Jupiter orbit, and can arrive at 6.95 U.A. from the Sun. If we consider two orbits with $L_1 = L_2 = L$ and $E_1 = E_2 = E$, their eccentricity is $\sqrt{5/32} \approx 0.40$ and the ratio between the major and the minor semi-axis is approximately 2.33.

vii. We have fixed the attractive center, therefore we are not considering here a three body problem, in which we can fix only the center of mass. The planetary three body system is usually described as a perturbation of the system obtained in the canonical heliocentric variables neglecting the terms of order $m_1 m_2$ (see e.g. [9]). This unperturbed system is a system of two particles moving independently on Keplerian orbits, with respect the position of the large body. The standard gravitational parameters are $G(M+m_i)$, and can be different for the two particles. Our results also hold in this case, with minor modifications.

viii. We have done some preliminary numerical simulations for this model, with $n > 2$ particles. For $n = 3$, if initially all the particles can collide with the others, in the elastic case one of the particle leaves the system after few collisions; in the inelastic case one of the particle stops to interact with the others after few collisions, and the orbits of the others two particles converge. A system of a large amount of colliding particles exhibits a complex behavior: a certain number of particles (decreasing with the parameter of inelasticity ε) leaves the system, the others particles asymptotically separate in non interacting clusters of one particle or two colliding particles. Let us note that for some different inelastic model (somewhat artificial), it can be proved the existence of "ringlets" i.e. the existence of a state of n particle which does not cease to interact, and whose orbits converge (see [7]).

Proof We prove the theorem in the case of elastic collisions, for which E, L and EL^2 are conserved quantities, and then the condition (11) is invariant for the dynamics. In the case of inelastic collisions, the thesis follows from the fact that L is conserved and E and EL^2 can only decrease, then the condition (11) is invariant for the dynamics also in this case.

The proof follows from the fact that if EL^2 is sufficiently close to -0.5, and the orbits intersect, then the two orbits are elliptic, as we now show.

The two orbits intersect if $\mathbf{x}_1 = \mathbf{x}_2$ for some value of the anomalies ϑ_1 and ϑ_2. Using eq. (5), this condition is expressed by the equalities

$$\vartheta_2 - \vartheta_1 = \omega_1 - \omega_2 = \delta\omega \quad \text{and} \quad L_1^2(1 + e_2 \cos\vartheta_2) = L_2^2(1 + e_1 \cos\vartheta_1).$$

Inserting in the last equation that $\vartheta_2 = \vartheta_1 + \delta\omega$, we obtain an equation in the unknown ϑ_2 that can be solved if and only if

$$e_1 e_2 \cos\delta\omega \le 1 + L_2^2 E_1 + L_1^2 E_2 \tag{12}$$

Let us define the set of the values of $(\delta L, \delta E)$ for which two orbits of parameters L_1, E_1 and L_2, E_2 intersect if the angle between the periapsides is $\delta\omega = \eta$:

$$I_\eta = \{(\delta L, \delta E) \in A |\, e_1 e_2 \cos\eta \le 1 + L_2^2 E_1 + L_1^2 E_2\} \tag{13}$$

By definition

$$I_{\eta_1} \subset I_{\eta_2} \quad \text{if } \eta_1 < \eta_2,$$

and in particular $I_\eta \subset I_\pi$ if $\eta \in [0, \pi]$. This implies that, if we rotate two intersecting orbits, in such a way that the two periapsides become in opposition ($\delta\omega = \pi$), we obtain two orbits which intersect. The intersection condition is invariant for the dynamics, then all the values $(\delta L, \delta E)$ during the evolution are in the set

$$I_\pi = \{(\delta L, \delta E) \in A |\, e_1 e_2 \ge -(1 + L_2^2 E_1 + L_1^2 E_2)\} \tag{14}$$

Therefore, the set I_π is invariant for the dynamics.

Now we show that I_π is contained in the region in which both the orbits are elliptic, if EL^2 is sufficiently small. The set I_π as defined in (14) is the union of the set of values of δL and δE in A which solve

$$e_1^2 e_2^2 = (1 + 2E_1 L_1^2)(1 + 2E_2 L_2^2) \ge (1 + L_2^2 E_1 + L_1^2 E_2)^2 \tag{15}$$

provided that

$$1 + L_2^2 E_1 + L_1^2 E_2 \le 0, \tag{16}$$

and the set of values which solve

$$1 + L_2^2 E_1 + L_1^2 E_2 \ge 0, \tag{17}$$

Note that this last equation identifies the region $I_{\pi/2}$, i.e. the region of intersecting orbits with perpendicular semi-major axis, and it is equivalent to

$$\delta E(\mu_1 \mu_2 \delta L + (\mu_1 - \mu_2)) \le 1 + 2EL^2 - (\mu_1 - \mu_2) E \delta L, \tag{18}$$

Eq. (15) is equivalent to

$$\delta E^2 (\mu_1 L_1^2 + \mu_2 L_2^2)^2 - 2\delta E (L_1^2 - L_2^2)(1 + E(\mu_1 L_1^2 + \mu_2 L_2^2)) + E^2 (L_1^2 - L_2^2)^2 \le 0 \tag{19}$$

which can be solved if

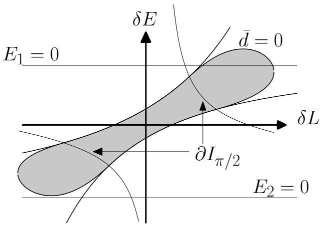
$$(L_1^2 - L_2^2)^2 (1 + 2E(\mu_1 L_1^2 + \mu_2 L_2^2)) \ge 0 \tag{20}$$

If $|L_1| \ne |L_2|$ this condition is equivalent to $1 + 2E(L^2 + \mu_1 \mu_2 \delta L^2) \ge 0$ then the set I_π is non void if and only if $1 + 2EL^2 \ge 0$, and it is bounded by the condition

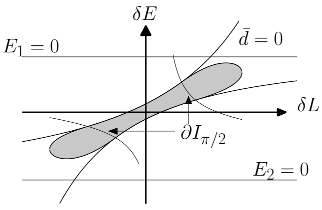
$$\delta L^2 \le \frac{1 - 2|E|L^2}{2|E|\mu_1 \mu_2} \tag{21}$$

The boundary of the region identified by the inequality (19) is given by the functions

$$\delta E = \frac{L_1^2 - L_2^2}{(\mu_1 L_1^2 + \mu_2 L_2^2)^2} \left(1 + E(\mu_1 L_1^2 + \mu_2 L_2^2) \pm \sqrt{1 + 2E(\mu_1 L_1^2 + \mu_2 L_2^2)}\right) \tag{22}$$



(a) $EL^2 = -0.41$



(b) $EL^2 = -0.445$

Fig. 2 $\mu_1 = 0.45$: the set I_π for $EL^2 = -0.41$ and $EL^2 = -0.445$. Over the line $E_1 = 0$, the orbit 1 is hyperbolic.

which have the sign of $L_1^2 - L_2^2$.

In figure 2 we show the region I_π, identified by the inequality (18) (region $I_{\pi/2} \subset I_\pi$) and the inequality (19) (region $I_\pi \backslash I_{\pi/2}$). In figure 2 (a), the value of EL^2 is -0.41, and the region I_π intersects the region $E_1 \geq 0$ (i.e. $\delta E > |E|/\mu_2$). Then, after a collision, one of the outgoing orbits can become hyperbolic. In figure 2 (b), the value of EL^2 is smaller and the invariant region I_π is completely contained in the region $\delta E \in (-|E|/\mu_2, |E|/\mu_1)$ in which both the orbits are elliptic. Then, whatever are the details of the collisions, the two orbits remain elliptic, with $e_i \leq c_i < 1$ for some constants c_1, c_2, and $|L_1|$, $|L_2|$ are bounded via (21) and (6).

Now we will show that the behavior of the system is driven by EL^2: there exists a critical value which separates the two cases. Let us define

$$\bar{d}(\delta L, \delta E) = \frac{L_1^2}{1+e_1} - \frac{L_2^2}{1-e_2}. \tag{23}$$

This quantity is the distance from the periapsis of the orbits 1 and the apoapsis of the orbit 2, in the case of $\omega = \pi$. The value of $\bar{d}$ is 0 on the boundary of I_π in the first quadrant, out of $I_{\pi/2}$ (see fig. 2). Then the critical value of E e L is such that

$$E_1 = 0,\ e_1 = 1,\ \bar{d}(\delta E, \delta L) = 0,\ \frac{\partial \bar{d}}{\partial \delta L}(\delta E, \delta L) = 0$$

(the gradient of $\bar{d}(\delta E, \delta L)$ is vertical in the point of tangency to the line $E_1 = 0$). By deriving $\bar{d}$ with respect to δL we obtain

$$\partial_{\delta L}\bar{d} = 2\mu_2 \frac{L_1}{1+e_1} + 2\mu_1 \frac{L_2}{1-e_2} - 2\mu_2 E_1 L_1 \frac{L_1^2}{e_1(1+e_1)^2} + 2\mu_1 E_2 L_2 \frac{L_2^2}{e_2(1-e_2)^2} \tag{24}$$

which, using $-2E_i L_i^2 = 1 - e_i^2$, becomes

$$\partial_{\delta L}\bar{d} = 2\mu_2 \frac{L_1}{1+e_1} + 2\mu_1 \frac{L_2}{1-e_2} \mu_2 L_1 \frac{1-e_1^2}{e_1(1+e_1)^2} - \mu_1 L_2 \frac{1-e_2^2}{e_2(1-e_2)^2} = \frac{\mu_2}{e_1} L_1 - \frac{\mu_1}{e_2} L_2. \tag{25}$$

The condition $\partial_{\delta L}\bar{d} = 0$ and the definition of L in eq. (6) allow us to calculate L_1, L_2 in terms of e_1, e_2:

$$\begin{aligned} L_1 &= \mu_1 e_1 L/(\mu_1^2 e_1 + \mu_2^2 e_2) \\ L_2 &= \mu_2 e_2 L/(\mu_1^2 e_1 + \mu_2^2 e_2) \end{aligned} \tag{26}$$

Using these expressions in $\bar{d} = 0$ with $e_1 = 1$ we obtain the following equation for e_2:

$$2\mu_2^2 e_2^2 = \mu_1^2(1-e_2),$$

which has only one solution in $(0,1)$, given by eq. (10). Substituting this value in the expression of L_2, we obtain the critical values of EL^2 as in (11), imposing $1+E_2L_2^2 = e_2^2$, with $E_2 = E/\mu_1$ and $e_1 = 1$:

$$EL^2 = -(1-e_2^2)(\mu_1^2 + \mu_2^2 e_2)^2/(2\mu_2 e_2^2).$$

Note that we can find a similar condition for which the region I_π is tangent to the line $E_2 = 0$, but in the case $\mu_1 \le \mu_2$ this second critical value of EL^2 is greater than the previous, then it can be ignored. □

4 The invariant region for two bodies in $\mathbb{R}^2$

In this section, the case of two bodies and the case of point particles which interact by means of a short range potential are analyzed.

Theorem 2 *We consider two circular bodies in $\mathbb{R}^2$ of radii R_1 and R_2 interacting by means of elastic or inelastic collisions. If $EL^2 < \sigma(\mu_1,\mu_2)$, with σ as defined in eq.* (9)*, and $D = R_1 + R_2$ is sufficiently small, then the two bodies remain on elliptic orbits.*

Proof We show that if two orbits have points whose distance is less than or equal to D, and $EL^2 < \sigma$ and D is sufficiently small, then the orbits are elliptic. We remark that the condition on the distance is preserved by the dynamics.

Fixed L_1, L_2, E_1, E_2, assuming that the two bodies can collide, we need to distinguish two situations.

If it exists $\delta\omega$ such that the orbits intersect, then $(\delta L, \delta E) \in I_\pi$.

If the orbits do not intersect for any $\delta\omega$, one of them, named orbit 2, is contained in the other. In this case, $0 < \min_{\delta\omega} \min_{\vartheta_1,\vartheta_2} |\mathbf{x}_1 - \mathbf{x}_2| \le D$, and the minimum is reached for $\delta\omega = \pi$; then $0 < \bar{d}(\delta L, \delta E) \le D$, where $\bar{d}$ is defined as in eq. (23).

Since $EL^2 < \sigma$, the distance between the level set $\bar{d} = 0$ (the boundary of I_π) and the critical lines $E_1 = 0$, $E_2 = 0$, is strictly positive. Then, if D is sufficiently small, the invariant set $I_\pi \cap \{(\delta L, \delta E) | \bar{d} \le D\}$ does not intersect the region in which $E_1 \ge 0$ or $E_2 \ge 0$, and this proves the theorem. □

Remark. In not-scaled units of measure, the smallness condition on D becomes a smallness condition on kD/L^2 (see remark (iv) after Theorem 1).

In figure 3 we show the level sets of $\bar{d}$. In figure 3 (a), $EL^2 = -0.445$ and the invariant set I_π and the values of $\bar{d}$ in the complementary region are shown: the black

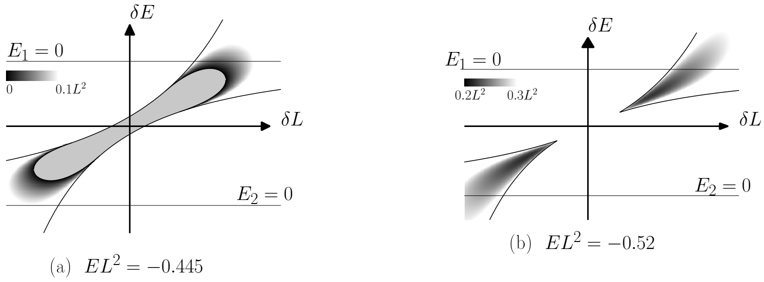


Fig. 3 $\mu_1 = 0.45$: the values of $\bar{d}$ in grayscale, in the case $EL^2 = -0.445$ and $EL^2 = -0.52$.

color corresponds to $\bar{d} = 0$, the white color corresponds to $\bar{d} \geq L^2/10$, while the grays correspond to the values $\bar{d} \in (0, L^2/10)$. The critical value of D is approximately $0.034L^2$. Figure 3 (b) is relative to the case $EL^2 = -0.52$, in which the set I_π is void. Nevertheless, the set $\bar{d} \leq D$ is invariant and is contained in the region of elliptic orbits if D is sufficiently small. In the graphics, the black color corresponds to $\bar{d} = 0.2L^2$, the white to $\bar{d} \geq 0.3L^2$. In this case, the critical value for D is approximately $0.25L^2$.

We are not able to give a simple expression for the critical values of EL^2 and D, but for the case $\mu_1 = \mu_2 = 1/2$, in which the particles have the same mass.

Theorem 3 *If $\mu_1 = \mu_2 = 1/2$, the conditions on EL^2 and D are*

$$EL^2 < -\frac{(1-\xi^2)(1+\xi)^2}{16\xi^2} \quad \textit{where } \xi = \gamma - \sqrt{\gamma^2 - \gamma + 1} \quad \textit{with } \gamma = 2L^2/D > 1$$

Proof We proceed as in the proof of Theorem 1. Assuming $\mu_1 = \mu_2 = 1/2$ and $EL^2 < -27/64$, the critical condition $\partial_{\delta L}\bar{d}(\delta E, \delta L) = 0$ allows us to obtain the value of L_1 and L_2 as in eq. (26), which becomes

$$\begin{aligned} L_1 &= 2e_1 L/(e_1 + e_2) \\ L_2 &= 2e_2 L/(e_1 + e_2) \end{aligned} \tag{27}$$

Using these values in $\bar{d} = D$ with $e_1 = 1$, we obtain the following equation for e_2

$$2L^2(1 - 2e_2) = D(1 - e_2^2),$$

which is solved in $(0,1)$ by

$$e_2 = \gamma - \sqrt{\gamma^2 - \gamma + 1} \quad \text{where} \quad \gamma = 2L^2/D \quad \text{with} \quad D < 2L^2.$$

The corresponding value of EL^2 is given by $EL^2 = -(1-e_2^2)(1+e_2)^2/(16e_2^2)$. □

In figure 4 we plot these values in function of D/L^2. Let us recall that for $EL^2 < -0.5$ the two orbits do not intersect.

The last extension we consider in the two dimensional case is that of two point particles interacting by means of a force of symmetric potential energy $V(|\mathbf{x}_1 - \mathbf{x}_2|)$, with a compact support.

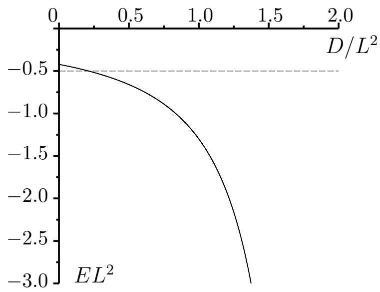


Fig. 4 $\mu_1 = \mu_2 = 0.5$: the critical value of EL^2 in function of D/L^2.

Theorem 4 *Let us consider two point particles interacting by means of a force of potential energy $V = V(|\mathbf{x}_1 - \mathbf{x}_2|)$, such that $V(r) = 0$ if $r \geq D$ for some $D > 0$ and $V(r) \geq -U$ with $U \geq 0$.*

If $EL^2 < \sigma(\mu_1, \mu_2)$, with σ as in eq. (9)*, and and D and U are sufficiently small, then the particles remains on bounded orbits.*

Proof The specific angular momentum $L = \mu_1 L_1 + \mu_2 L_2$ is conserved at any time also in this case, because the potential energy is symmetric. If $|\mathbf{x}_1 - \mathbf{x}_2| \geq D$ the energy of the interaction is zero and the specific energy of the system is exactly

$$E = \mu_1 \left(\frac{\mathbf{v}_1^2}{2} - \frac{1}{|\mathbf{x}_1|} \right) + \mu_2 \left(\frac{\mathbf{v}_2^2}{2} - \frac{1}{|\mathbf{x}_2|} \right)$$

Therefore, when the two particles leave the region of the interaction, we can apply Theorem 2, concluding that the particles remain on elliptic orbits until the next interaction.

To achieve the proof, we have to discuss the motion of the particles during the interaction, i.e. when their distance is less than D. If $|\mathbf{x}_1 - \mathbf{x}_2| < D$, we can consider the two 'osculating' Keplerian orbits, i.e. the Keplerian orbits which correspond to the two couple position-velocity $(\mathbf{x}_1, \mathbf{v}_1)$, and $(\mathbf{x}_2, \mathbf{v}_2)$. The specific energies of this two orbits are

$$E_1 = \mathbf{v}_1^2/2 - 1/|\mathbf{x}_1|, \quad E_2 = \mathbf{v}_2^2/2 - 1/|\mathbf{x}_2|.$$

These quantities are not the specific energies of the two particles (because the contribution of the interaction is not zero), but verify

$$\tilde{E} = \mu_1 E_1 + \mu_2 E_2 = E - V(|\mathbf{x}_1 - \mathbf{x}_2|)/(m_1 + m_2) \leq E + U/(m_1 + m_2)$$

by the conservation of the total energy. If $EL^2 < \sigma$ and U is sufficiently small, we have also that $\tilde{E}L^2 < \sigma$, then we can apply Theorem 2, using $\tilde{E}$ instead of E. Therefore, if D is sufficiently small, as in the hypothesis of Theorem 2, the two osculating orbits are elliptic, with bounded value of L_1, L_2, and $e_i \leq c_i < 1$, as follows from the compactness of the set $I_\pi \cup \{\bar{d}(\delta L, \delta E) \leq D\} \subset (-L/\mu_1, L/\mu_2) \times (E/\mu_1, -E/\mu_2)$. $\square$

5 The invariant region for two bodies in $\mathbb{R}^3$

Here we discuss the three dimensional case. We indicate with $\mathbf{L}_i = \mathbf{x}_i \wedge \mathbf{v}_i$ the vector which express the specific angular momentum of the particle i in the position $\mathbf{x}_i$ with velocity $\mathbf{v}_i$, and with $\mathbf{L}$ the specific angular momentum of the whole system $\mathbf{L} = \mu_1 \mathbf{L}_1 + \mu_2 \mathbf{L}_2$, which is a conserved vector. In this case, they hold the analogous of Theorems 1, 2, 4, where the role of L is played by $|\mathbf{L}|$. We summarize these results in the following theorem.

Theorem 5

1. *The invariant region for the orbital dynamics in $\mathbb{R}^3$, defined as in section 3, is given by*
$$E|\mathbf{L}|^2 < \sigma$$
with $\sigma = \sigma(\mu_1, \mu_2)$ defined as in eq. (9).
2. *The invariant region for the collisional dynamics of two hard spheres in $\mathbb{R}^3$, of radii R_1 and R_2, with $D = R_1 + R_2$, is given by $E|\mathbf{L}|^2 < \sigma$ with D sufficiently small.*
3. *The invariant region for two point particles interacting by means a potential energy V as in Theorem 4, is given by $E|\mathbf{L}|^2 < \sigma$ with D and U sufficiently small.*

Proof We prove the theorem starting from the last case, which includes the others as particular ones. As in the proof of Theorem 4, we define

$$E_1 = 1/2\mathbf{v}_1^2 - 1/|\mathbf{x}_1|, \quad E_2 = 1/2\mathbf{v}_2^2 - 1/|\mathbf{x}_2|, \quad \tilde{E} = \mu_1 E_1 + \mu_2 E_2$$

and we note that

$$\begin{array}{lll} \tilde{E} = E & \text{if} & |\mathbf{x}_1 - \mathbf{x}_2| \geq D \\ \tilde{E} \leq E + U/(m_1 + m_2) & \text{if} & |\mathbf{x}_1 - \mathbf{x}_2| < D. \end{array}$$

We indicate with $\tilde{L}_i = |\mathbf{L}_i|$ the modulus of the angular momentum of the orbit of the particle i, and we define

$$\tilde{L} = \mu_1 \tilde{L}_1 + \mu_2 \tilde{L}_2$$

which verifies

$$|\mathbf{L}| = |\mu_1 \mathbf{L}_1 + \mu_2 \mathbf{L}_2| \leq \tilde{L} \quad \text{and} \quad E\tilde{L}^2 \leq E|\mathbf{L}|^2$$

(E is negative). Moreover,

$$\tilde{E}\tilde{L}^2 \leq \left(E + \frac{U}{m_1 + m_2}\right)\tilde{L}^2$$

then, if $E|\mathbf{L}|^2 < \sigma$ as in the hypothesis, for U sufficiently small, it also holds

$$\tilde{E}\tilde{L}^2 < \sigma \tag{28}$$

Let us indicate with o_i the Keplerian orbit identified by the position $\mathbf{x}_i$ and the velocity $\mathbf{v}_i$; its energy is E_i and its angular momentum is $\mathbf{L}_i$. If the particles can interact, o_1 and o_2 have points at distance less than D. We consider the orbits $\tilde{o}_1$ and $\tilde{o}_2$ we obtain rigid rotating in $\mathbb{R}^3$, around the Sun, o_1 and o_2, in such that $\tilde{o}_1$ and $\tilde{o}_2$ are in the same plane, and the periapsides of $\tilde{o}_1$ and $\tilde{o}_2$ are in opposition. The energy and the eccentricity of the orbits $\tilde{o}_i$ are the same of the orbits o_i, while we can identify the angular momentum of $\tilde{o}_i$ with the positive scalar quantity $\tilde{L}_i$. This two planar orbits intersect or have points at distance less that D. In the first case, we can apply Theorem 1 using (28), and we can conclude that the $\tilde{o}_1$, $\tilde{o}_2$, and then o_1 and o_2 are elliptic. In the second case, we can apply Theorem 2, and we can again conclude that, if D is sufficiently small, the orbits are elliptic.

The case of spherical bodies can be considered as a particular case, in which $V = +\infty$, if $|\mathbf{x}_1 - \mathbf{x}_2| \leq D$. Now $U = 0$ and we have only to require a sufficiently small value of D. Finally, the case of the orbital dynamics can be considered as the particular case in which $D = 0$. The hypothesis $E|\mathbf{L}|^2 < \sigma$ is then sufficient to achieve the thesis. □

Remarks.

i. The values of D in Theorem 2 can be very large with respect to the scale L^2 of the semi-axis of the orbits, but, if it is so, the system is in the region $EL^2 < -1/2$, in which the orbits do not intersect. Therefore, the system is made of two particles which can interact only by means of grazing collisions.
ii. It can be interesting to analyze the case of Theorem 4 when V is unbounded from below. The particles can leave the system only if their distance remains less than D, and, in this case, we can expect that the center of mass of the two particles moves on an approximately elliptic orbit. On the other hand, there are no a priori bounds on the kinetic energy or on the position of this center of mass.
iii. It can be also interesting to analyze the case of $n > 2$ particles, with positive radii. In particular it can be expected that there exist stable ringlets in which the collisions are grazing.

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