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Sharp L^p estimates for Schrödinger groups on spaces of homogeneous type

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Abstract. We prove an L^p estimate

$$\|e^{-itL}\varphi(L)f\|_p \lesssim (1+|t|)^s\|f\|_p, \quad t\in\mathbb{R}, \quad s=n\Big|\frac{1}{2}-\frac{1}{p}\Big|$$

for the Schrödinger group generated by a semibounded, self-adjoint operator L on a metric measure space $\mathcal{X}$ of homogeneous type (where n is the doubling dimension of $\mathcal{X}$). The assumptions on L are a mild $L^{p_0}\to L^{p_0'}$ smoothing estimate and a mild $L^2\to L^2$ off-diagonal estimate for the corresponding heat kernel e^{-tL}. The estimate is uniform for φ varying in bounded sets of $\mathscr{S}(\mathbb{R})$, or more generally of a suitable weighted Sobolev space.

We also prove, under slightly stronger assumptions on L, that the estimate extends to

$$\|e^{-itL}\varphi(\theta L)f\|_p \lesssim (1+\theta^{-1}|t|)^s\|f\|_p, \quad \theta>0, \quad t\in\mathbb{R},$$

with uniformity also for θ varying in bounded subsets of $(0,+\infty)$. For nonnegative operators uniformity holds for all $\theta>0$.

1. Introduction

Bounds in L^p for the Schrödinger group $e^{it\Delta}$ have applications in harmonic analysis and to nonlinear dispersive equations. The group itself is not bounded in L^p for $p\neq 2$, but $(1-\Delta)^{-s}e^{it\Delta}$ is L^p bounded for s sufficiently large. A sharp estimate can be written if one introduces a frequency cutoff $\varphi\in C_c^\infty(\mathbb{R}^n)$: for all $1\le p\le\infty$, $k\in\mathbb{Z}$, $t\in\mathbb{R}$, we have

$$\|e^{it\Delta}\varphi(2^{-k}(-\Delta)^{1/2})f\|_{L^p} \lesssim (1+2^{2k}|t|)^s\|f\|_{L^p}, \quad s=n\Big|\frac{1}{2}-\frac{1}{p}\Big|, \tag{1.1}$$

see [4], [30], [21].

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This result can be regarded as an elementary example of L^p estimates with loss of derivatives for FIOs, in the spirit of [28]. However, our goal here is to extend (1.1) in a different direction, namely, to Schrödinger groups e^{itL} generated by a semibounded, self-adjoint operator L on a metric measure space $\mathcal{X}$ endowed with a doubling measure. This framework covers a large variety of situations which go far beyond the classical FIO setting.

Many properties of L and functions of L can be deduced from suitable estimates on the corresponding heat kernel e^{-tL}. A common assumption in the Euclidean case (see [17], [15]) is the Gaussian upper estimate

$$|e^{-tL}(x,y)| \lesssim t^{-n/m} \exp\big(-b\big(t^{-1/m}|x-y|\big)^{m/(m-1)}\big), \qquad t>0,\ x,y\in\mathbb{R}^n,$$

for some $b>0$, $m>1$. This includes Schrödinger operators perturbed with an electromagnetic potential (in this case $m=2$: see [6], [7] for some applications), and fractional Laplacians $(-\Delta)^{m/2}$ with m even. Note that these operators are already outside the reach of the classical theory of singular operators.

In order to include more general operators, one can weaken the assumptions on the heat kernel. In [13] we proposed, in the Euclidean case, to replace the Gaussian upper estimate with a weak $L^{p_0} \to L^{p_0'}$ smoothing estimate on dyadic cubes, and an even weaker off-diagonal $L^2 \to L^2$ algebraic decay (see (1.4), (1.5) below). These conditions are much more inclusive, as discussed in Remark 1.1 below, but they still allow to recover the estimate (1.1) at least in the restricted range $p \in [p_0, p_0']$.

Here we study the more general situation of metric measure spaces of homogeneous type. More precisely, in the following we shall assume that $(\mathcal{X}, d, \mu)$ is a metric space with distance d, equipped with a nonnegative Borel measure μ which satisfies the *doubling property*: there exists a constant $c_1>0$ such that

$$\mu(B(x,2r)) \le c_1\, \mu(B(x,r)) \tag{1.2}$$

for all $x\in\mathcal{X}$ and $r>0$, where $B(x,r)$ is the open ball of radius r and center x. We recall that the doubling property (1.2) implies the existence of $C>0$ and $n>0$ such that

$$\mu(B(x,\lambda r)) \le C\,\lambda^n \mu(B(x,r)), \quad \forall \lambda>0.$$

We shall also assume that $\mathcal{X}$ satisfies a *reverse doubling condition*: there exist $\kappa\in[0,n]$ and $C>0$ such that for all $x\in\mathcal{X}$, $0<r<\operatorname{diam}(\mathcal{X})/2$ and $1\le\lambda<\operatorname{diam}(\mathcal{X})/(2r)$, one has

$$C\,\lambda^\kappa \mu(B(x,r)) \le \mu(B(x,\lambda r)) \tag{1.3}$$

where $\operatorname{diam}(\mathcal{X}) = \sup_{x,y\in\mathcal{X}} d(x,y)$. Note that the reverse doubling condition is always satisfied with $\kappa=0$, thus (1.3) is restrictive only when $\kappa\in(0,n]$.

It was proved in [10] that it is always possible, for each $\nu\in\mathbb{Z}$, to define an almost covering $\mathcal{D}_\nu$ of open sets, with diameter $\simeq 2^{-\nu}$, which are called *dyadic cubes* and enjoy properties very similar to the standard dyadic cubes in $\mathbb{R}^n$; see Lemma 2.1 below for precise definitions and more details.

In this setting, we consider an operator L on $L^2(\mathcal{X})$ satisfying the following assumption, where 1_Q denotes the characteristic function of the cube Q:

Assumption ($\mathbf{L}_0$): L is a self-adjoint operator on $L^2(\mathcal{X})$ with $L + M_0 \geq 0$ for some constant $M_0 \geq 0$, satisfying the following estimate. There exist $p_0 \in [1, 2)$, $m_1, m_2 > 0$ and $C \geq 0$ such that for all $t > 0$ and $\nu \in \mathbb{Z}$ with either $2^{-\nu} \leq t^{1/m_1} < 2^{-\nu+1}$, $0 < t < 1$ or $2^{-\nu} \leq t^{1/m_2} < 2^{-\nu+1}$, $t \geq 1$ we have

$$\sum_{Q\in\mathcal{D}_\nu} \|1_Q e^{-tL} 1_{Q'}\|_{p_0\to 2} + \sum_{Q\in\mathcal{D}_\nu} \|1_Q e^{-tL} 1_{Q'}\|_{2\to p_0'} \leq C\, e^{M_0 t} (t_{>1}^{n/m_1} + t_{\leq 1}^{\kappa/m_2})^{1/2-1/p_0} \tag{1.4}$$

for all $Q' \in \mathcal{D}_\nu$, where $t_{>1} = t \cdot 1_{(1,+\infty)}(t)$ and $t_{\leq 1} = t \cdot 1_{(-\infty,1]}(t)$, and

$$\sup_{Q'\in\mathcal{D}_\nu} \sum_{Q\in\mathcal{D}_\nu} (1+2^\nu \mathrm{dist}(Q, Q'))^N \|1_Q e^{-tL} 1_{Q'}\|_{2\to 2} \leq C\, e^{M_0 t}, \quad N = \lfloor n/2 \rfloor + 1. \tag{1.5}$$

Remark 1.1. The previous assumptions include of course the typical Gaussian upper estimates for Schrödinger operators on $\mathbb{R}^n$. Indeed, in the particular case $m_1 = m_2 = m$, conditions (1.4) and (1.5) are a direct consequence of the following estimate:

$$\|1_{B(x,t^{1/m})} e^{-tL} 1_{B(y,t^{1/m})}\|_{p_0\to p_0'} \leq C\, e^{M_0 t}\, \mu(B(x, t^{1/m}))^{-1/p_0+1/p_0'} \exp\Big(-\frac{d(x, y)^{m/(m-1)}}{ct^{1/(m-1)}}\Big), \tag{1.6}$$

for all $t > 0$, and all $x, y \in \mathcal{X}$. Note that the presence of an exponentially growing factor $e^{M_0 t}$ allows to include some interesting cases like non-positive Schrödinger operators $-\Delta + V(x)$, see [29].

However, the converse implication may be false. An example is given by the fractional fractional Laplacian $(-\Delta)^\alpha, \alpha > 0$ on $\mathbb{R}^n$. It is well known that the kernel $e^{-t(-\Delta)^\alpha}(x, y)$ of $e^{-t(-\Delta)^\alpha}$ has a upper bound

$$e^{-t(-\Delta)^\alpha}(x, y) \lesssim \frac{1}{t^{n/2\alpha}} \Big(1 + \frac{|x-y|}{t^{1/2\alpha}}\Big)^{-(n+2\alpha)}$$

for all $x, y \in \mathbb{R}^n$ and $t > 0$. As a consequence, that for $2\alpha > \lfloor n/2 \rfloor + 1$, $(-\Delta)^\alpha$ satisfies (1.4) and (1.5) for $p_0 = 1$ and $m = 2\alpha$, but not (1.6). See for example [13].

Moreover, the estimate (1.6) does not imply the condition (1.4). However, if we assume in addition that $(\mathcal{X}, d, \mu)$ satisfies the *non-collapsing* condition

$$\mu(B(x, 1)) \gtrsim 1, \quad \forall x \in \mathcal{X}, \tag{1.7}$$

then (1.4) is a consequence of (1.6).

Then we can prove:

Theorem 1.2. *Assume L satisfies* $(\mathbf{L}_0)$. *Let $p \in [p_0, p_0']$ and $s = n|1/2 - 1/p|$. Then the estimate*

$$\|e^{-itL}\varphi(L)f\|_p \lesssim (1 + |t|)^s\, \|f\|_p, \quad t \in \mathbb{R},$$

holds uniformly for φ in bounded subsets of $\mathscr{S}(\mathbb{R})$.

Remark 1.3. The previous result is still valid for functions φ of Sobolev regularity. More precisely, the estimate is true and uniform in φ provided the following norm:

$$\sum_{j\le n+1} \|\langle\lambda\rangle^{2+n+j+n/m_1}\,\varphi^{(j)}(\lambda)\|_{L^2}. \tag{1.8}$$

remains bounded. This condition is not sharp; see Remark 2.13 for further details.

We now examine a few directions in which one can relax the assumptions of Theorem 1.2. In order to do this we introduce some definitions. The *amalgam space* $X^{1,p}_\nu$, with $1\le p\le\infty$ and $\nu\in\mathbb{Z}$, is the space of measurable functions on $\mathcal{X}$ such that the following norm is finite:

$$\|f\|_{X^{1,p}_\nu} := \sum_{Q\in\mathcal{D}_\nu} \|f\|_{L^p(Q)}. \tag{1.9}$$

Moreover, we say that $w\colon \mathcal{X}\times\mathcal{X}\to\mathbb{R}$ is a *weight function* if it is equivalent to the distance function, in the sense that

$$K_0^{-1}\,d(x,y)\le |w(x,y)|\le K_0\,d(x,y) \tag{1.10}$$

for some constant $K_0>0$.

Let w be a weight function and let $\mathscr{D}(w)$ be any topological vector space associated to w satisfying the following conditions:

(i) $\mathscr{D}(w)$ is dense in $L^2(\mathcal{X})$ (w.r.t. the $L^2(\mathcal{X})$ norm);

(ii) $w(x,\cdot)^N f\in\mathscr{D}(w)$ for all $f\in\mathscr{D}(w)$, $x\in\mathcal{X}$ and $N\in\mathbb{N}$.

Denote by $\mathscr{D}'(w)$ the dual space of $\mathscr{D}(w)$.

Remark 1.4. In applications, in Subsections 3.1–3.10, we will choose $w(x,y)=d(x,y)$ and $\mathscr{D}(w)=L^2_c(\mathcal{X})$ which is a space of all functions in L^2 with compact support. In Subsection 3.13, as $X=\mathbb{R}^n$ we will choose $\mathscr{D}(w)=C^\infty_0(\mathbb{R}^n)$. We will not recall this in Section 3.

Denoting by w_x the multiplication operator by the function $w(x,\cdot)$, the commutators $\mathrm{Ad}^{k+1}_x(T)\colon\mathscr{D}(w)\to\mathscr{D}'(w)$ of order k of an $L^2(X)$-bounded linear operator T with the weight w are defined as follows:

$$\mathrm{Ad}^0_x(T)=I,\quad \mathrm{Ad}^1_x(T)=[w_x,T],\quad \mathrm{Ad}^{k+1}_x(T)=[w_x,\mathrm{Ad}^k_x(T)].$$

In view of the applications, we shall also consider a more general kind of vector valued weight functions $w=(w_1,\dots,w_\ell)\colon\mathcal{X}\times\mathcal{X}\to\mathbb{R}^\ell$, defined again by condition (1.10) (where now $|w|=(w_1^2+\dots+w_\ell^2)^{1/2}$). In the vector valued case $\mathrm{Ad}^k_x(T)$ will denote the ℓ-tuple of commutators with $w_1,\dots,w_\ell$, that is to say we define, for $j=1,\dots,\ell$,

$$\mathrm{Ad}^0_{j,x}(T)=I,\quad \mathrm{Ad}^1_{j,x}(T)=[w_{j,x},T],\quad \mathrm{Ad}^{k+1}_{j,x}(T)=[w_{j,x},\mathrm{Ad}^k_x(T)].$$

(where $w_{j,x}$ is multiplication by $w_j(x,\cdot)$) and $\mathrm{Ad}^k_x(T):=(\mathrm{Ad}^k_{1,x}(T),\dots,\mathrm{Ad}^k_{\ell,x}(T))$. Note that the simplest choice of a weight satisfying (1.10) is given by the distance function itself, with $\ell=1$.

We can now state our second set of assumptions on L.

Assumption (L): L is a self-adjoint operator on $L^2(\mathcal{X})$ with $L + M_0 \geq 0$ for some constant $M_0 \geq 0$, satisfying the following estimates. There exist $p_0 \in [1, 2)$, $m_1, m_2 > 0$ and $C \geq 0$ such that for all $t > 0$ and $\nu \in \mathbb{Z}$ with either $2^{-\nu} \leq t^{1/m_1} < 2^{-\nu+1}$, $0 < t < 1$ or $2^{-\nu} \leq t^{1/m_2} < 2^{-\nu+1}$, $t \geq 1$ we have

$$\|e^{-tL}\|_{X^{1,p_0}_\nu \to X^{1,2}_\nu} + \|e^{-tL}\|_{X^{1,2}_\nu \to X^{1,p_0'}_\nu} \leq C\, e^{M_0 t}(t_{>1}^{n/m_1} + t_{\leq 1}^{\kappa/m_2})^{1/2-1/p_0} \tag{1.11}$$

where $t_{>1} = t \cdot 1_{(1,+\infty)}(t)$ and $t_{\leq 1} = t \cdot 1_{(-\infty,1]}(t)$. Moreover, there exists a weight function $w(x, y)$ and a constant $M_1 > M_0$ such that the resolvent $R(z) = (L+z)^{-1}$ satisfies, for all $x \in \mathcal{X}$,

$$\|\mathrm{Ad}^k_x(R(M_1))\|_{2\to 2} \leq C, \quad 0 \leq k \leq \lfloor n/2 \rfloor + 1. \tag{1.12}$$

Remark 1.5. The reason why condition (1.12) is interesting, besides being much weaker than (1.5), is that it is very easy to check directly for differential operators, and even some pseudodifferential ones, in the Euclidean setting. See Section 3.13.

Then we can prove:

Theorem 1.6. *Assume L satisfies* **(L)**. *Let $p \in [p_0, p_0']$ and let $s = n|1/2 - 1/p|$. Then the estimate*

$$\|e^{-itL}\varphi(L)f\|_p \lesssim (1 + |t|)^s\, \|f\|_p, \quad t \in \mathbb{R},$$

holds uniformly for φ in bounded subsets of $\mathscr{S}(\mathbb{R})$.

Remark 1.7. Comparing the two sets of assumptions we see that

$$\text{Assumption } (\mathbf{L}_0) \implies \text{Assumption } (\mathbf{L}).$$

Indeed, the implication

$$\text{condition (1.4)} \implies \text{condition (1.11)}$$

is obviously true. On the other hand, one has

$$\text{condition (1.5)} \implies \text{condition (1.12), with } w(x, y) = d(x, y),$$

but this is more delicate and will be proved in Propositions 2.5 and 2.6 below.

One notices that estimate (1.1) for the standard Laplacian is uniform also for rescaling in frequency $\sim 2^k$, $k \in \mathbb{Z}$. This is a direct consequence of the scaling properties of $\mathbb{R}^n$ and its Lebesgue measure, which are not available on a general metric measure space $\mathcal{X}$. Uniformity in frequency is an important property, especially useful when doing dyadic analysis on Sobolev or Besov spaces generated by the operator L. We can recover uniformity under slightly stronger assumptions on the operator L.

Assumption $(\mathbf{L}_1)$: L is a self-adjoint operator on $L^2(\mathcal{X})$, with $L + M_0 \geq 0$ for some constant $M_0 \geq 0$, satisfying condition (1.11) with $m_1 = m_2 = m > 0$. Moreover, there exists a weight function $w(x,y)$ such that the resolvent $R(z) = (L+z)^{-1}$ satisfies, for all $0 \leq j \leq \lfloor n/2 \rfloor + 1$,

$$\|\mathrm{Ad}_x^j(R(M))\|_{2\to 2} \leq C(M-M_0)^{-1-j/m}, \quad \forall M > M_0,\ x \in \mathcal{X}. \tag{1.13}$$

Remark 1.8. Note that when $m_1 = m_2$, the following implication holds:

$$\text{Assumption } (\mathbf{L}_0) \implies \text{Assumption } (\mathbf{L}_1) \text{ (with } w(x,y) = d(x,y))$$

(compare with Remark 1.7). This is proved in Propositions 2.5 and 2.6 below.

Under this assumption we can prove:

Theorem 1.9. *Assume L satisfies Assumption $(\mathbf{L}_1)$. Let $p \in [p_0, p_0']$ and let $s = n|1/2 - 1/p|$. Then we have*

$$\|e^{-itL}\varphi(\theta L)f\|_p \lesssim (1+\theta|t|)^s \|f\|_p, \quad t \in \mathbb{R},$$

and the estimate is uniform for φ in bounded subsets of $\mathscr{S}(\mathbb{R})$ and θ in bounded subsets of $(0,+\infty)$. In the special case when $\kappa = n$ and $M_0 = 0$ the estimate is uniform for all $\theta > 0$.

Remark 1.10. Like for Theorems 1.2 and 1.6, the previous estimate is valid and uniform in the more general case of functions φ varying in any bounded subset for the weighted Sobolev norm (1.8).

As an intermediate step in the proof of the previous theorems, we obtain uniform L^p estimates for operators of the form $\varphi(L)$ which are of independent interest, see Theorem 2.14. (This result can be recovered from the statement of Theorem 1.9 choosing $t = 0$).

Our results are based on a commutator argument and a reduction to amalgam spaces, following the methods of Jensen–Nakamura [19]. The adaptation of the argument from [19] to a multi-scale setting was introduced in [13] and was inspired by the ideas of [32]. Moreover, our approach can be adapted to study the L^p-boundedness for Schrödinger group on an open subset of the space of homogeneous type $\mathcal{X}$.

We finally consider a self-adjoint operator L on $L^2(\Omega)$, where Ω is an open subset of $\mathcal{X}$. This case can not be reduced to the previous results since Ω may not satisfy the doubling condition. However, if we assume that $L + M_0 \geq 0$ for some $M_0 \geq 0$ and the kernel $p_t(x,y)$ of heat semigroup e^{-tL} satisfies the following estimate: $\exists C \geq 0$, $m > 1$ such that

$$|p_t(x,y)| \leq \frac{Ce^{M_0 t}}{\mu(B(x,t^{1/m}))} \exp\Big(-\frac{d(x,y)^{m/(m-1)}}{ct^{1/(m-1)}}\Big) \tag{1.14}$$

for all $t > 0$ and $x, y \in \Omega$, then we can prove:

Theorem 1.11. *Let L be a nonnegative self-adjoint operator on $L^2(\Omega)$, where Ω is an open subset of $\mathcal{X}$. Assume that L satisfies (1.14). Let $p \in [1,\infty]$ and let $s = n|1/2 - 1/p|$. Then we have*

$$\|e^{-itL}\varphi(\theta L)f\|_{L^p(\Omega)} \lesssim (1+\theta|t|)^s \|f\|_{L^p(\Omega)}, \quad t \in \mathbb{R},$$

and the estimate is uniform for φ in bounded subsets of $\mathscr{S}(\mathbb{R})$ and θ in bounded subsets of $(0,+\infty)$. In the special case $\kappa = n$ and $M_0 = 0$ the estimate is uniform for all $\theta > 0$.

The proofs of the theorems, and some additional estimates, are given in the next section. The third, and final, section of the paper is devoted to an extensive list of applications: we consider Laplace–Beltrami operators on Riemannian manifolds with or without Gaussian heat kernel bounds; the operator associated to the Sierpinski gasket; Hörmander type operators generated by vector fields on homogeneous groups; Bessel operators; Schrödinger operators with potentials on manifolds; Euclidean Schrödinger operators with singular potentials of inverse square type; the sub-Laplacian on Heisenberg groups; and Dirichlet Laplacian on open connected domains. The list is not exhaustive and is intended to show the variety of possible applications and the generality of Assumption **(L)**.

2. Proof of the theorems

With the notation $V(x,r) = \mu(B(x,r))$, the doubling property (1.2) implies the existence of $C > 0$ and $n > 0$ such that

$$V(x,\lambda r) \le C\lambda^n V(x,r), \quad \forall \lambda > 0,\ x \in \mathcal{X}, \tag{2.1}$$

and

$$V(x,r) \le C\Big(1 + \frac{d(x,y)}{r}\Big)^n V(y,r), \quad \forall r > 0,\ x,y \in \mathcal{X}. \tag{2.2}$$

As a consequence of (2.2), we have $V(x,r) \simeq V(y,r)$ when $d(x,y) \le r$.

We recall the fundamental covering lemma from [10].

Lemma 2.1. *There exists a collection of open sets $\{Q_\tau^k \subset \mathcal{X} : k \in \mathbb{Z}, \tau \in I_k\}$, where I_k denotes certain (possibly finite) index sets depending on k, and constants $\rho \in (0,1)$ $c_0 \in (0,1]$ and $C_0, C_1 \in (0,\infty)$ such that*

(i) *$\mu(\mathcal{X} \setminus \cup_\tau Q_\tau^k) = 0$ for all $k \in \mathbb{Z}$;*

(ii) *if $\ell \ge k$ and $\tau \in I_\ell, \beta \in I_k$, then either $Q_\tau^\ell \subset Q_\beta^k$ or $Q_\tau^\ell \cap Q_\beta^k = \emptyset$;*

(iii) *for $k \in \mathbb{Z}$, $\tau \in I_k$ and each $\ell < k$, there exists a unique $\tau' \in I_\ell$ such that $Q_\tau^k \subset Q_{\tau'}^\ell$;*

(iv) *the diameters of the sets satisfy $\mathrm{diam}\,(Q_\tau^k) \le C_1\rho^k$;*

(v) *for $k \in \mathbb{Z}$, $\tau \in I_k$ there exists $x_{Q_\tau^k} \in X$ such that*

$$B(x_{Q_\tau^k}, c_0\rho^k) \subset Q_\tau^k \subset B(x_{Q_\tau^k}, C_0\rho^k).$$

Remark 2.2. (a) The constants ρ, c_0 and C_0 are inessential for our purposes, thus, without loss of generality, we may assume that $\rho = a_0 = 1/2$ and $C_0 = 1$. We then fix a collection of open sets in Lemma 2.1 and denote this collection by $\mathcal{D}$. We call these open sets the *dyadic cubes* in $\mathcal{X}$ and $x_{Q^k_\tau}$ the *center* of the cube Q^k_τ. We also write $\mathcal{D}_\nu := \{Q^\nu_\tau : \tau \in I_\nu\}$ for each $\nu \in \mathbb{Z}$. We have then $\ell_Q := \operatorname{diam} Q \sim 2^{-\nu}$ for all $Q \in \mathcal{D}_\nu$.

(b) From the doubling property (2.1), there exists a constant C such that for any $x \in \mathcal{X}$ and $k \in \mathbb{N}$ there are at most $C2^{kn}$ dyadic cubes in $\mathcal{D}_0$ which cover the ball $B(x, 2^k)$.

2.1. Amalgam spaces

For $1 \le p, q \le \infty$ and $\nu \in \mathbb{Z}$, we define the space $X^{p,q}_\nu$ as the vector space of all measurable functions $f : \mathcal{X} \to \mathbb{C}$ such that the following norm is finite:

$$\|f\|_{X^{p,q}_\nu} := \Big(\sum_{Q \in \mathcal{D}_\nu} \|f\|^p_{L^q(Q)} \Big)^{1/p}, \tag{2.3}$$

with the usual modification when $p = \infty$. We also write $X^{p,q} = X^{p,q}_0$.

The following embedding holds.

Proposition 2.3. *For $1 \le p \le q \le \infty$ and $\nu \in \mathbb{Z}$ we have*

$$\|f\|_{X^{p,q}} \le C\big(1 + 2^{-\nu n(1/p-1/q)}\big) \|f\|_{X^{p,q}_\nu} ,$$

where C depends only on the constant c_1 in the doubling property (1.2).

Proof. The proof of this proposition is elementary and we leave it to the reader. □

Recall that $\mathrm{Ad}^j_x(T)$ denotes the j-th order commutator of an operator T with the weight function $w_x(\cdot) = w(x, \cdot)$, $w : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^\ell$, satisfying (1.10).

Theorem 2.4. *Let T be a bounded operator on $L^2(\mathcal{X})$. Assume that Ad^k_z can be extended to be a bounded operator on $L^2(\mathcal{X})$ and there exists some constant $B_0 \ge 1$ so that*

$$\|\mathrm{Ad}^k_z(T)\|_{2\to2} \le B^k_0$$

for all $0 \le k \le \lfloor n/2 \rfloor + 1$ and all $z \in \mathcal{X}$.

Then for $1 \le p \le 2$ we have

$$\|T\|_{X^{p,2}\to X^{p,2}} \le C\, B_0^{n(1/p-1/2)} ,$$

where C is a constant depending only on n, $\|T\|_{2\to2}$ and K_0 (from (1.10)).

Proof. We first note that the L^2-boundedness of T implies

$$\|T\|_{X^{2,2}_{\nu_0}\to X^{2,2}_{\nu_0}} \le C.$$

Hence, by interpolation it suffices to prove that

$$\|T\|_{X^{1,2}\to X^{1,2}} \le C\, B_0^{n/2}.$$

To prove this, let $w = (w_1, \ldots, w_\ell)$ be the weight function and recall that $\mathrm{Ad}^k_{j,z}(T)$ denotes the commutator of order k with multiplication by $w_{j,z} := w_j(z, \cdot)$. We use a combinatorial identity from Lemma 3.1 in [19] and we write

$$w^m_{j,z} T = \sum_{k=0}^{m} c_{m,k}\, \mathrm{Ad}^k_{j,z}(T)\, w^{m-k}_{j,z}, \quad j = 1, \ldots, \ell,$$

where $c_{m,k}$ are appropriate constants. Denote also by d_z the multiplication operator by $d(z, \cdot)$. Then, we have for every x_Q with $Q \in \mathcal{D}_0$ and $N, m \in \mathbb{N}$ with $0 \le m \le N \le \lfloor n/2 \rfloor + 1$,

$$\begin{aligned}\| |w_{j,x_Q}|^m T [1 + d_{x_Q}]^{-N} \|_{2\to 2} &\le \sum_{k=0}^{m} c_{m,k}\, \|\mathrm{Ad}^k_{j,x_Q}(T)\|_{2\to 2}\, \|w^{m-k}_{j,x_Q} [1 + d_{x_Q}]^{-N}\|_{2\to 2} \\ &\le C\, B_0^m,\end{aligned}$$

since $|w_j|$ is dominated by d. Summing over $j = 1, \ldots, \ell$ and recalling (1.10) we obtain, for $0 \le N \le \lfloor n/2 \rfloor + 1$,

$$\big\| [1 + d_{x_Q}]^N\, T [1 + d_{x_Q}]^{-N} \big\|_{2\to 2} \le C\, B_0^N.$$

This implies

$$\big\| d^N_{x_Q} T\, 1_Q \big\|_{2\to 2} \le C\, B_0^N, \quad \forall Q \in \mathcal{D}_0. \tag{2.4}$$

Let $f \in \mathscr{D}(w)$. For each cube Q, write $f_Q = f\, 1_Q$. Then we have

$$\|Tf\|_{X^{1,2}} = \sum_{Q' \in \mathcal{D}_0} \|1_{Q'} T f\|_2 \le \sum_{Q \in \mathcal{D}_0} \sum_{Q' \in \mathcal{D}_0} \|1_{Q'} T f_Q\|_2.$$

Let now α be a constant which will be precised later; for the moment we assume only $\alpha \ge 2\, \sup_{Q \in \mathcal{D}_0} \operatorname{diam} Q$. For each $Q \in \mathcal{D}_0$ we can write

$$\sum_{Q' \in \mathcal{D}_0} \|1_{Q'} T f_Q\|_2 = I + II, \tag{2.5}$$

where

$$\begin{aligned} I &= \sum_{Q' : d(x_Q, x_{Q'}) > \alpha} d(x_Q, x_{Q'})^{-N} d(x_Q, x_{Q'})^{N} \|1_{Q'} T f_Q\|_2, \\ II &= \sum_{Q' : d(x_Q, x_{Q'}) \le \alpha} \|1_{Q'} T f_Q\|_2. \end{aligned}$$

On the other hand, by Remark 2.2 we get

$$\sharp\{Q' \in \mathcal{D}_0 : d(x_Q, x_{Q'}) \le \alpha\} \lesssim \alpha^n. \tag{2.6}$$

This, in combination with Hölder's inequality, implies that

$$\begin{aligned} II &\le \Big(\sum_{Q' : d(x_Q, x_{Q'}) \le \alpha} 1 \Big)^{1/2} \Big(\sum_{Q' : d(x_Q, x_{Q'}) \le \alpha} \|1_{Q'} T f_Q\|_2^2 \Big)^{1/2} \\ &\lesssim \alpha^{n/2}\, \|T f_Q\|_2 \lesssim \alpha^{n/2} \|T\|_{2\to 2}\, \|f_Q\|_2. \end{aligned}$$

On the other hand, for $x' \in Q'$ we have $d(x_Q, x') \geq d(x_Q, x_{Q'}) - \mathrm{diam}(Q')$, thus if $d(x_Q, x_{Q'}) \geq \alpha$ we have $d(x_Q, x') \gtrsim d(x_Q, x_{Q'})$ by the assumption on α. Then we can write

$$I \leq \Big(\sum_{Q': d(x_Q, x_{Q'}) \geq \alpha} d(x_Q, x_{Q'})^{-2N} \Big)^{1/2} \Big(\sum_{Q': d(x_Q, x_{Q'}) \geq \alpha} d(x_Q, x_{Q'})^{2N} \|1_{Q'} T f_Q\|_2^2 \Big)^{1/2}$$

$$\lesssim \Big(\sum_{Q': d(x_Q, x_{Q'}) \geq \alpha} d(x_Q, x_{Q'})^{-2N} \Big)^{1/2} \|d(\cdot, x_Q)^N T f_Q\|_2 \,,$$

which along with (2.4) and (2.6) yields

$$I \lesssim \alpha^{-N+n/2} B_0^N \|f_Q\|_2$$

provided that $N > n/2$.

Inserting the estimates of I and II into (2.5) and taking $\alpha = CB_0$ for a suitable C (depending only on $\sup_{Q \in \mathcal{D}_0} \mathrm{diam}\, Q$), we obtain

$$\sum_{Q' \in \mathcal{D}_0} \|1_{Q'} T f_Q\|_2 \lesssim (1 + \|T\|_{2\to 2}) \cdot B_0^{n/2} \|f_Q\|_2.$$

Therefore,

$$\|Tf\|_{X^{1,2}} \lesssim (1 + \|T\|_{2\to 2}) \cdot B_0^{n/2} \|f\|_{X^{1,2}}, \quad f \in \mathscr{D}(w).$$

Since $X^{1,2} \hookrightarrow L^2(X)$, $\mathscr{D}(w)$ is dense in $X^{1,2}$. It follows

$$\|Tf\|_{X^{1,2}} \lesssim (1 + \|T\|_{2\to 2}) \cdot B_0^{n/2} \|f\|_{X^{1,2}}, \quad f \in X^{1,2}.$$

On the other hand, since T is bounded on L^2, we have

$$\|Tf\|_{X^{2,2}} \lesssim \|T\|_{2\to 2} \|f\|_{X^{2,2}}.$$

Interpolating between the two estimates, we get the claim. □

We conclude this section by proving that assumption (1.5) implies (1.12) and (1.13), as stated in the Introduction.

Proposition 2.5. *Let the weight function be $w(x,y) = d(x,y)$. Assume that L is a self-adjoint operator in $L^2(\mathcal{X})$ with $L + M_0 \geq 0$ for some $M_0 \in \mathbb{R}$, satisfying the following condition: there exist $p_0 \in [1,2)$, $m_1, m_2 > 0$ and $C \geq 0$ such that for all $t > 0$ and $\nu \in \mathbb{Z}$ with either $2^{-\nu} \leq t^{1/m_1} < 2^{-\nu+1}, 0 < t < 1$ or $2^{-\nu} \leq t^{1/m_2} < 2^{-\nu+1}, t \geq 1$ we have*

$$\sup_{Q' \in \mathcal{D}_\nu} \sum_{Q \in \mathcal{D}_\nu} (1 + 2^\nu \mathrm{dist}(Q, Q'))^N \|1_Q e^{-tL} 1_{Q'}\|_{2\to 2} \leq C e^{M_0 t}, \quad N = \lfloor n/2 \rfloor + 1. \tag{2.7}$$

Then there exist $C_1 \geq 0$ such that for all t and ν as above we have

$$\|\mathrm{Ad}_x^k(e^{-tL})\|_{2\to 2} \leq C_1 e^{M_0 t} 2^{-k\nu}, \quad 0 \leq k \leq \lfloor n/2 \rfloor + 1, \quad x \in \mathcal{X}. \tag{2.8}$$

Proof. By considering the nonnegative operator $\widetilde{L} = L + M_0$ instead of L, we see that we can assume $M_0 = 0$. If $p_t(x, y)$ is the kernel of the heat semigroup e^{-tL} we obtain the representation

$$\mathrm{Ad}_z^k(e^{-tL})f(x) = \int_{\mathcal{X}} (d_z(x) - d_z(y))^k \, p_t(x, y) \, f(y) \, d\mu(y)$$

and our goal is to prove that the operator

$$Af(x) = 2^{\nu k} \int_{\mathcal{X}} (d_z(x) - d_z(y))^k \, p_t(x, y) \, f(y) \, d\mu(y)$$

for $0 < t < 1$ and $2^{-\nu} \le t^{1/m_1} < 2^{-\nu+1}$, or for $t \ge 1$ and $2^{-\nu} \le t^{1/m_2} < 2^{-\nu+1}$, satisfies $\|A\|_{2\to2} \le C$ with C independent of ν.

We shall now prove the estimate

$$\sup_{Q' \in \mathcal{D}_\nu} \sum_{Q \in \mathcal{D}_\nu} \|1_Q A 1_{Q'}\|_{2\to2} \le C_1 \tag{2.9}$$

with constants independent of ν. This implies the dual estimate

$$\sup_{Q' \in \mathcal{D}_\nu} \sum_{Q \in \mathcal{D}_\nu} \|1_{Q'} A 1_Q\|_{2\to2} \le C_1$$

and by the Schur test for sequences the two estimates together imply that A is bounded on $X_\nu^{p,2}$ with norm not larger than C_1, for all $p \in [1, \infty]$ and all $\nu \in \mathbb{Z}$. Since $L^2 = X_\nu^{2,2}$ for all $\nu \in \mathbb{Z}$, this concludes the proof.

It remains to prove (2.9). We write the kernel of $1_Q A 1_{Q'}$ as

$$1_Q A 1_{Q'}(x, y) = 2^{\nu k} \, (d_z(x) - d_z(y))^k \, 1_Q(x) \, p_t(x, y) \, 1_{Q'}(y),$$

and we use the estimate

$$|d_z(x) - d_z(y)| \le d(x_Q, x_{Q'}) + d(x, x_Q) + d(y, x_{Q'}), \quad x \in Q, \; y \in Q'$$

where $Q \subset B(x_Q, 2^{-\nu})$ and $Q' \subset B(x_{Q'}, 2^{-\nu})$ according to Remark 2.2. We now expand

$$|1_Q A 1_{Q'}(x, y)|$$
$$\le \sum_{\alpha+\beta+\gamma=k} \frac{k!}{\alpha!\beta!\gamma!} (2^\nu d(x_Q, x_{Q'}))^\alpha (2^\nu d(x, x_Q))^\beta 1_Q(x) p_t(x, y) 1_{Q'}(y) (2^\nu d(y, x_{Q'}))^\gamma.$$

We have trivially

$$\|(2^\nu d(x, x_Q))^\beta 1_Q\|_{2\to2} \le C \quad \text{and} \quad \|(2^\nu d(y, x_{Q'}))^\gamma 1_Q\|_{2\to2} \le C,$$

and recalling assumption (2.7) we see that the proof is concluded. □

From condition (2.8) it is fairly easy to deduce (1.12), thus concluding the proof of the implication (1.5) ⇒ (1.12), (1.13).

Proposition 2.6. *Let the weight function be $w(x,y) = d(x,y)$. Assume L satisfies (2.8) and $L + M_0 \geq 0$. Then for all $M > M_0$ we have, for all $z \in \mathcal{X}$ and $0 \leq k \leq \lfloor n/2 \rfloor + 1$,*

$$\|\mathrm{Ad}_z^k((L+M)^{-1})\|_{2\to 2} \lesssim (M - M_0)^{-1-k/m_1} + (M - M_0)^{-1-k/m_2},$$

with a constant independent of z and M.

Proof. By spectral calculus we can represent $R = (M + L)^{-1}$ in the form

$$R = (M+L)^{-1} = \int_0^\infty e^{-Mt} e^{-tL}\, dt$$

which implies

$$\mathrm{Ad}_z^k(R) = \int_0^\infty e^{-Mt}\, \mathrm{Ad}_z^k(e^{-tL})\, dt.$$

By assumption (2.8), since $2^{-\nu} \simeq t^{1/m_2}$ for $t < 1$ and $2^{-\nu} \simeq t^{1/m_1}$ for $t > 1$, we obtain

$$\|\mathrm{Ad}_z^k(R)\|_{2\to 2} \lesssim \int_0^1 e^{(M_0-M)t}\, t^{k/m_2}\, dt + \int_1^{+\infty} e^{(M_0-M)t}\, t^{k/m_1}\, dt\,,$$

and the claim follows easily. □

2.2. Estimates for the heat semigroup

The following result gives an estimate for the semigroups e^{-tL} on almagam spaces which plays an important role in the sequel.

Proposition 2.7. *For every $t > 0$ we have*

$$\|e^{-tL}\|_{L^{p_0} \to X^{p_0,2}} \leq C e^{M_0 t}\big(t^{-\frac{n}{m_1}(1/p_0 - 1/2)} + t^{\frac{n-\kappa}{m_2}(1/p_0-1/2)}\big),$$

where C depends only on the constants C in assumption (1.11) and c_1 in (1.2).

Proof. By redefining $\widetilde{L} = L + M_0$, we see that it is not restrictive to assume $M_0 = 0$. Now fix $\nu \in \mathbb{Z}$ and $t > 0$ such that either $2^{-\nu} \leq t^{1/m_1} < 2^{-\nu+1}, 0 < t < 1$ or $2^{-\nu} \leq t^{1/m_2} < 2^{-\nu+1}, t \geq 1$. By assumption (1.11), using duality we have

$$\|e^{-tL}\|_{X_\nu^{\infty,p_0} \to X_\nu^{\infty,2}} \leq C\left(2^{\nu\kappa(1/p_0-1/2)} + 2^{\nu n(1/p_0-1/2)}\right)$$

and interpolating with (1.11) we have, for all $1 \leq p \leq \infty$,

$$\|e^{-tL}\|_{X_\nu^{p,p_0} \to X_\nu^{p,2}} \leq C\left(2^{\nu\kappa(1/p_0-1/2)} + 2^{\nu n(1/p_0-1/2)}\right)$$

We choose $p = p_0$ and notice that $X_\nu^{p_0,p_0} = L^{p_0}$; thus we have proved

$$\|e^{-tL}\|_{L^{p_0} \to X_\nu^{p_0,2}} \leq C\left(2^{\nu\kappa(1/p_0-1/2)} + 2^{\nu n(1/p_0-1/2)}\right)$$

By the embedding in Proposition 2.3 this implies

$$\begin{aligned}\|e^{-tL}\|_{L^{p_0} \to X^{p_0,2}} &\leq C\left(2^{\nu\kappa(1/p_0-1/2)} + 2^{\nu n(1/p_0-1/2)}\right)\left(1 + 2^{-\nu n(1/p_0-1/2)}\right)\\ &\simeq 2^{\nu(\kappa-n)(1/p_0-1/2)} + 2^{\nu n(1/p_0-1/2)},\end{aligned}$$

and recalling the conditions on t, we obtain the claim. □

As a consequence we obtain the following result.

Proposition 2.8. *Let $M > M_0$ and $\gamma = \frac{n}{m_1}(1/p_0 - 1/2) + \epsilon$, with $\epsilon > 0$. Then*

$$\|(M+L)^{-\gamma} f\|_{X^{p_0,2}} \leq C\big(\epsilon^{-1} + (M-M_0)^{\gamma + \frac{n-\kappa}{m_2}(1/p_0-1/2)}\big)\|f\|_{p_0},$$

where C depends only on the constants C in assumption (1.11) and c_1 in (1.2).

Proof. It is sufficient to apply Minkowski's inequality and Proposition 2.7 to the standard representation

$$(M+L)^{-\gamma} = \frac{1}{\Gamma(\gamma)}\int_0^\infty t^\gamma\, e^{-Mt}\, e^{-tL}\,\frac{dt}{t}. \qquad \square$$

2.3. Estimate of $\varphi(L)$

We shall now prove that if φ is in a suitable weighted Sobolev space then $\varphi(L)$ is bounded on L^p. The proof will be achieved through a series of Lemmas, some of which are of independent interest.

In the following, L is an operator satisfying Assumption **(L)**, and we can take R as the resolvent operator

$$R = (M_1 + L)^{-1}$$

with $M_1 > M_0$ as in **(L)**.

Lemma 2.9. *We have the estimate*

$$\|e^{-i\xi R} f\|_{X^{p_0,2}} \leq c(n)\, C\,(1+|\xi|)^{n(1/p_0-1/2)}\,\|f\|_{X^{p_0,2}}, \quad \xi\in\mathbb{R},$$

where C is the constant in assumption (1.12) and $c(n)$ depends only on n.

Proof. From

$$e^{-i\xi R} w_z(\cdot)\, e^{i\xi R} - w_z(\cdot) = \int_0^\xi \partial_s (e^{-isR} w_z(\cdot) e^{isR})\, ds$$

we obtain the formula

$$\mathrm{Ad}_z(e^{-i\xi R}) = -i\int_0^\xi e^{-isR}\,\mathrm{Ad}_z(R) e^{-i(\xi-s)R}, ds$$

and by (1.12) we get

$$\|\mathrm{Ad}_z^1(e^{-i\xi R})\|_{2\to 2} \leq C|\xi|.$$

Using repeatedly this identity and proceeding by induction we obtain

$$\|\mathrm{Ad}_z^k(e^{-i\xi R})\|_{2\to 2} \leq C\,(1+|\xi|)^k, \quad k = 0,\dots,\lfloor n/2\rfloor + 1,$$

uniformly in $z\in\mathfrak{X}$, and by Theorem 2.4 we obtain the claim. $\square$

Lemma 2.10. *For any sufficiently smooth function ψ on $\mathbb{R}$ we have the estimate*

$$\|\psi(R) f\|_{X^{p_0,2}} \leq c(n)\, C\, \|(1+|\xi|)^{n(1/p_0-1/2)}\,\widehat{\psi}(\xi)\|_{L^1}\,\|f\|_{X^{p_0,2}}$$

with $c(n)$ and C as in Lemma 2.9.

Proof. It is sufficient to use the identity

$$\psi(R) = (2\pi)^{-1} \int e^{i\xi R}\, \widehat{\psi}(\xi)\, d\xi$$

and apply the previous result. □

We introduce a seminorm for functions $\psi\colon \mathbb{R} \to \mathbb{C}$, depending on the constant $M_1 \geq 0$ and on the integer $N \geq 0$:

$$\|\psi\|_N := \|\psi\|_{L^2(-M_1,\infty)} + \textstyle\sum_{j=0}^{N} \|(\lambda + M_1)^{j+N} \partial^j \psi(\lambda)\|_{L^2(-M_1,\infty)}.$$

Lemma 2.11. *Let $N = \lfloor n/p_0 \rfloor + 1$ and let $\psi\colon \mathbb{R} \to \mathbb{C}$. Then we have*

$$\|(L + M_1)^2 \psi(L) f\|_{X^{p_0,2}} \leq c(n)\, C\, \|\psi\|_N \cdot \|f\|_{X^{p_0,2}},$$

with $c(n)$ and C as in Lemma 2.9.

Proof. Define $\rho(\xi) := 0$ for $\xi \leq 0$, and

$$\rho(\xi) := \xi^{-2} \cdot \psi(1/\xi - M_1) \quad \text{for } \xi > 0$$

and note that $(\lambda + M_1)^2 \psi(\lambda) = \rho((M_1 + \lambda)^{-1})$ for λ in the spectrum of L, so that $(L + M_1)^2 \psi(L) = \rho(R)$. By the previous result we get

$$\|(L + M_1)^2 \psi(L) f\|_{X^{p_0,2}} \leq c(n)\, C\, \|(1 + |\xi|)^{n(1/p_0 - 1/2)}\, \widehat{\rho}(\xi)\|_{L^1}\, \|f\|_{X^{p_0,2}}.$$

It remains to estimate the norm of ρ. We proceed as follows:

$$\|(1 + |\xi|)^{n(1/p_0 - 1/2)}\, \widehat{\rho}(\xi)\|_{L^1} \lesssim \|(1 + |\xi|)^N\, \widehat{\rho}\,\|_{L^2} = \|\rho\|_{H^N(\mathbb{R}^+)}.$$

We note the elementary identity for $\xi > 0$, $k \geq 0$,

$$\partial_\xi^k \rho(\xi) = \textstyle\sum_{j=0}^{k} c_{j,k} \cdot \partial^j \psi(1/\xi - M_1) \cdot \xi^{-(j+k+2)}$$

(for suitable constants $c_{j,k}$). This gives

$$\|\partial_\xi^k \rho\|_{L^2(0,\infty)} \leq c(n) \textstyle\sum_{j=0}^{k} \|(\lambda + M_1)^{j+k} \partial^j \psi(\lambda)\|_{L^2(-M_1,\infty)}.$$

Using the last estimate for $k = 0$ and $k = N$ we obtain

$$\|\rho\|_{H^N(\mathbb{R}^+)} \leq c(n) \|\psi\|_{L^2(-M_1,\infty)} + c(n) \textstyle\sum_{j=0}^{N} \|(\lambda + M_1)^{j+N} \partial^j \psi(\lambda)\|_{L^2(-M_1,\infty)}$$

and we obtain the claim. □

Lemma 2.12. *Let $N = \lfloor n/p_0 \rfloor + 1$, $\beta \geq 0$ with $\beta + 2 > \frac{n}{m_1}(1/p_0 - 1/2)$ and $\psi\colon \mathbb{R} \to \mathbb{C}$. Then for $p \in [p_0, p_0']$ we have the estimate*

$$\|\psi(L) f\|_{L^p} \lesssim \|f\|_{L^p}.$$

The norm of $\psi(L)\colon L^p \to L^p$ can be estimated by

$$C\,(1 + (M_1 - M_0)^{\beta + 2 + \frac{n-\kappa}{m_2}(1/p_0 - 1/2)}) \|(\lambda + M_1)^\beta \psi(\lambda)\|_N\,,$$

where C depends on c_1 in the doubling property (1.2), on $\sup_{Q \in \mathfrak{Q}_0} \mu(Q)$ and on the constants in Assumption **(L)**, *but is independent of M_0, M_1, ψ.*

Proof. We apply the previous lemma to the function $\widetilde{\psi}(\lambda) = (\lambda + M_1)^{\beta+2}\psi(\lambda)$:

$$\|\widetilde{\psi}(L)f\|_{X^{p_0,2}} = \|(L+M_1)^2\psi(L)(L+M_1)^\beta f\|_{X^{p_0,2}} \\ \leq c(n)\, C \, \|\!|(\lambda+M_1)^\beta\psi\|\!|_N \, \|f\|_{X^{p_0,2}}.$$

Since $\psi(L) = \widetilde{\psi}(L)R^{\beta+2}$, we can write, using Proposition 2.8,

$$\|\psi(L)f\|_{X^{p_0,2}} \leq \|\widetilde{\psi}(L)\|_{X^{p_0,2}\to X^{p_0,2}}\|R^{\beta+2}f\|_{X^{p_0,2}} \lesssim \|\widetilde{\psi}(L)\|_{X^{p_0,2}\to X^{p_0,2}}\|f\|_{L^{p_0}},$$

where the implicit constant has the form

$$C\,(1+(M_1-M_0)^{\beta+2+\frac{n-\kappa}{m_2}(1/p_0-1/2)}),$$

with C depending on c_1 in the doubling property (1.2) and on the constants in Assumption **(L)**, but independent of M_0, M_1. Since $X^{p_0,2}$ is continuously embedded in L^{p_0} with embedding norm $\leq \sup_{Q\in\mathcal{Q}_0}\mu(Q)^{1/p_0-1/2}$, we have proved that $\psi(L)$ is bounded on L^{p_0} with the same norm. By duality and interpolation, we conclude the proof. □

Remark 2.13. The dependence on ψ of the norm of $\psi(L)$ is particularly interesting. The quantity $\|\!|(\lambda+M_1)^\beta\psi(\lambda)\|\!|_N$ is uniformly bounded if ψ varies in a bounded subset of $C_c^\infty(\mathbb{R})$ or of $\mathscr{S}(\mathbb{R})$, and M_1 is bounded. More generally, we can write

$$\|\!|(\lambda+M_1)^\beta\psi(\lambda)\|\!|_N \lesssim \|\lambda^\beta\psi(\lambda-M_1)\|_{L^2(\mathbb{R}^+)}+\textstyle\sum_{j=0}^N \|\lambda^{j+N+\beta}\partial^j\psi(\lambda-M_1)(\lambda)\|_{L^2(\mathbb{R}^+)}$$

and we see that the quantity is uniform for ψ varying in any bounded subset of a suitable weighted Sobolev space, provided M_1 is bounded (which is always the case in our applications). For instance, we can take the weighted Sobolev space with norm

$$\sum_{j\leq n+1}\|\langle\lambda\rangle^{2+n+j+n/m_1}\psi^{(j)}(\lambda)\|_{L^2}. \tag{2.10}$$

Theorem 2.14. *Under Assumption* **(L)**, *the following estimate holds: for all* $p\in[p_0,p_0']$,

$$\|\varphi(L)f\|_{L^p}\leq C\,\|f\|_{L^p},$$

and the estimate is uniform for φ in bounded subsets of $\mathscr{S}(\mathbb{R})$ (or, more generally, in bounded subsets for the norm (2.10)).

If the stronger Assumption **(L₁)** *holds, then for all $\theta>0$ we have*

$$\|\varphi(\theta L)f\|_{L^p}\leq C\,\|f\|_{L^p},$$

and the estimate is uniform for φ in bounded subsets of $\mathscr{S}(\mathbb{R})$ (or, more generally, in bounded subsets for the norm (2.10)) and θ in bounded subsets of $(0,+\infty)$. If in addition we assume $\kappa=n$ and $M_0=0$, then the estimate is uniform for all $\theta>0$.

Proof. The first claim is just a special case of the previous lemma. Thus we assume that $(\mathbf{L}_1)$ holds and we focus on the second claim. Clearly it is sufficient to prove the result for all $\theta > 0$ of the form

$$\theta = 2^{-m\gamma} \quad \text{for some } \gamma \in \mathbb{Z}.$$

Thus we fix a $\theta = 2^{-m\gamma} > 0$ and define a new metric measure space $(\overline{\mathcal{X}}, \overline{d}, \overline{\mu})$ by multiplying d and μ by fixed constants, as follows:

$$\overline{\mathcal{X}} = \mathcal{X}, \quad \overline{d} = 2^{\gamma} d, \quad \overline{\mu} = 2^{n\gamma}\mu.$$

Note the relation

$$\|u\|_{L^p(\overline{\mathcal{X}}, d\overline{\mu})} = 2^{n\gamma/p} \|u\|_{L^p(\mathcal{X}, d\mu)}.$$

Writing

$$\overline{\mathcal{D}}_\nu = \mathcal{D}_{\nu+\gamma}$$

we see that the $\overline{\mathcal{D}}_\nu$ form a collection of dyadic cubes for the space $\overline{\mathcal{X}}$, and with respect to the new distance $\overline{d}$ we have diam $Q \sim 2^{-\nu}$ for all $Q \in \overline{\mathcal{D}}_\nu$. Then if we define the amalgam spaces $\overline{X}^{p,q}_\nu$ as in (2.3) but with $\overline{\mathcal{D}}_\nu$ instead of $\mathcal{D}_\nu$ and with the $L^q(Q)$ norms computed in the measure $\overline{\mu}$, we get

$$\|f\|_{\overline{X}^{p,q}_\nu} = 2^{n\gamma/q} \|f\|_{X^{p,q}_{\nu+\gamma}}.$$

Next, we denote by $\overline{L}$ the operator θL, which is self-adjoint on $L^2(\overline{\mathcal{X}})$ and satisfies $\overline{L} + \overline{M_0} \geq 0$ with $\overline{M_0} = \theta M_0$. To prove the claim, it will be sufficient to prove that the operator $\overline{L}$ satisfies the conditions of Assumption $(\mathbf{L})$, with constants independent of θ in the prescribed range. By the first part of the theorem, the claim will follow.

Fix a $t > 0$ and $\nu \in \mathbb{Z}$ as in condition (1.11) with $m_1 = m_2 = m$, i.e.,

$$2^{-\nu} \leq t^{1/m} < 2^{-\nu+1}.$$

Consider the first term in (1.11) (the second one is handled in a similar way):

$$\|e^{-t\overline{L}}\|_{\overline{X}^{1,p_0}_\nu \to \overline{X}^{1,2}_\nu} = \|e^{-(\theta t)L}\|_{X^{1,p_0}_{\nu+\gamma} \to X^{1,2}_{\nu+\gamma}} \cdot 2^{n\gamma(1/2-1/p_0)};$$

using assumption (1.11) (with $m_1 = m_2 = m$), since $2^{-(\nu+\gamma)} \leq (\theta t)^{1/m} = 2^{-\gamma} t < 2^{-(\nu+\gamma)+1}$ we get

$$\leq C\, e^{M_0(\theta t)}((\theta t)^{n/m} \wedge (\theta t)^{\kappa/m})^{1/2-1/p_0} \cdot 2^{n\gamma(1/2-1/p_0)}$$

$$= Ce^{\overline{M}_0 t}(t^{n/m} \wedge t^{\kappa/m} \cdot 2^{(n-\kappa)\gamma})^{1/2-1/p_0}.$$

Thus we see that the operator $\overline{L}$ also satisfies condition (1.11) with $m_1 = m_2 = m$. Note that the estimate is uniform in γ provided $\gamma \geq \gamma_0$ for some fixed γ_0, or equivalently, provided θ is bounded from above; moreover, $\overline{M}_0$ is also uniformly bounded from above. It is also clear that if $\kappa = n$ and $M_0 = 0$ the condition is uniform for all $\gamma \in \mathbb{Z}$, i.e., for all $\theta > 0$.

It remains to check condition (1.12); we choose as weight function and the space of "test functions"

$$\overline{w}(x,y) = 2^\gamma w(x,y), \quad \text{and} \quad \mathscr{D}(\overline{w}) = \mathscr{D}(w).$$

Writing $\overline{\mathrm{Ad}}_x^j$ for the commutators with the new weight function $\overline{w}$, we have

$$\overline{\mathrm{Ad}}_x^j((\overline{L}+M)^{-1}) = 2^{m\gamma}\, 2^{j\gamma}\, \mathrm{Ad}_x^j((L+2^{m\gamma}M)^{-1})).$$

By (1.13) we have then

$$\|\overline{\mathrm{Ad}}_x^j((\overline{L}+M)^{-1})\|_{2\to 2} \leq C\, 2^{m\gamma}\, 2^{j\gamma}\, (2^{m\gamma}M - M_0)^{-1-j/m}$$

provided $2^{m\gamma}M > M_0$. Now, if $\gamma \geq \gamma_0$ is bounded from below, we can choose $M = M_1 = 2^{-m\gamma_0}(M_0+1)$ and we get

$$\leq C\, 2^{m\gamma}\, 2^{j\gamma}\, (2^{m(\gamma-\gamma_0)})^{-1-j/m} \leq C'$$

for some constant independent of γ. Note that if $M_0 = 0$ we have

$$\|\overline{\mathrm{Ad}}_x^j((\overline{L}+M)^{-1})\|_{2\to 2} \leq C\, 2^{m\gamma}\, 2^{j\gamma}\, (2^{m\gamma}M)^{-1-j/m} = C\, M^{-1-j/m}$$

for all $M > 0$, thus we can pick simply $M_1 = 1$ without restrictions on $\gamma \in \mathbb{Z}$. The proof is concluded. ☐

2.4. Proof of Theorems 1.6, 1.9 and 1.11

We keep using the notation

$$\mathrm{Ad}_x(A) = [w_x, A], \quad \mathrm{Ad}_x^k(A) = [w_x, \mathrm{Ad}_x^{k-1}(A)]$$

for a generic operator A and a $\mathbb{R}^\ell$ valued weight function $w_x(\cdot) = w(x,\cdot)$.

Lemma 2.15. *For any $k \geq 1$ and $z \in \mathfrak{X}$, the following identities hold:*

$$\begin{aligned}
\mathrm{Ad}_z(R^{2k}e^{-itL}) &= \sum_{\alpha=0}^{k} R^\alpha \mathrm{Ad}_z(R) e^{-itL} R^{2k-\alpha-1} + \sum_{\alpha=0}^{k} R^{2k-\alpha-1} e^{-itL} \mathrm{Ad}_z(R) R^\alpha \\
&\quad + i\int_0^t e^{-isL} R^{k-1} \mathrm{Ad}_z(R) R^{k-1} e^{i(s-t)L} ds, \\
\mathrm{Ad}_z(R^{2k+1}e^{-itL}) &= \sum_{\alpha=0}^{k} R^\alpha \mathrm{Ad}_z(R) e^{-itL} R^{2k-\alpha} + \sum_{\alpha=0}^{k+1} R^{2k-\alpha} e^{-itL} \mathrm{Ad}_z(R) R^\alpha \\
&\quad + i\int_0^t e^{-isL} R^{k-1} \mathrm{Ad}_z(R) R^{k} e^{i(s-t)L} ds.
\end{aligned}$$

Proof. The first identity is proved by induction on k.

It can be verified that

$$\mathrm{Ad}_z(Re^{-itL}R) = \mathrm{Ad}_z(R)e^{-itL}R + e^{-itL}R\mathrm{Ad}_z(R) + i\int_0^t e^{-isL}\mathrm{Ad}_z(R)e^{i(s-t)L} ds$$

which is the first formula for $k=1$. To prove the step $k\to k+1$ we write

$$\mathrm{Ad}_z(R\cdot(R^{2k}e^{-itL})\cdot R)=\mathrm{Ad}_z(R)e^{-itL}R^{2k+1}+R^{2k+1}e^{-itL}\mathrm{Ad}_z(R)+I,$$

where

$$I=R\cdot\mathrm{Ad}_z(R^{2k}e^{-itL})\cdot R,$$

and using the inductive assumption for the case k we easily obtain the claim. The second formula is deduced from the first one writing

$$\mathrm{Ad}_z((R^{2k}e^{-itL})\cdot R)=\mathrm{Ad}_z(R^{2k}e^{-itL})R+R^{2k}e^{-itL}\mathrm{Ad}_z(R).\qquad\square$$

Lemma 2.16. *For $0\le\ell\le k$ and $1\le k\le\lfloor n/2\rfloor+1$ we have*

$$\|\mathrm{Ad}_z^\ell(R^{2k}e^{-itL})\|_{2\to2}\le C(1+|t|)^\ell,$$

with C independent of $z\in\mathcal{X}$ and $t\in\mathbb{R}$.

Proof. We proceed by induction on $k=1,\dots,\lfloor n/2\rfloor+1$. When $k=1$, recalling the formulas from the previous Lemma and assumption (1.12), we obtain the claim immediately. Assume now the result is true for a certain k and let us prove it for $k+1$. If $\ell=1$ the estimate follows again from the first identity in the previous Lemma. If the estimate is true for some $\ell<k$, we prove it for $\ell+1$ writing

$$\mathrm{Ad}_z^{\ell+1}(R^{2k}e^{-itL})=\mathrm{Ad}_z^\ell(\mathrm{Ad}_z(R^{2k}e^{-itL})),$$

expanding the term $\mathrm{Ad}_z(R^{2k}e^{-itL})$ via the first identity of the previous lemma, and distributing the adjoint via the formula

$$\mathrm{Ad}_z^\ell(A_1\dots A_n)=\sum_{j_1+\dots+j_n=\ell}\frac{\ell!}{j_1!\cdots j_n!}\mathrm{Ad}_z^{j_1}(A_1)\dots\mathrm{Ad}_z^{j_n}(A_n).$$

It is easy to check that all the terms obtained are bounded operators on L^2, either using the inductive assumption or (1.12). The proof is concluded. $\square$

Lemma 2.17. *Let $k=\lfloor n/2\rfloor+1$. Then we have the estimates*

$$\|R^{2k}e^{-itL}\|_{X^{p_0,2}\to X^{p_0,2}}\le C(1+|t|)^{n(1/p_0-1/2)}$$

and, for all $p\in[p_0,p_0']$ and $\beta>\frac{n}{m_1}(1/p_0-1/2)$,

$$\|R^{2k+\beta}e^{-itL}\|_{L^p\to L^p}\le C(1+|t|)^{n|1/p-1/2|}.$$

Proof. The first result is a direct application of Lemma 2.16 and Theorem 2.4. Moreover, by Proposition 2.8 we have

$$\begin{aligned}\|R^{2k+\beta}e^{-itL}\|_{X^{p_0,2}\to L^{p_0}}&\lesssim\|R^{2k}e^{-itL}\|_{X^{p_0,2}\to X^{p_0,2}}\,\|R^\beta\|_{X^{p_0,2}\to L^{p_0}}\\&\lesssim(1+|t|)^{n(1/p_0-1/2)}\end{aligned}$$

and by the embedding $X^{p_0,2}\subset L^{p_0}$ we obtain

$$\|R^{2k+\beta}e^{-itL}\|_{L^{p_0}\to L^{p_0}}\lesssim(1+|t|)^{n(1/p_0-1/2)}.$$

Finally, by duality and interpolation, we obtain the second claim. $\square$

We can now conclude the proof of our main results (Theorems 1.6 and 1.9).

Theorem 2.18. *Assume that L satisfies* **(L)**. *Let $p \in [p_0, p_0']$ and $s = n|1/2-1/p|$. Then we have the following estimate:*

$$\|e^{-itL}\varphi(L)f\|_p \lesssim (1+|t|)^s \, \|f\|_p, \quad t \in \mathbb{R},$$

uniformly for φ in bounded subsets of $\mathscr{S}(\mathbb{R})$ (or, more generally, in bounded subsets for the norm (2.10)).

If Assumption $(\mathbf{L}_1)$ *holds, we have*

$$\|e^{-itL}\varphi(\theta L)f\|_p \lesssim (1+\theta^{-1}|t|)^s \, \|f\|_p, \quad \theta > 0, \quad t \in \mathbb{R},$$

and the estimate is uniform for θ in bounded subsets of $(0,+\infty)$ and φ in bounded subsets of $\mathscr{S}(\mathbb{R})$ (or, more generally, in bounded subsets for the norm (2.10)). If in addition we assume $\kappa = n$ and $M_0 = 0$, the estimate is uniform also for all $\theta > 0$.

Proof. For the first claim it is sufficient to write

$$e^{-itL}\varphi(L) = (I+L)^{2k+\beta}e^{-itL} \cdot (I+L)^{-2k-\beta}\varphi(L)$$

and use the previous lemma and Lemma 2.12. The second claim is proved by a rescaling argument exactly as in the proof of Theorem 2.14. □

Proof of Theorem 1.11. Since the proof is quite similar to that of Theorem 1.9, we just sketch the main steps.

Denote by $p_t^{\mathfrak{X}}(x,y)$ the kernel of $1_\Omega e^{-tL} 1_\Omega$, regarded as an operator on functions defined on the entire space $\mathfrak{X}$. Then it is easy to see that

$$p_t^{\mathfrak{X}}(x,y) = \begin{cases} p_t(x,y), & \text{if } x,y \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

This, along with (1.14), implies

$$|p_t^{\mathfrak{X}}(x,y)| \le \frac{C\, e^{M_0 t}}{\mu(B(x,t^{1/m}))} \exp\Big(- \frac{d(x,y)^{m/(m-1)}}{ct^{1/(m-1)}} \Big)$$

for all $t > 0$ and $x, y \in \mathfrak{X}$.

As a consequence, the assumption (1.4) and (1.5) hold true with $m_1 = m_2 = m$ and $1_\Omega e^{-tL} 1_\Omega$ taking place of e^{-tL}.

Arguing similarly to the proof of Proposition 2.5, $\nu \in \mathbb{Z}$ with $2^{-\nu} \le t^{1/m} < 2^{-\nu+1}$, there exist $C \ge 0$ such that for all t and ν as above we have

$$\|\mathrm{Ad}_x^k(1_\Omega e^{-tL} 1_\Omega)\|_{2\to 2} \le C\, e^{M_0 t}\, 2^{-k\nu}, \quad 0 \le k \le \lfloor n/2 \rfloor + 1, \quad x \in \mathfrak{X}.$$

We then argue as in the proof of Proposition 2.6 to find that for all $M > M_0$ we have, for all $z \in \mathfrak{X}$ and $0 \le k \le \lfloor n/2 \rfloor + 1$,

$$\|\mathrm{Ad}_z^k(1_\Omega (L+M)^{-1} 1_\Omega)\|_{2\to 2} \lesssim (M - M_0)^{-1-k/m}$$

with a constant independent of z, M, where $w(x,y) = d(x,y)$.

The argument used in the proof of Proposition 2.8 allows us to obtain that for $M > M_0$ and $\gamma = n/(2m) + \epsilon$, with $\epsilon > 0$. Then

$$\|1_\Omega (M+L)^{-\gamma} 1_\Omega f\|_{X^{1,2}} \le C\left(\epsilon^{-1} + (M-M_0)^{\gamma + \frac{n-\kappa}{2} m}\right) \|f\|_1.$$

Fix $M_1 > M_0$ and set $R = (M_1 + L)^{-1}$. Then we can verify that

$$\mathrm{Ad}_z(1_\Omega e^{-i\xi R} 1_\Omega) = -i \int_0^\xi 1_\Omega e^{-isR}\, \mathrm{Ad}_z(1_\Omega R 1_\Omega)\, e^{-i(\xi - s)R}\, 1_\Omega \, ds.$$

Hence, similarly to Lemma 2.9, we obtain

$$\|1_\Omega e^{-i\xi R} 1_\Omega f\|_{X^{1,2}} \le C(1+|\xi|)^{n/2} \|f\|_{X^{1,2}}, \quad \xi \in \mathbb{R}.$$

This, along with the identity

$$1_\Omega \psi(R) 1_\Omega = (2\pi)^{-1} \int 1_\Omega e^{i\xi R}\, 1_\Omega \widehat{\psi}(\xi)\, d\xi,$$

implies that

$$\|1_\Omega \psi(R) 1_\Omega f\|_{X^{1,2}} \le C\|(1+|\xi|)^{n/2} \widehat{\psi}(\xi)\|_{L^1} \|f\|_{X^{1,2}}$$

for any sufficiently smooth function ψ on $\mathbb{R}$.

Arguing similarly as in Theorem 2.14, for all $\theta > 0$ we have

$$\|1_\Omega \varphi(\theta L) 1_\Omega f\|_{L^p} \le C\|f\|_{L^p}$$

and the estimate is uniform for φ in bounded subsets of $\mathscr{S}(\mathbb{R})$. Moreover, if $\kappa = n$, then the estimate is uniform for all $\theta > 0$.

As this stage, arguing, mutatis mutandis, as in the proof of Theorem 1.9 we obtain that for any $p \in [1, \infty]$ and $s = n|1/2 - 1/p|$,

$$\|1_\Omega e^{-itL} \varphi(\theta L) 1_\Omega f\|_p \lesssim (1 + \theta|t|)^s \|f\|_p, \quad t \in \mathbb{R},$$

and the estimate is uniform for φ in bounded subsets of $\mathscr{S}(\mathbb{R})$ and $0 < \theta \le \theta_0$, for any fixed $\theta_0 > 0$. If, in addition, $\kappa = n$ and M_0, then the estimate is uniform for all $\theta > 0$. This completes our proof. □

3. Applications

Our framework is sufficiently general to include a large variety of applications; in this section we survey a few of the most interesting cases.

3.1. Laplace–Beltrami operators with a Gaussian heat kernel bound

Let $\mathcal{X}$ be a complete connected non-compact n-dimensional Riemannian manifold. The geodesic distance and the Riemannian measure are denoted by d and μ, respectively. The Laplace–Beltrami operator $L = -\Delta$ on $\mathcal{X}$ is nonnegative and self-adjoint.

We assume that the Riemannian measure μ satisfies the volume doubling property (1.2) and the *non-collapsing* condition

$$\mu(B(x,1)) \geq c$$

for all $x \in \mathcal{X}$ and for some fixed constant $c > 0$.

It is well-known (see [23]) that if the Ricci curvature of $\mathcal{X}$ is non-negative, then the heat kernel of the heat semigroup e^{-tL} satisfies the estimate

$$e^{-tL}(x,y) \lesssim \frac{1}{\mu(B(x,\sqrt{t}))} \exp\Big(-\frac{d(x,y)^2}{ct}\Big). \tag{3.1}$$

It can be verified that the Gaussian upper bound (3.1) implies (1.5). Moreover, the upper bound (3.1) also yields that for $\nu \in \mathbb{Z}$ and $2^{-\nu} \leq t^{1/2} < 2^{-\nu+1}$, we have

$$\sum_{Q\in\mathcal{D}_\nu} \|1_Q e^{-tL} 1_{Q'}\|_{1\to\infty} \leq C\,\mu(Q')^{-1}, \quad \text{for all } Q' \in \mathcal{D}_\nu.$$

This, in combination with the non-collapsing condition and (2.1), implies that

$$\sum_{Q\in\mathcal{D}_\nu} \|1_Q e^{-tL} 1_{Q'}\|_{1\to\infty} \leq C(1+2^{\nu n}), \quad \text{for all } Q' \in \mathcal{D}_\nu,$$

and this proves (1.4) and (1.5).

Hence, Assumption $(\mathbf{L}_0)$ is satisfied with $m_1 = m_2 = 2$ and $p_0 = 1$.

3.2. Laplace–Beltrami operators without Gaussian heat kernel bound

Let $\mathcal{X}$ be a complete connected non-compact Riemannian manifold. The geodesic distance and the Riemannian measure are denoted by d and μ, respectively. We assume that the Riemannian measure μ satisfies the volume doubling property (1.2) and the *non-collapsing* condition (1.7).

Let $L = -\Delta$ be the non-negative Laplace–Beltrami operator on $\mathcal{X}$. We assume that the kernel $e^{-tL}(x,y)$ of the semigroup e^{-tL} satisfies the following sub-Gaussian heat kernel upper estimate with exponent $m > 0$:

$$e^{-tL}(x,y) \leq \begin{cases} \frac{C}{\mu(B(x,\sqrt{t}))} \exp\Big(\frac{d(x,y)^2}{ct}\Big), & 0 < t < 1, \\ \frac{C}{\mu(B(x,t^{1/m}))} \exp\Big(\frac{d(x,y)^{m/(m-1)}}{ct^{1/(m-1)}}\Big), & t \geq 1, \end{cases} \tag{3.2}$$

for all $x, y \in \mathcal{X}$.

Typical examples that satisfy (1.2), (1.7) and (3.2) include certain fractal manifolds and infinite connected locally finite graphs. For further details, we refer to [3], [8].

By a similar argument as in Subsection 3.1 one can prove that L satisfies Assumption $(\mathbf{L}_0)$ with $m_1 = 2$, $m_2 = m$ and $p_0 = 1$.

3.3. Sierpinski gasket SG in $\mathbb{R}^n$

Let $\mathcal{X}$ be the unbounded Sierpinski gasket SG in $\mathbb{R}^n$. Let d be the induces metric on SG and μ be the Hausdorff measure on SG of dimension $\alpha = \log_2(n+1)$. It is well-known that the Hausdorff measure μ satisfies the doubling property (1.2); moreover,

$$\mu(B(x,r)) \lesssim r^\alpha, \tag{3.3}$$

for all $x \in \mathcal{X}$ and $r > 0$.

It was also proved in [2] that SG admits a local Dirichlet form $\mathcal{E}$ which generates a nonnegative self-adjoint operator L; moreover, the kernel $e^{-tL}(x,y)$ of e^{-tL} satisfies the sub-Gaussian estimate

$$e^{-tL}(x,y) \lesssim \frac{1}{t^{\alpha/m}} \exp\Big(- \frac{d(x,y)^{m/(m-1)}}{ct^{1/(m-1)}}\Big)$$

where $m = \log_2(n+3)$ is called the *walk dimension*.

Note that the assumption (1.5) is a direct consequence of the kernel upper bound above whereas the assumption (1.4) is a consequence of the same kernel upper bound, the doubling property (1.2) and (3.3). Theorefore, L satisfies Assumption $(\mathbf{L}_0)$ with $m_1 = m_2 = m$ and $p_0 = 1$.

3.4. Homogeneous groups

Let $\mathbf{G}$ be a Lie group of polynomial growth and let $X_1, \dots, X_k$ be a system of left-invariant vector fields on $\mathbf{G}$ satisfying the Hörmander condition. We define the Laplace operator L on $L^2(\mathbf{G})$ by

$$L = -\sum_{i=1}^k X_i^2. \tag{3.4}$$

Denote by d the distance associated with the system $X_1, \dots, X_k$, and let $B(x,r)$ be the corresponding balls. Then (see [34]) there exist positive numbers $d, D \geq 0$ such that

$$\mu(B(x,r)) \sim \begin{cases} r^d, & r \leq 1, \\ r^D, & r > 1. \end{cases} \tag{3.5}$$

Hence $(\mathbf{G}, d, \mu)$ satisfies the doubling property (1.2).

The group $\mathbf{G}$ is called a *homogeneous group* (see [16]) if there exists a *family of dilations* $(\delta_t)_{t>0}$ on $\mathbf{G}$, that is to say, a one-parameter group ($\delta_t \circ \delta_t = \delta_{ts}$) of automorphisms of $\mathbf{G}$ determined by

$$\delta_t Y_j = t^{d_j} Y_j,$$

where $Y_1, \dots, Y_\ell$ is a linear basis of the Lie algebra of $\mathbf{G}$ and $d_j \geq 1$ for $1 \leq j \leq \ell$. We say that the operator L defined by (3.4) is *homogeneous* if $\delta_t X_i = tY_i$ for

$1 \le i \le k$. It well known that the heat kernel of the heat semigroup e^{-tL} satisfies the estimate

$$e^{-tL}(x,y) \lesssim \frac{1}{\mu(B(x,\sqrt{t}))} \exp\Big(-\frac{d(x,y)^2}{ct}\Big).$$

This upper bound together with (3.5) implies that L satisfies (1.4) and (1.5) with $m_1 = m_2 = 2$ and $p_0 = 1$, and hence L satisfies Assumption $(\mathbf{L}_0)$ with $m_1 = m_2 = 2$ and $p_0 = 1$.

3.5. Bessel operators

Let $\mathcal{X} = ((0,\infty)^m, d\mu(x))$ where $d\mu(x) = d\mu_1(x_1)\cdots d\mu_n(x_m)$ and $d\mu_k = x_k^{\alpha_k} dx_k$, $\alpha_k > -1$, for $k = 1, \dots, m$ (dx_j being the one dimensional Lebesgue measure). We endow $\mathcal{X}$ with the distance d defined for $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m) \in \mathcal{X}$ as

$$d(x,y) := |x-y| = \Big(\sum_{k=1}^m |x_k - y_k|^2\Big)^{1/2}.$$

Then it is clear that

$$\mu(B(x,r)) \sim r^m \prod_{k=1}^m (x_k + r)^{\alpha_k}.$$

Note that this estimate implies the doubling property (1.2) with $n = m + \alpha_1 + \cdots + \alpha_n$ and the non-collapsing condition (1.7).

For an element $x \in \mathbb{R}^m$, unless specified otherwise, we shall write x_k for the k-th component of x, $k = 1, \dots, m$. Moreover, for $\lambda \in \mathbb{R}^m$, we write $\lambda^2 = (\lambda_1^2, \dots, \lambda_m^2)$.

We consider the second order Bessel differential operator

$$L = -\Delta - \sum_{k=1}^m \frac{\alpha_k}{x_k} \frac{\partial}{\partial x_k},$$

whose system of eigenvectors is defined by

$$E_\lambda(x) := \prod_{k=1}^n E_{\lambda_k}(x_k), \quad E_{\lambda_k}(x_k) := (x_k\lambda_k)^{-(\alpha_k-1)/2} J_{(\alpha_k-1)/2}(x_k\lambda_k), \quad \lambda, x \in \mathcal{X},$$

where $J_{(\alpha_k-1)/2}$ is the Bessel function of the first kind of order $(\alpha_k - 1)/2$ (see [22]). It is known that $L(E_\lambda) = |\lambda|^2 E_\lambda$. Moreover, the functions E_{λ_k} are eigenfunctions of the one-dimension Bessel operators

$$L_k = -\frac{\partial^2}{\partial {x_k}^2} - \frac{\alpha_k}{x_k}\frac{\partial}{\partial x_k}$$

and indeed $L_k(E_{\lambda_k}) = \lambda_k^2 E_{\lambda_k}$ for $k = 1, \dots, m$.

It is well known that L is nonnegative and self-adjoint; moreover, the kernel $e^{-tL}(x,y)$ of e^{-tL} satisfies the Gaussian estimate

$$e^{-tL}(x,y) \lesssim \frac{1}{\mu(B(x,\sqrt{t}))} \exp\Big(-\frac{d(x,y)^2}{ct}\Big). \tag{3.6}$$

Hence, the Gaussian upper bound (3.6), along with the doubling and the non-collapsing properties imply Assumption $(\mathbf{L}_0)$ with $m_1 = m_2 = m$ and $p_0 = 1$.

3.6. Schrödinger operators with real potentials on manifolds

Let $\mathcal{X}$ be a complete connected non-compact Riemannian manifold. The geodesic distance and the Riemannian measure are denoted by d and μ, respectively. We assume that the Riemannian measure μ satisfies the doubling property (1.2) and the non-collapsing condition (1.7). We also assume that the heat kernel $p_t(x,y)$ of the Laplace–Beltrami operator $-\Delta$ satisfies the standard Gaussian upper bound

$$p_t(x,y) \le \frac{C}{\mu(B(x,\sqrt{t}))} \exp\Big(-\frac{d^2(x,y)}{ct}\Big). \tag{3.7}$$

We now consider the Schrödinger operator $L = -\Delta + V$, $V \in L^1_{\rm loc}(\mathcal{X})$. If the potential V is nonnegative, then the kernel of the semigroup $\{e^{-tL}\}_{t>0}$ generated by L satisfies the same Gaussian bound (3.7); in the general case, we must impose some conditions on the negative part of V. Denote by V^+ and V^- the positive and negative parts of V, respectively. We define

$$\mathcal{Q}(u,v) = \int_{\mathcal{X}} \nabla u \nabla v\, d\mu + \int_{\mathcal{X}} V^+ u v\, d\mu - \int_{\mathcal{X}} V^- u v\, d\mu$$

with domain

$$\mathcal{D}(\mathcal{Q}) = \Big\{ u \in W^{1,2}(\mathcal{X}) : \int_{\mathcal{X}} V^+ u^2\, d\mu < \infty \Big\}.$$

Then we assume that the positive part $V^+ \in L^1_{\rm loc}$ and the negative part V^- satisfy the following condition:

$$\int_{\mathcal{X}} V^- u^2\, d\mu \le \alpha \Big[\int_{\mathcal{X}} |\nabla u|^2\, d\mu + \int_{\mathcal{X}} V^+ u^2\, d\mu \Big], \quad \forall u \in \mathcal{D}(\mathcal{Q}),$$

for some $\alpha \in (0,1)$.

It was proved in [1], Theorem 3.4, that for any $(\frac{2}{1-\sqrt{1-\alpha}})' < p_0 < 2$ there exist $C, c > 0$ and $\beta > 0$ such that

$$\begin{aligned} &\|1_{B(x,r)} e^{-sL} 1_{B(y,r)}\|_{p_0 \to p_0'} \\ &\quad \le C \mu(B(x,r))^{-1/p_0 + 1/p_0'} \Big(\max\Big(\frac{r}{\sqrt{s}}, \frac{\sqrt{s}}{r}\Big)\Big)^{\beta} \exp\Big(-\frac{\mathrm{dist}(B(x,r),B(y,r))^2}{ct}\Big) \end{aligned}$$

for all $r, s > 0$ and $x, y \in \mathcal{X}$.

This, in combination with the volume doubling property (1.2) and the non-collapsing condition (1.7), implies that Assumption $(\mathbf{L}_0)$ is satisfied with $m_1 = m_2 = 2$ and any $(\frac{2}{1-\sqrt{1-\alpha}})' < p_0 < 2$.

3.7. Schrödinger operators with inverse-square potentials

Consider the following Schrödinger operators with inverse square potential on $\mathbb{R}^n$, $n \geq 3$:

$$\mathcal{L}_a = -\Delta + \frac{a}{|x|^2} \quad \text{with} \quad a \geq -\Big(\frac{n-2}{2}\Big)^2.$$

Set

$$\sigma := \frac{n-2}{2} - \frac{1}{2}\sqrt{(n-2)^2 + 4a}.$$

The Schrödinger operator $\mathcal{L}_a$ is understood as the Friedrichs extension of $-\Delta + \frac{a}{|x|^2}$ defined initially on $C_c^\infty(\mathbb{R}^n \backslash \{0\})$. The condition $a \geq -((n-2)/2)^2$ guarantees that $\mathcal{L}_a$ is nonnegative. It is well known that $\mathcal{L}_a$ is self-adjoint and the extension may not be unique as $-((n-2)/2)^2 \leq a < 1 - ((n-2)/2)^2$. For further details, we refer the readers to [20], [27], [33]. For the corresponding heat kernel, we have the following result.

Theorem 3.1 ([26], [24])**.** *Assume $n \geq 3$ and $a \geq -((n-2)/2)^2$. Then there exist two positive constants C and c such that for all $t > 0$ and $x, y \in \mathbb{R}^n \backslash \{0\}$,*

$$|e^{-t\mathcal{L}_a}(x,y)| \leq C\Big(1 + \frac{\sqrt{t}}{|x|}\Big)^{\sigma}\Big(1 + \frac{\sqrt{t}}{|y|}\Big)^{\sigma} t^{-n/2} e^{-\frac{|x-y|^2}{ct}}.$$

Set $n_\sigma = n/\sigma$ if $\sigma > 0$ and $n_\sigma = \infty$ if $\sigma \leq 0$. From Theorem 3.1 and Theorem 3.1 in [5], for any $n'_\sigma < p \leq q < n_\sigma$ there exist $C, c > 0$ such that for every $t > 0$, any measurable subsets $E, F \subset \mathbb{R}^n$, and all $f \in L^p(E)$, we have:

$$\|e^{-t\mathcal{L}_a} f\|_{L^q(F)} \leq C\, t^{-\frac{n}{2}(1/p - 1/q)}\, e^{-\frac{d(E,F)^2}{ct}}\, \|f\|_{L^p(E)}.$$

Hence, with the standard dyadic systems in $\mathbb{R}^n$, this implies that Assumption $(\mathbf{L}_0)$ is satisfied with $m_1 = m_2 = 2$ and any $n'_\sigma < p_0 < 2$. Moreover, in this situation the reverse doubling condition (1.3) is valid with $\kappa = n$.

3.8. Fourth-order Schrödinger operators with singular potentials

Consider the following Schrödinger operator with singular potentials on $\mathbb{R}^n$ with $n \geq 5$:

$$L = (-\Delta)^2 - \frac{c}{|x|^4}$$

where $c < (N(N-4)/4)^2$.

It was proved in [18] that for any $2n/(n+4) < p \leq q < 2n/(n-4)$, there exist $C, c > 0$ such that for every $t > 0$, any measurable subsets $E, F \subset \mathbb{R}^n$, and all $f \in L^p(E)$, we have

$$\|e^{-tL} f\|_{L^q(F)} \leq C\, t^{-\frac{n}{4}(1/p - 1/q)}\, e^{-\frac{d(E,F)^{4/3}}{ct^{1/3}}}\, \|f\|_{L^p(E)}.$$

Hence, with the standard dyadic systems in $\mathbb{R}^n$, this implies that Assumption $(\mathbf{L}_0)$ is satisfied with $m_1 = m_2 = 4$ and $p_0 = 2n/(n+4)$. Moreover, in this situation the reverse doubling condition (1.3) is valid with $\kappa = n$.

3.9. Schrödinger operators with singular potentials on $L^2((0,\infty), r^{n-1}dr)$

Let $(X, d, \mu) = ((0,\infty), |\cdot|, r^{n-1}dr)$ with $n > 2$. We now consider the following operator on $L^2(X)$:

$$Lf = -\frac{d^2}{dr^2}f - \frac{n-1}{r}\frac{d}{dr}f + \frac{a}{r^2}f$$

where $a > -(n-2)^2/4$. It was proved in [25] that for any $p'_a < p \le q < p_a$ there exist $C, c > 0$ such that for every $t > 0$, any measurable subsets $E, F \subset X := (0,\infty)$, and all $f \in L^p(E)$, we have

$$\|e^{-tL}f\|_{L^q(F)} \le C\, t^{-\frac{n}{2}(1/p-1/q)}\, e^{-\frac{d(E,F)^2}{ct}}\, \|f\|_{L^p(E)}$$

where $p_a = n/\sigma$ and $\sigma = (n-2)/2 - \sqrt{(n-2)^2/4 + c}$ as $a < 0$ and $p_a = 1$ if $a > 0$.

Hence, with the standard dyadic systems in $(0,\infty)$, this implies that Assumption $(\mathbf{L}_0)$ is satisfied with $m_1 = m_2 = 2$ and any $a'_a < p_0 < 2$.

3.10. Sub-Laplacian operators on Heisenberg groups

Let $\mathbb{H}^d$ be a $(2d+1)$-dimensional Heisenberg group. Recall that a $(2d+1)$-dimensional Heisenberg group is a connected and simply connected nilpotent Lie group with the underlying manifold $\mathbb{R}^{2d} \times \mathbb{R}$. The group structure is defined by

$$(x,s)(y,t) = \Big(x + y, s + t + 2\sum_{j=1}^{d}(x_{d+j}\, y_j - x_j\, y_{d+j})\Big)$$

The homogeneous norm on $\mathbb{H}^d$ is defined by

$$|(x,t)| = (|x|^4 + |t|^2)^{1/4} \quad \text{for all } (x,t) \in \mathbb{H}^d.$$

See for example [31].

This norm satisfies the the triangle inequality and hence induces a left-invariant metric $d((x,t),(y,s)) = |(-x,-t)(y,s)|$. Moreover, there exists a positive constant C such that $|B((x,t),r)| = Cr^n$, where $n = 2d+2$ is the homogeneous dimension of $\mathbb{H}^d$ and $|B((x,t),r)|$ is the Lebesgue measure of the ball $B((x,t),r)$. Obviously, the triplet $(\mathbb{H}^d, d, dx)$ satisfies the doubling condition (1.2), the reverse doubling condition (1.3) with $\kappa = n$, and the *non-collapsing* condition (1.7).

A basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}^d$ is given by

$$X_{2d+1} = \frac{\partial}{\partial t}, \quad X_j = \frac{\partial}{\partial x_j} + 2x_{d+j}\frac{\partial}{\partial t}, \quad X_{d+j} = \frac{\partial}{\partial x_{d+j}} - 2x_j\frac{\partial}{\partial t}, \quad j = 1, \dots, d.$$

The sub-Laplacian $\Delta_{\mathbb{H}^d}$ is defined by

$$\Delta_{\mathbb{H}^d} = -\sum_{j=1}^{2d} X_j^2.$$

Furthermore, it is well known that the sub-Laplacian $\Delta_{\mathbb{H}^d}$ satisfies the Gaussian upper bound

$$e^{-t\Delta_{\mathbb{H}^d}}((x,u),(y,s)) \le \frac{C}{t^{n/2}} \exp\Big(-\frac{d((x,u),(y,s)^2}{ct}\Big).$$

In $\mathbb{H}^d$, we consider the standard dyadic system consists of the cubes

$$2^{-k}((0,1]^{2d}+j)\times 4^{-k}((0,1]+\ell), \quad k\in\mathbb{Z},\ j\in\mathbb{Z}^{2d},\ \ell\in\mathbb{Z}.$$

Hence, the Gaussian upper bound yields the assumption $(\mathbf{L}_0)$ with $m_1 = m_2 = 2$ and $p_0 = 1$.

3.11. Dirichlet Laplacians on open domains

Let $\mathcal{X} = (\mathbb{R}^n, |\cdot|, dx)$. Then $\mathcal{X}$ is a space of homogeneous type satisfying (1.3) with $\kappa = n$ and the *non-collapsing* condition (1.7).

Let Ω be a connected open subset of $\mathbb{R}^n$. Note that Ω may not satisfy the doubling condition. Let Δ_D be Dirichlet Laplacian on the domain Ω. It is well known that the semigroup kernel $e^{-t\Delta_D}(x,y)$ of $e^{-t\Delta_D}$ satisfies the Gaussian upper bound

$$e^{-t\Delta_D}(x,y) \le \frac{1}{(4\pi t)^{n/2}} \exp\Big(-\frac{|x-y|^2}{4t}\Big),$$

for all $t>0$ and all $x,y\in\Omega$.

Hence, all assumptions in Theorem 1.11 are satisfied with $\mathcal{X} = (\mathbb{R}^n, |\cdot|, dx)$, $L = \Delta_D$ and $\kappa = n$.

3.12. Schrödinger operators with singular potentials

For our last example, we recall the definition of the *Kato class* K_n of potentials. The measurable function $V\colon\mathbb{R}^n\to\mathbb{R}$ belongs to K_n if the following conditions are satisfied:

1. If $n\ge 3$, $\displaystyle\limsup_{\alpha\downarrow 0}\sup_x \int_{|x-y|\le\alpha} |x-y|^{2-n} V(x)\,dx = 0.$

2. If $n = 2$, $\displaystyle\limsup_{\alpha\downarrow 0}\sup_x \int_{|x-y|\le\alpha} \log(|x-y|^{-1}) V(x)\,dx = 0.$

3. If $n = 1$, $\displaystyle\sup_x \int_{|x-y|\le 1} V(x)\,dx < \infty.$

Moreover, we say that $V\in K_{n,\mathrm{loc}}$ if $1_B V\in K_n$ for all balls B.

We consider a Schrödinger operator of the form $L = -\Delta + V(x)$ on $\mathbb{R}^n$, $n\ge 1$. We assume that the positive part V_+ of V is in $K_{n,\mathrm{loc}}$ while the negative part V_- is in K_n. Then the results of [29] (see in particular Proposition B.6.7) imply that L can be realized as a semibounded self-adjoint operator in $L^2(\mathbb{R}^n)$, and that the heat kernel e^{-tL} satisfies

$$|e^{-tL}(x,y)| \le C\,t^{-n/2}\,e^{M_0 t}\,e^{-\frac{|x-y|^2}{ct}}, \quad \text{with } C,c>0. \tag{3.8}$$

Thus Assumption $(\mathbf{L}_1)$ is satisfied, with $M_0 \geq 0$. If in addition we assume that the negative part satisfies

$$\sup_x \int |x-y|^{2-n}\, V_-(y)\, dy < 2\, \frac{\pi^{n/2}}{\Gamma(n/2-1)} \tag{3.9}$$

in dimension $n \geq 3$ (or $V_- = 0$ in dimensions 1, 2) then in [14] it is proved that one can take $M_0 = 0$, so that the uniform estimates of Theorem 1.9 apply.

Moreover, one can consider the same operator L with Dirichlet boundary conditions on $L^2(\Omega)$, for an open subset Ω of $\mathbb{R}^n$. If we assume for simplicity $V \geq 0$, then by the maximum principle we obtain that the heat kernel is nonnegative and satisfies again the upper Gaussian estimate (3.8), with $M_0 = 0$ i.e., all the assumptions of the second part of Theorem 1.11 are satisfied.

Similar results can be proved for the magnetic Schrödinger operators of the form $(i\nabla + A(x))^2 + V(x)$, using the heat kernel estimates proved in [12], and for elliptic operators with fully variable coefficients on exterior domains, via the results of [7]. We omit the details.

3.13. Magnetic Schrödinger operator

Consider the magnetic Schrödinger operator on $\mathbb{R}^n$ defined by

$$L = (i\nabla + A(x))^2 + V(x),$$

with magnetic potential $A = (A_1, \ldots, A_n)$ and electric potential $V(x)$.

If we choose as weight function $w(x,y) = x - y\colon \mathbb{R}^{2n} \to \mathbb{R}^n$ and $\mathscr{D}(w) = C_0^\infty(\mathbb{R}^n)$, then we have

$$\mathrm{Ad}_x^1(L) = 2\nabla + 2iA, \quad \mathrm{Ad}_x^2(L) = (2, \ldots, 2) \quad \text{and} \quad \mathrm{Ad}_x^j(L) = 0 \quad \text{for } j \geq 2.$$

The vector of operators $\mathrm{Ad}_x^2(L)R$ is obviously bounded on L^2; since R is also bounded from $L^2(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$, if the magnetic potential satisfies

$$\|Af\|_{L^2} \lesssim \|f\|_{H^1}$$

then also $\mathrm{Ad}_x^1(L)R$ is bounded on L^2. Moreover, by elementary computations one can write $\mathrm{Ad}_x^k(R)$ as a linear combination of terms

$$R\, \mathrm{Ad}_x^{k_1}(L)\, R\, \mathrm{Ad}_x^{k_2}(L)\, \ldots R\, \mathrm{Ad}_x^{k_N}(L)\, R$$

with $k \geq k_i, N \geq 1$ and $k_1 + \cdots + k_N = k$. For instance, in dimension $n \geq 3$ it is sufficient to assume that $|A| \leq C + C|x|^{-1}$, thanks to Hardy's inequality.

As a consequence, it follows the condition $(\mathbf{L})$.

References

[1] Assaad, J. and Ouhabaz, E. M.: Riesz transforms of Schrödinger operators on manifolds. *J. Geom. Anal.* **22** (2012), no. 4, 1108–1136.

[2] Barlow, M. T.: Diffusions on fractals. In *Lectures on Probability Theory and Statistics (Saint-Flour, 1995)*, 1–121. Lect. Notes Math. 1690, Springer, Berlin, 1998.

[3] BARLOW, M., COULHON, T. AND GRIGOR'YAN, A.: Manifolds and graphs with slow heat kernel decay. *Invent. Math.* **144** (2001), no. 3, 609–649.

[4] BRENNER, P., THOMÉE, V. AND WAHLBIN, L. B.: *Besov spaces and applications to difference methods for initial value problems.* Lecture Notes in Mathematics 434, Springer-Verlag, Berlin-New York, 1975.

[5] BUI, T. A., D'ANCONA, P., DUONG, X. T., LI, J. AND LY, F. K.: Weighted estimates for powers and smoothing estimates of Schrödinger operators with inverse-square potentials. *J. Differential Equations* **262** (2017), no. 3, 2771–2807.

[6] CACCIAFESTA, F. AND D'ANCONA, P.: Weighted L^p estimates for powers of selfadjoint operators. *Adv. Math.* **229** (2012), no. 1, 501–530.

[7] CASSANO, B. AND D'ANCONA, P.: Scattering in the energy space for the NLS with variable coefficients. *Math. Ann.* **366** (2016), no 1-2, 479–543.

[8] CHEN, L., COULHON, T., FENEUIL, J. AND RUSS, E.: Riesz transforms for $1 \leq p \leq 2$ without Gaussian heat kernel bound. *J. Geom. Anal.* **27** (2017), no. 2, 1489–1514.

[9] CHEN, Z.-Q. KIM, P. AND SONG, R.: Heat kernel estimates for the Dirichlet fractional Laplacian. *J. Eur. Math. Soc. (JEMS)* **12** (2010), no. 5, 1307–1329.

[10] CHRIST, M.: A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. *Colloq. Math.* **61** (1990), no. 2, 601–628.

[11] COIFMAN, R. R. AND WEISS, G.: Extensions of Hardy spaces and their use in analysis. *Bull. Amer. Math. Soc.* **83** (1977), no. 4, 569–645.

[12] D'ANCONA, P., FANELLI, L., VEGA, L. AND VISCIGLIA, N.: Endpoint Strichartz estimates for the magnetic Schrödinger equation. *J. Funct. Anal.* **258** (2010), no. 10, 3227–3240.

[13] D'ANCONA, P. AND NICOLA, F.: Sharp L^p estimates for Schrödinger groups. *Rev. Mat. Iberoam.* **32** (2016), no. 3, 1019–1038.

[14] D'ANCONA, P. AND PIERFELICE, V.: On the wave equation with a large rough potential. *J. Funct. Anal.* **227** (2005), no. 1, 30–77.

[15] DUONG, X. T., OUHABAZ, E. M. AND SIKORA, A.: Plancherel-type estimates and sharp spectral multipliers. *J. Funct. Anal.* **196** (2002), no. 2, 443–485.

[16] FOLLAND, G. AND STEIN, E. M.: *Hardy spaces on homogeneous groups.* Mathematical Notes 28, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982.

[17] HEBISCH, W.: A multiplier theorem for Schrödinger operators. *Colloq. Math.* **60/61** (1990), no. 2, 659–664.

[18] GREGORIO, F. AND MILDNER, S.: Fourth-order Schrödinger type operator with singular potentials. *Arch. Math. (Basel)* **107** (2016), no. 3, 285–294.

[19] JENSEN, A. AND NAKAMURA, S.: L^p-mapping properties of functions of Schrödinger operators and their applications to scattering theory. *J. Math. Soc. Japan* **47** (1995), no. 2, 253–273.

[20] KALF, H., SCHMINCKE, U. W., WALTER, J. AND WÜST, R.: On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials. In *Spectral theory and differential equations (Proc. Sympos., Dundee, 1974; dedicated to Konrad Jörgens),* 182–226. Lecture Notes in Math. 448, Springer, Berlin, 1975.

[21] LANCONELLI, E.: Valutazioni in $L_p(\mathbb{R}^n)$ della soluzione del problema di Cauchy per l'equazione di Schrödinger. *Boll. Un. Mat. Ital.(4)* **1** (1968), 591–607.

[22] Lebedev, N. N.: *Special functions and their applications.* Dover Publications, New York, 1972.

[23] Li, P. and Yau, S. T.: On the parabolic kernel of the Schrödinger operator. *Acta Math.* **156** (1986), no. 3-4, 153–201.

[24] Liskevich, V. and Sobol, Z.: Estimates of integral kernels for semigroups associated with second-order elliptic operators with singular coefficients. *Potential Anal.* **18** (2003), no. 4, 359–390.

[25] Liskevich, V., Sobol, Z. and Vogt, H.: On the L_p theory of C^0-semigroups associated with second-order elliptic operators II. *J. Funct. Anal.* **193** (2002), no. 1, 55–76.

[26] Milman, P. D. and Semenov, Y. A.: Global heat kernel bounds via desingularizing weights. *J. Funct. Anal.* **212** (2004), no. 2, 373–398.

[27] Planchon, F., Stalker, J. and Tahvildar-Zadeh, A. S.: L^p estimates for the wave equation with the inverse-square potential. *Discrete Contin. Dyn. Syst.* **9** (2003), no. 2, 427–442.

[28] Seeger, A., Sogge, C. D. and Stein, E. M.: Regularity properties of Fourier integral operators. *Ann. of Math. (2)* **134** (1991), no. 2, 231–251.

[29] Simon, B.: Schrödinger semigroups. *Bull. Amer. Math. Soc. (N.S.)* **7** (1982), no. 3, 447–526.

[30] Sjöstrand, S.: On the Riesz means of the solutions of the Schrödinger equation. *Ann. Scuola Norm. Sup. Pisa CL. Sci. (3)* **24** (1970), 331–348.

[31] Stein, E. M.: *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* Princeton Mathematical Series 43, Monographs in Harmonic Analysis III, Princeton University Press, Princeton, NJ, 1993.

[32] Tao, T.: Low regularity semi-linear wave equations. *Comm. Partial Differential Equations* **24** (1999), no. 3-4, 599–629.

[33] Titchmarsh, E. C.: *Eigenfunction expansions associated with second-order differential equations.* University Press, Oxford, 1946.

[34] Varopoulos, N., Saloff-Coste, L. and Coulhon, T.: *Analysis and geometry on groups.* Cambridge Tracts in Mathematics 100, Cambridge University Press, Cambridge, 1992.

[35] Watson, G. N.: *A treatise on the theory of Bessel functions.* Cambridge University Press, Cambridge, 1966.

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