MIXING LENGTH SCALES OF LOW TEMPERATURE SPIN PLAQUETTES MODELS

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ABSTRACT. Plaquette models are short range ferromagnetic spin models that play a key role in the dynamic facilitation approach to the liquid glass transition. In this paper we perform a rigorous study of the thermodynamic properties of two dimensional plaquette models, the square and triangular plaquette models. We prove that for any positive temperature both models have a unique infinite volume Gibbs measure with exponentially decaying correlations. We analyse the scaling of three a priori different static correlation lengths in the small temperature regime, the mixing, cavity and multispin correlation lengths. Finally, using the symmetries of the model we determine an exact self similarity property for the infinite volume Gibbs measure.

1. INTRODUCTION

Providing a clear and deep understanding of the liquid glass transition and of the glassy state of matter remains an open challenge for condensed matter physicists (see [2] for a review on the various theoretical approaches). One of the theories of the glass transition, known as *dynamical facilitation* (DF), relies on two basic assumptions. The first paradigm is that the dominant relaxation mechanism when approaching the glass transition should be facilitated relaxation: a local region that relaxes allows (facilitates) a neighbouring region to relax as well. The second assumption is that slowing down should be due to a decreasing density of these local facilitation regions: mobility is sparse in a low temperature (dense) liquid. Assuming that any local relaxation events needs to be triggered by a nearby relaxation implies that mobility is essentially conserved and propagates through the system. These are strong assumptions which lack a clearcut experimental validation. Indeed, experimental tests are difficult in the absence of a systematic coarse-graining procedure to represent a molecular model with continuous degrees of freedom in terms of mobility variables. However, an undiscussed success of the DF scenario is that it has lead to the introduction of different models which display several key properties of glassy dynamics. In turn, these models have provided a deeper understanding of the dynamical heterogeneities that occur in supercooled liquids: fast and slow regions coexist and their typical size increases while decreasing temperature.

The first class of models that has been studied in the context of the DF scenario are the so called Kinetically Constrained Models (KCM), which feature trivial statics and constrained dynamics. KCM include for example the East and Friedrickson-Andersen model (see [11], [7] and [3] for some references to the physical and mathematical literature respectively). Two crucial difficulties in justifying KCM as models for the

liquid glass transition are the following: (i) it is not clear how kinetic constraints can truly emerge from the unconstrained dynamics of a many-body system; (ii) since KCM have a trivial thermodynamics they cannot account for the eventual growth of a static amorphous order. To cope with both problems, a second class of models has been introduced, the so called *plaquette models* [9, 19].

In this paper we focus on the two dimensional plaquette models: the *square plaquette model* (SPM) [9] on $\mathbb{Z}^2$ and the *triangular plaquette model* (TPM) [19] on the triangular lattice $\mathcal{T}$. Both SPM and TPM are systems of ± 1 spins with short range non disordered ferromagnetic interactions. In order to define the Hamiltonian and therefore the Gibbs measure, we need first to define the plaquettes and the plaquette variables. For the SPM the plaquettes are all the unit squares in $\mathbb{Z}^2$ while for the TPM they are all the downward-pointing unit side triangles in $\mathcal{T}$. For a given spin configuration $\sigma \in \{\pm 1\}^{\mathbb{Z}^2}$ or $\sigma \in \{\pm 1\}^{\mathcal{T}}$ and a given plaquette $\mathcal{P}$, the corresponding *plaquette variable* is defined as $\prod_{x \in \mathcal{P}} \sigma_x$. Then, the Hamiltonian is defined as $-1/2$ times the sum of all plaquette variables. Despite the non interacting form of the Hamiltonian in terms of the plaquette variables, the thermodynamics of the spin variables is non trivial. Indeed, the correspondence among the spin and plaquette variables is not one-to-one, and the spin ground state is highly degenerate (see [9] for an informal description of the involved symmetries). In particular, both models feature diverging static correlation lengths as the temperature tends to zero [1, 10].

The study of the dynamics of SPM and TPM under the natural single spin flip Monte Carlo or Metropolis dynamics has been the focus of several works in physics literature. Numerical simulations clearly indicate the occurrence of glassy dynamics at low temperature [1, 9, 10, 19]. In this regime the dynamics in terms of plaquette variables is usually described via an effective dynamics of free defects subject to kinetic constraints. Indeed, flipping a single spin changes the value of all the plaquette variables containing the corresponding site. Thus plaquettes with variable equal to -1, the so called defects, are stable when they are isolated. Furthermore, relaxation is dominated by flips occurring in the vicinity of defects. This shows that effective kinetic constraints can naturally emerge from many body interactions. Concerning the critical scaling of time scales, heuristic analysis and numerical simulations clearly indicate an Arrhenius scaling for SPM and a super Arrhenius scaling for TPM [1, 19]. This difference is due to the nature of the energy barriers that should be overcome to bring isolated defects together and annihilate them.

The main focus of our paper is a rigorous study of the thermodynamic properties of SPM and TPM. Our first result (cf. Theorem 1) proves that, for any positive temperature, both models have a unique infinite volume Gibbs measure with a strong form of spatial mixing. For SPM the uniqueness of the Gibbs measure was known [8, Theorem 3.2] but not the strong spatial mixing. Notice that for the SPM transfer matrix techniques allow for the exact calculation of the free energy at zero external field [6, 18]. The techniques we employ, partially adapted from [8], are robust and they can also be applied to cases in which it is not feasible to exactly evaluate the free energy.

We then analyse the scaling of three natural length scales in the small temperature regime. The largest scale, $\ell_c^{(mix)}$, measures the critical scale at which correlations decay

in the bulk and also close to the boundary, uniformly in the boundary condition. We prove that $\ell_c^{(mix)}$ scales as e^β modulo polynomial corrections in β (Theorem 1 and Remark 3.10).

The second scale, $\ell_c^{(cavity)}$, measures the minimal distance at which boundary conditions do not significantly affect the average of local observables located far from the boundary. For SPM we show that $\ell_c^{(cavity)}$ scales as e^β modulo polynomial corrections in β (Theorem 2). For TPM we find $e^{\frac{\ln 2}{\ln 3}\beta} \leqslant \ell_c^{(cavity)} \leqslant \beta^2 e^\beta$. Different types of cavity or point-to-set correlation lengths have been defined and measured in plaquette models and other glassy systems. These are highly relevant in connection with the key problem of measuring subtle correlations due to the growing of an amorphous order in the glass state (references in [2]).

Remark 1.1. *The scale $\ell_c^{(cavity)}$ does not correspond to the cavity correlation length that can be measured via static overlap functions but rather to the a priori larger length beyond which the cavity behaves as the bulk. In fact, for the SPM case, $\ell_c^{(cavity)}$ scales as the second crossover length in [10]. In turn, this length is expected to diverge as the dynamical length that can be extracted from the four point correlator (see also [1]) which is relevant for the study of dynamical heterogeneities. For the TPM it has been conjectured from numerical simulations that there exists a unique static and dynamic correlation length scaling with $e^{\frac{\ln 2}{\ln 3}\beta}$ [10]. This would suggest that our lower bound for $\ell_c^{(cavity)}$ is the correct bound. However, since we are taking the supremum over all boundary conditions in the definition of $\ell_c^{(cavity)}$ a larger scaling could occur. Notice also that our results prove that there is another critical length, $\ell_c^{(mix)}$, with a faster divergence.*

The third and smallest scale, $\ell_c^{(multispin)}$, is the correlation length for the product of spin variables in the infinite volume Gibbs measure. We find (cf. Theorem 2) that $\ell_c^{(multispin)} \sim e^{\beta/2}$ for the SPM and $\ell_c^{(multispin)} \sim e^{\frac{\ln 2}{\ln 3}\beta}$ for the TPM. Both scalings were previously derived in physical literature (see [1, 9]) for special sets of spins, namely located at the vertices of an equilateral triangle for the TPM and of a rectangle for the SPM.

Finally, using the symmetries of the model we determine an exact self similarity property for infinite volume Gibbs measure (cf. Theorem 3).

2. NOTATION AND MODELS

2.1. **Notation.** For an integer n we will write $[n]$ for the set $\{1, \ldots, n\}$ and for any finite set $\Lambda \subset \mathbb{Z}^d$, we will write $|\Lambda|$ for its cardinality. The ℓ_1-distance in $\mathbb{Z}^d$ will be denoted by $d(\cdot, \cdot)$ and the standard basis vectors by $\vec{e}_1, \ldots, \vec{e}_d$. Sometime we will write x_1, x_2 for the coordinates of a vertex $x \in \mathbb{Z}^2$. Given $\Lambda \subset \mathbb{Z}^d$ the basic configuration space will be $\Omega_\Lambda := \{-1, 1\}^\Lambda$. When $\Lambda = \mathbb{Z}^d$ we drop the corresponding suffix from our notation. Given $\sigma \in \Omega$ we let σ_x be its value at the site x, in the sequel the *spin at x*, and for any finite V, we let $\sigma_V = (\sigma_x)_{x \in V}$ and $[\sigma]_V := \prod_{x \in V} \sigma_x$. Given two disjoint subsets V, V' of $\mathbb{Z}^d$, and two configurations $\sigma \in \Omega_V$ and $\sigma' \in \Omega_{V'}$ we denote by $\sigma \otimes \sigma'$ the configuration in $\Omega_{V \cup V'}$ whose restrictions to V and V' coincide with σ and σ' respectively. For any $\sigma \in \Omega$ and vertex $x \in \mathbb{Z}^2$, σ^x will denote the configuration obtained from σ by flipping

the spin at x. Given a function $f : \Omega \mapsto \mathbb{R}$ the smallest set V such that f does not depend on σ_{V^c} will be called the support of f and it will be denoted by S_f. A function f is said to be *local* if its support is finite.

Finally we recall that $f(x) = O(g(x))$ as $x \to +\infty$ means that $|f(x)| \le Cg(x)$ for some constant C and any x large enough and that $f = \Omega(g)$ if $g = O(f)$. We will also write $f \asymp g$ if $f = O(g)$ and $g = O(f)$.

2.2. **Finite volume Gibbs measures.**

In order to define the finite volume Gibbs measure of our models with spin boundary conditions $\tau \in \Omega$, it will be convenient to first introduce the notation Ω_Λ^τ to denote those configurations $\sigma \in \Omega$ such that $\sigma_{\Lambda^c} = \tau_{\Lambda^c}$, where $\tau \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$. Let B_* be a finite subset of $\mathbb{Z}^d$, in the sequel referred to as the *fundamental plaquette*, and let $\mathcal{B} := \{B_* + x : x \in \mathbb{Z}^d\}$. Elements of the collection $\mathcal{B}$ will be called *plaquettes*. Given $\Lambda \subset \mathbb{Z}^d$ we also let $\mathcal{B}(\Lambda) = \{B \in \mathcal{B} : B \cap \Lambda \ne \emptyset\}$. Finally, given a finite set Λ and $\tau \in \Omega$, we define the Gibbs measure in Λ with boundary conditions τ and fundamental plaquette B_* as the positive probability measure $\mu_\Lambda^{\beta,\tau}$ on Ω_Λ^τ given by

$$\mu_\Lambda^{\beta,\tau}(\sigma) := \frac{e^{\frac{\beta}{2}\sum_{B \in \mathcal{B}(\Lambda)}[\sigma]_B}}{Z_\Lambda^{\beta,\tau}}, \qquad \sigma \in \Omega_\Lambda^\tau,$$

where $\beta > 0$ is the inverse temperature and the normalisation constant (partition function) $Z_\Lambda^{\beta,\tau}$ takes the form

$$Z_\Lambda^{\beta,\tau} = \sum_{\sigma \in \Omega_\Lambda^\tau} e^{\frac{\beta}{2}\sum_{B \in \mathcal{B}(\Lambda)}[\sigma]_B} . \tag{2.1}$$

In the sequel we will denote by $\mathrm{Cov}_\Lambda^{\beta,\tau}(f,g)$ the covariance w.r.t. $\mu_\Lambda^{\beta,\tau}$ of two functions $f, g : \Omega_\Lambda^\tau \mapsto \mathbb{R}$.

Using the fact that the fundamental plaquette B_* is finite, $\mu_\Lambda^{\beta,\tau}$ is the finite volume Gibbs measure in Λ with boundary conditions τ for the finite range, ± 1 spin model on $\mathbb{Z}^d$ with (formal) Hamiltonian $H(\sigma) = -\frac{1}{2}\sum_{B \in \mathcal{B}}[\sigma]_B$. As usual for spin systems we set $\|H\| = |B_*|/2$.

2.2.1. *Specific models.* In this paper we will concentrate on two models in two dimensions: the square plaquette model (SPM in the sequel), with fundamental plaquette $B_* \subset \mathbb{Z}^2$ equal to the unit square $\{0,1\}^2$ and the *triangular plaquette model* (TPM) with B_* equal to the right triangle with vertices the origin, $\vec{e}_2$ and $\vec{e}_1 + \vec{e}_2$ (see Figure 1). In the language of [8] both models are *trivial*, *i.e.* each $B \in \mathcal{B}$ is the translate of exactly one fundamental plaquette B_*. The SPM is also *factorizable* because B_* is a rectangle, *i.e.* of the form $B_* = B_*^{(1)} \times B_*^{(2)}$ with $B_*^{(i)} \subset \mathbb{Z}$.

Remark 2.1. *Usually the TPM is defined on the triangular lattice $\mathcal{T}$ (cf. e.g. [19]) in which the fundamental plaquette is any downward–pointing unit triangle. Using the bijection Φ between $\mathcal{T}$ and $\mathbb{Z}^2$ given by $\mathcal{T} \ni x = x_1\vec{a}_1 + x_2\vec{a}_2 \mapsto (x_1, x_2) \in \mathbb{Z}^2$, where $\vec{a}_1 = \vec{e}_1$ and $\vec{a}_2 = \cos(\frac{2}{3}\pi)\vec{e}_1 + \sin(\frac{2}{3}\pi)\vec{e}_2$, we see immediately that any unit downward–pointing triangle of $\mathcal{T}$ is transformed into a plaquette $B_* + x$, $x \in \mathbb{Z}^2$, where B_* is the*
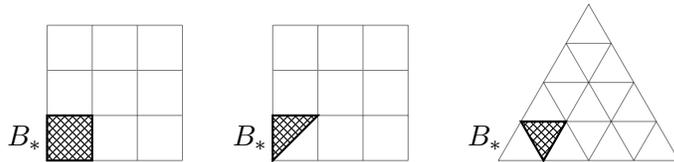
FIGURE 1. The fundamental plaquette for the SPM (left) and for the TPM in $\mathbb{Z}^2$ (center) and in the triangular lattice $\mathcal{T}$ (right).

right triangle above. In the sequel and if no confusion arises, we will work indifferently either in $\mathbb{Z}^2$ or in $\mathcal{T}$ according to the geometric convenience.

3. MAIN RESULTS

Our results are essentially twofold. The first set of results analyses the spatial mixing properties of the two models (decay of correlations, influence of boundary conditions and uniqueness of the infinite volume Gibbs measure). The second set of results discusses a self-similarity property of the unique infinite volume Gibbs measure.

We first prove that for all temperatures the SPM and the TPM satisfy the so called *strong mixing property on regular volumes* (we refer the reader to [12–14] for a thorough analysis of this property and for a critical discussion of its relation with the notion of complete analyticity of Dobrushin-Shlosman [4, 5], and to [16] for a simple and direct approach to prove it). As is well known the above property implies the existence of a unique infinite volume Gibbs measure $\mu^\beta = \lim_{\Lambda \uparrow \mathbb{Z}^2} \mu_\Lambda^{\beta,\tau}$ with exponentially decaying correlations and finite logarithmic Sobolev constant [13]. Already at this level a first critical scale $\ell_c^{(\text{mix})}$ appears, measuring the minimal distance ℓ at which two functions f, g with supports at distance ℓ have a small covariance in a box Λ containing their supports, uniformly in the boundary conditions τ.

Secondly at low temperature we analyse the dependence on β of two other natural critical length scales related to the general concept of spatial mixing. The first scale, $\ell_c^{(\text{cavity})}$, measures the minimal distance at which boundary conditions do not significantly affect the average of local observables located far from the boundary. The second scale, $\ell_c^{(\text{multispin})}$, is the minimal scale ℓ such that the infinite volume average of multispins separated one from each other by a distance at least ℓ becomes small. As we discuss in the appendix these three length scales are ordered:

$$\ell_c^{(\text{multispin})} = O(\ell_c^{(\text{cavity})}) = O(\beta \, \ell_c^{(\text{mix})}). \tag{3.1}$$

We will now define precisely the above scales and state the main results concerning their asymptotics for large β.

3.1. **Spatial Mixing Results.** We begin by recalling the notion of strong mixing on regular volumes.

Definition 3.1. *Given a positive integer ℓ, a finite set $\Lambda \subset \mathbb{Z}^2$ is said to be ℓ-regular if it is the disjoint union of squares of the form $Q_\ell + \ell x$, $x \in \mathbb{Z}^2$, where $Q_\ell = [\ell]^2$.*

Definition 3.2. *We say that strong mixing on ℓ-regular sets with positive constants C, m holds, in the sequel denoted $SM(\ell, C, m)$, if for all ℓ-regular sets Λ, all boundary conditions $\tau \in \Omega$ and all functions $f, g : \Omega_\Lambda^\tau \mapsto \mathbb{R}$ we have*

$$|\operatorname{Cov}_\Lambda^{\beta,\tau}(f, g)| \leq C\|f\|_\infty \|g\|_\infty |S_f||S_g|e^{-m\,d(S_f, S_g)}. \tag{3.2}$$

Remark 3.3. *The key point in the definition of strong mixing is the arbitrariness of the location of the functions f, g. Their supports could in fact be far from each other but close to the boundary of Λ. Thus strong mixing requires the exponential decay of correlations not only in the bulk but also close to the boundary. It is known [15] that in two dimensions bulk exponential decay (technically refereed to as* weak mixing*) implies strong mixing on regular sets, essentially because the boundary of a square is one dimensional and information cannot propagate in a one dimensional space. In higher dimensions there are example of models [21] with bulk exponential decay but no strong mixing property because of the occurrence of boundary phase transitions.*

As in the original work of Dobrushin-Shlosman [4, 5] in order to prove $SM(\ell, C, m)$ it is enough to verify that a certain *finite volume condition* on boxes of side ℓ holds (see again [12, 14, 16]). Following e.g. [16, Proposition 3.1 and Theorem 3.3], we have in fact the following result:

Proposition 3.4 ([16])**.** *Given x let*

$$h_x(\sigma) = \exp\Big(\frac{\beta}{2} \sum_{\substack{B \in \mathcal{B}\,:\\ B \ni x}} ([\sigma^x]_B - [\sigma]_B)\Big).$$

Let also

$$\varphi(\ell) = \sup_{\substack{x, y \notin Q_\ell \\ d(x,y) \geq \ell/4}} \sup_\tau |\operatorname{Cov}_{Q_\ell}^{\beta,\tau}(h_x, h_y)|.$$

Then there exist positive constants $\varepsilon_0, c_0 > 0$ independent of β such that the inequality

$$e^{4\beta\|H\|}\ell\,\varphi(\ell) \leq \varepsilon_0 \tag{3.3}$$

implies $SM(\ell, C, m)$ with $C = e^{c_0(\beta+1)}$ and $m = 1/\ell$.

Remark 3.5. *The special role played by the functions $\{h_x\}_{x \in \mathbb{Z}^2}$ in checking strong mixing is due to the following basic identity*

$$\mu_\Lambda^{\beta,\tau^x}(f) - \mu_\Lambda^{\beta,\tau}(f) = \frac{\operatorname{Cov}_\Lambda^{\beta,\tau}(h_x, f)}{\mu_\Lambda^{\beta,\tau}(h_x)} \quad \forall x \notin \Lambda. \tag{3.4}$$

Thus, if f has a small covariance with h_x, its expectation is not sensitive to a change in the boundary conditions at x.

Notice that in order to verify (3.3) we need some decay (at least $1/\ell$) of the function $\varphi(\ell)$ for ℓ large enough. The smallest scale at which the required decay takes place will be our first critical scale.

Definition 3.6. *We define the* strong mixing scale $\ell_c^{(mix)}$ *as the smallest integer ℓ such that* (3.3) *holds.*

Our first theorem then says that strong mixing on regular volumes holds for both SPM and TPM.

Theorem 1. *For both SPM and TPM $\ell_c^{(mix)} = O(\beta e^\beta)$ as $\beta \to \infty$. In particular both models have a unique infinite volume Gibbs measure μ^β with exponential decay of correlations and zero magnetisation $\mu^\beta(\sigma_0) = 0$.*

Remark 3.7. *It is easily seen that the SPM and TPM do not satisfy the Dobrushin-Shlosman stronger form of decay of correlations [5] at low temperatures, i.e. (3.2) for all finite sets Λ and not just for the ℓ-regular ones. Partition $\mathbb{Z}^2$ into the odd and even sub-latticesand consider the SPM on the even sub-lattice with plus boundary conditions on the odd one. The resulting system is clearly the standard ferromagnetic Ising model on $\mathbb{Z}^2$ for which a phase transition occurs at low temperature. Hence it is not possible to have (3.2) at low temperature for all subsets Λ of the even sub-lattice with plus boundary conditions on the odd one.*

For the TPM the role played by the odd sub-lattice for the SPM is played by the image under the mapping $\mathcal{T} \overset{\Phi}{\mapsto} \mathbb{Z}^2$ described in Remark 2.1 of the subset $\mathcal{O} = \mathcal{T} \setminus \mathcal{H}$, where $\mathcal{H}$ is the hexagonal tiling of the plane with each hexagon formed by the union of six triangles of $\mathcal{T}$ (cf. Figure 2). It is immediate to check that, if we fix all the spins at the vertices of $\mathcal{O}$
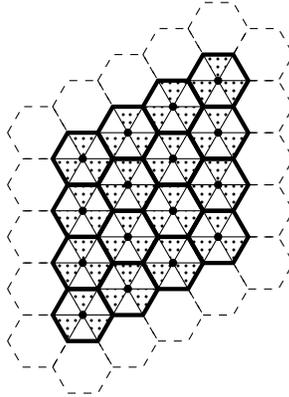

FIGURE 2. A piece of the triangular lattice $\mathcal{T}$ with the interaction plaquettes (dotted triangles). The black dots are the vertices of the set $\mathcal{O} \subset \mathcal{T}$ where the spins are fixed equal to $+1$. The remaining vertices form an hexagonal lattice (thick bonds). Clearly the interaction among the survival spins is a standard two-body ferromagnetic Ising interaction.

equal to $+1$, the remaining spins form a standard Ising model on the hexagonal grid and the same conclusion valid for the SPM holds.

In the previous section we encountered a first possible critical scale $\ell_c^{(mix)}$ as the smallest scale at which strong mixing takes place. Here we are going to define two other possible critical scales and we will prove results about their scaling behaviour in

β as $\beta \to \infty$. The first new scale, dubbed $\ell_c^{(\text{cavity})}$ is the scale at which the influence of the boundary conditions on bulk variables (*i.e.* whose support is far from the boundary) is small[1]. The second scale $\ell_c^{(\text{multispin})}$ concerns the smallness of the infinite volume average of the product of finitely many spins. In what follows the choice of small numbers $1/10, 1/5$ is somewhat arbitrary (one can replace them by $u, 2u$ with u small).

Definition 3.8. *Fix two concentric squares $V \subset \Lambda$ of side ℓ and 10ℓ respectively. Given $\tau, \tau' \in \Omega$, let $\psi(\ell; \tau, \tau')$ be the total variation distance[2] between the marginals on Ω_V of the measures $\mu_\Lambda^\tau, \mu_\Lambda^{\tau'}$ and let $\psi(\ell) := \sup_{\tau,\tau'} \psi(\ell; \tau, \tau')$. Then we set*

$$\ell_c^{(\text{cavity})} = \min\{\ell : \ \psi(\ell') \leq 1/10 \ \ \forall \ell' \geqslant \ell\}.$$

To define the second new scale, let $\mathbb{F}_\ell$ be the class of finite non-empty subsets $A \subset \mathbb{Z}^2$ such that $d(x,y) \geq \ell$ for all $x, y \in A$. Then we set

$$\ell_c^{(\text{multispin})} = \min\{\ell : \ \sup_{A \in \mathbb{F}_\ell} |\mu^\beta([\sigma]_A)| \leq 1/5\}.$$

Remark 3.9. *By the very definition of the variation distance (cf. [20, Proposition 4.5]), for $V \subset \Lambda$ as in Definition 3.8 and for any function $f : \Omega_\Lambda \mapsto \mathbb{R}$ with $\|f\|_\infty \leq 1$ and $S_f \subseteq V$, one has*

$$\max_{\tau,\tau'} |\mu_\Lambda^{\beta,\tau}(f) - \mu_\Lambda^{\beta,\tau'}(f)| \leq \max_{\tau,\tau'} 2\psi(\ell; \tau, \tau') = 2\psi(\ell).$$

Therefore any *observable in V becomes insensitive to the boundary conditions beyond the scale $\ell_c^{(\text{cavity})}$. It is a natural question whether there are special observables for which this phenomenon occurs on a shorter scale. As discussed in Section 5.0.1 a very natural candidate is the spin at the center of the box. In this case a naive analysis suggests that, under the plus b.c. and for large β, the average spin magnetisation becomes very small as soon as the side of the box becomes greater than $\left(\ell_c^{(\text{cavity})}\right)^{1/2}$. In Proposition 5.2 we prove that for the SPM this actually does not happen and that the correct scale is still $\ell_c^{(\text{cavity})}$. The same question remains open for the TPM.*

Theorem 2. *As $\beta \to \infty$ the following scaling holds:*

(SPM) $\ell_c^{(\text{cavity})} = O(\beta \ell_c^{(\text{mix})}), \qquad \ell_c^{(\text{cavity})} = \Omega(e^\beta), \qquad \ell_c^{(\text{multispin})} \asymp e^{\frac{\beta}{2}}.$

(TPM) $\ell_c^{(\text{cavity})} = O(\beta \ell_c^{(\text{mix})}), \qquad \ell_c^{(\text{cavity})} = \Omega(e^{\frac{\ln 2}{\ln 3}\beta}), \qquad \ell_c^{(\text{multispin})} \asymp e^{\frac{\ln 2}{\ln 3}\beta} .$

In particular, for both models $\ell_c^{(\text{cavity})} = O(\beta e^\beta)$.

Remark 3.10. *We can use the above result to derive a* lower bound *on $\ell_c^{(\text{mix})}$. We in fact get $\ell_c^{(\text{mix})} = \Omega(\beta^{-1}e^\beta)$ for the SPM. For the TPM we only get $\ell_c^{(\text{mix})} = \Omega(\beta^{-1}e^{\frac{\ln 2}{\ln 3}\beta})$. However one can prove that the magnetisation of the vertex x at the middle of one of the sides of the square of side length δe^β is very sensitive to the choice of the boundary conditions on the other three sides as long as δ is a small constant independent of β. By using the symmetries defined in Section 6.2, one can in fact construct a boundary condition τ such*

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[1]For the experts this is essentially the first scale at which *weak mixing* [12] starts to kick in.

[2]Given two probability measures μ, ν on a finite probability space Ω their total variation distance is $\max_{A \subset \Omega} |\mu(A) - \nu(A)|$

that τ is identically equal to $+1$ within distance $\delta e^\beta/2$ from x and $|m_x^\tau - m_x^+| = 2m_x^+$, where m_x^τ, m_x^+ denote the magnetisation at x with boundary conditions τ and all plus respectively. In turn $m_x^+ = \Omega(1)$ as $\beta \to \infty$ using Lemma 5.1. By proceeding as in Section 1.1 we easily conclude that $\ell_c^{(mix)} = \Omega(\beta^{-1}e^\beta)$. We conclude that $\ell_c^{(mix)}$ scales like e^β, up to polynomial corrections, for both the SPM and TPM.

Remark 3.11. *As the reader can check, our derivation of Theorem 1, as well as our upper bounds on $\ell_c^{(cavity)}$ and $\ell_c^{(multispin)}$ stated in Theorem 2, remain valid when working with generic Hamiltonians of the form $H(\sigma) = -\sum_{B \in \mathcal{B}} J(B)[\sigma]_B$, where $J(B)$ are arbitrary coupling constants with $\sup\{|J(B)| : B \in \mathcal{B}\} < \infty$.*

3.2. **Self-similarity of the infinite volume Gibbs measure.** In this section we establish an exact self–similarity property of the infinite volume Gibbs measure μ^β.

Let $\varphi(q,k) := \frac{1}{2} - \frac{1}{2}(1-2q)^k$ and $q(\beta) := \mu^\beta([\sigma]_{B_*} = -1)$. Clearly $q(\beta) = e^{-\beta/2}/(e^{-\beta/2} + e^{\beta/2})$.

Theorem 3.

(a) *For the SPM the marginal of μ^β on $\Omega_{\ell\mathbb{Z}^2}$, $\ell \in \mathbb{N}$, coincides with $\mu^{\beta'}$, where $\beta' = \beta'(\beta, \ell)$ is such that $q(\beta') = \varphi(q(\beta), \ell^2)$.*

(b) *For the TPM the marginal of μ^β on $\Omega_{\ell\mathbb{Z}^2}$, for $\ell = 2^n$ and $n \in \mathbb{N}$, coincides with $\mu^{\beta'}$, where β' is such that $q(\beta') = \varphi(q(\beta), \ell^\alpha)$ with $\alpha = \frac{\log 3}{\log 2}$.*

By explicit computations we have

$$\beta' = \log\left(\frac{1 + (1 - 2q(\beta))^k}{1 - (1 - 2q(\beta))^k}\right) \quad \text{where} \quad k = \begin{cases} \ell^2 & \text{in the SPM}, \\ \ell^\alpha & \text{in the TPM}. \end{cases} \tag{3.5}$$

3.3. **Extensions to trivial factorizable models.** We briefly discuss the extension of Theorems 1 and 2 to an arbitrary *factorizable trivial model* [8, 23] (FTM in the sequel) in $\mathbb{Z}^d$. These spin systems have a formal Hamiltonian like the one introduced in Section 2.2, namely $H(\sigma) = -\frac{1}{2}\sum_{B \in \mathcal{B}}[\sigma]_B$, where now a generic plaquette $B \in \mathcal{B}$ has the form $B = B_* + x$, with $B_* = B_*^{(1)} \times \cdots \times B_*^{(d)}$ and $B_*^{(i)} \subset \mathbb{Z}$. W.l.o.g. we can assume that $|B_*^{(i)}| > 1$ (otherwise the system breaks into lower dimensional independent subsystems) and that $\min\{k \in \mathbb{Z} : k \in B_*^{(i)}\} = 0$.

Like the SPM also a FTM has a unique Gibbs measure μ^β at all positive temperatures and it satisfies the Dobrushin-Shlosman uniqueness criterium [8]. In what follows we will work with the same length scales $\ell_c^{(mix)}, \ell_c^{(cavity)}$ and $\ell_c^{(multispin)}$ defined exactly as before and ordered as in (3.1) (cf. the proof given in the appendix).

Theorem 4. *Choose a FTM. Then as $\beta \to \infty$ we have:*

(i) $\ell_c^{(mix)} = O(\beta e^\beta)$;
(ii) $\ell_c^{(cavity)} = O(\beta \ell_c^{(mix)})$ *and* $\ell_c^{(cavity)} = \Omega(e^\beta)$;
(iii) $\ell_c^{(multispin)} = O(e^{\beta/d})$.

3.4. **Summary.** In conclusion we observe that the SPM and TPM have (at least) three a priori different critical length scales. On the first scale covariances in regular volumes start to decay (exponentially fast) uniformly in the boundary conditions *and* in the location of the observables. On the second scale the boundary conditions do not significantly affect the expectation of bulk observables while on the third scale the product of finitely many spins have small expectation w.r.t. to the unique infinite volume Gibbs measure.

The first scale has a scaling roughly e^{β} with a possible poly(β) pre-factor in both models. The second scale behaves similarly to the first one in the SPM, whereas for TPM we can only say that it is in between the first and third scale. Finally the third scale is deeply affected by the underlying geometry of the models and has a scaling $e^{\beta/2}$ for the SPM and $e^{\frac{\ln 2}{\ln 3}\beta}$ for the TPM. For both models this third length scale takes the form e^{β/d_f}, where d_f is the Hausdorff dimension of certain sets which are naturally associated to the fundamental plaquette B_*. These sets can be constructed by taking any collection of plaquettes in $\mathcal{B}$ such that the origin appears in exactly one plaquette and all other vertices of $\mathbb{Z}^2$ are include either twice or not at all, and then rescaling space. For the SPM this set is just a quadrant in $\mathbb{R}^2$, while for the TPM it is given by the Sierpinski gasket.

One could define a forth length scale as the smallest ℓ such that β' appearing in Theorem 3 is small, *i.e.* $\beta' \leqslant 1$. It turns out, by a simple computation from (3.5), that as expected, this length scales exactly as the multispin length scale.

4. Proof of Theorem 1

The proof of Theorem 1 starts with a simple general observation. Recall the definition of h_x given in Proposition 3.4 and let S_{h_x} be its support. Fix $\Lambda \subset \mathbb{Z}^2$ and $\varepsilon \in (0,1)$. Also fix two sites $x, y \notin \Lambda$ such that the $S_x^{\Lambda} := S_{h_x} \cap \Lambda$ and $S_y^{\Lambda} := S_{h_y} \cap \Lambda$ are both not empty.

Definition 4.1. *We say that $\mathcal{S} \subset \Lambda$ is a ε-screening set for x, y if the following holds:*

(i) *$\Lambda \setminus \mathcal{S}$ can be written as the disjoint union of two sets V_x, V_y such that $S_x^{\Lambda} \subset V_x$ and $S_y^{\Lambda} \subset V_y$;*

(ii) *there exists no plaquette $B \in \mathcal{B}(\Lambda)$ such that $B \cap V_x \neq \emptyset$ and $B \cap V_y \neq \emptyset$;*

(iii) *the partition functions $Z_{\mathcal{S}}^{\beta,\bullet}$ satisfies the bound:*

$$\sup_{\tau,\tau'} \frac{Z_{\mathcal{S}}^{\beta,\tau}}{Z_{\mathcal{S}}^{\beta,\tau'}} \leq 1 + \varepsilon.$$

The usefulness of the above definition appears in the next lemma:

Lemma 4.2. *In the above setting suppose that there exists an ε-screening set for x, y. Then*

$$\sup_{\tau} |\operatorname{Cov}_{\Lambda}^{\beta,\tau}(h_x, h_y)| \leq 4\|h_x\|_{\infty}^2 \, \varepsilon \leq 4e^{4\beta\|H\|} \, \varepsilon.$$

Proof. Let $\mathcal{S}$ be a ε-screening set and let V_x, V_y be the corresponding sets appearing in Definition 4.1 above. Using the DLR equations we begin with the simple bound:

$$|\operatorname{Cov}_{\Lambda}^{\beta,\tau}(h_x, h_y)| = |\mu_{\Lambda}^{\beta,\tau}((h_y - \mu_{\Lambda}^{\beta,\tau}(h_y))\mu_{\Lambda}^{\beta,\tau}(h_x \mid \sigma_{V_y}))|$$

$$\leq 2\|h_y\|_{\infty} \sup_{\tau,\xi,\xi'} |\mu_{\Lambda}^{\beta,\tau}(h_x \mid \sigma_{V_y} = \xi) - \mu_{\Lambda}^{\beta,\tau}(h_x \mid \sigma_{V_y} = \xi')|.$$

Next, given $\xi \in \Omega_{V_y}$ and using (ii) of Definition 4.1, we write

$$\mu_{\Lambda}^{\beta,\tau}\left(h_x \mid \sigma_{V_y} = \xi\right) = \frac{\sum_{\sigma \in \Omega_{V_x}} e^{-\frac{\beta}{2}\hat{H}_{V_x}(\sigma)} h_x(\sigma) Z_{\mathcal{S}}^{\beta,\tau \otimes \sigma \otimes \xi}}{\sum_{\sigma' \in \Omega_{V_x}} e^{-\frac{\beta}{2}\hat{H}_{V_x}(\sigma')} Z_{\mathcal{S}}^{\beta,\tau \otimes \sigma' \otimes \xi}},$$

where

$$\hat{H}_{V_x}(\sigma) = - \sum_{\substack{B \in \mathcal{B}(V_x) \\ B \cap \mathcal{S} = \emptyset}} [\sigma \otimes \tau]_B,$$

and $\tau \otimes \sigma \otimes \xi$ denotes the configuration whose restriction to Λ^c, V_x and V_y is equal to τ, σ and ξ respectively. Using (iii) of Definition 4.1 we get immediately that

$$(1 + \varepsilon)^{-1} \leq \frac{\mu_{\Lambda}^{\beta,\tau}\left(h_x \mid \sigma_{V_y} = \xi\right)}{\hat{\mu}_{V_x}^{\beta,\tau}(h_x)} \leq 1 + \varepsilon,$$

where $\hat{\mu}_{V_x}^{\beta,\tau}$ is the probability measure on Ω_{V_x} such that $\hat{\mu}_{V_x}^{\beta,\tau}(\sigma) \propto \exp(-\frac{\beta}{2}\hat{H}_{V_x}(\sigma))$. Since $\hat{\mu}_{V_x}^{\beta,\tau}(h_x)$ does not depend on ξ we can conclude that

$$|\mu_{\Lambda}^{\beta,\tau}(h_x \mid \sigma_{V_y} = \xi) - \mu_{\Lambda}^{\beta,\tau}(h_x \mid \sigma_{V_y} = \xi')| \leq 2\|h_x\|_{\infty} \varepsilon.$$

$\square$

For both models ε-screening sets will consist of the disjoint union of suitable translates of the n-dilated version of a special set T_*. For the SPM the set T_* will coincide with the fundamental plaquette B_*. For the TPM instead T_* will be the right triangle with vertices the origin, $\vec{e}_1$ and $\vec{e}_1 + \vec{e}_2$. Having that in mind, the next lemma becomes useful. Before stating it we need to modify a bit the notion of boundary conditions in order to also cover the case of (partial) free boundary conditions.

Given a set Λ consider a subset $\mathcal{P} \subset \mathcal{B}(\Lambda)$ with the property that $\mathcal{P} \ni B$ for any plaquette $B \subset \Lambda$. In other words $\mathcal{P}$ contains all the plaquettes inside Λ and possibly some of the "boundary" plaquettes which intersect Λ and Λ^c. Then we define

$$Z_{\Lambda}^{\tau,\mathcal{P}} = \sum_{\sigma \in \Omega_{\Lambda}^{\tau}} \exp\left(\frac{\beta}{2} \sum_{B \in \mathcal{P}} [\sigma]_B\right). \tag{4.1}$$

Lemma 4.3. *Let $\{\Lambda_i\}_{i=1}^k$ be a collection of disjoint subsets of $\mathbb{Z}^2$ and let $\Lambda = \cup_{i=1}^k \Lambda_i$. Fix $\varepsilon \in (0,1)$ and assume that*

$$\max_i \max_{\mathcal{P}} \sup_{\tau,\tau'} \frac{Z_{\Lambda_i}^{\tau,\mathcal{P}}}{Z_{\Lambda_i}^{\tau',\mathcal{P}}} \leq 1 + \varepsilon.$$

Then

$$\sup_{\tau,\tau'} \frac{Z_\Lambda^{\beta,\tau}}{Z_\Lambda^{\beta,\tau'}} \le (1+\varepsilon)^k.$$

Proof. For $i \in [k]$ let $\mathcal{P}_i = \mathcal{B}(\Lambda_i) \setminus \{B \in \mathcal{B}(\Lambda_i) : \exists j < i \text{ such that } B \in \mathcal{B}(\Lambda_j)\}$. Given $\sigma^{(i)} \in \Omega_{\Lambda_i}$, $i \in [k]$, let $\sigma^{(1)} \otimes \cdots \otimes \sigma^{(k)} \otimes \tau$ be the configuration equal to $\sigma^{(i)}$ in Λ_i, $i \in [k]$, and equal to τ outside Λ. With this notation

$$Z_\Lambda^{\beta,\tau} = \sum_{\sigma_k \in \Omega_{\Lambda_k}} \cdots \sum_{\sigma_1 \in \Omega_{\Lambda_1}} \prod_{i=1}^k \exp\Big(\frac{\beta}{2}\sum_{B \in \mathcal{P}_i} [\sigma^{(1)} \otimes \cdots \otimes \sigma^{(k)} \otimes \tau]_B\Big).$$

Notice that by construction, if $i \ge 2$ then $\sum_{B \in \mathcal{P}_i}[\sigma^{(1)} \otimes \cdots \otimes \sigma^{(k)} \otimes \tau]_B$ does not depend on $\sigma^{(1)} \ldots \sigma^{(i-1)}$. Thus

$$\frac{Z_\Lambda^{\beta,\tau}}{Z_\Lambda^{\beta,\tau'}} =$$

$$= \frac{\sum_{j=2}^k \sum_{\sigma^{(j)} \in \Omega_{\Lambda_j}} \left[\prod_{i=2}^k \exp\left(\frac{\beta}{2}\sum_{B \in \mathcal{P}_i}[\sigma^{(i)} \otimes \cdots \otimes \sigma^{(k)} \otimes \tau]_B\right)\right] Z_{\Lambda_1}^{\sigma^{(2)} \otimes \cdots \otimes \sigma^{(k)} \otimes \tau, \mathcal{P}_1}}{\sum_{j=2}^k \sum_{\sigma^{(j)} \in \Omega_{\Lambda_j}} \left[\prod_{i=2}^k \exp\left(\frac{\beta}{2}\sum_{B \in \mathcal{P}_i}[\sigma^{(i)} \otimes \cdots \otimes \sigma^{(k)} \otimes \tau']_B\right)\right] Z_{\Lambda_1}^{\sigma^{(2)} \otimes \cdots \otimes \sigma^{(k)} \otimes \tau', \mathcal{P}_1}}$$

$$\le \frac{\max_\tau Z_{\Lambda_1}^{\tau,\mathcal{P}_1}}{\min_\tau Z_{\Lambda_1}^{\tau,\mathcal{P}_1}} \times \frac{\sum_{j=2}^k \sum_{\sigma^{(j)} \in \Omega_{\Lambda_j}} \left[\prod_{i=2}^k \left(\frac{\beta}{2}\sum_{B \in \mathcal{P}_i}[\sigma^{(i)} \otimes \cdots \otimes \sigma^{(k)} \otimes \tau]_B\right)\right]}{\sum_{j=2}^k \sum_{\sigma^{(j)} \in \Omega_{\Lambda_j}} \left[\prod_{i=2}^k \exp\left(\frac{\beta}{2}\sum_{B \in \mathcal{P}_i}[\sigma^{(i)} \otimes \cdots \otimes \sigma^{(k)} \otimes \tau']_B\right)\right]}$$

$$\le (1+\varepsilon) \times \frac{\sum_{j=3}^k \sum_{\sigma^{(j)} \in \Omega_{\Lambda_j}} \left[\prod_{i=3}^k \exp\left(\frac{\beta}{2}\sum_{B \in \mathcal{P}_i}[\sigma^{(i)} \otimes \cdots \otimes \sigma^{(k)} \otimes \tau]_B\right)\right] Z_{\Lambda_2}^{\sigma^{(3)} \otimes \cdots \otimes \sigma^{(k)} \otimes \tau, \mathcal{P}_2}}{\sum_{j=3}^k \sum_{\sigma^{(j)} \in \Omega_{\Lambda_j}} \left[\prod_{i=3}^k \exp\left(\frac{\beta}{2}\sum_{B \in \mathcal{P}_i}[\sigma^{(i)} \otimes \cdots \otimes \sigma^{(k)} \otimes \tau']_B\right)\right] Z_{\Lambda_2}^{\sigma^{(3)} \otimes \cdots \otimes \sigma^{(k)} \otimes \tau', \mathcal{P}_2}}.$$

The proof is finished by iteration. $\qquad\square$

The next result provides good building blocks of ε-screening sets for large n.

Proposition 4.4. *Let $T_*^{(n)}$ be either the square $[n]^2$ for the SPM or the right triangle with vertices the origin, $n\vec{e}_1$ and $n(\vec{e}_1 + \vec{e}_2)$ for the TPM. Then*

$$\max_{\mathcal{P}} \sup_{\tau,\tau'} \frac{Z_{T_*^{(n)}}^{\tau,\mathcal{P}}}{Z_{T_*^{(n)}}^{\tau',\mathcal{P}}} \le 3\exp\Big(2(n+2)\tanh(\beta/2)^{\frac{n+1}{3}}\Big) - 2,$$

when the exponential in the r.h.s is bounded above by $4/3$.

The proof of the proposition is postponed to Section 4.0.1. We now conclude the proof of Theorem 1 by proving that for both models $\ell_c^{(\mathrm{mix})} = O(\beta e^\beta)$ for large β.

• *The SPM case.* Consider the square Q_ℓ and $x,y \in Q_\ell^c$ such that $S_{h_x} \cap Q_\ell \ne \emptyset$, $S_{h_y} \cap Q_\ell \ne \emptyset$ and $d(x,y) \ge \ell/4$. As can be easily checked, taking $\Lambda = Q_\ell$, conditions (i) and (ii) of Definition 4.1 are satisfied for suitable sets V_x, V_y when taking $\mathcal{S}$ equal to either $R_o + z$ or $R_v + z$ for a suitable $z \in \mathbb{Z}^2$, where $R_o = [\ell] \times [\ell/10]$ and $R_v = [\ell/10] \times [\ell]$, (w.l.o.g. we assume that $\ell/10 \in \mathbb{N}$). Clearly $R_o + z$ (the same for $R_v + z$) can be written

as the disjoint union of ten translates of the square $Q_{\ell/10}$ (see Figure 3). Therefore, if $\ell = c\lfloor \beta e^\beta \rfloor$ and using $\tanh(\beta/2) \sim \exp(-2e^{-\beta})$ for large β, we obtain from Lemma 4.3 and Proposition 4.4

$$\sup_{\tau,\tau'} \frac{Z_{R_o+z}^{\beta,\tau}}{Z_{R_o+z}^{\beta,\tau'}} \le (1 + \varepsilon(\beta))^{10},$$

with $\varepsilon(\beta) = O(\beta e^{\beta(1-\frac{2c}{30})})$ for $c > 15$. Hence using Lemma 4.2 and the same notation of Proposition 3.4 we get:

$$\lim_{\beta\to\infty} e^{4\beta\|H\|}\ell\varphi(\ell) \le \lim_{\beta\to\infty} C'\beta e^{17\beta}\left((1 + \varepsilon(\beta))^{10} - 1\right) = 0,$$

for c large enough, where we have used $\|H\| = 2$ for the SPM. In particular for the above choice of ℓ (3.3) holds for β large enough. $\qquad\square$

• *The TPM case.* We proceed exactly as for the SPM. Suppose w.l.o.g. that $R_o + z$ satisfies conditions (i) and (ii) of Definition 4.1 for the given x, y. Then we first divide each of the ten squares of side $\ell/10$ forming $R_o + z$ into two right triangles with oblique side along the $\pi/4$ direction and then take as ε-screening set the union of the lowest right triangles (see Figure 3). Using again Lemmas 4.2, 4.3 and Proposition 4.4 we conclude as in the SPM case.
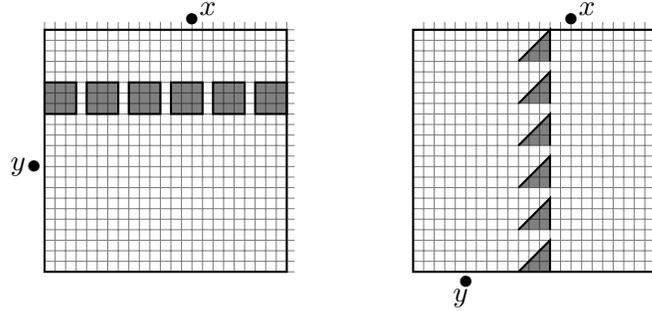

FIGURE 3. Examples of ε-screening sets for the SPM (left) and for the TPM (right)

4.0.1. *Proof of Proposition 4.4.* The starting point is a high temperature expansion of the partition function $Z_\Lambda^{\tau,\mathcal{P}}$ for any finite set Λ and $\mathcal{P} \subset \mathcal{B}(\Lambda)$. Our expansion is similar to but not exactly the same as the one in [8] (cf. Eq. (2.9) there).

Using

$$e^{(\beta/2)u} = \cosh(\beta/2)\left(1 + u\tanh(\beta/2)\right), \qquad u = \pm 1,$$

from (4.1) we get

$$Z_\Lambda^{\tau,\mathcal{P}} = \cosh(\beta/2)^{|\mathcal{P}|}\left(2^{|\Lambda|} + \sum_{\substack{\alpha\subset\mathcal{P}\\\alpha\neq\emptyset}} \tanh(\beta/2)^{|\alpha|} \sum_{\sigma\in\Omega_\Lambda^\tau} \prod_{B\in\alpha} [\sigma]_B\right). \tag{4.2}$$

Definition 4.5. *We say that $\alpha \subset \mathcal{B}(\Lambda)$ is a cycle in Λ (and write $\alpha \in \mathcal{K}(\Lambda)$) if for each $x \in \Lambda$ the cardinality of $\{B \in \alpha : x \in B\}$ is even. The empty set is a cycle.*

Remark 4.6. *Notice that since $\mathcal{B}(\Lambda)$ may contain plaquettes B such that $B \cap \Lambda^c \neq \emptyset$, there can be vertices $x \notin \Lambda$ with odd cardinality of $\{B \in \alpha : x \in B\}$.*

If $\alpha \subset \mathcal{P}$ is not a cycle, then there exists $x \in \Lambda$ such that the term $\prod_{B \in \alpha}[\sigma]_B$ can be written as $\sigma_x f(\sigma)$ for some function f with the property that $x \notin S_f$. Thus $\sum_{\sigma \in \Omega_\Lambda} \prod_{B \in \alpha}[\sigma]_B = 0$. As a consequence we can write $Z_\Lambda^{\tau,\mathcal{P}} = 2^{|\Lambda|} \cosh(\beta/2)^{|\mathcal{P}|} \mathcal{Z}_\Lambda^{\tau,\mathcal{P}}$, where

$$\mathcal{Z}_\Lambda^{\tau,\mathcal{P}} := 1 + 2^{-|\Lambda|} \sum_{\substack{\alpha \in \mathcal{K}(\Lambda) \\ \alpha \subset \mathcal{P},\, \alpha \neq \emptyset}} \tanh(\beta/2)^{|\alpha|} \sum_{\sigma \in \Omega_\Lambda^\tau} \prod_{B \in \alpha}[\sigma]_B . \tag{4.3}$$

Clearly

$$|\mathcal{Z}_\Lambda^{\tau,\mathcal{P}} - 1| \leqslant \sum_{\substack{\alpha \in \mathcal{K}(\Lambda) \\ \alpha \neq \emptyset}} |\tanh(\beta/2)|^{|\alpha|} .$$

In conclusion if $\sum_{\substack{\alpha \in \mathcal{K}(\Lambda) \\ \alpha \neq \emptyset}} |\tanh(\beta/2)|^{|\alpha|} \leqslant 1/3$ then

$$\max_{\mathcal{P}} \sup_{\tau,\tau'} \frac{\mathcal{Z}_\Lambda^{\tau,\mathcal{P}}}{\mathcal{Z}_\Lambda^{\tau',\mathcal{P}}} \leq 1 + 3 \sum_{\substack{\alpha \in \mathcal{K}(\Lambda) \\ \alpha \neq \emptyset}} |\tanh(\beta/2)|^{|\alpha|} .$$

We now take $\Lambda = T_*^{(n)}$. Then the statement of the proposition follows from the following key lemma.

Lemma 4.7. *For both models and for any $t \in (0,1)$ and $n \in \mathbb{N}$,*

$$\sum_{\substack{\alpha \in \mathcal{K}(T_*^{(n)}): \\ \alpha \neq \emptyset}} t^{|\alpha|} \leqslant \exp\{2(n+2)t^{\frac{n+1}{3}}\} - 1 . \tag{4.4}$$

Proof of Lemma 4.7.
• *The SPM case.* For any $0 \leqslant j \leqslant n$ call H_j the cycle $\alpha \in \mathcal{K}(T_*^{(n)})$ given by the horizontal stripe of plaquettes $(k,j) + B_*$, $0 \leqslant k \leqslant n$. Define similarly the vertical stripe of plaquettes $V_j = \{(j,k) + B_* : 0 \leqslant k \leqslant n\}$. [8, Proposition 4.4] implies that any cycle $\alpha \in \mathcal{K}(T_*^{(n)})$ can be written as $\alpha = \alpha(1)\Delta\alpha(2)$, where Δ is the symmetric differerence, $\alpha(1)$ is a union of cycles of the form H_j, $\alpha(2)$ is a union of cycles of the form V_j. For each $\alpha \in \mathcal{K}(T_*^{(n)})$ we fix once and for all a decomposition $\alpha = \alpha(1)\Delta\alpha(2)$ as above, with $\alpha(1)$ having minimal cardinality. These special decompositions are called *economic* in [8], where it is proved that they satisfy the bound $|\alpha| \geqslant \frac{|\alpha(1)|}{3} + \frac{|\alpha(2)|}{3}$ (see formula (4.31) there, apart a typo in the sign). Thus we conclude that:

$$\sum_{\alpha \in \mathcal{K}(T_*^{(n)})} t^{|\alpha|} \leqslant \sum_{\alpha \in \mathcal{K}(T_*^{(n)})} t^{\frac{|\alpha(1)|}{3}} t^{\frac{|\alpha(2)|}{3}} \leqslant \Big[\sum_{A \subset \{0,1,\dots,n\}} t^{|A|\frac{n+1}{3}} \Big]^2 = (1 + t^{\frac{n+1}{3}})^{2(n+1)} .$$
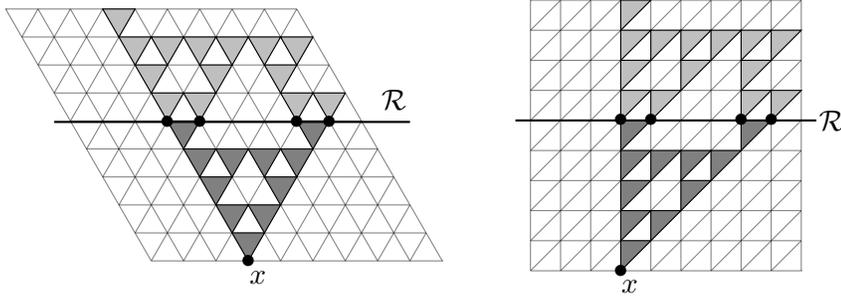
FIGURE 4. Left: A piece of the triangular lattice $\mathcal{T}$ with the line $\mathcal{R}$ and the Pascal's triangle $\mathcal{P}_x$ in gray and light gray. The black dots denote the shadow of x on $\mathcal{R}$ and in dark gray are the plaquettes of $\mathcal{B}_{x,\mathcal{R}}$. The set consisting of $\{x\}$ and its shadow can be thought of as the sum of all the shadowed plaquettes (dark gray). Right: The same figure on $\mathbb{Z}^2$ under the change of basis Φ (see Remark 2.1).

Using that $1 + x \leqslant e^x$ we conclude.

• *The TPM case.* Given two sets of plaquettes $A, B \subset \{x + B_* : x \in \mathbb{Z}^2\}$, we define the sum $A + B$ as their symmetric difference $A \triangle B$. Moreover, we define the multiplication kA with $k \in \mathbb{F}_2 = \{0, 1\}$ (the field of integers mod 2) as $1A := A$ and $0A := \emptyset$. It is then clear that the family of subsets of $\mathcal{B}(T_*^{(n)})$ forms a vector space over $\mathbb{F}_2$. Let $\mathcal{P}_x$ be the family of plaquettes belonging to the binary Pascal's triangle rooted at x and developing up-wards on the triangular lattice (see Figure 4). For $i = -1, 0, \ldots, n$ we denote by P_i the intersection of the plaquettes contained in the Pascal's triangle rooted at $(i, -1)$ with the plaquettes intersecting $T_*^{(n)}$, i.e. (cf. Figure 5)

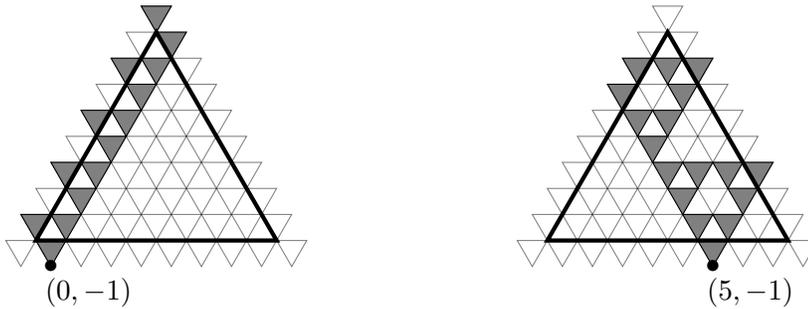$$P_i := \mathcal{P}_{(i,-1)} \cap \mathcal{B}(T_*^{(n)}) \,.$$

FIGURE 5. The region $T_*^{(8)}$ given by all the vertices on and inside the dark lines. The figures show all the plaquettes belonging to $\mathcal{B}(T_*^{(8)})$. Left: The plaquettes belonging to P_0 are marked in gray. Right: The plaquettes belonging to P_5 are marked in gray.

Lemma 4.8. *The cycle space $\mathcal{K}(T_*^{(n)})$ forms a vector space over the field $\mathbb{F}_2$ under the symmetric difference. Furthermore $P_{-1}, P_0, \ldots, P_n$ is a basis.*

The proof of the lemma is postponed to the appendix. Using the above lemma we can parametrize all cycles in $\mathcal{K}(T_*^{(n)})$ with $\sigma \in \{-1, 1\}^{[n]_-}$, where $[n]_- = \{-1, 0, \ldots, n\}$, via the bijection

$$\{-1, 1\}^{[n]_-} \ni \sigma \mapsto \alpha(\sigma) \in \mathcal{K}(T_*^{(n)}), \quad \text{where} \quad \alpha(\sigma) := \sum_{i \in [n]_-} \mathbb{1}(\sigma_i = -1) P_i . \qquad (4.5)$$

Unlike the SPM, we do not have a combinatorial method to efficiently bound from below the cardinality $|\alpha(\sigma)|$. We will now develop a probabilistic method that turns out to be effective. We sample $\sigma \in \{0, 1\}^{[n]_-}$ according to a Bernoulli field of parameter $1/2$, and so

$$\sum_{\substack{\alpha \in \mathcal{K}(T_n^*): \\ \alpha \neq \emptyset}} t^{|\alpha|} = 2^{n+2} \mathbb{E}\big[t^{|\alpha(\sigma)|}\big] - 1 . \qquad (4.6)$$

In order to control the size of the cycles in (4.6), we now give a useful characterisation of the plaquettes belonging to a cycle $\alpha(\sigma)$ in terms of the product of certain elements of σ, which is inspired by a discussion in [25]. For this purpose and inspired by [19], it is convenient to introduce the notion of the *shadow of a vertex* in the TMP.

Definition 4.9. *In the TPM on the triangular lattice $\mathcal{T}$ let $\mathcal{R}$ be an arbitrary horizontal line. For any vertex x lying below $\mathcal{R}$ let $\mathcal{B}_{x,\mathcal{R}}$ consists of all plaquettes $B \in \mathcal{P}_x$ between x and $\mathcal{R}$ included (recall $\mathcal{P}_x$ is the set of plaquettes forming an infinite Pascal's triangle rooted at x rotated by π). Then the shadow on $\mathcal{R}$ of a vertex x lying on $\mathcal{R}$ or below it, denoted $S_{x,\mathcal{R}}$, is x itself in the first case or the set of vertices on $\mathcal{R}$ belonging to an odd number of plaquettes contained in the family $\mathcal{B}_{x,\mathcal{R}}$ (cf. Figure 4) . For the TPM on $\mathbb{Z}^2$ one simply applies the mapping $\mathcal{T} \overset{\Phi}{\mapsto} \mathbb{Z}^2$ described in Remark 2.1 to the above geometric construction.*

We denote by $\mathcal{R}_y$ the horizontal line passing through $(0, y)$, *i.e.* the line at height y. Given $y \geqslant -1$, let $S(y) = \{x \in \mathbb{Z} : (x, y) \in S_{(0,-1),\mathcal{R}_y}\}$ be the projection on the first coordinate of the shadow of the vertex $(0, -1)$ on $\mathcal{R}_y$ (see Definition 4.9). We define the set $T^{(n)}$ as

$$T^{(n)} := \{z \in \mathbb{Z}^2 : z + B_* \in \mathcal{B}(T_*^{(n)})\} = T_*^{(n)} \cup \{(i, -1) : i \in [n]_-\} .$$

Given $z \in T^{(n)}$ we define

$$\mathcal{A}(z) := \{j \in [n]_- : z + B_* \in P_j\} .$$

Note that by definition of $\alpha(\sigma)$ we have

$$z \in B_* \in \alpha(\sigma) \Leftrightarrow [\sigma]_{\mathcal{A}(z)} = -1 . \qquad (4.7)$$

The following claim characterises the set $\mathcal{A}(z)$ and describes some of its properties:

Claim 4.10. *Given $z = (z_1, z_2) \in T^{(n)}$ we have $\mathcal{A}(z) = z_1 - S(z_2)$. Moreover, we have*

$$z_1, z_1 - z_2 - 1 \in \mathcal{A}(z) \subset [z_1 - z_2 - 1, z_1] . \qquad (4.8)$$
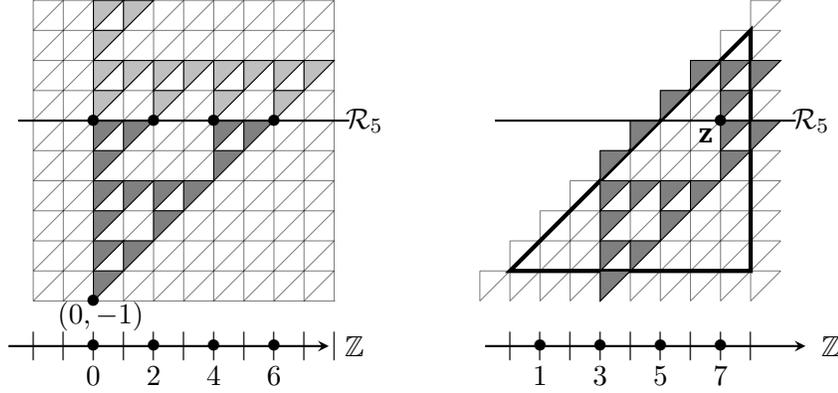
FIGURE 6. Left: The shadow of the vertex $(0, -1)$ on $\mathcal{R}_5$ and the construction of the set $S(5) = \{0, 2, 4, 6\}$. Right: a vertex $z = (7, 5) \in T_*^{(8)}$, the corresponding set $\mathcal{A}(z)$ (below), and the cycle P_3 in gray.

Proof. Given $y \geqslant -1$ it is trivial to check that $S(y) \subset [0, y + 1]$ and $0, y + 1 \in S(y)$. Writing $z = (z_1, z_2)$, it is clear from the construction that $z + B_* \in P_0$ iff $z_1 \in S(z_2)$. Fix now a generic $j \in [n]_-$. By translation we have that $z + B_* \in P_j$ if and only if $(z - (j, 0)) + B_* \in P_0$, which holds if and only if $z_1 - j \in S(z_2)$, i.e. $j \in z_1 - S(z_2)$. This proves that $\mathcal{A}(z) = z_1 - S(z_2)$.

By the initial observations on $S(y)$ we know that $S(z_2) \subset [0, z_2 + 1]$ and $0, z_2 + 1 \in S(z_2)$. Since $\mathcal{A}(z) = z_1 - S(z_2)$, we conclude that that $\mathcal{A}(z) \subset [z_1 - z_2 - 1, z_1]$ and that $z_1 - z_2 - 1, z_1 \in \mathcal{A}(z)$. $\qquad\qquad\square$

We now introduce special subsets of $T^{(n)}$ denoted by $\Gamma(j)$ with $j = -1, 0, 1, \ldots, n$ (see Figure 7). $\Gamma(j)$ is defined as

$$\Gamma(j) := \{(j, i) \,:\, -1 \leqslant i \leqslant j\} \cup \{(j + i, i - 1) \,:\, 1 \leqslant i \leqslant n - j\}. \qquad (4.9)$$

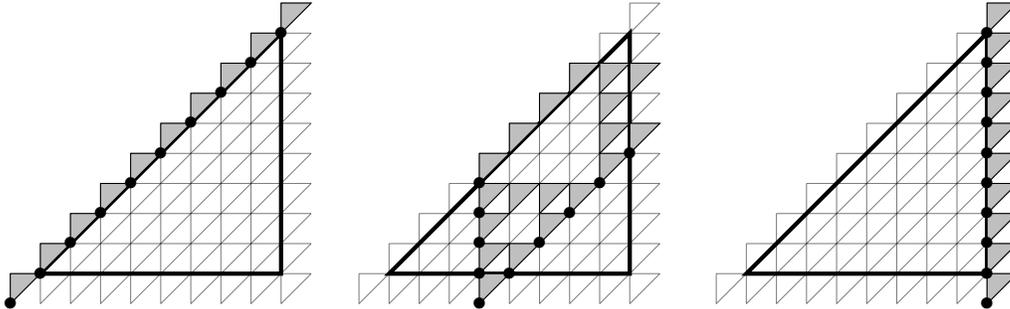Note that $|\Gamma(j)| = n + 2$ for all $j = -1, 0, \ldots, n$. Moreover, for any $z \in T^{(n)}$ we have

FIGURE 7. The marked vertices show the sets $\Gamma(-1)$ (left), $\Gamma(3)$ (middle) and $\Gamma(8)$(right) for $n = 8$. The light gray plaquettes correspond to P_{-1}, P_3 and P_8 respectively.

$$\big|\{j \in [n]_- \,:\, z \in \Gamma(j)\}\big| = \begin{cases} 1 & \text{if } z \in \{(i,-1) \,:\, i \in [n]_-\}, \\ 2 & \text{otherwise.} \end{cases}$$

Now, using these sets and (4.7), we can bound the size of each cycle $\alpha(\sigma) \in \mathcal{K}(T_*^{(n)})$ as follows,

$$\begin{aligned}
|\alpha(\sigma)| &= \sum_{z \in T^{(n)}} \mathbb{1}(z + B_* \in \alpha(\sigma)) \geqslant \frac{1}{2} \sum_{j \in [n]_-} \sum_{z \in \Gamma(j)} \mathbb{1}(z + B_* \in \alpha(\sigma)) \\
&= \frac{1}{2} \sum_{j \in [n]_-} \sum_{z \in \Gamma(j)} \mathbb{1}([\sigma]_{\mathcal{A}(z)} = -1)\,.
\end{aligned} \tag{4.10}$$

It is convenient to write $t = e^{-\varepsilon}$ with $\varepsilon > 0$ (in the applications we have $\varepsilon \approx 2e^{-\beta}$). Using (4.10) and Jensen inequality we can write

$$\begin{aligned}
\mathbb{E}\big[t^{|\alpha(\sigma)|}\big] = \mathbb{E}\big[e^{-\varepsilon|\alpha(\sigma)|}\big] &\leqslant \mathbb{E}\Big[\exp\Big\{-\frac{\varepsilon}{2} \sum_{j \in [n]_-} \sum_{z \in \Gamma(j)} \mathbb{1}([\sigma]_{\mathcal{A}(z)} = -1)\Big\}\Big] \\
&= \mathbb{E}\Big[\exp\Big\{-\frac{\varepsilon(n+2)}{2}\mathrm{Av}_j \sum_{z \in \Gamma(j)} \mathbb{1}([\sigma]_{\mathcal{A}(z)} = -1)\Big\}\Big] \\
&\leqslant \mathrm{Av}_j \mathbb{E}\Big[\exp\Big\{-\frac{\varepsilon(n+2)}{2} \sum_{z \in \Gamma(j)} \mathbb{1}([\sigma]_{\mathcal{A}(z)} = -1)\Big\}\Big]
\end{aligned} \tag{4.11}$$

To bound the last expectation from above we need a last and crucial technical fact.

Claim 4.11. *Let $\sigma \in \{-1,1\}^{\mathbb{Z}}$ be sampled according to the product Bernoulli$(\frac{1}{2})$ measure. Let $A_1, \ldots, A_m$ be finite subsets of $\mathbb{Z}$ such that*

$$A_k \setminus (A_1 \cup A_2 \cup \cdots \cup A_{k-1}) \neq \emptyset \qquad \forall k = 1, \ldots, m\,. \tag{4.12}$$

Then, for any $c \in \mathbb{R}$,

$$\mathbb{E}\Big[\exp\Big\{c \sum_{k=1}^{m} \mathbb{1}([\sigma]_{A_k} = -1)\Big\}\Big] = 2^{-m}(e^c + 1)^m\,.$$

We will postpone the proof of Claim 4.11 until the end of this section. We now have all the tools required to conclude the proof of Lemma 4.7 for the TPM. In order to apply Claim 4.11 to bound from above the r.h.s. of (4.11), we first show that we may order the sets $\mathcal{A}(z)$ for $z \in \Gamma(j)$ such that they satisfy (4.12), *i.e.* each set is nonempty and any set is not included in the union of its predecessors. We index the vertices's in $\Gamma(j)$ as follows (recall (4.9)),

$$z^{(i)} = \begin{cases} (j, -2+i) & \text{if } 1 \leqslant i \leqslant j+2\,, \\ (i-2, i-j-3) & \text{if } j+3 \leqslant i \leqslant n+2\,. \end{cases}$$

Then the sets $\mathcal{A}(z^{(i)})$, $i = 1, \ldots, n+2$, satisfy condition (4.12) by Claim 4.10, in particular by (4.8). Indeed, by Claim 4.10, $\mathcal{A}(z^{(i)})$, $i = 1, \ldots, n+2$, equals

$$j - S(-1)\,, \ j - S(0)\,, \ \ldots\,, \ j - S(j)\,, \ j+1 - S(0)\,, \ j+2 - S(1)\,, \ \ldots\,, \ n - S(n-j-1)\,.$$

Using (4.8) one gets that the following ordered family of points

$$0\,,\ j-1\,,\ j-2\,,\ \ldots\,,-1\,,\ j+1\,,\ j+2\,,\ \ldots\,,n$$

has the property the the i–th point belongs to $\mathcal{A}(z^{(i)})$ but it does not belong to $\mathcal{A}(z^{(i')})$ for $i' < i$. As a consequence, by applying Claim 4.11,

$$\mathbb{E}\Big[\exp\Big\{-\frac{\varepsilon(n+2)}{2}\sum_{z\in\Gamma(j)}\mathbb{1}([\sigma]_{\mathcal{A}(z)}=-1)\Big\}\Big]=2^{-n-2}(e^{-\frac{\varepsilon(n+2)}{2}}+1)^{n+2}\,. \qquad (4.13)$$

By combining (4.6), (4.11) and (4.13) we conclude that (recall that $1+x\leqslant e^x$)

$$\sum_{\substack{\alpha\in\mathcal{K}(T_*^{(n)}):\\ \alpha\neq\emptyset}}t^{|\alpha|}=2^{n+2}\mathbb{E}\big[t^{|\alpha(\sigma)|}\big]-1\leqslant(e^{-\frac{\varepsilon(n+2)}{2}}+1)^{n+2}-1\leqslant\exp\{(n+2)e^{-\frac{\varepsilon(n+2)}{2}}\}-1$$

$$(4.14)$$

as required since $t=e^{-\varepsilon}$. $\qquad\square$

Proof of Claim 4.11. We define $G_k:=\cup_{r=1}^k A_r$ and define $\mathcal{F}_k$ as the σ–algebra generated by $\{\sigma_i\}_{i\in G_k}$. Due to (4.12) we can decompose A_m as $A_m=B_m\cup C_m$ with $B_m:=A_m\setminus G_{m-1}\neq\emptyset$ and $C_m:=A_m\cap G_{m-1}$, where B_m and C_m are disjoint. In particular $[\sigma]_{A_m}=[\sigma]_{B_m}[\sigma]_{C_m}$ and $[\sigma]_{C_m}$ is $\mathcal{F}_{m-1}$–measurable. Observe that if $\{X_i\}_{i=1}^n$ are i.i.d ±1 random variables and $q:=\mathbb{P}(X_i=-1)$ then $\mathbb{E}\left(\prod_{i=1}^n X_i\right)=(1-2q)^n$, *i.e.*

$$\mathbb{P}\big(\prod_{i=1}^n X_i=1\big)=\frac{1}{2}+\frac{(1-2q)^n}{2}\,. \qquad (4.15)$$

Hence, using (4.15) and $\mathbb{P}(\sigma_i=1)=\mathbb{P}(\sigma_i=-1)=\frac{1}{2}$, we have

$$\mathbb{E}\Big[e^{c\mathbb{1}([\sigma]_{A_m}=-1)}\,|\,\mathcal{F}_{m-1}\Big]=\mathbb{E}\Big[e^{c\mathbb{1}([\sigma]_{B_m}=-[\sigma]_{C_m})}\,|\,\mathcal{F}_{m-1}\Big]=\frac{e^c+1}{2}\,, \qquad (4.16)$$

where the last identity follows by integrating over σ_i with $i\in B_m$.

Note that $[\sigma]_{A_k}$ is $\mathcal{F}_{m-1}$–measurable for $k\leqslant m-1$. The by conditioning on $\mathcal{F}_{m-1}$, and applying (4.16), we get

$$\mathbb{E}\Big[\exp\Big(c\sum_{k=1}^m\mathbb{1}([\sigma]_{A_k}=-1)\Big)\Big]$$

$$=\mathbb{E}\Big[\exp\Big(c\sum_{k=1}^{m-1}\mathbb{1}([\sigma]_{A_k}=-1)\Big)\mathbb{E}\Big[\exp\Big(c\mathbb{1}([\sigma]_{A_m}=-1)\Big)\,|\,\mathcal{F}_{m-1}\Big]\Big] \qquad (4.17)$$

$$=\frac{e^c+1}{2}\mathbb{E}\Big[\exp\Big(c\sum_{k=1}^{m-1}\mathbb{1}([\sigma]_{A_k}=-1)\Big)\Big]\,.$$

The result now follows by iteration. $\qquad\square$

5. PROOF OF THEOREM 2

Given $\Lambda_1, \Lambda_2, \ldots, \Lambda_n \subset \mathbb{Z}^2$ we define their sum $\Lambda_1 + \cdots + \Lambda_n$ as the set of vertices of $\mathbb{Z}^2$ belonging to an odd number of Λ_i's (*i.e.* for set addition we take the symmetric difference). Notice that

$$[\sigma]_{\Lambda_1 + \cdots + \Lambda_n} = \prod_{i=1}^{n} [\sigma]_{\Lambda_i} . \tag{5.1}$$

We also define Λ, Λ' to be *equivalent* and write $\Lambda \sim \Lambda'$, if there exist a finite family of plaquettes $\{B_i\}_{i=1}^{n}$ such that $\Lambda = \Lambda' + \sum_{i=1}^{n} B_i$. When $\Lambda \sim \emptyset$ we say that $\{B_i\}_{i=1}^{n}$ is a *plaquettes decomposition* of Λ if $\Lambda = B_1 + B_2 + \cdots + B_n$. The decomposition is *minimal* if each B_i appears only once. It is straightforward to show that the minimal decomposition is unique. The importance of the above construction is justified by the following result proved in [23, Section 4.4]. For any finite set $A \subset \mathbb{Z}^2$ the multispin average $\mu^\beta([\sigma]_A)$ satisfies

$$\mu^\beta([\sigma]_A) = \begin{cases} 0 & \text{if } A \not\sim \emptyset, \\ \tanh(\beta/2)^n & \text{if } A \sim \emptyset, \end{cases} \tag{5.2}$$

where $n = n(A)$ is the size of the minimal plaquettes decomposition of A.

5.0.1. *Asymptotics of $\ell_c^{(\text{cavity})}$ as $\beta \to \infty$.* For both models the *upper bound* on $\ell_c^{(\text{cavity})}$ follows from (3.1) and Theorem 1. For the TPM the *lower bound* on $\ell_c^{(\text{cavity})}$ follows again from (3.1) and the lower bound on $\ell_c^{(\text{multispin})}$ in Theorem 2 (cf. below). Thus it remains to prove the lower bound $\ell_c^{(\text{cavity})} = \Omega(e^\beta)$ for the SPM. The following simple argument forms the basis of our approach.

Let $A \not\sim \emptyset$ be a finite set contained inside the square centered at the origin of side $\ell \geqslant \ell_c^{(\text{cavity})}$. Then, by definition, for all boundary conditions τ we must have $|\mu_\Lambda^{\beta;\tau}([\sigma]_A)| \leq 1/5$, where Λ is the square of side 10ℓ centered at the origin. Using the DLR equations we can write in fact

$$|\mu_\Lambda^{\beta;\tau}([\sigma]_A)| = \Big| \int d\mu^\beta(\tau') \left(\mu_\Lambda^{\beta;\tau}([\sigma]_A) - \mu_\Lambda^{\beta;\tau'}([\sigma]_A) \right) \Big|$$

$$\leq \int d\mu^\beta(\tau') |\mu_\Lambda^{\beta;\tau}([\sigma]_A) - \mu_\Lambda^{\beta;\tau'}([\sigma]_A)| \leq 2 \int d\mu^\beta(\tau') \psi(\ell;\tau,\tau') \leq 1/5,$$

where $\psi(\ell;\tau,\tau')$ is the variation distance introduced in Definition 3.8. Thus, in order to bound from below $\ell_c^{(\text{cavity})}$ by ℓ, it is enough to prove that for a suitably chosen boundary condition τ and $A \subset [-\frac{\ell}{2}, \frac{\ell}{2}]^2 \cap \mathbb{Z}^2$, we have $|\mu_\Lambda^{\beta;\tau}([\sigma]_A)| > 1/5$ if $\Lambda = [-5\ell, 5\ell]^2 \cap \mathbb{Z}^2$.

Our choice of τ will be the *all plus* boundary conditions. With this choice it is convenient to define $\mathcal{B}^+(\Lambda) = \{B \cap \Lambda : B \in \mathcal{B}(\Lambda)\}$ and to denote by B^+ its generic element. We will write $\mathcal{K}^+(\Lambda)$ for the family of cycles in $\mathcal{B}^+(\Lambda)$, *i.e.* collections of the B_i^+'s such that any point in Λ belongs to an even number of the B_i^+'s.

Lemma 5.1. *Let $A \subset \Lambda \subset \mathbb{Z}^2$ be non-empty finite sets and suppose that $A = B_1^+ + \cdots + B_n^+$ with $B_i^+ \neq B_j^+$ whenever $i \neq j$. Then*

$$\mu_\Lambda^{\beta;+}([\sigma]_A) \geqslant \tanh(\beta/2)^n .$$

Proof. Let $\alpha_A := \{B_1^+, \ldots, B_n^+\}$. By applying [8, eq. (2.12)] together with $|V_1 \triangle V_2| \leqslant |V_1| + |V_2|$ we get

$$\mu_\Lambda^{\beta,+}(\sigma_A) = \frac{\sum_{\alpha \in \mathcal{K}^+(\Lambda)} \tanh(\beta/2)^{|\alpha \triangle \alpha_A|}}{\sum_{\alpha \in \mathcal{K}^+(\Lambda)} (\tanh(\beta/2))^{|\alpha|}} \geqslant \tanh(\beta/2)^{|\alpha_A|} = \tanh(\beta/2)^n.$$

$\square$

The above bound is quite crude and one may suspect that for many choices of A it would be too pessimistic. That is indeed true as we will see shortly. Nevertheless it is enough to prove the sought lower bound on $\ell_c^{(\text{cavity})}$.

Choose $\Lambda = [-10\ell, 10\ell]^2 \cap \mathbb{Z}^2$ and let $A = \{(0,0), (0,1)\}$. Clearly $A \not\sim \emptyset$ and $A = B_1^+ + \cdots + B_{10\ell+1}^+$ where $B_i^+ = B_* + (i-1, 0)$ for $i \in [10\ell]$ and $B_{10\ell+1}^+ = \{(10\ell, 0), (10\ell, 1)\}$. Using the lemma $\mu_\Lambda^{\beta,+}(\sigma_A) \geqslant \tanh(\beta/2)^{10\ell+1}$ *i.e.* $\ell_c^{(\text{cavity})} = \Omega(e^\beta)$ by the argument given above. $\square$

The key feature of the above choice of A is that its minimal plaquettes decomposition requires a *linear* (in ℓ) number of plaquettes B^+. Another natural choice for A would be the origin so that $\mu^{\beta,+}([\sigma]_A)$ becomes the magnetisation at the origin under the plus boundary conditions. In this case the minimal plaquettes decomposition requires a $\Omega(\ell^2)$ number of plaquettes B^+. Reapplying the above strategy would however only produce a lower bound $\Omega(e^{\beta/2})$ on corresponding cavity length. A natural question is therefore whether the magnetisation at the origin starts to be (roughly) independent of the boundary conditions on scale $\approx e^\beta$ or on scale $e^{\beta/2}$ (or on some scale in between). The answer is provided in the next result whose technical proof is deferred to the appendix.

Proposition 5.2. *There exists $c > 0$ such that for any $\ell < ce^\beta$*

$$\liminf_{\beta \to \infty} \mu_{\Lambda_\ell}^{\beta,+}(\sigma_0) > 0,$$

where $\Lambda_\ell = [-\ell, \ell]^2 \cap \mathbb{Z}^2$.

5.0.2. *Asymptotics of $\ell_c^{(\text{multispin})}$ as $\beta \to \infty$.* Using (5.2) the required bound on $\ell_c^{(\text{multispin})}$ will follow once we are able to estimate the size $n(A)$ of the minimal plaquettes decomposition of a given finite set A. For this purpose, we now also define the *shadow of a vertex* for the SPM (recall Definition 4.9 for the TPM). We call positive (negative) *corner* any translation of the set $\{k\vec{e}_1 : k \geq 0\} \cup \{k\vec{e}_2 : k \geq 0\}$ (of the set $\{k\vec{e}_1 : k \leq 0\} \cup \{k\vec{e}_2 : k \leq 0\}$).

Definition 5.3. *For the SPM fix a corner $\mathcal{R}$ together with x belonging to the quadrant delimited by $\mathcal{R}$. If $x \in \mathcal{R}$ then we define the shadow $S_{x,\mathcal{R}}$ of x on $\mathcal{R}$ as x itself. Otherwise $S_{x,\mathcal{R}}$ are the three points on $\mathcal{R}$ which, together with x, form the vertices of a rectangle. In that case the set of plaquettes contained in the rectangle will be denoted by $\mathcal{B}_{x,\mathcal{R}}$.*

The next result gives an algorithmic characterisation of sets $A \sim \emptyset$ and of their minimal plaquettes decomposition in terms of the shadows of their elements.

Lemma 5.4. *Let $A \subset \mathbb{Z}^2$ be finite and let $\mathcal{R}$ be either a corner (positive or negative) in the SPM or an horizontal line in the TPM, such that every $x \in A$ belongs to the quadrant delimited by $\mathcal{R}$ in the first case or lies below $\mathcal{R}$ in the second case. Then*

(i) $A \sim \emptyset$ if and only if $\sum_{x \in A} S_{x,\mathcal{R}} = \emptyset$.
(ii) If $A \sim \emptyset$, then the minimal decomposition of A is given by all the plaquettes belonging to an odd number of the families $\{\mathcal{B}_{x,\mathcal{R}}\}_{x \in A}$ described in Definition 4.9 and 5.3.

Proof. (i) Fix a point $x \in A$ and a corner or a line $\mathcal{R}$ depending on the model such that x casts a shadow on $\mathcal{R}$. By definition $\{x\} \sim S_{x,\mathcal{R}}$ for any $x \in A$ so that $A = \sum_{x \in A} \{x\} \sim \sum_{x \in A} S_{x,\mathcal{R}}$. In order to conclude it is enough to prove that $\sum_{x \in A} S_{x,\mathcal{R}} \sim \emptyset$ iff $\sum_{x \in A} S_{x,\mathcal{R}} = \emptyset$. The "if" part is trivial. To prove the opposite implication suppose that $\sum_{x \in A} S_{x,\mathcal{R}} \neq \emptyset$ while $\sum_{x \in A} S_{x,\mathcal{R}} \sim \emptyset$. Then there exist $n \geqslant 1$ distinct plaquettes $B_1, \dots, B_n$ such that $\sum_{x \in A} S_{x,\mathcal{R}} = B_1 + \cdots + B_n$. By a minimality argument, one can easily check that the set $B_1 + \cdots + B_n$ cannot belong to $\mathcal{R}$, while $\sum_{x \in A} S_{x,\mathcal{R}}$ does, which gives a contradiction.

(ii) Suppose that $A \sim \emptyset$ and notice that $\{x\} = S_{x,\mathcal{R}} + \sum_{B \in \mathcal{B}_{x,\mathcal{R}}} B$. Hence, using $\sum_{x \in A} S_{x,\mathcal{R}} = \emptyset$ (cf. above), we get

$$A = \sum_{x \in A} \{x\} = \sum_{x \in A} \Big(S_{x,\mathcal{R}} + \sum_{B \in \mathcal{B}_{x,\mathcal{R}}} B\Big) = \sum_{x \in A} \sum_{B \in \mathcal{B}_{x,\mathcal{R}}} B \,.$$

The r.h.s. in the above expression gives a plaquettes decomposition of A. To get the minimal decomposition it is enough to remove all possible repetitions. $\square$

We now have all the necessary tools to analyse $\ell_c^{(\mathrm{multispin})}$.

• *The SPM case.* Using (5.2) and Lemma 5.4, if $A \subset \mathbb{Z}^2$ is a square of side length ℓ, then $A \sim \emptyset$ and $\mu^\beta([\sigma]_A) = \tanh(\beta/2)^{\ell^2}$. This implies immediately that $\ell_c^{(\mathrm{multispin})} = \Omega(e^{\beta/2})$. In order to get an upper bound on $\ell_c^{(\mathrm{multispin})}$, take $A \in \mathbb{F}_\ell$ such that $A \sim \emptyset$ (recall (5.2)) and, w.l.o.g., assume that A contains the origin and is contained in the half-space $\{x \in \mathbb{Z}^2 : x_2 \geq -x_1\}$. Since the minimal distance between the points of A is at least ℓ, any positive quadrant rooted at $y \in A$, $y \neq 0$, cannot share plaquettes with $W = \{0, \dots, \lfloor \ell/2 \rfloor\}^2$ (see Figure 8). We now take $\mathcal{R}$ as a negative corner such that A and W lies in the quadrant delimited by $\mathcal{R}$. Let $B_1, \dots, B_n$ be the minimal plaquettes decomposition of A w.r.t. $\mathcal{R}$ as described in Lemma 5.4. By construction all plaquettes of $\mathcal{B}_{0,\mathcal{R}}$ contained in W cannot belong to any other family $\mathcal{B}_{y,\mathcal{R}}$, $y \in A \setminus \{0\}$. Hence, $n \geqslant \lfloor \ell \rfloor^2 / 4$.

In conclusion (5.2) implies that there exists $c > 0$ such that

$$\mu^\beta([\sigma]_A) \leqslant \tanh(\beta/2)^{\frac{\lfloor \ell \rfloor^2}{4}} \leq \frac{1}{5}\,, \qquad \forall \ell \leq c\, e^{\beta/2}.$$

• *The TPM case.* W.l.o.g. and to simplify the notation we consider the TPM on the triangular lattice $\mathcal{T}$ with the ℓ_1-distance replaced by the graph distance and we define the family $\mathbb{F}_\ell$ accordingly. Let A consists of the vertices x, y, z of a downward pointing equilateral triangle with side length 2^k. Using Lemma 5.4 $A \sim \emptyset$ and its minimal
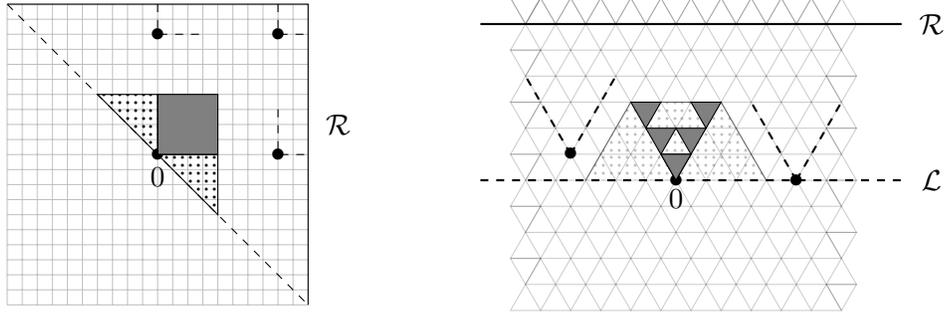
FIGURE 8. *Left*: a set $A \sim \emptyset$ in $\mathbb{F}_{\ell=8}$ (black dots) above the -45° diagonal, the square W (in gray) and the negative corner $\mathcal{R}$ in the construction for the SPM. *Right*: a portion of the set $A \in \mathbb{F}_{\ell=4}$ (black dots) above the line $\mathcal{L}$, the half-hexagon $\mathcal{H}$, the Pascal triangle W (in gray) and the line $\mathcal{R}$ in the construction for the TPM.

plaquettes decomposition is given by all the plaquettes lying in the truncated upward–pointing Pascal's triangle with vertices x, y, z. The number of these plaquettes equals 3^k while the minimal graph distance between points of A is equal to 2^k. Hence, using again (5.2), $\ell_c^{(\text{multispin})} = \Omega(e^{\beta\alpha})$, $\alpha = \frac{\log 2}{\log 3}$.

To prove the upper bound we proceed similarly to the SPM. Fix $A \subset \mathcal{T}$, $A \in \mathbb{F}_\ell$ and such that $A \sim \emptyset$ (recall (5.2)). We denote by k the largest integer such that $\ell \geq 2^k$ and, w.l.o.g., we assume that the origin of $\mathcal{T}$ belongs to A and that A is contained in the upper half graph delimited by the horizontal line $\mathcal{L}$ through the origin. Consider the half-hexagon $\mathcal{H}$ of side length 2^{k-1} centered at the origin and lying above $\mathcal{L}$ and let $\mathcal{R}$ be any horizontal line of $\mathcal{T}$ such that both A and $\mathcal{H}$ lie below $\mathcal{R}$. By construction, for any $x \in A$, $x \neq 0$, the upward Pascal triangle rooted at x cannot intersect the upward Pascal triangle W rooted at the origin and contained in $\mathcal{H}$. Lemma 5.4 implies the minimal plaquettes decomposition of A must contain all the plaquettes in W. In conclusion the cardinality of the minimal plaquettes decomposition of A is at least 3^{k-1} and the sought bound $\ell_c^{(\text{multispin})} = O(e^{\beta\alpha})$, $\alpha = \frac{\log 2}{\log 3}$, follows as before.

6. PROOF OF THEOREM 3

In order to prove Theorem 3 it is enough to establish a finite volume version of the result with free boundary conditions (equations (6.1) and (6.6)) and use the uniqueness of the thermodynamic limit.

Let us start by fixing some useful notation. Given a finite set $\Lambda \subset \mathbb{Z}^2$ let $\mathcal{B}^f(\Lambda) = \{B \in \mathcal{B}(\Lambda) : B \subset \Lambda\}$ and define the Gibbs measure on Λ with free boundary conditions as

$$\mu_\Lambda^{\beta,f}(\sigma) = \frac{e^{\frac{\beta}{2} \sum_{B \in \mathcal{B}^f(\Lambda)} [\sigma]_B}}{Z_\Lambda^{\beta,f}}, \qquad \sigma \in \Omega_\Lambda,$$

where $Z_\Lambda^{\beta,f}$ is the normalisation constant.

6.1. **The SPM case.** Given $\ell, N \in \mathbb{N}$ let $\Lambda_{\ell,N} = \{0, 1, \ldots, \ell N\}^2$ and $\Lambda^*_{\ell,N} = \Lambda_{\ell,N} \cap \ell\mathbb{Z}^2$. Our aim is to prove that

$$\mu^{\beta,f}_{\Lambda_{\ell,N}}(\{\sigma : \sigma_{\Lambda^*_{\ell,N}} = \eta\}) = \mu^{\beta',f}_{\Lambda_{1,N}}(\eta), \quad \forall \eta \in \Omega_{\Lambda^*_{\ell,N}}, \tag{6.1}$$

where $\beta' = \beta'(\beta, \ell)$ is as in the statement of Theorem 3. Above we have identified $\eta \in \Omega_{\Lambda^*_{\ell,N}}$ with a configuration in $\Omega_{\Lambda_{1,N}}$ using the bijection $\Lambda_{1,N} \ni x \mapsto \ell x \in \Lambda^*_{\ell,N}$.

Given $\sigma \in \Omega_{\Lambda_{\ell,N}}$, we denote by $p_x(\sigma)$ the ± 1 plaquette variable $[\sigma]_{B_x}$ associated to the plaquette $B_x := B_* + x$. Notice that the spin configuration $\sigma \in \Omega_{\Lambda_{\ell,N}}$ is completely determined once we assign its values along the South and West boundary of $\Lambda_{\ell,N}$ (*i.e.* the sites with at least one coordinate equal to zero) *and* all the plaquette variables associated to $\mathcal{B}^f(\Lambda_{\ell,N})$. It follows that the number of spin configurations compatible with an assignment of the plaquette variables equals $2^{2N\ell+1}$ and that

$$\mu^{\beta,f}_{\Lambda_{\ell,N}}(\sigma) = \frac{1}{2^{2N\ell+1}} \prod_{x \in [0, N\ell-1]^2 \cap \mathbb{Z}^2} \nu^\beta(p_x(\sigma)), \tag{6.2}$$

where $\nu^\beta(-1) = q(\beta)$ and $\nu^\beta(1) = 1 - q(\beta)$.

Given $\eta \in \Omega_{\Lambda^*_{\ell,N}}$ we introduce the *renormalised* plaquette variables $\hat{p}_x(\eta) = [\eta]_{\ell B_* + x}$, for $x \in \Lambda^*_{\ell,N}$. Again η is univocally determined by its values on the South and West boundary of $\Lambda^*_{\ell,N}$ and its renormalised plaquette variables. Notice also that it holds

$$\hat{p}_x(\sigma_{\Lambda^*_{\ell,N}}) = \prod_{y \in \{0,\ldots,\ell-1\}^2 + x} p_y(\sigma) \qquad \forall \sigma \in \Omega_{\Lambda_{\ell,N}}. \tag{6.3}$$

We can now state the basic lemma concerning the SPM Gibbs measure $\mu^{\beta,f}_{\Lambda_{\ell,N}}$:

Lemma 6.1. *Fix two configurations η, η' on $\Lambda^*_{\ell,N}$ such that their renormalised plaquette variables coincide. Then*

$$\mu^{\beta,f}_{\Lambda_{\ell,N}}(\{\sigma : \sigma_{\Lambda^*_{\ell,N}} = \eta\}) = \mu^{\beta,f}_{\Lambda_{\ell,N}}(\{\sigma : \sigma_{\Lambda^*_{\ell,N}} = \eta'\}).$$

Proof. We introduce a bijection T on $\Omega_{\Lambda_{\ell,N}}$ as the composition of the bijections T_x with x varying among the sites in the South and West boundary of $\Lambda^*_{\ell,N}$, defined as follows. If $x = (0, x_2)$, then T_x is obtained by flipping all spins on sites with second coordinate x_2. If $x = (x_1, 0)$ with $x_1 > 0$, then T_x is obtained by flipping all spins on sites with first coordinate x_1. To get the thesis it is enough to show that the bijection T satisfies two key properties:

  (i) $\forall \sigma \in \Omega_{\Lambda_{\ell,N}}$ it holds $\mu^{\beta,f}_{\Lambda_{\ell,N}}(\sigma) = \mu^{\beta,f}_{\Lambda_{\ell,N}}(T(\sigma))$;
  (ii) if $\sigma_{\Lambda^*_{\ell,N}} = \eta$ then $T(\sigma)_{\Lambda^*_{\ell,N}} = \eta'$.

Property (i) immediately follows noticing that $p_x(\sigma) = p_x(T(\sigma))$ and using (6.2). To prove property (ii) we notice, using (6.3), that $T(\sigma)_{\Lambda^*_{\ell,N}}$ has the same renormalised plaquette variables as η. These in turn coincide with the renormalised plaquette variables of η' by hypothesis. Furthermore T has been defined in such a way that, if σ agrees with η on the South-West boundary of $\Lambda^*_{\ell,N}$, then $T(\sigma)$ agrees with η' on the

same set. Therefore property (ii) follows by recalling that the renormalized spin configuration is determined uniquely by its values on the South-West boundary of $\Lambda_{\ell,N}^*$, and the renormalised plaquette variables. $\square$

We now claim that for any spin configuration $\eta \in \Omega_{\Lambda_{\ell,N}^*}$ it holds

$$\mu_{\Lambda_{\ell,N}}^{\beta,f}(\{\sigma : \sigma_{\Lambda_{\ell,N}^*} = \eta\}) = \frac{1}{2^{2N+1}} \prod_{x \in [0,N\ell-1]^2 \cap \ell\mathbb{Z}^2} \rho^\beta(\hat{p}_x(\eta)), \tag{6.4}$$

where $\rho^\beta(-1) = \varphi(q(\beta), \ell^2)) = q(\beta')$ and $\rho^\beta(1) = 1 - \rho^\beta(-1)$. Then (6.1) follows by comparing (6.4) with (6.2) where in the latter we take $\ell = 1$ and replace β by β'.

We are left with proving (6.4). It holds

$$\begin{aligned}
\mu_{\Lambda_{\ell,N}}^{\beta,f}(\{\sigma : \sigma_{\Lambda_{\ell,N}^*} = \eta\}) &= \frac{1}{2^{2N+1}} \mu_{\Lambda_{\ell,N}}^{\beta,f}(\{\sigma : \forall x \; \hat{p}_x(\sigma_{\Lambda_{\ell,N}^*}) = \hat{p}_x(\eta)\}) \\
&= \frac{1}{2^{2N+1}} \prod_x \mu_{\Lambda_{\ell,N}}^{\beta,f}(\{\sigma : \prod_{y \in \ell B_* + x} p_y(\sigma) = \hat{p}_x(\eta)\})
\end{aligned} \tag{6.5}$$

where x varies on $[0, N\ell - 1]^2 \cap \ell\mathbb{Z}^2$. Indeed, the first equality follows from Lemma 6.1 and the fact that the number of configurations η compatible with a given choice of renormalised plaquettes variables equals 2^{2N+1}. The second equality follows from (6.3). Then (6.4) is obtained by using equations (4.15) and (6.5).

6.2. **The TPM case.** We denote by $\mathcal{T}_{n,N}$ the triangular region in $\mathbb{Z}^2$ with vertices the origin, $2^{n+N}\vec{e}_2$ and $2^{n+N}(\vec{e}_1 + \vec{e}_2)$ (see Figure 9 left). Moreover we set $\mathcal{T}_{n,N}^* := \mathcal{T}_{n,N} \cap 2^n\mathbb{Z}^2$. Our aim is to prove that

$$\mu_{\mathcal{T}_{n,N}}^{\beta,f}(\{\sigma : \sigma_{\mathcal{T}_{n,N}^*} = \eta\}) = \mu_{\mathcal{T}_{1,N}}^{\beta',f}(\eta), \quad \forall \, \eta \in \Omega_{\mathcal{T}_{n,N}^*}. \tag{6.6}$$

where we identify $\eta \in \Omega_{\mathcal{T}_{n,N}^*}$ with a configuration on $\Omega_{\mathcal{T}_{1,N}}$ in the natural way.

We define the North and West boundary of $\mathcal{T}_{n,N}$ as $\partial_N \mathcal{T}_{n,N} := \{(x_1, x_2) \in \mathcal{T}_{n,N} : x_2 = 2^{n+N}\}$ and $\partial_W \mathcal{T}_{n,N} := \{(x_1, x_2) \in \mathcal{T}_{n,N} : x_1 = 0\}$, respectively. Note that $|\partial_N \mathcal{T}_{n,N}| = |\partial_W \mathcal{T}_{n,N}| = \mathcal{N}(n + N) + 1$, with $\mathcal{N}(i) = 2^i$.

Given $\sigma \in \Omega_{\mathcal{T}_{n,N}}$, we denote by $p_x(\sigma)$ the plaquette variable $[\sigma]_{B_*+x}$. Furthermore, given $\eta \in \Omega_{\mathcal{T}_{n,N}^*}$ we define the *renormalised* plaquette variable $\hat{p}_x(\eta) = [\eta]_{2^n B_* + x}$, for $x \in \mathcal{T}_{n,N}^*$. We observe that the spin configuration $\sigma \in \Omega_{\mathcal{T}_{n,N}}$ is completely determined once we assign its values along the West boundary *and* the plaquette variables for each plaquette in $\mathcal{B}^f(\mathcal{T}_{n,N})$. It follows that the number of spin configurations compatible with an choice of the plaquette variables is equal to $2^{\mathcal{N}(n+N)+1}$ and that

$$\mu_{\mathcal{T}_{n,N}}^{\beta,f}(\sigma) = \frac{1}{2^{\mathcal{N}(n+N)+1}} \prod_{x \in \mathcal{T}_{n,N} \setminus \partial_N \mathcal{T}_{n,N}} \nu^\beta(p_x(\sigma)), \tag{6.7}$$

where ν^β is defined as in the SPM case. Similarly, $\eta \in \Omega_{\mathcal{T}_{n,N}^*}$ is univocally determined by its values on $\partial_W \mathcal{T}_{n,N}^*$ and its renormalised plaquette variables.

Recall that $\mathcal{P}_x$ is the family of plaquettes belonging to the binary Pascal's triangle rooted at x and developing up-wards on the triangular lattice (see Figure 4). Then let

$\mathcal{P}_x^n$ be the set of plaquettes in $\mathcal{P}_x$ that are also contained in the triangle with vertices $x, x + 2^n \vec{e}_2, x + 2^n(\vec{e}_1 + \vec{e}_2)$ (see Figure 9 left). It is then simple to check that

$$\hat{p}_x(\sigma_{\mathcal{T}_{n,N}^*}) = \prod_{y: y + B_* \in \mathcal{P}_x^n} p_y(\sigma) \qquad \forall \sigma \in \Omega_{\mathcal{T}_{n,N}}. \tag{6.8}$$

We now define iteratively some geometric sets that we will need to define a bijection on $\Omega_{\mathcal{T}_{n,N}}$ analogously to what we did in proof of Lemma 6.1. Let $T_0 := \{(0,0), (0,1), (1,1)\}$ and $\ell_0 := 1$. Then $\forall i \geqslant 0$ we set

$$T_{i+1} := T_i \cup \{T_i + (\ell_i + 1)\vec{e}_1\} \cup \{T_i + (\ell_i + 1)(\vec{e}_1 + \vec{e}_2)\}, \qquad \ell_{i+1} := 2\ell_i + 1,$$

(which implies $\ell_i = 2^{i+1} - 1$) as in Figure 9 (right).

The following result can be easily proven inductively:

Claim 6.2. *Fix $i \geqslant 1$ and $\xi \in \Omega$. Consider the configuration ξ' obtained from ξ by flipping all spins in T_i. Then the plaquette variables of ξ and ξ' differ at exactly three plaquettes, those with bottom left corner at $(-1, -1)$, $(\ell_i, -1)$ and (ℓ_i, ℓ_i):*

$$[\xi]_{x+B_*} = \begin{cases} -[\xi]_{x+B_*} & \text{if } x \in \{(-1,-1), (\ell_i, -1), (\ell_i, \ell_i)\}, \\ [\xi]_{x+B_*} & \text{otherwise.} \end{cases}$$
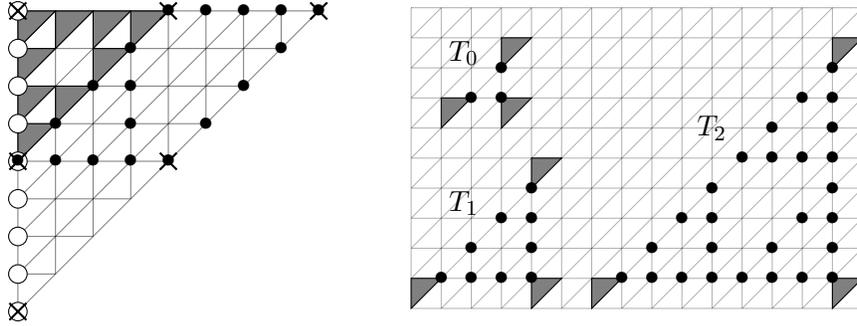

FIGURE 9. Left: The region $\mathcal{T}_{2,1}$. In grey are the plaquettes in $\mathcal{P}_x^2$ with $x = (0, 4)$, crosses correspond to the sites in $\mathcal{T}_{2,1}^* = \mathcal{T}_{2,1} \cap 2^2 \mathbb{Z}^2$, white circles are the sites in $\partial_W \mathcal{T}_{2,1}$, and the black circles are $T^{(4)}$. Right: Black circles correspond to sites belonging to T_0, T_1 and T_2 (modulo translations). The grey plaquettes are those on which the plaquette variable is changed when flipping all spins of T_i (see Claim 6.2).

Lemma 6.3. *Fix two spin configurations η, η' on $\mathcal{T}_{n,N}^*$ such that their renormalised plaquette variables coincide. Then*

$$\mu_{\mathcal{T}_{n,N}}^{\beta,f}(\{\sigma : \sigma_{\mathcal{T}_{n,N}^*} = \eta\}) = \mu_{\mathcal{T}_{n,N}}^{\beta,f}(\{\sigma : \sigma_{\mathcal{T}_{n,N}^*} = \eta'\}).$$

Proof. For $j \in [2^{N+n}]$, we define the bijection $T^{(j)}$ on $\Omega_{\mathcal{T}_{n,N}}$ as the map that flips all the spins in $(T_{n+N+1} + j\vec{e}_2) \cap \mathcal{T}_{n,N}$. Thanks to Claim 6.2 the plaquette variables associated

to $\mathcal{B}^f(\mathcal{T}_{n,N})$ are left invariant by $T^{(j)}$ and the only spin variable on the West boundary of $\mathcal{T}_{n,N}$ which is flipped by $T^{(j)}$ is on site $j\vec{e}_2$. We define a bijection T on $\Omega_{\mathcal{T}_{n,N}}$ as the composition of $T^{(j)}$ for all j such that $\eta_{(0,j)} \neq \eta'_{(0,j)}$. Then analogously to the SPM case (cf. proof of Lemma 6.2) we have (i) $\mu^{\beta}_{\mathcal{T}_{n,N}}(\sigma) = \mu^{\beta}_{\mathcal{T}_{n,N}}(T(\sigma))$, and (ii) if $\sigma_{\mathcal{T}^*_{n,N}} = \eta$ then $T(\sigma)_{\mathcal{T}^*_{n,N}} = \eta'$. Since T is a bijection, the thesis follows from (i) and (ii). $\qquad\square$

It follows from Lemma 6.3 together with (6.8) that for any $\eta \in \Omega_{\mathcal{T}^*_{n,N}}$

$$\mu^{\beta,f}_{\mathcal{T}_{n,N}}(\{\sigma : \sigma_{\mathcal{T}^*_{n,N}} = \eta\}) = \frac{1}{2^{\mathcal{N}(N)+1}} \prod_{x \in \bar{\mathcal{T}}_{n,N}} \mu^{\beta,f}_{\mathcal{T}_{n,N}}(\{\sigma : \prod_{y:y+B_* \in \mathcal{P}^n_x} p_y(\sigma) = \hat{p}_x(\eta)\}), \quad (6.9)$$

where $\bar{\mathcal{T}}_{n,N} = \mathcal{T}^*_{n,N} \setminus \partial_N \mathcal{T}_{n,N}$. Then since the number of plaquettes inside the Pascal's triangle of linear size $\ell = 2^n$ is $\ell^\alpha = 3^n$ ($\alpha = \log 3/\log 2$), it follows from (4.15) that

$$\mu^{\beta,f}_{\mathcal{T}_{n,N}}(\{\sigma : \prod_{y:y+B_* \in \mathcal{P}^n_x} p_y(\sigma) = \hat{p}_x(\eta)\}) = \rho^\beta(\hat{p}_x(\eta)), \quad (6.10)$$

where $\rho^\beta(-1) = \varphi(q(\beta), \ell^\alpha) = q(\beta')$ (cf. Theorem 3-(b)). Plugging (6.10) into (6.9) we get the analogous of (6.4). Due to (6.7) which is the analogous of (6.2), one can derive (6.6) as in the SPM case.

7. Proof of Theorem 4

(i) The proof is identical to the one given for the SPM if one takes as ε-screening sets large enough d-dimensional cubes. For these sets in fact the analog of Lemma 4.7 can be generalised (see also the arguments in the proof of [8, Proposition 4.5]).

(ii) The upper bound on $\ell_c^{(\text{cavity})}$ follows again from (3.1). To prove the lower bound we follow the pattern of the proof for the SPM. Let $V \subset \Lambda$ be the two concentric squares centered at the origin of side ℓ and 10ℓ respectively, and let $A = B_*^{(1)} \times \cdots \times B_*^{(d-1)} \times \{0\}$. For ℓ large enough, we have $A \subset V$. It is easy to check that $A \not\sim \emptyset$ so that $\mu^\beta([\sigma]_A) = 0$ (cf. [23, Section 4.4]). It is simple to exhibit a subset $I \subset \{0, 1, \ldots, 10\ell\}$ of cardinality at least $10\ell/|\text{diam}(B_*^{(d)})|$ such that $\{0\} = \sum_{i \in I}(B_*^{(d)} + i) \cap \{0, \ldots, 10\ell\}$. Then $A = \sum_{i \in I}(B_* + i\vec{e}_d) \cap \Lambda$ and Lemma 5.1, which continues to hold in our generality, implies that $\mu^+_\Lambda([\sigma]_A) \geq \tanh(\beta/2)^{|I|}$ i.e. $\ell_c^{(\text{cavity})} = \Omega(e^\beta)$.

(iii) Let $A \subset \mathbb{Z}^d$ be a finite set such that $\min\{d(x,y) : x,y \in A\} \geq \ell$. W.l.o.g. we assume that A contains the origin. Let $k_i = \max\{j \geqslant 0 : j \in B_*^{(i)}\}$ for $i \in [d]$, and let $\mathbb{L} = \{\sum_{i=1}^d a_i k_i \vec{e}_i : a \in \mathbb{Z}^d\}$. Clearly $\mathbb{L}$ is isomorphic to $\mathbb{Z}^d$. We set $A_1 = A \cap \mathbb{L}$ and $A_2 = A \setminus A_1$. We can compute $\mu^\beta([\sigma]_A)$ by first conditioning on the spins at the vertices of $\mathbb{Z}^d \setminus \mathbb{L}$ to get

$$\mu^\beta([\sigma]_A) = \mu^\beta\Big([\sigma]_{A_2}\, \mu^\beta\big([\sigma]_{A_1} \mid \sigma_{\mathbb{Z}^d \setminus \mathbb{L}}\big)\Big).$$

Observe now that, given the spins on the vertices of $\mathbb{Z}^d \setminus \mathbb{L}$, the law of the remaining spins is again a FTM with fundamental plaquette $B'_* = \{0,1\}^d$ (upon identifying $\mathbb{L}$ with $\mathbb{Z}^d$) and Hamiltonian $H'(\eta) = -\frac{1}{2}\sum_{a \in \mathbb{Z}^d} J_a[\eta]_{B'_*+a}$, where $J_a \in \{-1,1\}$ and the sign is determined by the conditioning spins.

Using [24] (cf. also [22, Formula 4.7]) we get that

$$|\mu^\beta\big([\sigma]_{A_1} \mid \sigma_{\mathbb{Z}^d\setminus\mathbb{L}} = \hat\sigma\big)| \leq \mu^\beta\big([\sigma]_{A_1} \mid \sigma_{\mathbb{Z}^d\setminus\mathbb{L}} \equiv +\big), \quad \forall\, \hat\sigma \in \{-1,1\}^{\mathbb{Z}^d\setminus\mathbb{L}}. \tag{7.1}$$

In the r.h.s. of (7.1) the average is computed w.r.t. the *ferromagnetic* trivial factorizable Hamiltonian H' corresponding to $J_a = +1 \,\forall\, a \in \mathbb{Z}^d$. In particular it is equal to zero if $A_1 \not\sim \emptyset$ and equal to $\tanh(\beta/2)^{n(A_1)}$ otherwise, where the equivalence relation $\sim$ is taken using as fundamental plaquette the hypercube B'_* and $n(A_1)$ is the size of the corresponding minimal plaquette decomposition (see (5.2)). By generalising the results of Section 5.0.2 for the SPM case, since $A_1 \neq \emptyset$ and $\min\{d(x,y) : x,y \in A_1\} \geq \ell$, one gets that $n(A_1) \geq c\ell^d$ for some constant c depending on the k_i's. It is immediate to conclude that $\ell_c^{(\text{multispin})} = O(e^{\beta/d})$.

APPENDIX A.

As anticipated in Section 3 we collect here the proof of the ordering of the three length scales $\ell_c^{(\text{mix})}, \ell_c^{(\text{cavity})}$ and $\ell_c^{(\text{multispin})}$ and of Proposition 5.2.

1.1. **Proof of (3.1).** We begin by proving that $\ell_c^{(\text{multispin})} = O(\ell_c^{(\text{cavity})})$. Let A be a finite set, with $|A| \geqslant 2$, such that $\min_{x,y \in A} d(x,y) \geq C\ell_c^{(\text{cavity})}$ and fix $x \in A$. Let also Λ be the square centered at x of side $10\ell_c^{(\text{cavity})}$. For C large enough no other points of A belong to Λ. Using the DLR equations we write

$$\mu^\beta([\sigma]_A) = \int d\mu^\beta(\tau)\, [\tau]_{A\setminus\{x\}}\, \mu_\Lambda^{\beta,\tau}(\sigma_x).$$

Recall now that $\mu^\beta(\sigma_x) = 0$. Thus

$$|\mu_\Lambda^{\beta,\tau}(\sigma_x)| \leq \Big| \int d\mu^\beta(\tau')\, \Big(\mu_\Lambda^{\beta,\tau}(\sigma_x) - \mu_\Lambda^{\beta,\tau'}(\sigma_x)\Big) \Big|$$

$$\leq \sup_{\tau,\tau'} \big|\mu_\Lambda^{\beta,\tau}(\sigma_x) - \mu_\Lambda^{\beta,\tau'}(\sigma_x)\big| \leq \frac{1}{5},$$

by construction and the definition of $\ell_c^{(\text{cavity})}$. The result follows immediately.

Next we prove that $\ell_c^{(\text{cavity})} = O(\beta\,\ell_c^{(\text{mix})})$. Fix two concentric squares $V \subset \Lambda$ of side ℓ and 10ℓ together with $f : \Omega_V \mapsto \mathbb{R}$ with $\|f\|_\infty \leq 1$. The triangular inequality and (3.4) (notice that $h_x \geq e^{-2\beta\|H\|}$) imply that

$$\big|\mu_\Lambda^{\beta,\tau}(f) - \mu_\Lambda^{\beta,\tau'}(f)\big| \leq e^{2\beta\|H\|} \sum_{x\in\Lambda^c} \sup_\xi \big|\operatorname{Cov}_\Lambda^{\beta;\xi}(h_x, f)\big|.$$

Choose now $\ell = C\beta\,\ell_c^{(\text{mix})}$. The definition of $\ell_c^{(\text{mix})}$ together with (3.2) and Proposition 3.4 implies that there exists a constant c such that the r.h.s. above is not larger than $e^{c\beta}e^{-\ell/\ell_c^{(\text{mix})}} \ll 1$ for $C \gg 1$.

1.2. **Proof of Proposition 5.2.** Let $\Lambda = [-\ell, \ell]^2 \cap \mathbb{Z}^2$ and recall the definition of the plaquettes family $\mathcal{B}^+(\Lambda)$ under plus boundary conditions and of the associated space of cycles $\mathcal{K}^+(\Lambda)$, given right before Lemma 5.1. In what follows it will useful to think of $\mathcal{K}^+(\Lambda)$ as $\mathbb{F}_2$–vector space, by taking the symmetric difference as summation. In particular, if $\mathcal{A}$ is a collections of cycles in $\mathcal{K}^+(\Lambda)$ then a plaquette B^+ will belong to $\sum_{\alpha \in \mathcal{A}} \alpha$ iff it belongs to an odd number of cycles in $\mathcal{A}$. In $\mathcal{K}^+(\Lambda)$ we consider the following collection $\mathcal{G}$ of special cycles $\{R_i, C_i\}_{i=0}^{2\ell+1}$, called row and column cycles respectively. Order the rows and columns of Λ from bottom to top and left to right. For $i, j \notin \{0, 2\ell+1\}$, the cycle $R_i(C_j)$ consists of all the plaquettes in $\mathcal{B}^+(\Lambda) \cap \mathcal{B}(\Lambda)$ whose lowermost (leftmost) vertex lies in the ith-row (j^{th}-column) of Λ. For $i = 0$ ($i = 2\ell + 1$) R_i consists of all the plaquettes in $\mathcal{B}^+(\Lambda)$ lying on the first (last) row. Similarly for C_0, C_ℓ.

The following are few elementary properties of $\mathcal{G}$ which we collect for convenience in a lemma whose proof is omitted.

Lemma A.1.

(a) *Let $\mathcal{A} \subset \mathcal{G}$ be nonempty and suppose that $\sum_{\alpha \in \mathcal{A}} \alpha = \emptyset$. Then $\mathcal{A} = \mathcal{G}$.*
(b) *Let $\mathcal{A}, \mathcal{B} \subset \mathcal{G}$. Suppose that $\sum_{\alpha \in \mathcal{A}} \alpha = \sum_{\alpha \in \mathcal{B}} \alpha$. Then $\mathcal{A} = \mathcal{B}$ or $\mathcal{B} = \mathcal{G} \setminus \mathcal{A}$.*
(c) *Given $\alpha' \in \mathcal{K}(\Lambda)$ there exists $\mathcal{A} \subset \mathcal{G}$ such that $\alpha' = \sum_{\alpha \in \mathcal{A}} \alpha = \sum_{\alpha \in \mathcal{G} \setminus \mathcal{A}} \alpha$.*

Corollary A.2. *Given $f : \mathcal{K}^+(\Lambda) \to \mathbb{R}$, we have the identity*

$$\sum_{\alpha \in \mathcal{K}^+(\Lambda)} f(\alpha) = \frac{1}{2} \sum_{W \subset \mathcal{G}} f(\alpha(W)),$$

where $\alpha(W) := \sum_{\alpha \in W} \alpha$.

We are now ready to write a workable formula for $\mu_\Lambda^+(\sigma_0)$. Let α_* denotes the family of plaquettes in $\mathcal{B}^+(\Lambda)$ contained in $[0, \ell] \times [0, -\ell] \cap \mathbb{Z}^2$ so that $\sum_{B^+ \in \alpha_*} B^+ = \{0\}$. Using [8, eq. (2.12)] and Corollary A.2 we have

$$\mu_\Lambda^+(\sigma_0) = \frac{\sum_{W \subset \mathcal{G}} \tanh(\beta/2)^{|\alpha(W) \triangle \alpha_*|}}{\sum_{W \subset \mathcal{G}} \tanh(\beta/2)^{|\alpha(W)|}} \equiv \frac{N(\beta)}{D(\beta)}. \tag{A.1}$$

Lemma A.3. *If $W \subset \mathcal{G}$ contains i row cycles and j column cycles then $|\alpha(W)| = (i + j)(2\ell + 2) - 2ij$. Moreover, if $W \subset \mathcal{G}$ consists of u rows among $R_0, R_1, \ldots, R_\ell$, v rows among $R_{\ell+1}, R_{\ell+2}, \ldots, R_{2\ell+1}$, j columns among $C_0, C_1, \ldots, C_\ell$ and k columns among $C_{\ell+1}, C_{\ell+2}, \ldots, C_{2\ell+1}$, then*

$$|\alpha(W) \triangle \alpha_*| = jL + vL - 2vj - 2uj - 2vk + 2uk.$$

Proof. For the first assertion we observe that each $\mathcal{B}^+(\Lambda)$–plaquette that appears in $\alpha(W)$ must be exactly either in a row cycle or in a column cycle of W. For the second assertion, we note that $|\alpha(W) \triangle \alpha_*|$ can be obtained from $|\alpha(W)| = (u + v + j + k)(2\ell + 2) - 2(u+v)(j+k)$ (as in the first assertion) by subtracting twice the number of $\mathcal{B}^+(\Lambda)$–plaquettes of $\alpha(W)$ inside $[0, \ell] \times [0, -\ell]$ (this number is $(u+k)(\ell+1) - 2uk$) and adding the number all $\mathcal{B}^+(\Lambda)$–plaquettes in $[0, \ell] \times [0, -\ell]$ (this number is $(\ell + 1)^2$). $\qquad \square$

For notation convenience let $L = 2\ell + 2$ and let $t = \tanh(\beta/2)$. As a consequence of the above lemma we get immediately

$$D(\beta) = \sum_{i=0}^{L} \sum_{j=0}^{L} \binom{L}{i} \binom{L}{j} t^{iL+jL-2ij} = \sum_{i=0}^{L} \binom{L}{i} (t^i + t^{L-i})^L \tag{A.2}$$

$$N(\beta) = t^{\frac{L^2}{4}} \sum_{u,v,j,k} \binom{L/2}{u} \binom{L/2}{v} \binom{L/2}{j} \binom{L/2}{k} t^{jL+vL-2vj-2uj-2vk+2uk} \tag{A.3}$$

$$= t^{\frac{L^2}{4}} \sum_{u=0}^{L/2} \sum_{v=0}^{L/2} \binom{L/2}{u} \binom{L/2}{v} (t^{2v} + t^{L-2v} + t^{2u} + t^{L-2u})^{L/2}. \tag{A.4}$$

In order to bound from below the ratio $N(\beta)/D(\beta)$ it is convenient to rewrite both expressions in a probabilistic fashion. Let X be a centered $Bin(L, 1/2)$ random variable and let Y, Z be i.i.d. centered $Bin(L/2, 1/2)$ random variables. Then (A.2) and a little algebra give

$$D(\beta) = 2^{2L} t^{L^2/2} \, \mathbb{E}\left(\left[\frac{t^X + t^{-X}}{2} \right]^L \right),$$

$$N(\beta) = t^{\frac{L^2}{2}} 2^{2L} \mathbb{E}\left(\left[\frac{t^{2Y} + t^{-2Y} + t^{2Z} + t^{-2Z}}{4} \right]^{L/2} \right).$$

Trivially $N(\beta) \geq 2^{2L} t^{\frac{L^2}{2}}$. Moreover $\cosh(x) \leq e^{x^2/2}$ implies that

$$D(\beta) \leq 2^{2L} t^{L^2/2} \, \mathbb{E}(e^{(\log t)^2 L X^2/2}).$$

Thus

$$\frac{N(\beta)}{D(\beta)} \geq \frac{1}{\mathbb{E}(e^{(\log t)^2 L X^2/2})} .$$

Next we bound from above the above expectation value. For any $\lambda > 0$ we write

$$\mathbb{E}(e^{(\log t)^2 L X^2/2}) \leq \lambda + \int_{\lambda}^{\infty} da \, \mathbb{P}(e^{(\log t)^2 L X^2/2} \geq a).$$

The Azuma-Hoeffding's inequality (cf. e.g. [17]) implies that $\mathbb{P}(|X| \geq \kappa) \leq 2e^{-2\kappa^2/L}$, so that

$$\mathbb{P}(e^{(\log t)^2 L X^2/2} \geq a) \leq 2 \exp\left(-4 \log a/(\log t)^2 L^2 \right) = 2a^{-4/((\log t)^2 L^2)}.$$

If $\gamma := \frac{4}{(\log t)^2 L^2} > 1$, then

$$\mathbb{E}(e^{\log(t)^2 L X^2/2}) \leq \lambda + \frac{2}{(\gamma - 1)\lambda^{\gamma-1}}, \tag{A.5}$$

which, after optimising over the free parameter λ, becomes

$$\mathbb{E}(e^{(\log t)^2 L X^2/2}) \leq 2^{1/\gamma} + \frac{2}{(\gamma - 1)2^{(\gamma-1)/\gamma}}.$$

Notice that $\gamma > 1$ is equivalent to $L < 2/|\log t| = e^{\beta}(1 + o(1))$. Recalling that $L = 2\ell + 2$ we conclude that for any $\delta \in (0, 1)$ and any $\ell < (1 - \delta)e^{\beta}/2$

$$\liminf_{\beta \to \infty} \mu_{\Lambda}^{+}(\sigma_0) = \liminf_{\beta \to \infty} \frac{N(\beta)}{D(\beta)} > 0$$

as required. $\qquad\square$

1.3. **Proof of Lemma 4.8.** To prove the first assertion it is enough to show that if $\alpha, \gamma \in \mathcal{K}(T_*^{(n)})$ then $\alpha + \gamma \in \mathcal{K}(T_*^{(n)})$. To this aim take $v \in T_*^{(n)}$ and call $\Delta_1, \Delta_2, \Delta_3$ the three triangular plaquettes containing v. We set $n_i := \mathbb{1}(\Delta_i \in \alpha)$ and $m_i := \mathbb{1}(\Delta_i \in \gamma)$. Since α and γ are cycles of T_n^* we have $n_1 + n_2 + n_3 \equiv 0 \mod 2$, and $m_1 + m_2 + m_3 \equiv 0 \mod 2$. If we call $k_i := \mathbb{1}(\Delta_i \in \alpha + \gamma)$, by definition of $\alpha + \gamma$ we have $k_i = n_i + m_i \mod 2$. We conclude that $k_1 + k_2 + k_3 \equiv 0 \mod 2$, hence v belongs to an even number of plaquettes in $\alpha + \gamma$. By the arbitrariness of $v \in T_*^{(n)}$ we conclude that $\alpha + \gamma$ is a cycle.

We now prove that $P_{-1}, P_0, \ldots, P_n$ is a basis of $\mathcal{K}(T_*^{(n)})$. We observe that the Pascal's triangle $\mathcal{P}_0$ (rooted at the origin 0 of $\mathbb{Z}^2$) is a cycle in $\mathbb{Z}^2 \setminus \{0\}$, that is every site in $\mathbb{Z}^2$ is contained in an even number of plaquettes of $\mathcal{P}_0$ except for the origin. It follows immediately that P_i is a cycle of $T_*^{(n)}$ (see Fig. 4). To prove that $P_{-1}, \ldots, P_n$ are linearly independent, suppose that $a_{-1}P_{-1} + \cdots + a_nP_n = \emptyset$ with $a_i \in \mathbb{F}_2$. By construction we have $(i, -1) + B_* \in P_j$ if and only if $i = j$. Hence $(i, -1) + B_*$ appears a_i times in the cycle $a_{-1}P_{-1} + \cdots + a_nP_n = \emptyset$, and therefore $a_i = 0$ for each $i = -1, 0, \ldots, n$.

It remains to show that $P_{-1}, P_0, \ldots, P_n$ generates $\mathcal{K}(T_*^{(n)})$. For this we will use the following result:

Claim A.4. *Every non-empty cycle in $\mathcal{K}(T_*^{(n)})$ contains a plaquette of the form $(i, -1) + B_*$ for some $i \in \{-1, 0, \ldots, n\}$.*

Before proving our claim, we conclude the proof of Lemma 4.8. Fix $\alpha \in \mathcal{K}(T_*^{(n)})$, let $I(\alpha) = \{i \in \{-1, 0, \ldots, n\} : (i, -1) + B_* \in \alpha\}$, and consider the cycle $\gamma := \alpha + \sum_{i \in I(\alpha)} P_i$. By construction $(i, -1) + B_* \notin \gamma$ for any $i = -1, 0, \ldots, n$. Therefore, by Claim A.4, $\gamma = \emptyset$. Equivalently, we have $\alpha = \sum_{i \in I(\alpha)} P_i$ as required.

Proof of Claim A.4. Suppose, for contradiction, that α is a non-empty cycle and $(i, -1) + B_* \notin \alpha$ for any $i = 0, \ldots, n$. Let $\mathcal{R}$ be the horizontal line passing through the lowest vertices contained in any plaquette of the cycle α. By assumption $\mathcal{R}$ lies on or above the line $\{k\vec{e}_1 : k \in \mathbb{Z}\}$, and therefore $\mathcal{R} \cap (\cup_{B \in \alpha} B) \subset T_*^{(n)}$ and is non empty. Fix $v \in \mathcal{R} \cap (\cup_{B \in \alpha} B)$, by the definition of a cycle v must belong to an even number of plaquettes in α. Since v belongs to exactly one plaquette rooted on $\mathcal{R}$, and two plaquette rooted below $\mathcal{R}$, α must contain at least one plaquette rooted below $\mathcal{R}$, which contradicts the minimality of $\mathcal{R}$. $\qquad\square$

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