On stable right-inversion of non-minimum-phase systems

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Abstract— **The paper deals with the characterization of a dummy 'output function' associated with the stable component of the zero-dynamics of a linear square multi-input multi-output system. With reference to the 4-Tank dynamics, it is shown how such a procedure, applied to the linear tangent model of a nonlinear plant, may be profitably applied to assure local stability in closed loop.**

Index Terms— **Algebraic/geometric methods; Linear systems; Stability of nonlinear systems**

I. INTRODUCTION

As well known, most control problems are concerned with partial cancellation of the dynamics which is achieved by forcing unobservability [1]–[8]. In the linear case, this is achieved by designing a feedback assigning part of the eigenvalues coincident with the zeros of the system so making the corresponding dynamics unobservable. Such an approach is at the basis of feedback linearization which is achieved, in general, by cancelling the so-called zero-dynamics whose stability is thus necessary for guaranteeing feasibility of the control system [9].

The idea of employing *factorization*, properly introduced in [10] for studying the zero-dynamics of sampled-data systems, and consequently partial dynamic cancellation has been formalized and developed in [11] to deal with feedback linearization of nonlinear single-input single-output (SISO) non-minimum phase systems (i.e., whose zero-dynamics are unstable). The design approach represents a first generalization to the nonlinear context of the idea of assigning part of the eigenvalues over part of the zeros of the transfer function of a linear system (partial zero-pole cancelation). When considering dynamical systems, stability of the feedback system can be achieved when only a stable component of the zero-dynamics is cancelled. Such a stable component can be identified, in the SISO case, by considering the output associated with the minimum-phase factorization of the transfer function of the linear tangent model at the origin. However, when dealing with MIMO systems identifying such a stable component and hence the corresponding dummy output is still challenging.

In this paper, the results in [11] are extended to the multi-input multi-output context by providing a systematic procedure for extracting, via factorization, a dummy output identifying the minumum-phase component of a dynamical system. In particular, focusing on linear time-invariant and right-invertible dynamics, we show how the Smith form can be suitably exploited for factorizing the matrix transfer function and extract, in the state-space representation, the output identifying the minimum-phase component of the original system. Then, the geometric relations among the original system and the one with the dummy output are investigated in terms of invariant subspaces and making reference to MIMO normal forms [12]. In particular, it is shown that the new output identifies the largest control-invariant subspace contained in the kernel of the original one maximizing unobservability while, at the same time, preserving stability of the closed loop. This allows the definition of systematic solutions with stability of a large variety of control problems dealing with right-inversion (e.g., disturbance decoupling, tracking). The case of nonlinear systems is sketched through the simulated example of a four tanks dynamics dealt with at an academic level. The case of square systems is dealt with as the extension to larger number of inputs and outputs follows the same lines.

The paper is organized as follows. In Section II, recalls on MIMO systems are given and the problem is formulated. In III, the procedure for constructing an output associated to the minimum-phase component is presented and applied to several control problems in Section IV. In Section V, the example of a four tank dynamics serves, at an academic level, for sketching the extension to nonlinear dynamics with linear output while conclusions and perspectives are in Section VI.

Notations: $\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural numbers including 0, respectively. $\mathbb{C}^+$ (resp. $\mathbb{C}^-$) denote the left-hand (resp. right-hand) side of the complex plane. $\mathrm{Mat}_{\mathbb{R}}(n,m)$ defines the group of real matrices of dimension $n \times m$ with, for short, $\mathrm{Mat}_{\mathbb{R}}(n) = \mathrm{Mat}_{\mathbb{R}}(n,n)$. Given a matrix $A \in \mathrm{Mat}_{\mathbb{R}}(n)$, $\sigma\{A\}$ defines its spectrum. For a sorted set of $a_i \in \mathbb{R}$ with $i = 1, \dots, n$, $\mathrm{diag}\{a_1, \dots, a_n\}$ defines a diagonal matrix with a_i being the diagonal elements. For a smooth vector field f, L_f denotes the Lie derivative operator, $\mathrm{L}_f = \sum_{i=1}^n f_i(\cdot)\frac{\partial}{\partial x_i}$.

II. PRELIMINARIES AND PROBLEM STATEMENT

Consider a linear time invariant (LTI) system of the form

$$\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx \tag{1b}$$

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with $u,\, y \in \mathbb{R}^2$, $x \in \mathbb{R}^n$, $B = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$, $C^\top = \begin{pmatrix} c_1^\top & c_2^\top \end{pmatrix}^\top$ and transfer function

$$P(s) = C(sI - A)^{-1}B. \tag{2}$$

The following standing assumptions are set.

A1. The pairs (A, B) and (C, A) are, respectively, controllable and observable.

A2. The system (1) is right-invertible [12], [13].

A3. The system (1) is partially minimum-phase; i.e., the zero polyonmial defined as

$$z(s) = \det \begin{pmatrix} sI - A & B \\ -C & \mathbf{0} \end{pmatrix} = z_u(s) z_s(s), \quad s \in \mathbb{C}$$

is non-Hurwitz with $z_s(s)$ denoting the corresponding Hurwitz component with roots in the left hand side of the complex plane.

In the following, general recalls for MIMO linear systems are given as instrumental for the problem we address.

A. The Smith McMillan form

Consider any $p \times p$ polynomial matrix $N(s)$, then there exist elementrary row and column operations, or corresponding unimodular matrices $\{L^{-1}(s), R^{-1}(s)\}$ such that

$$N(s) = L(s) N_{sm}(s) R(s) \tag{3}$$

with

$$N_{sm}(s) = \text{diag}\{\epsilon_1(s), \dots, \epsilon_p(s)\} \tag{4}$$

where $\{\epsilon_i(s)\}$ are unique monic polynomials verifying $\epsilon_i(s)$ is a factor of $\epsilon_{i+1}(s)$ for all $i = 1, \dots, p-1$. Moreover, by denoting as $\Delta_i(s)$ the greatest common divisor of all $i \times i$ minors of $N(s)$ for $i = 1, \dots, p$ one gets that $\epsilon_i(s) = \frac{\Delta_i(s)}{\Delta_{i-1}(s)}$ with $\Delta_0(s) = 1$ [14]. Note that, although $N_{sm}(s)$ is unique, $\{L(s), R(s)\}$ are not. Accordingly, one gets that the (rational) matrix transfer function (2) always admits a unique Smith form, that is

$$P(s) = L(s) N_{sm}(s) D^{-1}(s) R(s) \tag{5}$$

with $M(s) = N_{sm}(s) D^{-1}(s) = \text{diag}\{\frac{z_1(s)}{d_1(s)}, \dots, \frac{z_p(s)}{d_p(s)}\}$ where from **A1.**: $z(s) = z_1(s) \dots z_p(s) = z_u(s) z_s(s)$ corresponds to the zero-polynomial defined in **A3.**; $d(s) = d_1(s) \dots d_p(s)$ is the pole-polynomial with the property that $d_{i+1}(s)$ is a factor of $d_i(s)$. For the sake of notational simplicity, and without loss of generality, we will assume in the sequel that $p = 2$.

Remark 2.1: [14] Two matrices $sI - A$ and $sI - A_r$ possess the same Smith form if, and only if, A and A_r are similar. This easily extends to the case of two realizations (A, B, C) and (A_r, B_r, C_r) (with the same dimension) sharing the same transfer function $P(s)$.

B. Generalized normal forms and the zero-dynamics

As (1) is invertible [12, Chapter 9], one can pick constant $r_2 \ge r_1 > 0$ such that

$$c_i A^\ell B = 0, \quad \ell = 0, \dots r_i - 2, \quad c_i A^{r_i - 1} B \neq 0$$

and $\nu \ge 0$ such that there exist constant $\alpha_{r_2}, \dots, \alpha_{r_2+\nu-1} \in \mathbb{R}$ verifying for $j = 0, \dots, \nu - 1$

$$c_2 A^{r_2+j-1} B + c_1 A^{r_1} (\alpha_{r_2+j} I + \dots + \alpha_{r_2} A^{j-1}) B = 0.$$

and

$$M = \begin{pmatrix} c_1 A^{r_1 - 1} B \\ c_2 A^{r_2+\nu-1} B + c_1 A^{r_1} (\alpha_{r_2+\nu-1} I + \dots + \alpha_{r_2} A^{\nu-1}) B \end{pmatrix}$$
$$\det\{M\} \neq 0.$$

In this setting, one can define a coordinate transformation

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{\nu+2} \\ \eta \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_{\nu+2} \\ T_\eta \end{pmatrix} x, \quad z_i = \begin{pmatrix} z_{i,1} \\ \dots \\ z_{i,r_i} \end{pmatrix} \tag{6}$$

for $i = 1, 2$ and , for $j = 0, \dots, \nu - 1$ with

$$T_i = \begin{pmatrix} c_i^\top & \dots & (c_i A^{r_i - 1})^\top \end{pmatrix}^\top, \quad T_\eta B = 0$$
$$T_{j+3} = c_2 A^{r_2+j} + c_1 A^{r_1} (\alpha_{r_2+j} I + \dots + \alpha_{r_2} A^j)$$

such that

$$\begin{aligned}
\dot{z}_{i,\ell} &= z_{i,\ell+1}, \quad i = 1, 2, \ \ell = 1, \dots, r_i - 1 \\
\dot{z}_{1,r_1} &= c_1 A^{r_1} x + c_1 A^{r_1 - 1} B u \\
\dot{z}_{2,r_2} &= c_2 A^{r_2} x + c_2 A^{r_2 - 1} B u \\
&= z_3 - \alpha_{r_2} \big(c_1 A^{r_1} x + c_1 A^{r_1 - 1} B u \big) \\
\dot{z}_{j+3} &= c_2 A^{r_2+j+1} x + c_1 A^{r_1+1} (\alpha_{r_2+j} I + \dots + \alpha_{r_2} A^j) x \\
&= z_{j+4} - \alpha_{r_2+j+1} \big(c_1 A^{r_1} x + c_1 A^{r_1 - 1} B u \big) \\
\dot{z}_{\nu+2} &= c_2 A^{r_2+\nu} x + c_1 A^{r_1+1} (\alpha_{r_2+\nu-1} I + \dots + \alpha_{r_2} A^{\nu-1}) x \\
&\quad + \big(c_2 A^{r_2+\nu-1} + c_1 A^{r_1} (\alpha_{r_2+\nu-1} I + \dots + \alpha_{r_2} A^{\nu-1}) B \big) u
\end{aligned}$$

Accordingly, defining

$$R_1 z + S_1 \eta + \hat{b}_1 u := c_1 A^{r_1} T^{-1} \begin{pmatrix} z \\ \eta \end{pmatrix} + c_1 A^{r_1 - 1} B u$$

$$R_2 z + S_2 \eta := \big(c_2 A^{r_2+\nu} + c_1 A^{r_1+1} (\alpha_{r_2+\nu-1} I + \dots + \alpha_{r_2} A^{\nu-1}) \big) T^{-1} \begin{pmatrix} z \\ \eta \end{pmatrix}$$

$$\hat{b}_2 := \big(c_2 A^{r_2+\nu-1} + c_1 A^{r_1} (\alpha_{r_2+\nu-1} + \dots + \alpha_{r_2} A^{\nu-1}) B \big) u$$

one gets for $i = 1, 2$, $\ell_i = 1, \dots, r_i - 1$ and $j = 0, \dots, \nu - 2$

$$\begin{aligned}
\dot{z}_{1,\ell_1} &= z_{1,\ell_1+1}, \quad \dot{z}_{1,r_1} = R_1 z + S_1 \eta + \hat{b}_1 u \\
\dot{z}_{2,\ell_2} &= z_{2,\ell_2+1} \\
\dot{z}_{2,r_2} &= z_3 - \alpha_{r_2} \big(R_1 z + S_1 \eta + \hat{b}_1 u \big) \\
\dot{z}_{j+3} &= z_{j+4} - \alpha_{r_2+j+1} \big(R_1 z + S_1 \eta + \hat{b}_1 u \big) \\
z_{\nu+2} &= R_2 z + S_2 \eta + \hat{b}_2 u \\
\dot{\eta} &= P z + Q \eta \\
y_1 &= z_{1,1}, \quad y_2 = z_{2,1}
\end{aligned} \tag{7}$$

that is the MIMO normal form associated to (1).

As a straightforward consequence of (7), one gets that the zero-dynamics of (1) is $\dot{\eta} = Q\eta$ with $\eta \in \mathbb{R}^{n-r_1-r_2-\nu}$ with $n-r_1-r_2-\nu$ being the excess poles-zeros and $\sigma\{Q\} = \{s \in \mathbb{C} \text{ s.t. } z(s) = 0\}$ that is, the eigenvalues of Q correspond to the transmission zeros of (1).

From the properties described above it is hence immediate to state that the control

$$\begin{aligned} u =& M^{-1}(v - Lx) \\ \overset{(6)}{=}& M^{-1}v - M^{-1}\begin{pmatrix} R_1 z + S_1\eta \\ R_2 z + S_2 z \end{pmatrix} \end{aligned} \tag{8}$$

with $v = \text{col}(v_1, v_2)$ and

$$L = \begin{pmatrix} c_1 A^{r_1} \\ c_2 A^{r_2+\nu} + c_1 A^{r_1+1}(\alpha_{r_2+\nu-1} I + \cdots + \alpha_{r_2} A^{\nu-1}) \end{pmatrix}$$

achieves right-invertibility of (1); namely, one gets

$$\begin{aligned} \dot{z}_{1,\ell} =& z_{1,\ell+1}, \quad \dot{z}_{1,r_1} = v_1 \\ \dot{z}_{2,\ell} =& z_{2,\ell+1}, \quad \dot{z}_{2,r_2} = z_3 - \alpha_{r_2} v_1 \\ \dot{z}_{j+3} =& z_{j+4} - \alpha_{r_2+j+1} v_1 \\ z_{\nu+2} =& v_2 \\ \dot{\eta} =& Pz + Q\eta \\ y_1 =& z_{1,1}, \quad y_2 = z_{2,1}. \end{aligned} \tag{9}$$

Remark 2.2: In this setting, the largest control-invariant subspace contained in $\ker C$ is given by

$$\mathcal{V}^\star = \ker\begin{pmatrix} c_1 \\ \vdots \\ c_1 A^{r_1-1} \end{pmatrix} \cap \ker\begin{pmatrix} c_2 \\ \vdots \\ c_2 A^{r_2-1} \end{pmatrix}$$
$$\cap_{j=0}^{\nu-1} \ker\begin{pmatrix} c_2 A^{r_2} + \alpha_{r_2} c_1 A^{r_1} \\ \vdots \\ c_2 A^{r_2+\nu-1} + c_1 A^{r_1}(\alpha_{r_2+\nu-1} I + \ldots + \alpha_{r_2} A^{\nu-1}) \end{pmatrix}.$$

Accordingly, the feedback law (8) represents the friend of $\mathcal{V}^\star \subset \ker C$ and thus the one achieving maximum unobservability via zeros cancellation. In general, we refer to $\mathcal{V}^\star$ as the zero-dynamics subspace.

Remark 2.3: We note that if $\nu = 0$ one recovers the standard normal form issued when (1) possesses a well-defined relative degree with non-singular decoupling (and right-invertibility) matrix provided by

$$M = \begin{pmatrix} c_1 A^{r_1-1} B \\ c_2 A^{r_2-1} B \end{pmatrix}.$$

In general, as $\nu > 0$ the above form shows that non interaction (and input/output decoupling) cannot be achieved through static state-feedback. However, the same does not stand for disturbance decoupling.

III. Stable zero factorization of MIMO systems

In this section, we extend the approach proposed in [11] for extracting the minimum-phase component of a general non-minimum phase systems (1). The approach is based on *output factorization*; namely, starting from (1), we identify a new dummy output $y^s(t) = C_s x(t)$ corresponding to the stable component of the zero-dynamics associated to (1) and related to (1b) through the differential equation

$$y(t) = Z_u(\mathrm{d}) y^s(t) \tag{10}$$

with $\mathrm{d} = \frac{\mathrm{d}}{\mathrm{d}t}$ and a suitably defined two dimensional square differential matrix

$$Z_u(\mathrm{d}) = \begin{pmatrix} z_{1,1}(\mathrm{d}) & z_{1,2}(\mathrm{d}) \\ z_{2,1}(\mathrm{d}) & z_{2,2}(\mathrm{d}) \end{pmatrix}. \tag{11}$$

More in details, starting from (5), one can split the zero-matrix as $N_{sm}(s) = N_u(s)N_s(s)$ with $N_u(s) = \text{diag}\{z_{u,1}(s), z_{u,2}(s)\}$ and $N_s(s) = \text{diag}\{z_{s,1}(s), z_{s,2}(s)\}$ such that $z_u(s) = z_{u,1}(s)z_{u,2}(s), z_s(s) = z_{s,1}(s)z_{s,2}(s)$ containing, respectively, the zeros on the right and left hand side of the complex plane; that is $\det(N_s(s)) = z_s(s)$ and $\det(N_u(s)) = z_u(s)$. Accordingly, the (5) rewrites as

$$\begin{aligned} P(s) =& L(s)N_u(s)N_s(s)D^{-1}(s)R(s) \\ =& Z_u(s)P_s(s) \end{aligned} \tag{12}$$

with $Z_u(s) = L(s)N_u(s)$ and $P_s(s) = N_s(s)D^{-1}(s)R(s)$.

In particular, $Z_u(s)$ is a polynomial matrix in s whereas $P_s(s)$ is transfer function matrix.

Remark 3.1: When (1) possesses distinct poles with unitary algebraic multiplicity the term $N_s(s)D^{-1}(s)R(s)$ is improper as $(L(s), R(s))$ introduce poles at $s = \infty$ [14]. To handle this issue [14], one can compute a matrix $K(s)$ (the so-called *right divisor*) such that: $\tilde{P}_s(s) = K(s)N_s(s)D^{-1}(s)R(s)$ is proper and with the same poles as (2) and z_s as zeros polynomial; $\tilde{Z}_u(s) = L(s)N_u(s)K^{-1}(s)$ is a polynomial matrix in $s \in \mathbb{C}$. Accordingly, (12) reads

$$P(s) = \tilde{Z}_u(s)\tilde{P}_s(s) \tag{13}$$

such that $\tilde{P}_s(s)$ is strictly proper and verifying (Remark 2.1)

$$\tilde{P}_s(s) = C_s(sI - A)^{-1}B. \tag{14}$$

The computation of such $K(s)$ might not be an easy task and needs to be performed through a vis-a-vis study. From now on, for the sake of clarity, we shall assume $K(s) = \mathrm{I}$ although all the results to come hold true in general as illustrated through the case study.

Proposition 3.1: Consider the system (1) under Assumptions **A1** to **A3** and let $z(s) = z_u(s)z_s(s)$ be the zero-polynomial where $z_s(s)$ denotes the Hurwitz component. Let the transfer function $P(s)$ be of the form (13) with $P_s(s)$ as in (14) and $y^s = C_s x$ solution to (10). Then, the system

$$\dot{x} = Ax + Bu \tag{15a}$$
$$y^s = C_s x \tag{15b}$$

identifies the minimum-phase component of (1) with zero polynomial given by $z_s(s)$.

Proof: The proof is a straightforward consequence of the Smith McMillan form associated to (2) and (12). ∎

Remark 3.2: The new output (15b) can be computed as follows. Let $(\hat{A}, \hat{B}, \hat{C})$ with $\hat{A} \in \text{Mat}_{\mathbb{R}}(\hat{n})$ with $\hat{n} \geq n$ be a realization of (14) and $(\hat{A}_\star, \hat{B}_\star, \hat{C}_\star)$ with $\hat{A}_\star \in \text{Mat}_{\mathbb{R}}(n)$

the corresponding restriction onto the observable and controllable subspaces. Denote by T and $T_\star$ the non-singular transformations putting, respectively, (A, B) and $(\hat{A}_\star, \hat{B}_\star)$ into the *eigenvalues assignment* canonical form [13](Ch.5). Then, because $TAT^{-1} = T_\star \hat{A} T_\star^{-1}$ and $TB = T_\star \hat{B}_\star$ one gets $C_s = \hat{C}_\star T_\star^{-1} T$.

IV. Applications

A. Right-invertibility with stability

As a consequence of the factorization in Section III, because of Assumption **A2**, one gets the following result.

Lemma 4.1: Consider the system (1) under Assumptions **A1** to **A3** and let (15) identify its minimum-phase component with transfer function (14). Then, (15) is right-invertible with indexes (r_1^s, r_2^s, ν^s) verifying $r_1^s + r_2^s + \nu^s = n - deg(z_s(s))$.

Proof: The proof is a straightforward consequence of the factorization (12). Indeed, as $R(s)$ is unimodular, the result follows from [15]. ∎

Remark 4.1: From the result above it is always possible to choose $L(s)$ such that the components of the matrix $Z_u(s)$ in (11) are such that $\deg\{z_{1,1}(s)\} = r_1^s - r_1$, $\deg\{z_{1,2}(s)\} = r_2^s + \nu^s - r_1$, $\deg\{z_{2,1}(s)\} = r_2^s + \nu^s - r_1$, $\deg\{z_{2,2}(s)\} = r_2^s + \nu^s - r_2 - \nu$ with $\deg\{\cdot\}$ denoting the degree of the corresponding polynomial and (r_1, r_2, ν) being the invertibility indices associated to (1).

By virtue of the result above and Section II-B, right-invertibility of the stable component of (1) can be achieved through right-invertibility of the same system with dummy output (15b).

Proposition 4.1: Consider the system (1) under Assumptions **A1** to **A3**. Let (15) identify the minimum phase component of (1) with invertibility indices (r_1^s, r_2^s, ν^s). Then, there exist $\alpha_{r_2^s+j} \in \mathbb{R}$ with $j = 0, \dots, \nu^s - 1$ such that

$$c_2^s A^{r_2^s+j-1} B + c_1^s A^{r_1^s} \sum_{\ell=0}^{j-1} \alpha_{r_2^s+\ell} A^{j-\ell-1} B = 0 \qquad (16)$$

and

$$M_s = \begin{pmatrix} c_1^s A^{r_1^s - 1} B \\ c_2^s A^{r_2^s+\nu^s-1} B + c_1^s A^{r_1^s} \sum_{i=0}^{\nu^s-1} \alpha_{r_2^s+i} A^{\nu^s-i-1} B \end{pmatrix}$$

with $\det\{M_s\} \neq 0$. Accordingly, the feedback

$$u = M_s^{-1}(v - L_s x) \qquad (17)$$

with

$$L_s = \begin{pmatrix} c_1^s A^{r_1^s-1} \\ c_2^s A^{r_2+\nu} + c_1^s A^{r_1^s+1}(\alpha_{r_2^s+\nu^s-1} I + \dots + \alpha_{r_2^s} A^{\nu^s-1}) \end{pmatrix}$$

performs right-invertibility of the minimum-phase component of (1).

Proof: Along the lines of Section II-B, we introduce the coordinate transformation

$$\begin{pmatrix} z_1^s \\ z_2^s \\ z_3^s \\ \vdots \\ z_{\nu^s+2} \\ \eta \end{pmatrix} = \begin{pmatrix} T_1^s \\ T_2^s \\ T_3^s \\ \vdots \\ T_{\nu^s+2}^s \\ T_{\eta^s} \end{pmatrix} x, \quad z_i^s = \begin{pmatrix} z_{i,1}^s \\ \dots \\ z_{i,r_i^s}^s \end{pmatrix} \qquad (18)$$

for $i = 1, 2$ and with, for $j = 3, \dots, \nu^s + 2$ so that, under the control (17), (15) gets the form

$$\begin{aligned} \dot{z}_{1,\ell}^s &= z_{1,\ell+1}^s, \quad \dot{z}_{1,r_1}^s = v_1 \\ \dot{z}_{2,\ell}^s &= z_{2,\ell+1}^s, \quad \dot{z}_{2,r_2^s} = z_3^s - \alpha_{r_2^s} v_1 \\ \dot{z}_{j+3}^s &= z_{j+4}^s - \alpha_{r_2^s+j+1} v_1 \\ z_{\nu^s+2}^s &= v_2 \\ \dot{\eta}^s &= P^s z^s + Q^s \eta^s \\ y_1^s &= z_{1,1}^s, \quad y_2^s = z_{2,1}^s \end{aligned}$$

with $\dot{\eta}^s = Q^s \eta^s$ being the asymptotically stable zero-dynamics with $\sigma\{Q^s\} = \{s \in \mathbb{C} \text{ s.t. } z_s(s) = 0\} \subseteq \{s \in \mathbb{C} \text{ s.t. } z(s) = 0\}$. From (10), one gets in the new coordinates

$$y = Z_u(\mathrm{d}) y^s = Z_u(\mathrm{d}) \begin{pmatrix} z_{1,1}^s \\ z_{2,1}^s \end{pmatrix} = \hat{C} z^s$$

and thus the result. ∎

As a consequence, one gets the following result.

Proposition 4.2: Consider the systems (1) and (15) and let $\mathcal{V}^\star$ and $\mathcal{V}_s^\star$ be, respectively, the largest controlled (A,B)-invariant subspaces contained in $\ker\{C\}$ and $\ker\{C_s\}$. Then, $\mathcal{V}_s^\star \subset \mathcal{V}^\star$.

Proof: For the ease of the proof, assume that for (1) $\nu = 0$. By virtue of (10) one has, for $i = 1, 2$ and $j = 0, \dots, r_i - 1$

$$\begin{aligned} \mathrm{d}^j y_i(t) = c_i A^j x &= \mathrm{d}^j z_{i,1}(\mathrm{d}) y_2^s + \mathrm{d}^j z_{i,2}(\mathrm{d}) y_2^s \\ &= z_{i,1}(\mathrm{d}) c_1^s A^j x + \mathrm{d}^j z_{i,2}(\mathrm{d}) c_s^s A^j x \end{aligned}$$

with, by definition of r_1, r_2, $\nabla_u(Z_u(\mathrm{d}))\mathrm{d}^j y^s) = 0$. By exploiting (16), it is a matter of computations to deduce that

$$\underbrace{\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}}_{V} = N \underbrace{\begin{pmatrix} T_1^s \\ \vdots \\ T_{\nu^s+2}^s \end{pmatrix}}_{V_s}.$$

with T_i and T_j^s as in (6) and (18) for $i = 1, 2$ and $j = 1, \dots, \nu^s + 2$ and N being an upper triangular full rank matrix. From the equality above one gets that $\mathcal{V}_s^\star = \ker\{V_s\} \subset \ker\{V\} = \mathcal{V}^\star$ and thus the result. ∎

The feedback (17) is the one generating in closed loop the maximal unobservability constrained to stability; namely, (17) is the one canceling only the Hurwitz component of the zero dynamics of the original system (1). In other words, (17) is rendering only the stable component of $\mathcal{V}^\star$ feedback-invariant (that is $\mathcal{V}_s^\star$).

Remark 4.2: It must be noted that, albeit (1) possesses a well-defined relative degree (that is $\nu = 0$), when introducing the new dummy output issued from Proposition 3.1 through the Smith form, one might get $\nu_s \geq 0$. However, ν^s defines the order of the dynamic extension that is necessary (over the control input u_1) for recovering a well-defined relative degree $(\hat{r}_1^s, r_2^s)$.

Remark 4.3: By virtue of Assumption **A2** and Remark 2.3, one gets that the asymptotic tracking problem of a smooth reference $y_r(t) \in \mathbb{R}^2$, corresponding to $z_r^s(t)$ in the new coordinates, admits a solution with stability (under dynamical feedback if $\nu^s > 0$) by solving the equivalent problem over the partially minimum-phase system (15) and with respect to the dummy output reference $y_r^s(t) \in \mathbb{R}^2$.

B. DDP of non minimum-phase systems with stability

Let (1) be affected by a disturbance $w \in \mathbb{R}^2$ that is

$$\dot{x} = Ax + Bu + Pw \quad (19a)$$
$$y = Cx \quad (19b)$$

with $P \in \mathrm{Mat}_{\mathbb{R}}(n, 2)$. In this section, it is shown how disturbance decoupling (DDP) can be solved with stability by making use of the new output deduced in Section III. It is worth recalling that, in general and regardless stability, disturbance decoupling is solvable if and only if $\mathrm{Im}\{P\} \subset \mathcal{V}^\star \subset \ker\{C\}$. However, the corresponding solution guarantees stability of the closed loop if and only if the zero-dynamics associated to (19) is asymptotically stable. The next statement provides a new result ensuring the existence of a disturbance decoupling controller preserving stability of the internal dynamics.

Theorem 4.1: Consider (19) under assumptions **A1.** to **A3.** and the dummy output (15b) defined by Proposition 3.1. Then, output disturbance decoupling with stability for (19) is solvable for all P verifying $\mathrm{Im}\{P\} \subset \mathcal{V}_s^\star \subset \ker\{C\}$ where $\mathcal{V}_s^\star$ is the largest (A, B)-invariant subspace contained in $\ker\{C_s\}$. In addition, the corresponding feedback is (17).

Proof: The proof is a straightforward consequence of Proposition 4.2 ensuring $\mathcal{V}_s^\star \subset \mathcal{V}^\star \subset \ker\{C\}$. ■

Remark 4.4: By virtue of Remark 4.2, one gets that dynamical feedback extension is unnecessary for solving DDP whenever the system is right-invertible even if (15) do not possess a well-defined relative degree.

V. The 4-Tanks as an example

Consider the case of a 4-tanks system [16] given by

$$\dot{h} = f(h) + Bu \quad (20a)$$
$$y = Ch \quad (20b)$$

with $h = \mathrm{col}\{h_1, h_2, h_3, h_4\},\ f(h) = 2F(h)h$

$$F(h) = \begin{pmatrix} -p_1(h_1) & 0 & \frac{A_3}{A_1}p_3(h_3) & 0 \\ 0 & -p_2(h_2) & 0 & \frac{A_4}{A_2}p_4(h_4) \\ 0 & 0 & -p_3(h_3) & 0 \\ 0 & 0 & 0 & -p_4(h_4) \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{pmatrix}, \quad C = \kappa_t \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}^\top$$

$p_i(h_i) = \frac{c_i\sqrt{2gh_i}}{2A_ih_i}$. For the sake of compactness, let b_{ij} correspond to the element in position (i, j) of the input-state matrix B. In particular, h_i, A_i and c_i are, respectively, the level of water in the i^{th}-tank, its cross-section area and the cross-section of the outlet hole for $i = 1, 2, 3, 4$. The control signals u_j with $j = 1, 2$ correspond to the voltage applied to j^{th}-pump with k_ju_j being the corresponding flow. We consider the problem of locally asymptotically tracking the output of (20) to a desired $y_\star = (h_1^\star,\ h_2^\star)$ corresponding to make $h^\star = (h_1^\star, h_2^\star, h_3^\star, h_4^\star)^\top$ with

$$h_3^\star = \frac{(c_1\gamma_2\sqrt{h_1^\star} - c_2(1-\gamma_2)\sqrt{h_2^\star})^2}{c_3^2a_3^2\gamma_2^2}$$
$$h_4^\star = \frac{(c_2\gamma_1\sqrt{h_2^\star} - c_1(1-\gamma_1)\sqrt{h_1^\star})^2}{c_4^2a_4^2\gamma_1^2}$$

for $a_3 = \frac{\gamma_2}{1-\gamma_2} - \frac{1-\gamma_1}{\gamma_1}$ and $a_4 = \frac{\gamma_1}{1-\gamma_1} - \frac{1-\gamma_2}{\gamma_2}$ a locally asymptotically stable equilibrium for the closed-loop system under nonlinear feedback.

1) Analysis of the zero-dynamics: The vector relative degree of (20) is well defined and given by $r = (1\ 1)$ so that it exhibits a two-dimensional zero-dynamics. Accordingly, for investigating minimum-phaseness of (20), one computes the linear tangent model (LTM) at $h^\star$ of the form (1) with $x = h - h^\star$ and $A = 2F(h^\star)$ with corresponding transfer function matrix

$$P(s) = \kappa_t \begin{pmatrix} \frac{b_{11}}{s+p_1} & \frac{b_{32}\,p_3}{(s+p_1)\,(s+p_3)} \\ \frac{b_{41}\,p_4}{(s+p_2)\,(s+p_4)} & \frac{b_{22}}{s+p_2} \end{pmatrix} \quad (21)$$

$p_i = p_i(h_i^\star) > 0$ for $i = 1, 2, 3, 4$, Smith form as $M(s) = \mathrm{diag}\{\frac{1}{d(s)}, z(s)\}$, with pole-polynomial $d(s) = (s + p_1)(s + p_2)(s + p_3)(s + p_4)$ and zero-polynomial $z(s) = s^2 + (p_3 + p_4)s + \frac{p_3p_4}{b_{11}b_{22}}(b_{11}b_{22} - b_{32}b_{41})$. Thus, (20) is nonminimum-phase if $b_{11}b_{22} - b_{32}b_{41} < 0$ so that one can factorize $z(s) = (s-z_u)(s-z_s)$ for $z_u \in \mathbb{R}^+$ and $z_s \in \mathbb{R}^-$. As a consequence, if $b_{11}b_{22} - b_{32}b_{41} < 0$, output regulation to $y_\star$ cannot be achieved through classical right-inversion even if the relative degree is well-defined.

In the following we show how the procedure detailed in Section III allows to deduce a new output $y_s = C_sh$ and a nonlinear feedback locally solving the regulation problem with stability for (20).

2) The new dummy output: By virtue of Remark 3.1, because (A, B, C) possesses three distinct poles in general, one gets that the matrix $P_s(s) = \mathrm{diag}\{1, s - z_s\}\mathrm{diag}\{d(s), 1\}R(s)$ is improper for all choices of $(L(s), R(s))$. However, by suitably setting $K(s)$[1] so to make $\tilde{P}_s(s) = K(s)P_s(s)$ rational one gets the dummy output

$$y_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{b_{32}b_{41}p_4}{2b_{11}\beta} & \frac{b_{32}}{2} - \frac{b_{32}(p_3+p_4)}{2\beta} & -\frac{b_{22}}{2} - \frac{b_{22}(p_3+p_4)}{2\beta} & \frac{b_{32}\,p_4}{2\beta} \end{pmatrix} h \quad (22)$$

with $\beta = \sqrt{(p_3 + p_4)^2 - 4\frac{p_3p_4}{b_{11}b_{22}}(b_{11}b_{22} - b_{32}b_{41})}$ making the LTM model of (20a) minimum-phase.

3) Asymptotic tracking with stability: It is easily checked that, the nonlinear dynamics (20a) with output as in (22) possesses a well-defined relative degree $r_s = (1,\ 2)$ at $h^\star$. Also, it is a matter of computations to verify that (20a) with output (22) is locally minimum-phase with zero-dynamics $\dot{\eta}^s = q_s(0, \eta^s)$ verifying $\frac{\partial q_s}{\partial \eta_s}(0, \eta_\star^s) = z_s < 0$. At this point,

[1] For the sake of space, $(L(s), R(s), K(s))$ are reported at `https://hal.archives-ouvertes.fr/hal-02526676`.

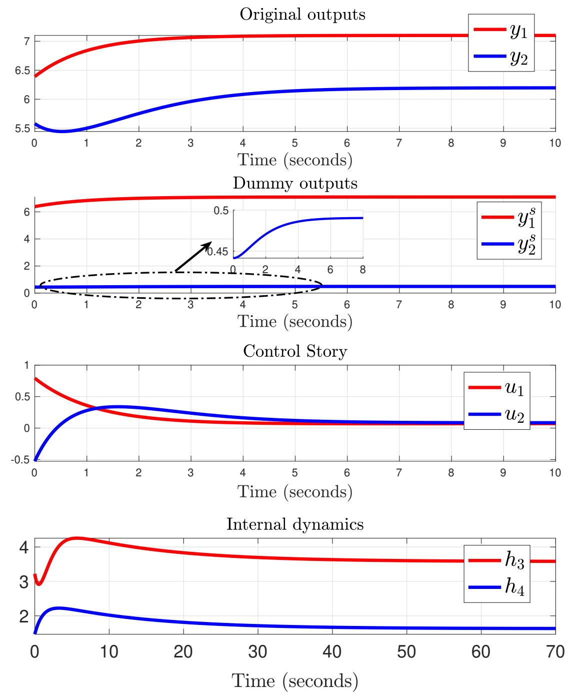

Fig. 1. The four tank model under stable dynamic inversion.

along the lines of Remark 4.3 and by exploiting the results in [9, Chapter 5], one gets that output tracking of (20) can be solved over the dummy output (22) by setting the constant $y^s_\star = (y^s_{1,\star}, y^s_{2,\star})^\top \in \mathbb{R}^2$ as solution to $y_\star = Z_u(\mathrm{d})y^s_\star$ which is given by construction as $y^s_\star = C_s h^\star$. Accordingly, for all $k_0, k_1 > 0$ the feedback

$$u = -M_s^{-1}(h)\begin{pmatrix} c_1^s f(h) + y_1^s - y_{1,\star}^s \\ \mathrm{L}_f c_2^s f(h) + k_1 c_2^s f(h) + k_0(y_2^s - y_{2,\star}^s) \end{pmatrix} \quad (23)$$

with decoupling matrix

$$M_s^{-1}(h) = \begin{pmatrix} c_1^s B \\ \mathrm{L}_f^2 c_2^s f(h) B) \end{pmatrix}$$

ensures local asymptotic regulation of y to the desired $y^\star$ while preserving internal stability.

4) Simulations: For completeness, simulations are reported in Figure 1 for the closed-loop system under the stabilizing feedback designed over the new dummy output highlighting the locally minimum-phase components of (20). Simulations are performed for the parameters fixed as in the Table below.

A_1 $[cm^2]$	28	A_3 $[cm^2]$	28
A_2 $[cm^2]$	32	A_4 $[cm^2]$	32
c_1 $[cm^2]$	0.071	c_3 $[cm^2]$	0.071
c_2 $[cm^2]$	0.057	c_4 $[cm^2]$	0.057
k_t $[V/cm]$	1	g $[cm/s^2]$	981
γ_1	0.43	γ_2	0.34
k_1	65.12	k_2	94.12

In addition, we fixed $y_\star = (7.1,\ 6.2)^\top$ corresponding to $h^\star = (7.1,\ 6.2,\ 3.58, 1.632)^\top$. In particular, with this choice of parameters, the plant is nonminum-phase with the zeros of LTM model at the desired equilibrium provided by $z_u = 0.018$ and $z_s = -0.0789$. The gains of the controller (23) are fixed as $(k_0, k_1) = (1, 2)$. Simulations report the story of the original and dummy outputs plus the real residual internal-dynamics of the feedback plant (that is the water levels of the third and fourth tank with respect to the real output) while proving the effectiveness of the proposed design approach.

VI. Conclusions

In this paper, a systematic procedure for controlling MIMO non-minimum phase systems has been proposed based on output factorization. In particular, recurring to the Smith-MacMillan form, a dummy output associated with the stable component of the zero-dynamics is exhibited to perform inversion of the minimum-phase component. The results locally apply to the case of nonlinear dynamics with linear outputs. Perspectives concern the extension of this methodology to the case of nonlinear output mappings.

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