Time evolution of an infinitely extended Vlasov system with singular mutual interaction

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Abstract

We study the time evolution of an infinitely extended system in the mean field approximation, governed by the Vlasov equation. This system is confined in an unbounded cylinder by an external force singular on the border. The mutual interaction is assumed singular at short distance as $1/r^\alpha$ with $\alpha < 2/3$ (or $\alpha < 1$ in case of an external Lorentz force) and with a short range. The initial density is assumed bounded. Differently from studies which assume initial data compact in space and/or in velocities, here we consider a system having infinite mass and an exponential bound on the velocities, according to the Maxwell-Boltzmann law.

Key words: Vlasov equation, infinitely extended plasma, singular interaction.
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1 Introduction, statement of the problem and main result

In the present paper we study the time evolution of a gas of particles in the mean field approximation, that is in the limit in which the mass of the particles goes to zero, while the number of particles per unit volume diverges, in such a way that the mass density stays finite. As well known, this system can be described by the Vlasov equation. We suppose that the particles mutually interact via a short range potential singular at the origin as $1/r^\alpha$, $0 < \alpha < 2/3$. We put this system in an unbounded cylinder, where

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it is confined by an external radial force singular on the boundary of the cylinder.

It is not obvious that this system cannot have a blow-up, that is a collapse of infinite mass in a finite region and/or an unbounded growth of the velocity. Actually for point particles the problem has been solved many years ago, when the problem of the dynamics of infinitely many particles has been faced (see [1, 2, 5, 6, 7, 11, 12, 18, 20, 21, 22, 27, 28, 29, 32, 33, 39, 45, 46, 47] and for a short review [9]). The solution depends on the dimensions and the shape of the region in which the motion happens (the unbounded cylinder case is treated in [6]). In three dimensions the problem has been solved for bounded interactions only [11, 18].

The mean field model has many features similar to the three dimensional infinite particles system, up to a scaling in the spatial distances. The singularity of the interaction is a real difficulty, too hard for point particle systems, but solvable in the Vlasov case, since the singularity is averaged by the mean field, as we will show in the present paper. For bounded interactions the construction of the mean field limit has been proved (for bounded total mass [4, 19, 34, 48] and for an unbounded total mass [5]), while for singular interactions this problem is open except for a particular case with weak singularity [24] (for a review see [26]). In the present paper we forget this problem and we assume the Vlasov equation as the physical model that we will study. We want to stress that, while a point particle cannot escape from the cylinder by the energy conservation, here a characteristic of the Vlasov system could be a priori pushed on the border of the cylinder by the force produced by the other particles. We will prove that this cannot happen.

Recently we have studied a similar problem in [14, 15] with a magnetic confinement, that is we have introduced the Lorentz force produced by an external magnetic field parallel to the symmetry axis of the cylinder, depending on the distance from the border and singular on it. While in [14, 15] we assumed initially bounded velocities, here we assume only an initial control on the large velocities according to the Maxwell-Boltzmann law. This weaker assumption appears more reasonable for an infinitely extended system and produces some non-trivial difficulties. Actually in [14, 15] we have studied the Coulomb singularity $\alpha = 1$, while here we confine ourselves to the case $\alpha < 2/3$, i.e. the assumption of unbounded velocities imposes a restriction on the possible singularities of the interaction. Moreover there is another reason for this restriction: as it is well known, the Lorentz force produced by a magnetic field cannot increase the energy of the system and this property is used in [14, 15], while here the confining external force can change the energy and this fact produces further difficulties in the proof. In this set up we also quote [35], in which it is treated the relativistic Vlasov-Maxwell system in one and a half dimension. We finally observe that, in case of magnetic confinement, our technique is more efficient and we can arrive to $\alpha < 1$. Unfortunately the more interesting case $\alpha = 1$ (Coulomb case) is not reachable, because this interaction is no more Lipschitz but only quasi-Lipschitz, and so the convergence of the partial dynamics (a cutoffed system introduced in Section 2) to the infinite one remains an open problem. In this case a study is in progress when we assume initial data with some decay of the spatial density.

The Vlasov system we consider is

$$\begin{cases} \partial_t f(x,v,t) + v \cdot \nabla_x f(x,v,t) + (E(x,t) + E_{ext}(x)) \cdot \nabla_v f(x,v,t) = 0 \\ E(x,t) = -\displaystyle\int_{\mathbb{R}^3} \nabla\Phi(|x-y|)\ \rho(y,t)\, dy \\ \rho(x,t) = \displaystyle\int_{\mathbb{R}^3} f(x,v,t)\ dv \\ E_{ext}(x) = -\nabla U(x) \\ f(x,v,0) = f_0(x,v), \end{cases} \tag{1.1}$$

where $f(x,v,t)$ denotes the mass distribution at point (x,v) of the phase-space at time t, $E(x,t)$ is the force produced by the system and $E_{ext}(x)$ is the confining external force.

Equation (1.1) is a conservation equation for the density f along the characteristics of the system, that is the solutions to the following problem:

$$\begin{cases} \dot{X}(t) = V(t) \\ \dot{V}(t) = E(X(t),t) + E_{ext}(X(t)) \\ (X(0),V(0)) = (x,v) \\ f(X(t),V(t),t) = f_0(x,v), \end{cases} \tag{1.2}$$

where $(X(t),V(t)) := (X(x,v,t),V(x,v,t))$ denote position and velocity at time t of a particle starting at time $t = 0$ from (x,v). Since f is time-invariant along the motion, it is:

$$\|f(t)\|_{L^\infty} = \|f_0\|_{L^\infty}. \tag{1.3}$$

As it is well known, solutions along the characteristics (1.2) produce a weak solution of the Vlasov equation (1.1), which becomes a classical solution when $f_0(x,v)$ is assumed smooth. We remark that the Lebesgue measure $dxdv$ is conserved along the motion.

We assume the potential Φ to be a not negative, twice differentiable function for $|x| > 0$, such that

$$\Phi(|x|) = \frac{1}{|x|^\alpha} \quad \text{if} \quad |x| < r_1 \quad \text{and} \quad \Phi(|x|) = 0 \quad \text{if} \quad |x| \geq r_0, \tag{1.4}$$

where $0 < r_1 < r_0 < \infty$.

For $\alpha = 1$ this equation have been studied in several papers, first when the total mass is finite and $E_{ext}(x) = 0$. The existence and uniqueness of the solution have been proved, initially assuming symmetries that reduce the problem to a two dimensional one, then for a real three dimensional system. See for instance for bounded mass [3, 30, 37, 40, 41, 49]; for a nice review of the mathematical results on this topic see also [23]. For particular cases with unbounded mass see [14, 15, 16, 25, 36, 42, 43, 44]. The extension of these results to the simpler case $\alpha < 1$ is trivial.

In the present paper we study a case of unbounded mass. The system evolves in an infinite cylinder D with symmetry axis directed along the x_1 direction,

$$D = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2^2 + x_3^2 < A^2\} \qquad A > 0. \tag{1.5}$$

We assume that U, the potential of the external confining force E_{ext}, is a positive non decreasing smooth function, together with its first and second derivatives, diverging on the border of D, and depending only on $r = \sqrt{x_2^2 + x_3^2}$. For concreteness we assume that $U(r)$ has the following form:

$$\begin{aligned} &U(r) = 0 \quad \text{for} \quad r < A - \nu \\ &U(r) = \frac{1}{(A-r)^\theta} \quad \text{for} \quad A - \frac{\nu}{2} < r < A, \quad \theta > 2. \end{aligned} \tag{1.6}$$

The requirement $\theta > 2$ is due to the iterative method and will be clear in the following (see the argument after (2.38)).
We define

$$D_0 = \{x \in \mathbb{R}^3 : r < (A - \nu)\} \qquad 0 < \nu < \frac{A}{2}, \tag{1.7}$$

to be the support of the initial density.

Theorem 1. *Let us fix an arbitrary positive time T. Assume in (1.4) that $\alpha < \frac{2}{3}$ and let $f_0(x, v) \in L_\infty$ be supported on D_0 and such that*

$$0 \le f_0(x, v) \le C_1 e^{-\lambda v^2} \tag{1.8}$$

for some positive constants C_1 and λ. Then there exists a solution to system (1.2) in $[0, T]$. This solution is supported on D and there exist two positive constants C_2 and $\bar\lambda < \lambda$ such that

$$0 \le f(x, v, t) \le C_2 e^{-\bar\lambda v^2}.$$

Moreover it is unique in the class of the characteristics distributed with $f(x, v, t) \le Ce^{-C' v^2}$ for positive constants C and C'.

The proof of the theorem will be given in the next Section. We stress that we are in presence of three difficulties: the infinite mass, the singularity of the external force, the initially unbounded velocities. In the sequel we shall see that the first two difficulties can be solved in analogy with [14, 15], while the third difficulty will be the main object of our analysis.

We anticipate the strategy of the proof. First we introduce a partial dynamics in which the initial distribution has compact support, being x in a cylinder centered at the origin and of length M and $v : |v| \leq N$. We prove, in analogy with [14, 15], that for any fixed time T we can find a bound on the velocities which is linear in N and consequently the support of the density increases in time at most linearly with N. The reason for limiting ourselves to $\alpha < \frac{2}{3}$ relies in (3.17), in which there is the condition for arriving at this result. At this point, by choosing $M = N^{\beta}$, with β large enough, we obtain the result by an iterative method (via a contraction), which allows to perform the limit $N \to \infty$.

The plan of the paper is the following: in Section 2 we give the proof of Theorem 1, by assuming some properties on the partial dynamics which are proved in Section 3. In Section 4 we state a result on a magnetically confined plasma, enlarging the range of α up to $\alpha < 1$, and in the Appendix we collect some proofs and technical tools.

Before starting with the proof of the theorem, we shortly discuss the case with $\alpha \geq 2/3$. We observe that the mean field approximation makes an average on the interaction with different particles, and so it makes sense for $\alpha < 2$ (integrability condition of the mutual force). When $1 < \alpha < 2$ the interaction is not Lipschitz nor quasi-Lipschitz, hence the partial dynamics may be not unique and the method fails. When $\alpha = 1$ the interaction is quasi-Lipschitz and so the partial dynamics could be defined. Actually this fact is not trivial due to the presence of an infinite external force confining the system in the cylinder. This problem has been studied in [13] in presence of a singular magnetic field that confines the system. Unfortunately, we are not able to pass from the partial dynamics to the infinite one, because of the quasi-Lipschitz property of the interaction which imposes a control on the growth of the maximal velocity which is beyond our ability. For $2/3 \leq \alpha < 1$ the problem with a confinement due to an external force like the one described above remains unsolved for mathematical difficulties, essentially due to the loss of control over the velocity component directed along the external force direction. In the present paper we study the case $\alpha < 2/3$, but we stress that the proof for $\alpha < 1/2$ would be much easier, as we observe in the Remark 2 in Section 3.

2 Proof of Theorem 1

For $a \in \mathbb{R}$, $L > 0$, we set

$$\mathcal{D}(a,L) = \{x \in D : |x_1 - a| < L\}, \qquad \mathcal{D}(L) = \mathcal{D}(0,L),$$

and

$$B(L) = \{v \in \mathbb{R}^3 : |v| < L\}.$$

We introduce the following sequence of cut-off problems (named *partial dynamics* in the following), for M and N positive integer numbers:

$$\begin{cases} \dot{X}^{M,N}(t) = V^{M,N}(t) \\ \dot{V}^{M,N}(t) = E^{M,N}(X^{M,N}(t),t) + E_{ext}(X^{M,N}(t)) \\ (X^{M,N}(0), V^{M,N}(0)) = (x,v) \end{cases} \tag{2.1}$$

$$E^{M,N}(x,t) = -\int_D \nabla\Phi(|x-y|)\ \rho^{M,N}(y,t)\,dy,$$

$$\rho^{M,N}(x,t) = \int_{\mathbb{R}^3} f^{M,N}(x,v,t)\,dv,$$

$$f^{M,N}(X^{M,N}(t), V^{M,N}(t), t) = f_0^{M,N}(x,v), \tag{2.2}$$

$$f_0^{M,N}(x,v) = f_0(x,v)\,\chi\left(x \in \mathcal{D}(M)\right)\,\chi\left(v \in B(N)\right),$$

with f_0 defined as in Theorem 1, $\chi(\cdot)$ denoting the characteristic function of the set $(\cdot)$. Such problem admits a unique solution over any arbitrarily fixed time interval $[0,T]$, since the density has compact support and hence the total charge is finite. An explicit proof of this fact is not written in the literature, but it can be easily achieved by using the techniques of the present paper, putting the total energy bounded by a constant (see Proposition 3).

We want to investigate the limit $M, N \to \infty$. To do so, we consider a relation $M = N^\beta$, with $\beta > 1$ to be fixed later, hence in the following we will drop the dependence on the index M.

Let us fix arbitrarily a time T. For any $t \in [0,T]$ we introduce the maximal velocity of a plasma particle,

$$\mathcal{V}^N(t) = \max\left\{\widetilde{C},\ \sup_{s\in[0,t]}\ \sup_{(x,v)} |V^N(s)|\right\}, \tag{2.3}$$

where $\widetilde{C}$ is a constant that will be chosen large enough, and the maximal displacement,

$$R^N(t) = r_0 + \int_0^t \mathcal{V}^N(s)\,ds, \tag{2.4}$$

where r_0 is the range of the interaction.

The following result on the partial dynamics is the core of the proof, and will be proved in Section 3:

Proposition 1.

$$\int_0^T |E^N(X^N(s),s)|ds \leq C\,\mathcal{V}^N(T)^\gamma, \qquad \gamma < 1. \tag{2.5}$$

As a consequence, the following holds:

Corollary 1.

$$\mathcal{V}^N(T) \leq CN \tag{2.6}$$

$$U(X^N(t)) \leq CN^2 \tag{2.7}$$

$$\rho^N(x,t) \leq CN^{3\gamma'}, \qquad \gamma' = \frac{(1+\gamma)}{2} < 1. \tag{2.8}$$

Proof. We introduce the quantity

$$\mathcal{E}^N(X^N(t),V^N(t),t) = \frac{|V^N(t)|^2}{2} + U(X^N(t)). \tag{2.9}$$

Then we have:

$$\begin{aligned}\frac{d\mathcal{E}^N}{dt} &= V^N(t)\cdot\left[E(X^N(t),t) + E_{ext}(X^N(t))\right] \\ &\quad + \nabla U(X^N(t))\cdot V^N(t) = V^N(t)\cdot E(X^N(t),t),\end{aligned}$$

and,

$$\frac{|V^N(t)|^2}{2} + U(X^N(t)) = \frac{|V(0)|^2}{2} + U(X(0)) + \int_0^t V^N(s)\cdot E(X^N(s),s)\,ds. \tag{2.10}$$

We remark that U is positive and the initial condition is such that $U(X(0)) = 0$. So by (2.10) we get

$$\frac{|V^N(t)|^2}{2} + U(X^N(t)) \leq \frac{|V(0)|^2}{2} + \mathcal{V}^N(t)\int_0^t |E(X^N(s),s)|\,ds. \tag{2.11}$$

Hence, by Proposition 1 and the assumptions on the initial data,

$$\frac{|V^N(t)|^2}{2} + U(X^N(t)) \leq C\left[|V(0)|^2 + (\mathcal{V}^N(T))^{1+\gamma}\right] \leq C\left[N^2 + (\mathcal{V}^N(tT)^{1+\gamma}\right].$$

Then

$$|V^N(t)|^2 \leq C\left[N^2 + (\mathcal{V}^N(T))^{1+\gamma}\right]$$

which implies (2.6), being $\gamma < 1$, while

$$U(X^N(t)) \leq C\left[N^2 + N^{1+\gamma}\right]$$

which implies (2.7).

Now we prove (2.8). We have:

$$\rho^N(X^N(t),t) \quad = \int f^N(X^N(t),V^N(t),t)\,dV^N(t) = \int f_0(x,v)\,dV^N(t).$$

From (2.11) and Proposition 1 it follows that

$$\begin{aligned} &v^2 = V(0)^2 \geq (V^N(t))^2 - 2\mathcal{V}^N(t)\int_0^t |E(X^N(s),s)|ds \geq \\ &(V^N(t))^2 - C_3N^{\gamma+1}. \end{aligned} \tag{2.12}$$

We decompose the integral as follows:

$$\begin{aligned} &\int f_0(x,v)\;dV^N(t) = \\ &\int_{|V^N(t)|\leq 2C_3N^{\frac{1+\gamma}{2}}} f_0(x,v)\;dV^N(t) + \int_{|V^N(t)|>2C_3N^{\frac{1+\gamma}{2}}} f_0(x,v)\;dV^N(t) \\ &\leq CN^{3\frac{1+\gamma}{2}} + C_1\int_{|V^N(t)|>2C_3N^{\frac{1+\gamma}{2}}} e^{-\lambda v^2}\;dV^N(t) \end{aligned}$$

Notice that, by (2.12), $|V^N(t)| > 2C_3N^{\frac{1+\gamma}{2}}$ implies, for C_3 sufficiently large,

$$v^2 \geq (V^N(t))^2 - \frac{(V^N(t))^2}{4C_3} \geq \frac{(V^N(t))^2}{2},$$

so that

$$\begin{aligned} &\int f_0(x,v)\;dV^N(t) \leq \\ &CN^{3\frac{1+\gamma}{2}} + C_1\int_{|V^N(t)|>2C_3N^{\frac{1+\gamma}{2}}} e^{-\lambda\frac{(V^N(t))^2}{2}}\;dV^N(t) \leq CN^{3\frac{1+\gamma}{2}} \end{aligned} \tag{2.13}$$

which implies the thesis. $\square$

Now we prove Theorem 1. We observe first that (2.7) ensures that the solutions to the partial dynamics remain in the cylinder for the fixed time interval $[0,T]$, since the potential stays bounded. We will keep this in mind in the following, and we will avoid to put the characteristic function $\chi(x \in D)$ in the spatial integrals.

Let us set, for k integer

$$\delta^N(x,v,t) = |X^N(x,v,t) - X^{N+1}(x,v,t)|$$

$$u_k^N(t) = \sup_{(x,v)\in\mathcal{D}(k)\times B(N)} \delta^N(x,v,t).$$

We fix a couple $(x,v) \in \mathcal{D}(k) \times B(N)$, with $k \leq N$. By the equations of motion in integral form it immediately follows

$$\begin{aligned}\delta^N(x,v,t) = &\left| \int_0^t dt_1 \int_0^{t_1} dt_2 \, \left[E^N\left(X^N(t_2),t_2\right) + E_{ext}(X^N(t_2))\right] \right. \\ &\left. - \int_0^t dt_1 \int_0^{t_1} dt_2 \, \left[E^{N+1}\left(X^{N+1}(t_2),t_2\right) + E_{ext}(X^{N+1}(t_2))\right] \right| \\ \leq \; &\int_0^t dt_1 \int_0^{t_1} dt_2 \, \left[\mathcal{F}_1(x,v,t_2) + \mathcal{F}_2(x,v,t_2) + \mathcal{F}_3(x,v,t_2)\right],\end{aligned} \tag{2.14}$$

where

$$\mathcal{F}_1(x,v,t) = \left|E^N\left(X^N(t),t\right) - E^N\left(X^{N+1}(t),t\right)\right|, \tag{2.15}$$

$$\mathcal{F}_2(x,v,t) = \left|E^N\left(X^{N+1}(t),t\right) - E^{N+1}\left(X^{N+1}(t),t\right)\right|, \tag{2.16}$$

and

$$\mathcal{F}_3(x,v,t) = \left|E_{ext}\left(X^N(t)\right) - E_{ext}\left(X^{N+1}(t)\right)\right|. \tag{2.17}$$

We warn that from now on we indicate by C any generic positive constant, depending possibly on the initial data and the fixed time T but not on N, and possibly changing from line to line.

We notice that, being the potential Φ short-range, (2.6) in Corollary 1 ensures that the characteristics starting from the set $\mathcal{D}(k) \times B(N)$, during the time interval $[0,T]$, at most interact with those starting from the set $\mathcal{D}(k_1) \times B(N)$, with

$$k_1 = k + r_0 + CTN. \tag{2.18}$$

We begin to treat $\mathcal{F}_2$, since the bound for $\mathcal{F}_1$ will be a by-product, as it will be evident. To simplify the notation we put $\bar{X} = X^{N+1}(t)$ and $d := u^N_{k_1}(t)$. We have:

$$\mathcal{F}_2(x,v,t) \leq \mathcal{F}_2'(x,v,t) + \mathcal{F}_2''(x,v,t), \tag{2.19}$$

where

$$\mathcal{F}_2'(x,v,t) = \left| \int_{|\bar{X}-y|\leq 2d} \nabla\Phi(|\bar{X}-y|) \, \left(\rho^N(y,t) - \rho^{N+1}(y,t)\right) \, dy \right| \tag{2.20}$$

and

$$\mathcal{F}_2''(x,v,t) = \left| \int_{|\bar{X}-y|> 2d} \nabla\Phi(|\bar{X}-y|) \, \left(\rho^N(y,t) - \rho^{N+1}(y,t)\right) \, dy \right|. \tag{2.21}$$

We estimate $\mathcal{F}_2'(x,v,t)$. We have two cases: $d < 1$ or $d \geq 1$. In the first case by (2.8) we get:

$$\begin{aligned}&\mathcal{F}_2'(x,v,t) \leq \int_{|\bar{X}-y|\leq 2d} \frac{\rho^N(y,t) + \rho^{N+1}(y,t)}{|\bar{X}-y|^{\alpha+1}} \, dy \leq \\ &CN^{3\gamma'} \int_{|\bar{X}-y|\leq 2d} \frac{1}{|\bar{X}-y|^{\alpha+1}} \, dy \leq CN^{3\gamma'} d^{2-\alpha} \leq CN^{3\gamma'} d\end{aligned} \tag{2.22}$$

while, in the second case we have

$$\mathcal{F}_2'(x,v,t) \leq CN^{3\gamma'} \int dy \frac{1}{|\bar{X}-y|^{\alpha+1}} \leq CN^{3\gamma'} \leq CN^{3\gamma'}\, d \tag{2.23}$$

since, we recall, the integral is taken over the cylinder D, and then it is finite, being $0<\alpha<1$. The definition of d implies that:

$$\mathcal{F}_2'(x,v,t) \leq CN^{3\gamma'}\, u_{k_1}^N(t). \tag{2.24}$$

As it regards the term $\mathcal{F}_2''$, we put

$$(Y^N(t), W^N(t)) = (X^N(y,w,t), V^N(y,w,t))$$

and

$$S^i(t) = \{(y,w) : |\bar{X} - Y^i(t)| \geq 2d\} \quad i = N, N+1.$$

By the invariance of the density along the characteristics and the Liouville theorem we have,

$$\begin{aligned}
&\mathcal{F}_2''(x,v,t) \leq \\
&\int dy \int dw \left| \frac{f_0^N(y,w)}{|\bar{X}-Y^N(t)|^{\alpha+1}} - \frac{f_0^{N+1}(y,w)}{|\bar{X}-Y^{N+1}(t)|^{\alpha+1}} \right| \left[\chi(S^N(t)) + \chi(S^{N+1}(t))\right] \\
&\leq \int_{S^N(t)} dy \int dw \left| \frac{\chi\left(|\bar{X}-Y^N(t)| \leq r_0|\right)}{|\bar{X}-Y^N(t)|^{\alpha+1}} - \frac{\chi\left(|\bar{X}-Y^{N+1}(t)| \leq r_0|\right)}{|\bar{X}-Y^{N+1}(t)|^{\alpha+1}} \right| f_0^N(y,w) \\
&+ \int_{S^{N+1}(t)} dy \int dw \frac{\chi\left(|\bar{X}-Y^{N+1}(t)| \leq r_0|\right)}{|\bar{X}-Y^{N+1}(t)|^{\alpha+1}} \left| f_0^N(y,w) - f_0^{N+1}(y,w) \right|.
\end{aligned} \tag{2.25}$$

By the Lagrange theorem the first term in (2.25) is bounded by

$$d \int_{S^N(t)} dy \int dw\, \frac{f_0^N(y,w)}{|\bar{X}-\xi(t)|^{\alpha+2}}\,, \tag{2.26}$$

where $\xi(t)$ is a point of the segment joining $Y^N(t)$ and $Y^{N+1}(t)$. Note that if $|\bar{X}-Y^N(t)| > 2d$, then

$$|\bar{X}-Y^{N+1}(t)| > |\bar{X}-Y^N(t)| - |Y^N(t)-Y^{N+1}(t)| > 2d - d = d.$$

This implies that $|\bar{X}-\xi(t)|$ is certainly bigger than $\frac{1}{2}|\bar{X}-Y^N(t)|$, hence by (2.8) the previous term can be bounded by (using once again the Liouville theorem)

$$
\begin{aligned}
& d\int_{S^N(t)} dy \int dw\, \frac{f_0^N(y,w)}{|\bar{X}-\xi(t)|^{\alpha+2}} \\
& \leq 2^{\alpha+2} d\int dy \int dw\, \frac{f_0^N(y,w)}{|\bar{X}-Y^N(t)|^{\alpha+2}} \\
& = 2^{\alpha+2} d\int dY^N(t) \int dW^N(t)\, \frac{f^N(Y^N(t),W^N(t),t)}{|\bar{X}-Y^N(t)|^{\alpha+2}} \\
& \leq CN^{3\gamma'} d\int dy\, \frac{1}{|\bar{X}-y|^{\alpha+2}} \leq CN^{3\gamma'} u_{k_1}^N(t).
\end{aligned} \tag{2.27}
$$

For the second term in (2.25) we have:

$$
\begin{aligned}
& \int_{S^{N+1}(t)} dy \int dw\, \frac{\chi\left(|\bar{X}-Y^{N+1}(t)| \leq r_0\right)}{|\bar{X}-Y^{N+1}(t)|^{\alpha+1}} \left| f_0^N(y,w) - f_0^{N+1}(y,w) \right| \\
& \quad \leq 2\int_{S^{N+1}(t)} dy \int dw\, f_0^{N+1}(y,w) \frac{\chi\left(|\bar{X}-Y^{N+1}(t)| \leq r_0\right)}{|\bar{X}-Y^{N+1}(t)|^{\alpha+1}} \\
& \qquad \left[\chi(N \leq |w| \leq N+1) + \chi(|y| > N^\beta)\right].
\end{aligned} \tag{2.28}
$$

At this point we notice that, being $x \in \mathcal{D}(k)$ with $k < N$, if $|y| > N^\beta$ then by (2.6) it is $|\bar{X}-Y^{N+1}(t)| \geq |\bar{X}-y| - CN > r_0$. By the change of variables $(\bar{y},\bar{w}) = (Y^N(t), W^N(t))$ we get for N sufficiently large, using once more (2.6):

$$
\begin{aligned}
& \int_{S^{N+1}(t)} dy \int dw\, \frac{\chi\left(|\bar{X}-Y^{N+1}(t)| \leq r_0\right)}{|\bar{X}-Y^{N+1}(t)|^{\alpha+1}} \left| f_0^N(y,w) - f_0^{N+1}(y,w) \right| \\
& \quad \leq 2C_1 e^{-\lambda N^2} \int d\bar{y} \int d\bar{w}\, \frac{\chi(|\bar{w}| \leq CN)}{|\bar{X}-\bar{y}|^{\alpha+1}} \\
& \quad \leq CN^3 e^{-\lambda N^2} \int d\bar{y}\, \frac{1}{|\bar{X}-\bar{y}|^{\alpha+1}} \leq Ce^{-\frac{\lambda}{2}N^2}.
\end{aligned} \tag{2.29}
$$

Therefore, collecting all the bounds (2.24), (2.25), (2.27) and (2.29) we have

$$
\sup_{(x,v)\in\mathcal{D}(k)\times B(N)} \mathcal{F}_2(x,v,t) \leq CN^{3\gamma'} u_{k_1}^N(t) + Ce^{-\frac{\lambda}{2}N^2}. \tag{2.30}
$$

A bound for $\mathcal{F}_1$ takes the same form, without the last exponential term, so that we have

$$
\sup_{(x,v)\in\mathcal{D}(k)\times B(N)} \{\mathcal{F}_1(x,v,t) + \mathcal{F}_2(x,v,t)\} \leq CN^{3\gamma'} u_{k_1}^N(t) + Ce^{-\frac{\lambda}{2}N^2}. \tag{2.31}
$$

For $\mathcal{F}_3(x,v,t)$ we have, by applying again the Lagrange theorem:

$$
\mathcal{F}_3(x,v,t) \leq \left[\left|U''(X^N(t))\right| + \left|U''(X^{N+1}(t))\right|\right] \delta^N(x,v,t). \tag{2.32}
$$

By (2.7) and by using the explicit form of U in (1.6), it's easily seen that

$$\left|U''(X^N(t))\right| \leq CN^{\frac{2(\theta+2)}{\theta}}, \tag{2.33}$$

hence

$$\sup_{(x,v)\in\mathcal{D}(k)\times B(N)} \mathcal{F}_3(x,v,t) \leq CN^{\frac{2(\theta+2)}{\theta}} u^N_{k_1}(t). \tag{2.34}$$

Finally, by (2.14), (2.30) and (2.34) we get:

$$u^N_k(t) \leq C\left(N^{3\gamma'} + N^{\frac{2(\theta+2)}{\theta}}\right)\int_0^t dt_1 \int_0^{t_1} dt_2\, u^N_{k_1}(t_2) + Ce^{-\frac{\lambda}{2}N^2}. \tag{2.35}$$

We can iterate now this relation, passing from k_1 to k_2 and in general from k_j to k_{j+1}, with $k_{j+1} = k_j + r_0 + CTN$, up to k_ℓ, where ℓ is the largest integer such that a trajectory with initial position in $\mathcal{D}(k_\ell)$ does not reach the boundary of $\mathcal{D}(M) = \mathcal{D}(N^\beta)$ in the time interval $[0,T]$.
Hence we can make ℓ steps with

$$\ell \leq \frac{M-k}{r_0 + CNT}. \tag{2.36}$$

Once chosen k, we can fix

$$\ell = \mathrm{Intg}\left(C\frac{N^{\beta-1}}{r_0+T}\right),$$

where $\mathrm{Intg}(x)$ is the integer part of x. Since the maximal displacement is bounded by CTN, we obtain

$$u^N_k(t) \leq Ce^{-\frac{\lambda}{2}N^2}\sum_{s=0}^{\ell-1}(CN^\sigma)^s\frac{t^{2s}}{(2s)!} + CTN(CN^\sigma)^\ell\frac{t^{2\ell}}{(2\ell)!}, \tag{2.37}$$

putting

$$\sigma = \max\left\{3\gamma', \frac{2(\theta+2)}{\theta}\right\}.$$

The first term in (2.37) is bounded by

$$Ce^{-\frac{\lambda}{2}N^2 + (CN^\sigma)^{1/2}\,t} \leq e^{-\frac{\lambda}{4}N^2} \tag{2.38}$$

for N large enough since, being $\gamma < 1$ and $\theta > 2$, it is $\sigma < 4$. The second term, by the choice of ℓ, is bounded by $N^{-CN^{\beta-1}}$, provided that $\beta > 1+\sigma/2$ and N is large.

We obtain analogously a similar bound for $|V^N(x,v,t) - V^{N+1}(x,v,t)|$, arriving at

$$\begin{aligned}\sup_{\{(x,v,t)\in\mathcal{D}(k)\times B(N)\times(0,T)\}} &\left\{|X^N(x,v,t) - X^{N+1}(x,v,t)|\right.\\ &\left.+\,|V^N(x,v,t) - V^{N+1}(x,v,t)|\right\}\\ &\leq Ce^{-\frac{\lambda}{4}N^2}\end{aligned}$$

for N large enough. Thus the sequence $\big(X^N(x,v,t),\, V^N(x,v,t)\big)$ converges uniformly over

$$\mathcal{D}\left(\frac{N^\beta}{2}\right) \times B(N) \times [0,T].$$

Let us denote the limit of the sequence by $(X(x,v,t), V(x,v,t))$. We have to prove that this couple, during the time interval $[0,T]$, (i) satisfies (1.2), (ii) is unique, (iii) $X(t)$ remains confined in the cylinder and (iv) the related spatial distribution is gaussian in the velocities.

(i) We need a N-uniform estimate of the displacement $|X^N(x,v,t) - x|$. To this purpose, arguing in analogy with [10], we observe that all the previous results hold whenever we choose the initial condition in $\mathcal{D}(a, N^\beta) \times B(N)$ instead of $\mathcal{D}(N^\beta) \times B(N)$ being, we recall, $\mathcal{D}(a, N^\beta) = \{x \in D : |x_1 - a| < N^\beta\}$. Now we fix a couple (x,v) and a such that $|x_1 - a| \le \frac{|v|^\beta}{2}$. Moreover we fix $N_0 = \mathrm{Intg}(|v| + \bar{C})$, being $\bar{C}$ a suitably large constant. By this choice it is $(x,v) \in \mathcal{D}(a, N_0^\beta) \times B(N_0)$. We have

$$|X^N(x,v,t) - x| \le |X^{N_0}(x,v,t) - x| + \sum_{k=N_0+1}^{N} |X^k(x,v,t) - X^{k-1}(x,v,t)|. \tag{2.39}$$

From (2.6) it follows $|X^{N_0}(x,v,t) - x| \le CN_0 t$, while by the preceding arguments the sum is converging as $N \to \infty$, hence

$$|X^N(x,v,t) - x| \le CN_0 t \le C(|v| + 1). \tag{2.40}$$

Analogously, for the velocities we get

$$\begin{aligned} |V^N(x,v,t) - v| \le &|V^{N_0}(x,v,t) - v| + \sum_{k=N_0+1}^{N} |V^k(x,v,t) - V^{k-1}(x,v,t)| \\ &\le C(|v| + 1). \end{aligned} \tag{2.41}$$

These estimates allow to give a bound on the field E. Indeed from (2.40) it follows

$$\begin{aligned} |E^N(x,t)| &= \int dy \int dw \frac{f^N(y,w,t)}{|x-y|^{\alpha+1}} \chi\left(|x-y| \le r_0\right) \\ &\le C_1 \int dy \int dw \frac{e^{-\lambda w^2}}{|x - Y^N(t)|^{1+\alpha}} \chi\left(|x - Y^N(t)| \le r_0\right) \\ &\le C_1 \sum_{k=0}^{\infty} \int_{k \le |w| \le k+1} dw\, e^{-\lambda w^2} \int dy\, \frac{1}{|x - Y^N(t)|^{1+\alpha}} \\ &\le C_1 \sum_{k=0}^{\infty} e^{-\lambda k^2} \int_{|w| \le k+1} dw \int dy\, \frac{1}{|x - Y^N(t)|^{1+\alpha}}. \end{aligned} \tag{2.42}$$

By (2.41), if $|w| \leq k$, then $|W^N(t)| \leq C(k+1)$ so that, by a change of variables, we get

$$\begin{aligned} |E^N(x,t)| \leq & C_1 \sum_{k=0}^{\infty} e^{-\lambda k^2} \int_{|w| \leq C(k+1)} dw \int dy \frac{1}{|x-y|^{1+\alpha}} \leq \\ & C \sum_{k=0}^{\infty} e^{-\lambda k^2} k^3 \leq C. \end{aligned} \tag{2.43}$$

Notice that this estimate is uniform in N and thus it proves that the limit field $E(x,t)$ is finite. Moreover (2.43) induces a bound on the external field E_{ext}. Indeed, from (2.10), (2.41) and (2.43) it follows

$$U(X^N(t)) \leq \frac{|v|^2}{2} + \int_0^t V^N(s) \cdot E(X^N(s), s)\, ds \leq \frac{|v|^2}{2} + C(1+|v|).$$

The explicit expression for U in (1.6) implies that

$$|E_{ext}(X^N(t))| \leq C(|v|^2 + |v|)^{1+\frac{1}{\theta}}. \tag{2.44}$$

Since

$$X^N(t) = x + vt + \int_0^t dt_1 \int_0^{t_1} dt_2 \, [E(X^N(t_2), t_2) + E_{ext}(X^N(t_2))],$$

the N-uniform estimates (2.43) and (2.44) allow to pass to the limit under the time integrals and to prove that the limit functions $X(t), V(t)$ satisfy equations (1.2).

(ii) The uniqueness can be proved in the same way we bounded the term $\mathcal{F}_1(x,v,t)$, by putting two different solutions $X(x,v,t)$ and $X'(x,v,t)$ in place of $X^N(t)$ and $X^{N+1}(t)$.

(iii) The confinement of the characteristics $X(t)$ is proven by (2.44), which shows that for any fixed characteristic the external force stays bounded over $[0,T]$ and hence $X(t)$ does not reach the boundary of the cylinder.

(iv) Let $\bar\lambda > 0$. Again by (2.10), (2.41) and (2.43) we get

$$\begin{aligned} & f(X(t), V(t), t) e^{\bar\lambda V(t)^2} = f_0(x,v) e^{\bar\lambda V(t)^2} \leq \\ & C_1 e^{-(\lambda - \bar\lambda) v^2} e^{\bar\lambda (V(t)^2 - v^2)} \leq C_1 e^{-(\lambda-\bar\lambda)v^2} e^{\bar\lambda C(1+|v|)} \leq C \end{aligned} \tag{2.45}$$

for any $\bar\lambda$ sufficiently smaller than λ.

3 Proof of Proposition 1.

In all the following estimates the constants will not depend on N, so from now on we will omit the index N.

We define here the local energy, which is a key tool in our analysis. For $\mu \in \mathbb{R}$ and $R > 0$ we define the function,

$$\varphi^{\mu,R}(x) = \varphi\left(\frac{|x_1 - \mu|}{R}\right); \tag{3.1}$$

φ is assumed to be smooth for technical purposes, as it will be clear in the following, and is such that:

$$\varphi(r) = 1 \quad \text{if} \quad r \in [0, 1] \tag{3.2}$$

$$\varphi(r) = 0 \quad \text{if} \quad r \in [2, +\infty) \tag{3.3}$$

$$-2 \leq \varphi'(r) \leq 0. \tag{3.4}$$

Now we define

$$\begin{aligned} W(\mu, R, t) = & \frac{1}{2}\int dx\, \varphi^{\mu,R}(x) \int dv\, |v|^2 f(x, v, t) \\ & + \frac{1}{2}\int dx\, \varphi^{\mu,R}(x)\rho(x, t) \int dy\, \rho(y, t)\Phi(|x - y|) \\ & + \int dx\, \varphi^{\mu,R}(x)\rho(x, t)U(x). \end{aligned} \tag{3.5}$$

The function W, already introduced in [10] without any external potential U, can be seen as a sort of mollified version of the energy of a bounded region interacting with the rest of the system, and it will be the most important tool to deal with the unboundedness of the plasma.

We define,

$$Q(R, t) = \max\left\{1, \sup_{\mu \in \mathbb{R}} W(\mu, R, t)\right\} \tag{3.6}$$

and

$$Q(t) = \sup_{s \in [0,t]} Q(R(s), s). \tag{3.7}$$

Remark 1. *By the properties of the potentials Φ and U and by the assumptions on the initial conditions it is*

$$W(\mu, R, 0) \leq CR.$$

We state the most important result on the local energy, whose proof is given in the Appendix:

Proposition 2. *There exists a constant C independent of N such that*

$$Q(R(t),t) \leq CQ(R(t),0).$$

As consequence of Remark 1 we have:

Corollary 2.

$$Q(R(t),t) \leq CR(t). \tag{3.8}$$

Now we give a first estimate on E, which will be refined in the following Proposition 4 (such bound is analogous to the one given in [14, 15] for $\alpha = 1$).

Proposition 3. *There exist constants C_3 and C, independent of N, such that:*

$$|E(x,t)| \leq C_3 \mathcal{V}(t)^{\frac{5\alpha-1}{3}} Q(R(t),t)^{\frac{2-\alpha}{3}} \quad \textit{if} \ \ \frac{1}{5} < \alpha < \frac{2}{3}; \tag{3.9}$$

$$|E(x,t)| \leq CQ(R(t),t)^{\frac{3}{5}} \log \mathcal{V}(t) \quad \textit{if} \ \ \alpha \leq \frac{1}{5}; \tag{3.10}$$

Remark 2. *Notice that since by definition $R(t) \leq C\mathcal{V}(t)$, by Corollary 2 it follows*

$$|E(x,t)| \leq C\mathcal{V}(t)^{\frac{4\alpha+1}{3}}. \tag{3.11}$$

Hence for $\alpha < 1/2$ we get a bound on $|E(x,t)|$ which is less than linear in $\mathcal{V}(t)$, and this would be sufficient to prove Proposition 1. Hence the successive efforts will be addressed to enlarge the range of α up to $\frac{2}{3}$.

Proof. We premise an estimate on the spatial density: for any $\mu \in \mathbb{R}$ and any positive number R it is

$$\int_{|\mu - x| \leq R} dx \ \rho(x,t)^{\frac{5}{3}} \leq CW(\mu, R, t). \tag{3.12}$$

Indeed:

$$\rho(x,t) \leq \int_{|v|\leq a} dv f(x,v,t) + \frac{1}{a^2} \int_{|v|>a} dv \ v^2 f(x,v,t) \leq$$
$$Ca^3 + \frac{1}{a^2} \int dv \ v^2 f(x,v,t).$$

By minimizing over a, taking the power $5/3$ of both members and integrating over the set $\{x : |\mu - x_1| \leq R\}$ we get (3.12).

Let us start with the case $\alpha > \frac{1}{5}$. For any positive $b < r_0$ it is

$$|E(x,t)| \leq \mathcal{J}_0(x,t) + \mathcal{J}_1(x,t), \tag{3.13}$$

with

$$\mathcal{J}_0(x,t) = C\int_{0<|x-y|\leq b} dy\ \frac{\rho(y,t)}{|x-y|^{1+\alpha}},$$

$$\mathcal{J}_1(x,t) = C\int_{b<|x-y|\leq r_0} dy\ \frac{\rho(y,t)}{|x-y|^{1+\alpha}}.$$

We estimate the terms in (3.13). We have

$$\mathcal{J}_0(x,t) \leq C\|\rho(t)\|_{L^\infty} b^{2-\alpha} \leq C\mathcal{V}(t)^3 b^{2-\alpha}.$$

By (3.12) we get:

$$\mathcal{J}_1(x,t) \leq C\left(\int_{|x-y|\leq r_0} dy\ \rho(y,t)^{\frac{5}{3}}\right)^{\frac{3}{5}}\left(\int_{b<|x-y|\leq r_0} \frac{1}{|x-y|^{\frac{5}{2}(1+\alpha)}} dy\right)^{\frac{2}{5}}$$

$$\leq CW(x_1,r_0,t)^{\frac{3}{5}}\left[b^{\frac{1}{2}-\frac{5}{2}\alpha} + r_0^{\frac{1}{2}-\frac{5}{2}\alpha}\right]^{\frac{2}{5}} \leq CQ(R(t),t)^{\frac{3}{5}} b^{\frac{1}{5}-\alpha}.$$

Hence

$$\mathcal{J}_0(x,t) + \mathcal{J}_1(x,t) \leq C\left(\mathcal{V}(t)^3 b^{2-\alpha} + Q(R(t),t)^{\frac{3}{5}} b^{\frac{1}{5}-\alpha}\right).$$

The minimum value in b is attained at

$$b = C\left(\frac{\left(\alpha - \frac{1}{5}\right) Q(R(t),t)^{\frac{3}{5}}}{(2-\alpha)\mathcal{V}(t)^3}\right)^{\frac{5}{9}} \tag{3.14}$$

so that we get

$$\mathcal{J}_0(x,t) + \mathcal{J}_1(x,t) \leq C\,\mathcal{V}(t)^{\frac{5\alpha-1}{3}} Q(R(t),t)^{\frac{2-\alpha}{3}} \tag{3.15}$$

which, by (3.13), proves (3.9). To prove (3.10), let us consider first the case $\alpha = \frac{1}{5}$. By the same procedure used before we get

$$\mathcal{J}_0(x,t) + \mathcal{J}_1(x,t) \leq C\mathcal{V}(t)^3 b^{2-\alpha} + Q(R(t),t)^{\frac{3}{5}} \log b.$$

The minimum is attained at

$$b = \left[\frac{Q(R(t),t)^{\frac{3}{5}}}{\mathcal{V}(t)^3}\right]^{\frac{1}{2-\alpha}},$$

which implies the thesis. The case $\alpha < \frac{1}{5}$ is immediately proved as follows:

$$
\begin{aligned}
|E(x,t)| &\leq C \int_{|x-y|\leq r_0} dy \; \frac{\rho(y,t)}{|x-y|^{1+\alpha}} \\
&\leq CQ(R(t),t)^{\frac{3}{5}} \left(\int_{|x-y|\leq r_0} \frac{1}{|x-y|^{\frac{5}{2}(1+\alpha)}} dy \right)^{\frac{2}{5}} \leq CQ(R(t),t)^{\frac{3}{5}}.
\end{aligned}
$$

□

Now we need to control the time average of the field E, that is

$$
\langle E \rangle_{\bar{\Delta}} := \frac{1}{\bar{\Delta}} \int_t^{t+\bar{\Delta}} |E(X(s),s)| \, ds
$$

over a suitably small time interval $\bar{\Delta}$. We have to fix some parameters which will be used in what follows, precisely:

$$
\begin{aligned}
&\eta \in \left(\frac{2-\alpha^2}{8-5\alpha}, 1-\alpha \right) \\
&\eta' \in \left(\frac{2-\alpha^2}{8-5\alpha}, \eta \right) \\
&\bar{\eta} \in \left(0, 1-\alpha+\eta-(1-\eta)\left(2-\frac{\alpha}{2-\alpha}\right)+\eta' \right) \\
&\delta \in (0,\bar{\eta}].
\end{aligned}
\tag{3.16}
$$

Notice that in order not to have an empty interval for $\bar{\eta}$, we must require that $\eta' > \frac{2-\alpha^2}{8-5\alpha}$ (see the following eqn. (3.66)), and analogously for η, so that the condition that has to be fulfilled is

$$
\frac{2-\alpha^2}{8-5\alpha} < 1-\alpha, \tag{3.17}
$$

which is satisfied for $\alpha < 2/3$ and gives the range over which we are able to prove Theorem 1.

By Remark 2, from now on we will assume $\frac{1}{2} < \alpha < \frac{2}{3}$.

We define a time interval

$$
\Delta_1 := \frac{\mathcal{V}(T)^{\eta'}}{4C_3 \mathcal{V}(T)^{\frac{5\alpha-1}{3}} Q(T)^{\frac{1}{3}(2-\alpha)}} \tag{3.18}
$$

where C_3 is the constant in (3.9), and for any positive integer ℓ we set:

$$
\Delta_\ell = \Delta_{\ell-1}\mathcal{G} = \ldots = \Delta_1 \mathcal{G}^{\ell-1}, \tag{3.19}
$$

denoting by

$$
\mathcal{G} = \operatorname{Intg}\left(\mathcal{V}(T)^{\delta} \right). \tag{3.20}
$$

Putting for brevity

$$\mathcal{V} := \mathcal{V}(T) \qquad Q := Q(T)$$

we have the following result:

Proposition 4. *Suppose the following estimate holds:*

$$\langle E\rangle_{\Delta_\ell} \le C\left[\mathcal{V}^{\frac{4}{3}\alpha-\frac{2}{3}+\eta}Q^{\frac{(2-\alpha)}{3}}\log\mathcal{V} + \frac{\mathcal{V}^{\frac{5\alpha-1}{3}}Q^{\frac{(2-\alpha)}{3}}}{\mathcal{V}^{\bar\eta}\,\mathcal{V}^{\delta(\ell-1)}}\right]. \tag{3.21}$$

Then, there exists a positive number $\bar\Delta$ *such that:*

$$\langle E\rangle_{\bar\Delta} \le C_4\mathcal{V}(T)^\gamma \tag{3.22}$$

for any $t\in[0,T]$ *such that* $t \le T-\bar\Delta$*.*

Proof. By Corollary 2 it is $Q \le CR(T) \le C\mathcal{V}$, hence from (3.21) it follows:

$$\langle E\rangle_{\Delta_\ell} \le C\left[\mathcal{V}^{\alpha+\eta}\log\mathcal{V} + \frac{\mathcal{V}^{\frac{4\alpha+1}{3}}}{\mathcal{V}^{\bar\eta}\,\mathcal{V}^{\delta(\ell-1)}}\right]. \tag{3.23}$$

At this point we define $\bar\ell$ as the smallest integer such that

$$\delta(\ell-1) > \frac{4\alpha-2}{3} \tag{3.24}$$

and, being $\eta < 1-\alpha$, we obtain estimate (3.22) with $\bar\Delta = \Delta_{\bar\ell}$ from (3.23). □

We remark that the time interval $\bar\Delta$ is of the order

$$\bar\Delta \approx \frac{C}{\mathcal{V}^{1-\eta'}}. \tag{3.25}$$

Proposition 4 enables us to conclude the proof of Proposition 1. We divide the interval $[0,T]$ by n subintervals $[t_{i-1},t_i]$, $i=1,...,n$, with $t_0=0$, $t_n=T$ and $\frac{1}{2}\bar\Delta \le t_i - t_{i-1} \le \bar\Delta$. Hence it is:

$$\int_0^T |E(X(s),s)|\,ds \le \sum_{i=1}^n \int_{t_{i-1}}^{t_i} |E(X(s),s)|\,ds \le C\sum_{i=1}^n \bar\Delta\langle E\rangle_{\bar\Delta}, \tag{3.26}$$

and by (3.22) we get:

$$\int_0^T |E(X(s),s)|\,ds \le C\sum_{i=1}^n \bar\Delta\,\mathcal{V}^\gamma \le C\,T\,\mathcal{V}^\gamma \tag{3.27}$$

with $\gamma < 1$, which proves the Proposition. □

3.1 Proof of (3.21)

We prove here that the assumption made in Proposition 4 is verified. Before starting with the proof of (3.21) we give some preliminary results.
Let us consider two solutions of the partial dynamics, $(X(t), V(t))$ and $(Y(t), W(t))$. By Proposition 3 and the definition (3.19) of Δ_ℓ the following lemmas can be stated, whose proofs are given in the Appendix. We will assume, for Lemma 2, to be in the region $A/2 \le r < A$ (A is the radius of the cylinder D), in order to avoid an unessential singularity of the polar coordinates, while for $r < A/2$ the proof becomes trivial, since the external field is zero.

Lemma 1. *Let $t \in [0, T]$ such that $t + \Delta_\ell \in [0, T]$ $\forall \ell \le \bar{\ell}$. Then*

$$\text{If} \qquad |V_1(t) - W_1(t)| \le \mathcal{V}^\eta$$

then

$$\sup_{s \in [t, t+\Delta_\ell]} |V_1(s) - W_1(s)| \le 2\mathcal{V}^\eta. \tag{3.28}$$

$$\text{If} \qquad |V_1(t) - W_1(t)| \ge \mathcal{V}^\eta$$

then

$$\inf_{s \in [t, t+\Delta_\ell]} |V_1(s) - W_1(s)| \ge \frac{1}{2}\mathcal{V}^\eta. \tag{3.29}$$

Using cylindrical coordinates we put $V = (V_1, V_r, V_\tau)$, where V_r is the radial velocity component and V_τ the transversal velocity component.

Lemma 2. *Let $t \in [0, T]$ such that $t + \Delta_\ell \in [0, T]$ $\forall \ell \le \bar{\ell}$.*
For any $\mathcal{V}^* \in [\mathcal{V}^\eta, \mathcal{V}]$,

$$\text{If} \qquad |V_\tau(t)| \le \mathcal{V}^*$$

then

$$\sup_{s \in [t, t+\Delta_\ell]} |V_\tau(s)| \le 2\mathcal{V}^*. \tag{3.30}$$

$$\text{If} \qquad |V_\tau(t)| \ge \mathcal{V}^*$$

then

$$\inf_{s \in [t, t+\Delta_\ell]} |V_\tau(s)| \ge \frac{1}{2}\mathcal{V}^*. \tag{3.31}$$

Lemma 3. *Let $t \in [0,T]$ such that $t + \Delta_\ell \in [0,T]$ $\forall \ell \leq \bar{\ell}$, and assume that $|V_1(t) - W_1(t)| \geq h\mathcal{V}^\eta$ for some $h \geq 1$. Then it exists $t_0 \in [t, t+\Delta_\ell]$ such that for any $s \in [t, t+\Delta_\ell]$ it holds:*

$$|X(s) - Y(s)| \geq \frac{h\mathcal{V}^\eta}{4}|s - t_0|.$$

Lemma 4. *There exists a positive constant C such that, for any $\mu \in \mathbb{R}$ and for any couple of positive numbers $R, R' : R < R'$ we have:*

$$W(\mu, R', t) < C\frac{R'}{R}Q(R,t).$$

Now we are ready to start the proof of (3.21). It is based on an inductive procedure, whose steps are the following:
step i) we prove (3.21) for $\ell = 1$;
step ii) we show that if (3.21) holds for $\ell - 1$ it holds also for ℓ;
Proof of step *i*).

We show that the following estimate holds:

$$\langle E \rangle_{\Delta_1} \leq C\left[\mathcal{V}^{\frac{4}{3}\alpha - \frac{2}{3} + \eta}Q^{\frac{2-\alpha}{3}}\log\mathcal{V} + \frac{\mathcal{V}^{\frac{5\alpha-1}{3}}Q^{\frac{2-\alpha}{3}}}{\mathcal{V}^{\bar{\eta}}}\right]. \tag{3.32}$$

For any $t \in [0,T]$ such that $t + \Delta_1 \leq T$, we consider the time evolution of the system over the time interval $[t, t+\Delta_1]$. For any $s \in [t, t+\Delta_1]$ we set

$$(Y(s), W(s)) := (Y(s,t,y,w), W(s,t,y,w))$$

being

$$Y(t) = y, \qquad W(t) = w.$$

The time-invariance of f and of the measure $dydw$ along the characteristics allows to write, by the change of variables $(y,w) \to (Y(s), W(s))$:

$$|E(X(s),s)| \leq \int dydw\ \frac{f(y,w,s)}{|X(s)-y|^{\alpha+1}} = \int dydw\ \frac{f(y,w,t)}{|X(s)-Y(s)|^{\alpha+1}}. \tag{3.33}$$

We decompose the phase space in the following way. We define

$$T_1 = \{y : |y - X(t)| \leq 2R(T)\} \tag{3.34}$$

$$S_1 = \{w : |v_1 - w_1| \leq \mathcal{V}^\eta\} \tag{3.35}$$

$$S_2 = \{w :\ |w_\tau| \leq \mathcal{V}^\eta\} \tag{3.36}$$

$$S_3 = \{w :\ |v_1 - w_1| > \mathcal{V}^\eta\} \cap \{w : |w_\tau| > \mathcal{V}^\eta\}. \tag{3.37}$$

We have

$$|E(X(s),s)| \le \sum_{j=1}^{3} \mathcal{I}_j(X(s)), \tag{3.38}$$

where for any $s \in [t, t+\Delta_1]$

$$\mathcal{I}_j(X(s)) = \int_{T_1 \cap S_j} dydw\ \frac{f(y,w,t)}{|X(s)-Y(s)|^{\alpha+1}}, \qquad j=1,2,3.$$

Let us start with $\mathcal{I}_1$. Putting $(Y(s),W(s)) = (\bar{y},\bar{w})$, by the invariance of f along the trajectories, Lemma 1 implies

$$\mathcal{I}_1(X(s)) \le \int_{T_1' \cap S_1'} d\bar{y}d\bar{w}\ \frac{f(\bar{y},\bar{w},s)}{|X(s)-\bar{y}|^{\alpha+1}}, \tag{3.39}$$

where $T_1' = \{\bar{y} : |\bar{y} - X(s)| \le 4R(T)\}$ and $S_1' = \{\bar{w} : |V_1(s) - \bar{w}_1| \le 2\mathcal{V}^{\eta}\}$. Now it is

$$\begin{aligned}\mathcal{I}_1(X(s)) \le &\int_{T_1' \cap S_1' \cap \{|X(s)-\bar{y}| \le \varepsilon\}} d\bar{y}d\bar{w} \frac{f(\bar{y},\bar{w},s)}{|X(s)-\bar{y}|^{\alpha+1}} + \\ &\int_{T_1' \cap S_1' \cap \{|X(s)-\bar{y}| > \varepsilon\}} d\bar{y}d\bar{w} \frac{f(\bar{y},\bar{w},s)}{|X(s)-\bar{y}|^{\alpha+1}}.\end{aligned} \tag{3.40}$$

Notice that

$$\begin{aligned}&\int_{S_1'} dwf(y,w,s) \le C\mathcal{V}^{\eta} \int_{|w_\perp| \le a} dw_\perp + \\ &\int_{|w_\perp|>a} dw_\perp \int dw_1\ f(y,w,s) \le \\ &Ca^2\mathcal{V}^{\eta} + \frac{1}{a^2}\int dw|w|^2 f(y,w,s) = Ca^2\mathcal{V}^{\eta} + \frac{1}{a^2}K(y,s)\end{aligned}$$

where $w_\perp = (0,w_2,w_3)$ and $K(y,s) = \int dw|w|^2 f(y,w,s)$. Minimizing in a we obtain

$$\int_{S_1'} dwf(y,w,s) \le C\mathcal{V}^{\frac{\eta}{2}} K(y,s)^{\frac{1}{2}}. \tag{3.41}$$

Hence, setting

$$\rho_1(y,s) = \int_{S_1'} dwf(y,w,s),$$

by (3.41) and Lemma 4 we get

$$\begin{aligned}\left(\int_{T_1'} dy\ \rho_1(y,s)^2\right)^{\frac{1}{2}} \le\ & C\mathcal{V}^{\frac{\eta}{2}}\left(\int_{T_1'} dy\ K(y,s)\right)^{\frac{1}{2}} \le \\ & C\mathcal{V}^{\frac{\eta}{2}}\sqrt{W(X_1(s),4R(s),s)} \le C\mathcal{V}^{\frac{\eta}{2}}\sqrt{Q}.\end{aligned} \tag{3.42}$$

Going back to (3.40), this bound implies:

$$\begin{aligned}\mathcal{I}_1(X(s)) \le{}& C\mathcal{V}^2\mathcal{V}^\eta\varepsilon^{2-\alpha}+\\ &\left(\int_{T_1'} dy\,\rho_1(y,s)^2\right)^{\frac12}\left(\int_{T_1'\cap\{|X(s)-y|>\varepsilon\}} dy\,\frac{1}{|X(s)-y|^{2+2\alpha}}\right)^{\frac12}\le\\ &C\left(\mathcal{V}^{2+\eta}\varepsilon^{2-\alpha}+\mathcal{V}^{\frac{\eta}{2}}\sqrt{\frac{Q}{\varepsilon^{2\alpha-1}}}\right).\end{aligned}$$

Minimizing in ε we obtain:

$$\mathcal{I}_1(X(s)) \le C\,\mathcal{V}^{\frac43\alpha-\frac23+\frac{\alpha+1}{3}\eta}\,Q^{\frac13(2-\alpha)}. \tag{3.43}$$

For $\mathcal{I}_2$ we obtain obviously the same bound,

$$\mathcal{I}_2(X(s)) \le C\,\mathcal{V}^{\frac43\alpha-\frac23+\frac{\alpha+1}{3}\eta}\,Q^{\frac13(2-\alpha)}. \tag{3.44}$$

For the third term we cover $S_3\cap T_1$ by means of the sets $A_{h,k}$ and $B_{h,k}$, with $k=0,1,2,...,m$ and $h=1,2,...,m'$, defined in the following way:

$$\begin{aligned}A_{h,k}=\big\{(y,w,s):\ & h\mathcal{V}^\eta<|v_1-w_1|\le(h+1)\mathcal{V}^\eta,\\ &\beta_{k+1}<|w_\tau|\le\beta_k,\ |X(s)-Y(s)|\le l_{h,k}\big\}\end{aligned} \tag{3.45}$$

$$\begin{aligned}B_{h,k}=\big\{(y,w,s):\ & h\mathcal{V}^\eta<|v_1-w_1|\le(h+1)\mathcal{V}^\eta,\\ &\beta_{k+1}<|w_\tau|\le\beta_k,\ |X(s)-Y(s)|> l_{h,k}\big\}\end{aligned} \tag{3.46}$$

where:

$$\beta_k=\frac{\mathcal{V}}{2^k}\qquad l_{h,k}=\frac{2^{\frac{k}{2-\alpha}}Q^{\frac13}}{h\mathcal{V}^{\frac43}}. \tag{3.47}$$

Since we are in S_3, it is immediately seen that

$$m<(1-\eta)\log_2\mathcal{V},\quad m'<2\mathcal{V}^{1-\eta}. \tag{3.48}$$

Consequently we put

$$\mathcal{I}_3(X(s))\le\sum_{h=1}^{m'}\sum_{k=0}^{m}\left(\mathcal{I}_3'(h,k)+\mathcal{I}_3''(h,k)\right), \tag{3.49}$$

being

$$\mathcal{I}_3'(h,k)=\int_{T_1\cap A_{h,k}}\frac{f(y,w,t)}{|X(s)-Y(s)|^{\alpha+1}}\,dydw \tag{3.50}$$

and

$$\mathcal{I}_3''(h,k)=\int_{T_1\cap B_{h,k}}\frac{f(y,w,t)}{|X(s)-Y(s)|^{\alpha+1}}\,dydw. \tag{3.51}$$

By adapting Lemma 1 and Lemma 2 to this context, it is easily seen that $\forall\,(y,w,s) \in A_{h,k}$ it holds:

$$(h-1)\mathcal{V}^\eta < |V_1(s) - W_1(s)| \le (h+2)\mathcal{V}^\eta,$$

and

$$\frac{\beta_{k+1}}{2} < |W_\tau(s)| \le 2\beta_k.$$

Hence setting

$$\begin{aligned} A'_{h,k} = \big\{(\bar{y},\bar{w},s) :\ &(h-1)\mathcal{V}^\eta < |V_1(s) - \bar{w}_1| \le (h+2)\mathcal{V}^\eta, \\ &\frac{\beta_{k+1}}{2} < |\bar{w}_\tau| \le 2\beta_k,\ |X(s) - \bar{y}| \le l_{h,k}\big\}, \end{aligned} \tag{3.52}$$

we have

$$\mathcal{I}'_3(h,k) \le \int_{T'_1 \cap A'_{h,k}} \frac{f(\bar{y},\bar{w},s)}{|X(s)-\bar{y}|^{\alpha+1}}\, d\bar{y} d\bar{w}. \tag{3.53}$$

By the choice of the parameters β_k and $l_{h,k}$ made in (3.47) we have:

$$\begin{aligned} \mathcal{I}'_3(h,k) \le C\, l_{h,k}^{2-\alpha} \int_{A'_{h,k}} d\bar{w} \le C l_{h,k}^{2-\alpha} \beta_k \mathcal{V} \int_{A'_{h,k}} d\bar{w}_1 \le \\ C\, l_{h,k}^{2-\alpha} \beta_k \mathcal{V}^{1+\eta} \le \frac{C}{h^{2-\alpha}} \mathcal{V}^{\frac{4}{3}\alpha - \frac{2}{3} + \eta} Q^{\frac{1}{3}(2-\alpha)}. \end{aligned} \tag{3.54}$$

Hence by (3.48)

$$\sum_{h=1}^{m'} \sum_{k=0}^{m} \mathcal{I}'_3(h,k) \le C\, \mathcal{V}^{\frac{4}{3}\alpha - \frac{2}{3} + \eta} Q^{\frac{1}{3}(2-\alpha)} \log \mathcal{V}. \tag{3.55}$$

Now we pass to $\mathcal{I}''_3(h,k)$, for which we need to make the time average over the interval $[t, t+\Delta_1]$. Setting

$$B'_{h,k} = \{(y,w) : (y,w,s) \in B_{h,k} \quad \text{for some} \quad s \in [t, t+\Delta_1]\} \tag{3.56}$$

we have,

$$\begin{aligned} \int_t^{t+\Delta_1} \mathcal{I}''_3(h,k)\, ds \le \int_t^{t+\Delta_1} ds \int_{T'_1 \cap B'_{h,k}} dydw \frac{f(y,w,t)}{|X(s)-Y(s)|^{\alpha+1}} \le \\ \int_{T'_1 \cap B'_{h,k}} dydw\, f(y,w,t) \int_t^{t+\Delta_1} ds\, \frac{\chi(B_{h,k})}{|X(s)-Y(s)|^{\alpha+1}}. \end{aligned} \tag{3.57}$$

By Lemma 3, putting $a = \frac{4\, l_{h,k}}{h\mathcal{V}^\eta}$ we have,

$$
\begin{aligned}
&\int_t^{t+\Delta_1} ds\, \frac{\chi(B_{h,k})}{|X(s)-Y(s)|^{\alpha+1}} = \\
&\int_t^{t+\Delta_1} \frac{\chi(|X(s)-Y(s)| > l_{h,k})}{|X(s)-Y(s)|^{\alpha+1}}\, ds \le \\
&\int_{\{s:|s-t_0|\le a\}} \frac{\chi(|X(s)-Y(s)| > l_{h,k})}{|X(s)-Y(s)|^{\alpha+1}}\, ds + \\
&\int_{\{s:|s-t_0|> a\}} \frac{\chi(|X(s)-Y(s)| > l_{h,k})}{|X(s)-Y(s)|^{\alpha+1}}\, ds \le \\
&\frac{1}{l_{h,k}^{\alpha+1}} \int_{\{s:|s-t_0|\le a\}} ds + \left[\frac{4}{h\mathcal{V}^\eta}\right]^{\alpha+1} \int_{\{s:|s-t_0|> a\}} \frac{1}{|s-t_0|^{\alpha+1}}\, ds \le \\
&\quad \frac{2a}{(l_{h,k})^{\alpha+1}} + 2\left[\frac{4}{h\mathcal{V}^\eta}\right]^{\alpha+1} \int_a^{+\infty} \frac{1}{s^{\alpha+1}}\, ds = \frac{C}{(l_{h,k})^\alpha h \mathcal{V}^\eta}.
\end{aligned}
\tag{3.58}
$$

Moreover,

$$
\int_{T_1'\cap B_{h,k}'} f(y,w,t)\, dy dw \le \frac{C}{\beta_k^2} \int_{T_1'\cap B_{h,k}'} w^2 f(y,w,t)\, dy dw,
\tag{3.59}
$$

so that

$$
\int_t^{t+\Delta_1} \mathcal{I}_3''(h,k)\; ds \le \frac{C}{(\beta_k)^2\, (l_{h,k})^\alpha\, h\, \mathcal{V}^\eta} \int_{T_1'\cap B_{h,k}'} w^2 f(y,w,t)\, dy dw.
\tag{3.60}
$$

Taking into account (3.47) and (3.48), it is

$$
\begin{aligned}
&\frac{C}{(\beta_k)^2\, (l_{h,k})^\alpha\, h\, \mathcal{V}^\eta} \le C \frac{2^{2k}}{2^{\frac{k\alpha}{2-\alpha}} h^{1-\alpha}}\, Q^{-\frac{\alpha}{3}}\, \mathcal{V}^{\frac{4}{3}\alpha-2-\eta} \le \\
&C \frac{1}{h^{1-\alpha}}\, Q^{-\frac{\alpha}{3}}\, \mathcal{V}^{\frac{4}{3}\alpha-2-\eta+(1-\eta)(2-\frac{\alpha}{2-\alpha})} \le \\
&C\, Q^{-\frac{\alpha}{3}}\, \mathcal{V}^{\frac{4}{3}\alpha-2-\eta+(1-\eta)(2-\frac{\alpha}{2-\alpha})}
\end{aligned}
\tag{3.61}
$$

since $\alpha < 1$. Now it is:

$$
\int_{T_1'\cap B_{h,k}'} w^2 f(y,w,t)\, dy dw \le \int_{T_1'\cap C_{h,k}} w^2 f(y,w,t)\, dy dw
\tag{3.62}
$$

where

$$
\begin{aligned}
C_{h,k} = \big\{ w:\ &(h-1)\mathcal{V}^\eta < |v_1 - w_1| \le (h+2)\mathcal{V}^\eta, \\
&\beta_{k+1} < |w_\tau| \le \beta_k \big\},
\end{aligned}
\tag{3.63}
$$

so that,

$$\sum_{h=1}^{m'}\sum_{k=0}^{m}\int_{T_1'\cap B_{h,k}'} w^2 f(y,w,t)\,dydw \le \\ C\int_{T_1'} K(y,t)\,dy \le C\,W(X_1(T),5R(T),T) \le CQ \tag{3.64}$$

by Lemma 4.

Hence from (3.60), (3.61) and (3.64) it follows:

$$\sum_{h=1}^{m'}\sum_{k=0}^{m}\int_t^{t+\Delta_1} \mathcal{I}_3''(h,k)\,ds \le C\,Q^{1-\frac{\alpha}{3}}\,\mathcal{V}^{\frac{4}{3}\alpha-2-\eta+(1-\eta)(2-\frac{\alpha}{2-\alpha})}. \tag{3.65}$$

By multiplying and dividing by Δ_1 defined in (3.66) we obtain,

$$\sum_{h=1}^{m'}\sum_{k=0}^{m}\int_t^{t+\Delta_1} \mathcal{I}_3''(h,k)\,ds \le \\ C\mathcal{V}^{\frac{5\alpha-1}{3}}Q^{\frac{2-\alpha}{3}}\,\frac{Q^{1-\frac{\alpha}{3}}\,\mathcal{V}^{\frac{4}{3}\alpha-2-\eta+(1-\eta)(2-\frac{\alpha}{2-\alpha})}}{\mathcal{V}^{\eta'}}\,\Delta_1 \le \\ C\mathcal{V}^{\frac{5\alpha-1}{3}}Q^{\frac{2-\alpha}{3}}\mathcal{V}^{\alpha-1-\eta+(1-\eta)\left(2-\frac{\alpha}{2-\alpha}\right)-\eta'}\Delta_1, \tag{3.66}$$

where we have used the bound $Q^{1-\frac{\alpha}{3}} \le C\mathcal{V}^{1-\frac{\alpha}{3}}$. By the choice of the parameters made in (3.16), it is

$$\alpha-1-\eta+(1-\eta)\left(2-\frac{\alpha}{2-\alpha}\right)-\eta' < -\bar{\eta} < 0.$$

Hence

$$\sum_{h=1}^{m'}\sum_{k=0}^{m}\int_t^{t+\Delta_1} \mathcal{I}_3''(h,k)\,ds \le \frac{\mathcal{V}^{\frac{5\alpha-1}{3}}Q^{\frac{2-\alpha}{3}}}{\mathcal{V}^{\bar{\eta}}} \tag{3.67}$$

Finally the bounds (3.43), (3.44), (3.49), (3.55) and (3.67) imply:

$$\sum_{j=1}^{3}\int_t^{t+\Delta_1} \mathcal{I}_j(X(s))\,ds \le C\Delta_1\left[\mathcal{V}^{\frac{4}{3}\alpha-\frac{2}{3}+\eta}Q^{\frac{2-\alpha}{3}}\log\mathcal{V} + \frac{\mathcal{V}^{\frac{5\alpha-1}{3}}Q^{\frac{2-\alpha}{3}}}{\mathcal{V}^{\bar{\eta}}}\right]. \tag{3.68}$$

Hence by (3.38) and (3.68), we have

$$\int_t^{t+\Delta_1} |E(X(s),s)|\,ds \le C\Delta_1\left[\mathcal{V}^{\frac{4}{3}\alpha-\frac{2}{3}+\eta}Q^{\frac{2-\alpha}{3}}\log\mathcal{V} + \frac{\mathcal{V}^{\frac{5\alpha-1}{3}}Q^{\frac{2-\alpha}{3}}}{\mathcal{V}^{\bar{\eta}}}\right],$$

so that we have proved (3.21) for $\ell = 1$.

Proof of step *ii*).

In the previous step we have seen that, starting from estimate (3.9), we arrive at (3.21) on Δ_1. Let us now assume that (3.21) holds at level $\ell - 1$ over an interval of size $\Delta_{\ell-1}$, $\ell > 1$. Then it holds over an interval of size Δ_ℓ (see Remark 3 in the Appendix). In particular we get, analogously to (3.66),

$$\begin{aligned}&\sum_{h=1}^{m'}\sum_{k=0}^{m}\int_t^{t+\Delta_\ell} \mathcal{I}_3''(h,k)\,ds \leq \\ &C\,\frac{\mathcal{V}^{\frac{5\alpha-1}{3}}Q^{\frac{1}{3}(2-\alpha)}}{\mathcal{V}^{\eta'}\mathcal{V}^{\delta(\ell-1)}}Q^{1-\frac{\alpha}{3}}\,\mathcal{V}^{\frac{4}{3}\alpha-2-\eta+(1-\eta)(2-\frac{\alpha}{2-\alpha})}\Delta_\ell\end{aligned} \tag{3.69}$$

and consequently

$$\langle E\rangle_{\Delta_\ell} \leq C\left[\mathcal{V}^{\frac{4}{3}\alpha-\frac{2}{3}+\eta}Q^{\frac{1}{3}(2-\alpha)}\log\mathcal{V} + \frac{\mathcal{V}^{\frac{5\alpha-1}{3}}Q^{\frac{1}{3}(2-\alpha)}}{\mathcal{V}^{\bar\eta}\,\mathcal{V}^{\delta(\ell-1)}}\right] \tag{3.70}$$

which proves the second step. Hence (3.21) is proved.

4 Magnetic confinement

We consider the same system, in which the external confining force is the Lorentz force, that is we consider the equation

$$\begin{cases}\partial_t f(x,v,t) + v\cdot\nabla_x f(x,v,t) + (E(x,t) + v\wedge B(x))\cdot\nabla_v f(x,v,t) = 0\\ E(x,t) = -\displaystyle\int_{\mathbb{R}^3}\nabla\Phi(|x-y|)\;\rho(y,t)\,dy\\ \rho(x,t) = \displaystyle\int_{\mathbb{R}^3} f(x,v,t)\;dv\\ f(x,v,0) = f_0(x,v)\end{cases} \tag{4.1}$$

and the related characteristics equation

$$\begin{cases}\dot X(t) = V(t)\\ \dot V(t) = E(X(t),t) + V(t)\wedge B(X(t))\\ (X(0),V(0)) = (x,v)\\ f(X(t),V(t),t) = f_0(x,v),\end{cases} \tag{4.2}$$

We assume that the external magnetic field B is such that $B(x) = (h(r^2),0,0)$ where, we recall, $r^2 = x_2^2 + x_3^2$ and $h(r^2)$ is a non-negative, smooth function, diverging together with its primitive as $r \to A$. The following result holds:

Theorem 2. *Let us fix an arbitrary positive time T. Assume in (1.4) that $\alpha < 1$ and let $f_0(x,v) \in L_\infty$ be supported on D_0 and such that*

$$0 \leq f_0(x,v) \leq C_1 e^{-\lambda v^2} \tag{4.3}$$

for some positive constants C_1 and λ. Then there exists a solution to system (4.2) in $[0,T]$. This solution is supported on D and there exist two positive constants C_2 and $\bar{\lambda} < \lambda$ such that

$$0 \leq f(x,v,t) \leq C_2 e^{-\bar{\lambda} v^2}.$$

Moreover it is unique in the class of the characteristics distributed with $f(x,v,t) \leq Ce^{-C'v^2}$ for some couple of constants C and C'.

We do not give the proof of this theorem, as it can be deduced from the proof of Theorem 1, but we stress that the magnetic confinement is easier to deal with, since the magnetic force $v \wedge B$ does not change the modulus of the velocity. Hence also in this case we arrive at estimates (2.40), (2.41) and (2.43) for the solutions and the auto-induced field E. These estimates allow to prove the confinement of the plasma, whose proof, analogous to the one in [14], we give for completeness.

We write by components equations (4.2), putting $R^2(t) = X_2^2(t) + X_3^2(t)$, obtaining

$$(V_2X_2 + V_3X_3)h(R^2) = \dot{V}_2X_3 - \dot{V}_3X_2 + X_2E_3 - X_3E_2.$$

Let H be a primitive of h. By integrating in time we get

$$\begin{aligned} &\frac{1}{2}\int_0^t \frac{d}{ds} H(R^2(s))ds = \frac{1}{2}\left[H(R^2(t)) - H(R^2(0))\right] = \\ &\int_0^t ds \left[\dot{V}_2(s)X_3(s) - \dot{V}_3(s)X_2(s) + X_2(s)E_3(s) - X_3(s)E_2(s)\right]. \end{aligned} \tag{4.4}$$

Now we integrate by parts the right hand side of (4.4).

$$\begin{aligned} &\int_0^t ds \left[\dot{V}_2(s)X_3(s) - \dot{V}_3(s)X_2(s) + X_2(s)E_3(s) - X_3(s)E_2(s)\right] = \\ &[V_2(s)X_3(s) - V_3(s)X_2(s)]_0^t + \int_0^t [X_2(s)E_3(X(s),s) - X_3(s)E_2(X(s),s)]\, ds. \end{aligned} \tag{4.5}$$

By (2.40), (2.41) and (2.43), formula (4.5) shows that the right hand side in (4.4) is bounded by a function of $|v|$ and hence, for a fixed characteristic, bounded. This implies that also the left hand side has to be bounded, which gives us the confinement, since the assumptions on f_0 imply that $H(R^2(0)) \leq C$.

5 Appendix

Proof of Lemma 4.

It follows from the definition of the function $\varphi^{\mu,R}$ that, for any $\mu \in \mathbb{R}$ and any couple R, R' such that $0 < R < R'$, it is:

$$\varphi^{\mu,R'}(x) = \varphi\left(\frac{|x_1 - \mu|}{R'}\right) \leq \sum_{i \in \mathbb{Z}:|i| \leq \frac{R'}{R}} \varphi\left(\frac{|x_1 - (\mu + iR)|}{R}\right).$$

Hence, since all terms in the function W are positive, we have:

$$W(\mu, R', t) \leq \sum_{i \in \mathbb{Z}:|i| \leq \frac{R'}{R}} W(\mu + iR, R, t) \leq C\left(\frac{R'}{R}\right) Q(R, t).$$

Proof of Proposition 2.

For any s and t such that $0 \leq s < t \leq T$ we define

$$R(t, s) = R(t) + \int_s^t \mathcal{V}(\tau)\, d\tau. \tag{5.1}$$

Then, it is:

$$R(t, t) = R(t) \quad \text{and} \quad R(t, 0) = R(t) + \int_0^t \mathcal{V}(\tau) \leq 2R(t). \tag{5.2}$$

Let $(X(s), V(s))$ and $(Y(s), W(s))$ be two characteristics starting at time $s = 0$ from (x, v) and (y, w) respectively. Since the flow preserves the measure in the phase space and f is invariant along the characteristics we have:

$$\begin{aligned} W(\mu, R(t,s), s) &= \frac{1}{2}\int dx \int dv\, \varphi^{\mu,R(t,s)}(X(s))\, |V(s)|^2 f_0^N(x, v) + \\ \frac{1}{2}\int dxdv &\Big[\ \varphi^{\mu,R(t,s)}(X(s)) f_0^N(x, v) \int dydw f_0^N(y, w) \Phi(|X(s) - Y(s)|)\Big] \\ &+ \int dxdv\, \varphi^{\mu,R(t,s)} f_0^N(x, v)\, U(X(s)). \end{aligned} \tag{5.3}$$

Deriving the function W with respect to the time s we get:

$$\partial_s W(\mu, R(t, s), s) = A_1(t, s) + A_2(t, s) \tag{5.4}$$

with

$$\begin{aligned} A_1(t, s) = \int dxdv\ &\varphi^{\mu,R(t,s)}(X(s)) f_0^N(x, v)\Big[V(s) \cdot E(X(s), s) + \\ &\frac{1}{2}\int dydw f_0^N(y, w) \nabla\Phi(|X(s) - Y(s)|) \cdot (V(s) - W(s))\Big] \end{aligned} \tag{5.5}$$

and

$$A_2(t,s) = \frac{1}{2}\int dxdv\, f_0^N(x,v)\, \partial_s \left[\varphi^{\mu,R(t,s)}(X(s))\right]$$
$$\left[V^2(s) + \int dydw\, f_0^N(y,w)\, \Phi(|X(s)-Y(s)|) + U(X(s))\right]. \tag{5.6}$$

We show that $A_2(t,s)$ is negative. In fact, the sum in square brackets is positive, on the other hand, by the definition of the function φ it is:

$$\partial_s \left[\varphi^{\mu,R(t,s)}(X(s))\right] =$$
$$\varphi'\left(\frac{|X_1(s)-\mu|}{R(t,s)}\right)\left[\frac{X_1(s)-\mu}{|X_1(s)-\mu|}\cdot\frac{V_1(s)}{R(t,s)} - \frac{\partial_s R(t,s)}{R^2(t,s)}|X_1(s)-\mu|\right].$$

Now, $\varphi'(r) \neq 0$ only if $1 \leq r \leq 2$ and by definition $\partial_s R(t,s) = -\mathcal{V}(s)$, so that:

$$-\frac{\partial_s R(t,s)}{R^2(t,s)}|X_1(s)-\mu| \geq \frac{\mathcal{V}(s)}{R(t,s)}.$$

Hence

$$\frac{X_1(s)-\mu}{|X_1(s)-\mu|}\cdot\frac{V_1(s)}{R(t,s)} - \frac{\partial_s R(t,s)}{R^2(t,s)}|X_1(s)-\mu| \geq \frac{-|V_1(s)|+\mathcal{V}(s)}{R(t,s)} \geq 0.$$

Thus, being $\varphi' \leq 0$, we have proved that

$$A_2(t,s) \leq 0. \tag{5.7}$$

In the term A_1 noticing that $\nabla\Phi(|x-y|)$ is an odd function, by the change of variables $(x,v) \to (y,w)$ we obtain:

$$\begin{aligned} A_1(t,s) = & -\frac{1}{2}\int dxdv \int dydw\; f_0^N(x,v) f_0^N(y,w)\; \left[\varphi^{\mu,R(t,s)}(X(s))\right. \\ & \left.\nabla\Phi(|X(s)-Y(s)|)\cdot(V(s)+W(s))\right] \\ = & -\frac{1}{2}\int dxdv \int dydw\; f_0^N(x,v) f_0^N(y,w) \times \\ & \left\{\nabla\Phi(|X(s)-Y(s)|)\cdot V(s)\left[\varphi^{\mu,R(t,s)}(X(s)) - \varphi^{\mu,R(t,s)}(Y(s))\right]\right\}. \end{aligned}$$

By the definition of $\varphi^{\mu,R(t,s)}$ it follows

$$|\varphi^{\mu,R(t,s)}(X(s)) - \varphi^{\mu,R(t,s)}(Y(s))| \leq 2\frac{|X(s)-Y(s)|}{R(t,s)},$$

and then:

$$\begin{aligned} |A_1(t,s)| \leq & \frac{\mathcal{V}(s)}{R(t,s)}\int dxdv \int dydw\; f_0^N(x,v) f_0^N(y,w) \\ & |\nabla\Phi(|X(s)-Y(s)|)|\;\; |X(s)-Y(s)| \left[\chi_{B(s)}(x,v) + \chi_{\bar{B}(s)}(y,w)\right] \end{aligned}$$

where, by the definition of φ,

$B(s) = \{x : |X_1(s) - \mu| \leq 2R(t,s)\}$ and $\bar{B}(s) = \{y : |Y_1(s) - \mu| \leq 2R(t,s)\}$.

By symmetry we have:

$$|A_1(t,s)| \leq 2\frac{\mathcal{V}(s)}{R(t,s)} \int dxdv \int dydw \; f_0^N(x,v) f_0^N(y,w) \\ |\nabla\Phi(|X(s) - Y(s)|)| \; |X(s) - Y(s)| \; \chi_{B(s)}(x,v).$$

Since

$$r\,|\nabla\Phi(|r|)| \leq C\,\Phi(|r|),$$

we have,

$$\begin{aligned} |A_1(t,s)| &\leq C\,\frac{\mathcal{V}(s)}{R(t,s)} \int dxdv \int dydw \; f_0^N(x,v) f_0^N(y,w)\,\Phi(|X(s) - Y(s)|) \times \\ &\qquad \chi_{B(s)}(x,v) \\ &\leq C\,\frac{\mathcal{V}(s)}{R(t,s)}\,W(\mu, 3R(t,s), s), \end{aligned}$$

as for

$$|X_1(s) - \mu| \leq 2R(t,s) \qquad \text{it results} \qquad |x_1 - \mu| \leq 3R(t,s).$$

Hence by Lemma 4,

$$|A_1(t,s)| \leq C\frac{\mathcal{V}(s)}{R(t,s)} Q(R(t,s), s). \tag{5.8}$$

Going back to (5.4), we see that (5.7) and (5.8) imply:

$$\partial_s W(\mu, R(t,s), s) \leq C\frac{\mathcal{V}(s)}{R(t,s)} Q(R(t,s), s). \tag{5.9}$$

Notice that

$$\int_0^t \frac{\mathcal{V}(s)}{R(t,s)} ds = -\int_0^t \frac{\partial_s R(t,s)}{R(t,s)} ds = \log\frac{R(t,0)}{R(t,t)} \leq \log 2,$$

so that, by integrating in s both members and taking the supremum over μ in (5.9) we get, by the Gronwall lemma,

$$Q(R(t,s), s) \leq CQ(R(t,0), 0).$$

The thesis follows by putting $s = t$, since by (5.2) $Q(R(t,t), t) = Q(R(t), t)$, while the monotonicity of the function Q and Lemma 4 imply $Q(R(t,0), 0) \leq Q(2R(t), 0) \leq CQ(R(t), 0)$.

Remark 3. *We premise the following remark to the proofs of Lemmas 1, 2, and 3. We have,*

$$\langle E\rangle_{\Delta_\ell} \le \langle E\rangle_{\Delta_{\ell-1}}, \qquad \forall\, \ell \le \bar{\ell}. \tag{5.10}$$

In fact, $\Delta_\ell = \mathcal{G}\Delta_{\ell-1}$*, hence recalling (3.20),*

$$\Big[t, t+\Delta_\ell\Big] = \bigcup_{i=1}^{\mathcal{G}} \Big[t+(i-1)\Delta_{\ell-1}, t+i\Delta_{\ell-1}\Big] \tag{5.11}$$

and so,

$$\frac{1}{\Delta_\ell}\int_t^{t+\Delta_\ell} |E(X(s),s)|\, ds \le \max_i \frac{1}{\Delta_{\ell-1}}\int_{t+(i-1)\Delta_{\ell-1}}^{t+i\Delta_{\ell-1}} |E(X(s),s)|\, ds, \tag{5.12}$$

whence we get (5.10), since the estimate (3.21) is built with the maximal time T*.*

Proof of Lemma 1.

We give first the proof for $\ell = 1$, that is $\Delta_\ell = \Delta_1$.
Since the external force gives no contribution to the first component of the velocity, by (3.9) and (3.66) we get, for any $s \in [t, t+\Delta_1]$,

$$\begin{aligned}|V_1(s) - W_1(s)| \le |V_1(t) - W_1(t)| +& \\ &\int_t^{t+\Delta_1} \Big[|E(X(s),s)| + |E(Y(s),s)|\Big] ds \le \\ &\mathcal{V}^\eta + 2C_3\mathcal{V}^{\frac{5\alpha-1}{3}} Q^{\frac{1}{3}(2-\alpha)}\Delta_1 \le 2\mathcal{V}^\eta.\end{aligned}$$

Analogously we prove the second statement:

$$\begin{aligned}|V_1(s) - W_1(s)| \ge |V_1(t) - W_1(t)| -& \\ &\int_t^{t+\Delta_1} \Big[|E(X(s),s)| + |E(Y(s),s)|\Big] ds \ge \\ &\mathcal{V}^\eta - 2C_3\mathcal{V}^{\frac{5\alpha-1}{3}} Q^{\frac{1}{3}(2-\alpha)}\Delta_1 \ge \frac{\mathcal{V}^\eta}{2}.\end{aligned}$$

We show now that Lemma 1 holds true also over a time interval Δ_ℓ, $\ell > 1$, supposing for E the estimate (3.21) at level $\ell - 1$. Proceeding as before we get by Remark 1, for any $s \in [t, t+\Delta_\ell]$,

$$\begin{aligned}&|V_1(s) - W_1(s)| \le |V_1(t) - W_1(t)| + \\ &\int_t^{t+\Delta_\ell} \Big[|E(X(s),s)| + |E(Y(s),s)|\Big] ds \le \\ &\mathcal{V}^\eta + C\left[\mathcal{V}^{\frac{4}{3}\alpha - \frac{2}{3}+\eta} Q^{\frac{1}{3}(2-\alpha)}\log\mathcal{V} + \frac{\mathcal{V}^{\frac{5\alpha-1}{3}} Q^{\frac{1}{3}(2-\alpha)}}{\mathcal{V}^{\bar{\eta}}\,\mathcal{V}^{\delta(\ell-2)}}\right] \frac{\mathcal{V}^{\delta(\ell-1)}\mathcal{V}^{\eta'}}{4C_3\mathcal{V}^{\frac{5\alpha-1}{3}} Q^{\frac{1}{3}(2-\alpha)}} \le \\ &\mathcal{V}^\eta + C\log\mathcal{V}\;\mathcal{V}^{\frac{4}{3}\alpha-\frac{2}{3}+\eta+\frac{4\alpha-2}{3}+\eta'-\frac{5\alpha-1}{3}} + C\mathcal{V}^{\delta-\bar{\eta}+\eta'} \le \\ &\mathcal{V}^\eta + C\,\mathcal{V}^{\eta'} \le 2\mathcal{V}^\eta,\end{aligned}$$

using (3.24), recalling that $\eta' < \eta$, $\delta \in (0, \bar{\eta}]$, and $\mathcal{V}$ is sufficiently large.

We proceed analogously for the lower bound.

Proof of Lemma 2.

The equation of motion in cylindrical coordinates, ϕ-component, is

$$\frac{1}{\varrho}\frac{d}{dt}\left(\varrho^2\dot{\phi}\right) = E_\phi, \tag{5.13}$$

where E_ϕ is the transversal component of the self-generated field, being zero by definition the same component of the external field (ϱ and ϕ represent polar coordinates in the plane $x_1 = const$).

As said before, we assume to be in the region $A/2 \leq \varrho < A$ (A is the radius of the cylinder D), in order to avoid an unessential singularity of the polar coordinates, while for $\varrho < A/2$ the proof becomes trivial switching to cartesian coordinates, since the external field is zero.

Denoting by $V_\phi = \varrho\dot{\phi}$, by (5.13) we have, for any $s \in [t, t + \Delta_1]$,

$$|V_\phi(s)| \leq |V_\phi(t)| + C\int_t^s d\tau\, |E(X(\tau), \tau)|. \tag{5.14}$$

The proof proceeds in the same way as for Lemma 1, since $\mathcal{V}^* \in [\mathcal{V}^\eta, \mathcal{V}]$.

Proof of Lemma 3.

We treat first the case $\ell = 1$, that is $\Delta_\ell = \Delta_1$.

Let $t_0 \in [t, t + \Delta_1]$ be the time at which $|X_1(s) - Y_1(s)|$ has the minimum value. We put $\Gamma(s) = X_1(s) - Y_1(s)$. Moreover we define the function

$$\bar{\Gamma}(s) = \Gamma(t_0) + \dot{\Gamma}(t_0)(s - t_0).$$

Since the external force does not act on the first component of the velocity it is:

$$\ddot{\Gamma}(s) - \ddot{\bar{\Gamma}}(s) = E_1(X(s), s) - E_1(Y(s), s)$$
$$\Gamma(t_0) = \bar{\Gamma}(t_0), \quad \dot{\Gamma}(t_0) = \dot{\bar{\Gamma}}(t_0)$$

from which it follows

$$\Gamma(s) = \bar{\Gamma}(s) + \int_{t_0}^s d\tau \int_{t_0}^\tau d\xi\, \left[E_1(X(\xi), \xi) - E_1(Y(\xi), \xi)\right].$$

By (3.9) and (3.66),

$$\int_{t_0}^s d\tau \int_{t_0}^\tau d\xi\, |E_1(X(\xi), \xi) - E_1(Y(\xi), \xi)| \leq 2C_3\mathcal{V}^{\frac{5\alpha-1}{3}} Q^{\frac{2-\alpha}{3}} \frac{|s-t_0|^2}{2} \leq$$
$$C_3\mathcal{V}^{\frac{5\alpha-1}{3}} Q^{\frac{2-\alpha}{3}} \Delta_1 |s - t_0| \leq \mathcal{V}^{\eta'} \frac{|s-t_0|}{4}. \tag{5.15}$$

Hence,

$$|\Gamma(s)| \geq |\bar{\Gamma}(s)| - \mathcal{V}^{\eta'} \frac{|s - t_0|}{4}. \tag{5.16}$$

Now we have:

$$|\bar{\Gamma}(s)|^2 = |\Gamma(t_0)|^2 + 2\Gamma(t_0)\dot{\Gamma}(t_0)(s - t_0) + |\dot{\Gamma}(t_0)|^2|s - t_0|^2.$$

We observe that $\Gamma(t_0)\dot{\Gamma}(t_0)(s - t_0) \geq 0$. Indeed, if $t_0 \in (t, t + \Delta_1)$ then $\dot{\Gamma}(t_0) = 0$ while if $t_0 = t$ or $t_0 = t + \Delta_1$ the product $\Gamma(t_0)\dot{\Gamma}(t_0)(s - t_0) \geq 0$. Hence

$$|\bar{\Gamma}(s)|^2 \geq |\dot{\Gamma}(t_0)|^2|s - t_0|^2.$$

By Lemma 1 (adapted to this context with a factor $h \geq 1$), since $t_0 \in [t, t + \Delta_1]$ it is

$$|\dot{\Gamma}(t_0)| \geq h\frac{\mathcal{V}^\eta}{2}$$

hence

$$|\bar{\Gamma}(s)| \geq h\frac{\mathcal{V}^\eta}{2}|s - t_0|$$

and finally by (5.16),

$$|\Gamma(s)| \geq h\frac{\mathcal{V}^\eta}{4}|s - t_0|.$$

From this the thesis follows, since obviously $|X(s) - Y(s)| \geq |\Gamma(s)|$.

We note that the same proof works also considering the interval $[t, t+\Delta_\ell]$, $\ell > 1$, and for E the estimate (3.21) at level $\ell - 1$. In fact we have for the product (see at the end of the proof of Lemma 1),

$$\langle E \rangle_{\Delta_{\ell-1}} \Delta_\ell \leq C\mathcal{V}^{\eta'}$$

which, used in (5.15), allows to achieve the proof.

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References

[1] Alexander, R.: Time evolution for infinitely many hard spheres. Comm. Math. Phys. **49**, 217–232 (2076)

[2] Bahn, C., Park, Y.M., Yoo, H.J.: Nonequilibrium dynamics of infinite particle systems with infinite range interactions. J. Math. Phys. **40**, 4337–4358 (1999)

[3] Batt, J., Rein, G.: Global classical solutions of a periodic Vlasov-Poisson system in three dimensions. C.R. Acad. Sci. Paris **313** S.I., 411–416 (1991)

[4] Braun, W., Hepp, K.: The Vlasov dynamics and its fluctuations in the 1/N limit of interacting classical particles. Comm. Math. Phys. **56**, 101–113 (1977)

[5] Buttà, P., Caglioti, E., Marchioro, C.: On the Long Time Behavior of Infinitely Extended Systems of Particles Interacting via Kac Potentials. J. Stat. Phys. **108**, 317–339 (2002)

[6] Buttà, P., Caprino, S., Cavallaro, G., Marchioro, C.: On the dynamics of infinitely many particles with magnetic confinement. Boll. Unione Mat. Ital. Ser. VIII Sez. B **9**, 371–395 (2006)

[7] Buttà, P., Cavallaro, G., Marchioro, C.: Time evolution of two dimensional systems with infinitely many particles mutually interacting via very singular forces. J. Stat. Phys **108**, 317–339 (2012)

[8] Buttà, P., Cavallaro, G., Marchioro, C.: Dynamics of an infinitely extended hard core system. Rep. Math. Phys. **72**, 369–377 (2013)

[9] Buttà, P., Cavallaro, G., Marchioro, C.: Mathematical models of viscous friction. To appear on Lecture Notes in Math.

[10] Caglioti, E., Caprino, S., Marchioro, C., Pulvirenti, M.: The Vlasov equation with infinite mass. Arch. Ration. Mech. Anal. **159**, 85–108 (2001)

[11] Caglioti, E., Marchioro, C., Pulvirenti, M.: Non-equilibrium dynamics of three-dimensional infinite particle systems. Comm. Math. Phys. **215**, 25–43 (2000)

[12] Calderoni, P., Caprino, S.: Time evolution of infinitely many particles: an existence theorem. J. Stat. Phys. **28**, 815-833 (1982)

[13] Caprino, S., Cavallaro, G., Marchioro, C.: Time evolution of a Vlasov-Poisson plasma with magnetic confinement. Kinet. Relat. Models **5**, 729–742 (2012)

[14] Caprino, S., Cavallaro, G., Marchioro, C.: On a magnetically confined plasma with infinite charge. SIAM J. Math. Anal. **46**, 133–164 (2014)

[15] Caprino, S., Cavallaro, G., Marchioro, C.: Remark on a magnetically confined plasma with infinite charge. Rend. Mat. Appl. **35**, 69–98 (2014)

[16] Caprino, S., Cavallaro, G., Marchioro, C.: Time evolution of a Vlasov-Poisson plasma with infinite charge in $\mathbb{R}^3$. Comm. Partial Differential Equations **40**, 357–385 (2015)

[17] Caprino, S., Marchioro, C., Pulvirenti, M.: On the two dimensional Vlasov-Helmholtz equation with infinite mass. Comm. Partial Differential Equations **27**, 791–808 (2002)

[18] Cavallaro, G., Marchioro, C., Spitoni, C.: Dynamics of infinitely many particles mutually interacting in three dimensions via a bounded superstable long-range potential. J. Stat. Phys. **120**, 367–416 (2005)

[19] Dobrushin, R.L.: Vlasov equations. Funct. Anal. Appl. **13**, 115–123 (1979)

[20] Dobrushin, R.L., Fritz, J.: Non equilibrium dynamics of one-dimensional infinite particle system with hard-core interaction. Comm. Math. Phys. **55**, 275–292 (1977)

[21] Fritz, J.: Some remark on non equilibrium dynamics of infinite particle systems. J. Stat. Phys. **34**, 539–556 (1985)

[22] Fritz, J., Dobrushin, R.L.: Non-equilibrium dynamics of two-dimensional infinite particle systems with a singular interaction Comm. Math. Phys. **57**, 67–81 (1977)

[23] Glassey, R.: The Cauchy problem in kinetic theory. SIAM: Philadelphia, PA, (1996)

[24] Hauray, M., Jabin, P.-E.: N-particles approximation of the Vlasov equations with singular potential. Arch. Ration. Mech. Anal. **183**, 489–524 (2007)

[25] Jabin, P.-E.: The Vlasov-Poisson system with infinite mass and energy. J. Stat. Phys. **103**, 1107–1123 (2001)

[26] Jabin, P.-E.: A review of the mean field limits for Vlasov equations. Kinet. Relat. Models **7**, 661–711 (2014)

[27] Lanford, O.E.: Classical Mechanics of one-dimensional systems with infinitely many particles. I An existence theorem. Comm. Math. Phys. **9**, 176–191 (1968)

[28] Lanford, O.E.: Classical Mechanics of one-dimensional systems with infinitely many particles. II Kinetic Theory. Comm. Math. Phys. **11**, 257–292 (1969)

[29] Lanford, O.E.: Time evolution of large classical systems. Moser ed., Lecture Notes in Phys. **38**, Berlin-Heidelberg-New York: Springer-Verlag (1975)

[30] Lions, P.L., Perthame, B.: Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. Invent. Math. **105**, 415–430 (1991)

[31] Marchioro, C., Pellegrinotti, A., Presutti, E.: Existence of time evolution in ν-dimensional Statistical Mechanics. Comm. Math. Phys. **40**, 175–185 (1975)

[32] Marchioro, C., Pellegrinotti, A., Pulvirenti, M.: Remarks on the existence of non-equilibrium dynamics. Proceedings Esztergom Summer School, Colloq. Math. Soc. Janos Bolyai **27**, 733–746 (1978)

[33] Marchioro, C., Pulvirenti, M.: Time evolution of infinite one-dimensional Coulomb systems. J. Stat. Phys. **27**, 809–822 (1982)

[34] Neunzert, H.: An introduction to the nonlinear Boltzmann-Vlasov equation. In: Proceedings of Kinetic Theories and the Boltzmann Equation (Moltecatini, 1981), Lecture Notes in Math. **1048**, 60–110 (1984)

[35] T.T. Nguyen, T.V. Nguyen, W. Strauss, Global magnetic confinement for the 1.5D Vlasov-Maxwell system, Kinet. Relat. Models **8** 153–168 (2015)

[36] Pankavich, S.: Global existence for the three dimensional Vlasov-Poisson system with steady spatial asymptotics. Comm. Partial Differential Equations **31**, 349–370 (2006)

[37] Pfaffelmoser, K.: Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data. J. Differential Equations **95**, 281–303 (1992)

[38] Presutti, E., Pulvirenti, M., Tirozzi, B.: Time evolution of infinite classical systems with singular, long-range, two-body interaction. Comm. Math. Phys. **47**, 81–95 (1976)

[39] Pulvirenti, M.: On the time evolution of states of infinitely extended particle systems. J. Stat. Phys. **27**, 693–713 (1982)

[40] Rein, G.: Growth Estimates for the solutions of the Vlasov-Poisson System in the Plasma Physics Case. Math. Nachr. **191**, 269–278 (1998)

[41] Schaeffer, J.: Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions. Comm. Partial Differential Equations **16** (8-9), 1313–1335 (1991)

[42] Schaeffer, J.: The Vlasov-Poisson system with steady spatial asymptotics. Comm. Partial Differential Equations **28** (8-9), 1057–1084 (2003)

[43] Schaeffer, J.: Steady spatial asymptotics for the Vlasov-Poisson system. Math. Methods Appl. Sci. **26** (8-9), 273–296 (2003)

[44] Schaeffer, J.: Global existence for the Vlasov-Poisson system with steady spatial asymptotic behavior. Kinet. Relat. Models **5**, 129–153 (2012)

[45] Sigmund-Shultze, R: On non-equilibrium dynamics of multidimensional infinite particle systems in the translational invariant case. Comm. Math. Phys. **100**, 487–501 (1985)

[46] Sinai, Ya: Construction of the dynamics for one-dimensional systems of Statistical Mechanics. Sov. Theor. Math. Phys. **12**, 487–501 (1973)

[47] Sinai, Ya: The construction of cluster dynamics of dynamics systems in Statistical Mechanics Vest. Moskow Univ. Sez I Math. Mech. **29**, 152–176 (1974)

[48] Spohn, H.: On the Vlasov Hierarchy. Math. Methods Appl. Sci. **3**, 445–455 (1981)

[49] Wollman, S.: Global in time solution to the three-dimensional Vlasov-Poisson system. J. Math. Anal. Appl. **176** (1), 76–91 (1993)