On a Vlasov-Poisson plasma confined in a torus by a magnetic mirror

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Abstract

We study the time evolution of a Vlasov-Poisson plasma moving in a torus, in which it is confined by an unbounded external magnetic field. This field depends on the distance from the border of the torus, is tangent to the border and singular on it. We prove the existence and uniqueness of the solution, and also its confinement inside the torus for all times, i.e. the external field behaves like a magnetic mirror.

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1 Introduction

Recently the time evolution of a Vlasov-Poisson plasma confined in a cylinder by a magnetic field has been studied in [3, 4, 5]. More precisely the authors have assumed the presence of an external magnetic field parallel to the axis of the cylinder, smooth inside and singular on the border. It is proved in [3] (for finite total mass) and [4, 5] (for infinite total mass), in spite of the fact that the mutual interaction could become very large, that the magnetic mirror effect happens, i.e. each element of the plasma is rejected by the border and the plasma remains confined inside the cylinder. In the present paper we show that the same effect happens also if the plasma is contained in a torus, that is a region in which the border has a non vanishing curvature. The analysis in the present situation needs some non trivial improvements, due to the geometry of the region. For reader's convenience the present paper is self-contained, even if some parts follow directly [3]. The idea lies always in the *rectification* of the characteristics of the plasma

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(that is, a fluid-element's trajectory is close to a straight line in a short time interval) along the lines of the magnetic field. We show that a control of the time average of a plasma particle's velocity along the direction of the magnetic field allows to obtain the result. The principal difficulty is that the magnetic field could destroy the rectification property near the border, where its intensity becomes very large. Another motivation to the present paper is that in [3] it is stated that a confining effect happens also when the magnetic field is orthogonal to the symmetry axis of the cylinder, without an explicit proof, which actually is not straightforward, and it can be achieved analogously to the present one.

The existence of the solution of the Vlasov-Poisson equation in three dimensions is not obvious, because of the singular nature of the interaction, that could produce an infinite growth of the velocity in a finite time. This behavior has been excluded twenty years ago by a clever use of time averages of the electric field. A central point in the proof was a *rectification* of the motion of a characteristic of the Vlasov fluid during a small time interval Δ. The interval Δ has to be chosen large enough to benefit by the time average, but also small enough such that the rectification property for speedy particles holds. In this way it has been proved that initially bounded velocities remain bounded for any time interval $[0, T]$. This approach and other ones are discussed in the papers [2], [10], [13, 14, 16] and [17] in the case of finite total mass. See also [9] for a review. The various papers on the argument also improve the dependence on T. Moreover we mention that the time evolution of a Vlasov-Poisson plasma in presence of singular external forces has been studied in some recent papers [6, 7, 8, 11, 15], while the "one and one-half" dimensional relativistic Vlasov-Maxwell system in a bounded domain with magnetic confinement has been studied in [12].

The main difference between the present paper and [3] lies in the fact that here the magnetic lines are no more straight lines, which forces us to use curvilinear coordinates, and consequently to choose a smaller time interval Δ with respect to [3], for techical reasons which will be clear in the sequel. For this fact we take less advantage by the time average (in the limit, the point estimate in time is not good for our purposes), and to compensate it we need some more refined partitions of the phase-space. For the sake of concreteness we study the problem with a particular choice of the magnetic field, but other choices are possible. Moreover a posteriori it will be evident that the proof applies to a generic region containing the plasma, provided that the external magnetic field is tangent to the border and singular on it.

The plan of the paper is the following: in Section 2 we define the model and give the main results, and in Section 3 we give the proofs. The Appendix is devoted to some technical tools.

2 Statement of the problem and results

Consider a torus $\mathbb{T}^3$ such that $x = (x_1, x_2, x_3) \in \mathbb{T}^3$ if

$$\left(R - \sqrt{x_1^2 + x_2^2}\right)^2 + x_3^2 = r^2, \tag{2.1}$$

with $R > r_0 > 0$, and $r \in [0, r_0]$. In toroidal coordinates the equations are:

$$\begin{cases} x_1 = (R + r\cos\alpha)\cos\theta \\ x_2 = (R + r\cos\alpha)\sin\theta \\ x_3 = r\sin\alpha \\ 0 \le \alpha < 2\pi, \quad 0 \le \theta < 2\pi. \end{cases} \tag{2.2}$$

We study a charged plasma (with charges of the same sign) initially strictly contained in $\mathbb{T}^3$, and moving via the Vlasov-Poisson equation coupled with a magnetic external field $B(x)$. Let $f(x, v, t)$ be the charge distribution (or equivalently mass) of the plasma particles at the phase point (x, v) and time t. The evolution equations for the plasma are:

$$\begin{cases} \partial_t f(x, v, t) + v \cdot \nabla_x f(x, v, t) + (E(x, t) + v \wedge B(x)) \cdot \nabla_v f(x, v, t) = 0 \\ E(x, t) = \dfrac{1}{4\pi} \displaystyle\int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \rho(y, t)\, dy \\ \rho(x, t) = \displaystyle\int_{\mathbb{R}^3} f(x, v, t)\, dv \\ f(x, v, 0) = f_0(x, v). \end{cases} \tag{2.3}$$

We choose the external magnetic field $B(x)$ of the form

$$B(x) = \nabla \wedge A(x), \quad A(x) = \frac{a(r)}{R + r\cos\alpha}\, \hat{e}_\theta, \tag{2.4}$$

where $a(r)$ is a smooth function for $0 \le r < r_0$, which becomes singular for $r \to r_0$, and $\hat{e}_\theta$ is the unit vector tangent to the border of the torus in the direction of increasing θ (and fixed α).
From (2.2) one obtains

$$\begin{cases} \hat{e}_r = \cos\alpha\cos\theta\, \hat{c}_1 + \cos\alpha\sin\theta\, \hat{c}_2 + \sin\alpha\, \hat{c}_3 \\ \hat{e}_\theta = -\sin\theta\, \hat{c}_1 + \cos\theta\, \hat{c}_2 \\ \hat{e}_\alpha = -\sin\alpha\cos\theta\, \hat{c}_1 - \sin\alpha\sin\theta\, \hat{c}_2 + \cos\alpha\, \hat{c}_3 \end{cases} \tag{2.5}$$

where $\hat{e}_\alpha$ is the unit vector defined analogously to $\hat{e}_\theta$ (consequently orthogonal to $\hat{e}_\theta$), $\hat{e}_r = \hat{e}_\theta \wedge \hat{e}_\alpha$, and $\hat{c}_1$, $\hat{c}_2$, $\hat{c}_3$ are the unit vectors of the cartesian axes x_1, x_2, x_3. From (2.4) we have (for the curl in toroidal coordinates see for instance [1])

$$B(x) = \frac{a'(r)}{R + r\cos\alpha}\,\hat{e}_\alpha. \tag{2.6}$$

We will see that the properties assumed for $a(r)$ assure that the plasma remains confined inside the torus for all positive times.

Equation (2.3) is a conservation equation for the density f along the characteristics of the system, i.e. the solutions to the following problem:

$$\begin{cases} \dot{X}(t) = V(t) \\ \dot{V}(t) = E(X(t), t) + V(t) \wedge B(X(t)) \\ (X(t'), V(t')) = (x, v), \end{cases} \tag{2.7}$$

where we have used the simplified notation

$$(X(t), V(t)) = (X(t, t', x, v), V(t, t', x, v)) \tag{2.8}$$

to represent a characteristic at time t passing at time $t' < t$ through the point (x, v). Hence we have

$$\|f(t)\|_{L^\infty} = \|f(0)\|_{L^\infty}. \tag{2.9}$$

Moreover this dynamical system preserves the measure of the phase space (Liouville's theorem).

A remark, that will play an important role in the sequel, is the conservation of total energy. In fact the magnetic force $V(t) \wedge B(X(t))$ does not change the modulus of the velocity, since

$$\frac{d}{dt}V^2 = 2V \cdot \dot{V} = 2V \cdot (E + V \wedge B) = 2V \cdot E, \tag{2.10}$$

and the quantity

$$\mathcal{E} = \frac{1}{2}\int v^2 f(x, v, t)\, dxdv + \frac{1}{2}\int \frac{\rho(x, t)\,\rho(y, t)}{|x - y|}\, dxdy \tag{2.11}$$

is invariant under the evolution (2.7).
We denote by $\mathcal{S}_t$ the support in (x, v) of $f(x, v, t)$ for any $t \in [0, T]$, $T > 0$ being the positive arbitrarily fixed time, and we set:

$$P(T) := P = \max\left\{ \sup_{t\in[0,T]} \sup_{(x,v)\in S_t} |V(t)|, 1 \right\}. \tag{2.12}$$

Our main result is the following:

Theorem 1. *Let $T > 0$ be arbitrarily fixed and assume that $f_0 \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ is a positive function having compact support on the set*

$$\mathcal{S}_0 = \left\{(x, v) : x \in \mathbb{T}^3, \quad d(x, \partial\mathbb{T}^3) > \delta, \quad |v| \leq P_0\right\}, \tag{2.13}$$

for positive constants $P_0 > 0$, $\delta \in (0, r_0)$, being $d(x, \partial\mathbb{T}^3)$ the distance of a point x from the border of the torus. Then there exists a solution to system (2.7) over the interval $[0, T]$, which is supported for all times on the set

$$\mathcal{S}_t = \left\{(x, v) : x \in \mathbb{T}^3, \quad d(x, \partial\mathbb{T}^3) > \delta(t), \quad |v| \leq P(t)\right\},$$

for a suitable continuos function $\delta(t) \in (0, r_0)$.
Moreover this solution is unique in the class of the characteristics distributed with $f(x, v, t)$ and supported on S_t.

We remark that the assumptions on f_0 imply that initially the energy is bounded, and hence

$$\mathcal{E}(t) = \mathcal{E}(0) \leq C. \tag{2.14}$$

From now on we will indicate by C any positive constant, depending only on conserved quantities and possibly on T, and changing from line to line. Some constants will be numbered, in order to quote them in the paper. We set

$$\|f_0\|_{L^\infty} = C_1. \tag{2.15}$$

The fundamental estimate which we need to prove Theorem 1 is stated in the following Theorem.

Theorem 2. *In the hypotheses of Theorem 1 it is:*

$$P(T) \leq C.$$

As it is well known, Theorem 2 implies existence and uniqueness of the solution to system (2.7) globally in time and similarly to system (2.3) if the initial datum f_0 is assumed to be smooth.

We note that other choices of the magnetic field are possible, for example it can be taken directed along $\hat{e}_\theta$ and singular on the border of the torus. We have considered the form (2.6) which gives more difficulties in the analysis, other choices can be studied following the same lines of the present paper.

3 Proofs

We remark that the proof of Theorem 2 would be trivial if we could obtain an a priori bound on the electric field $|E(x, t)| \leq CP^\alpha$, with $0 \leq \alpha < 1$.

Unfortunately we are not able to obtain an estimate so sharp and we are only able to bound the time average of the electric field, as we will show in the sequel.

We need to state some preliminary Lemmas whose proofs are postponed in the Appendix. First of all we state two classical well known estimates, which we prove for the sake of completeness.

Lemma 1. *In the hypotheses of Theorem 1 we have:*

$$\left(\int \rho(x,t)^{\frac{5}{3}} dx\right)^{\frac{3}{5}} \leq C \tag{3.1}$$

Lemma 2. *In the hypotheses of Theorem 1 we have:*

$$\sup_{t\in[0,T]} \sup_{(x,v)} |E(x,t)| \leq C_2 P^{\frac{4}{3}}. \tag{3.2}$$

We define now the short time interval Δ by which we divide the total time interval $[0,T]$. We need to choose a Δ smaller than in [3], precisely we put

$$\Delta := \min\left\{\frac{C_3}{P^{2-\gamma}}, T\right\} \tag{3.3}$$

where $\gamma \in (0, \frac{1}{8})$, and

$$C_3 = \frac{1}{4\left(2 + \frac{1}{R-r_0} + C_2\right)}.$$

It will be useful to study the evolution of a single characteristic of the plasma in toroidal coordinates, whence system (2.7) becomes

$$\begin{cases} -\dot{\alpha}^2 r + \ddot{r} - (R + r\cos\alpha)\dot{\theta}^2 \cos\alpha = E_r + a'(r)\,\dot{\theta} \\ (R + r\cos\alpha)\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\cos\alpha - 2\,\dot{\alpha}\,\dot{\theta}\,r\sin\alpha = E_\theta - \dfrac{a'(r)\,\dot{r}}{R + r\cos\alpha} \\ \ddot{\alpha}\,r + 2\,\dot{\alpha}\,\dot{r} + (R + r\cos\alpha)\dot{\theta}^2 \sin\alpha = E_\alpha, \end{cases} \tag{3.4}$$

where we have omitted the dependence on t of (r, θ, α), and we denote by $(E_r, E_\theta, E_\alpha)$ the components of the electric field in toroidal coordinates. Note that the components of the velocity in such coordinates are

$$v_r = \dot{r}, \qquad v_\theta = (R + r\cos\alpha)\dot{\theta}, \qquad v_\alpha = r\dot{\alpha}, \tag{3.5}$$

and we remark that eq.s (3.4) show that the component v_α is not directly affected by the magnetic field.

We consider two characteristics, solutions of (3.4),

$$\Big(r_1(t), \theta_1(t), \alpha_1(t);\ \dot r_1(t), \dot\theta_1(t), \dot\alpha_1(t)\Big), \tag{3.6}$$

which corresponds in cartesian coordinates to $(X(t), V(t))$, and

$$\Big(r_2(t), \theta_2(t), \alpha_2(t);\ \dot r_2(t), \dot\theta_2(t), \dot\alpha_2(t)\Big), \tag{3.7}$$

corresponding in cartesian coordinates to $(Y(t), W(t))$.
In the following Lemmas we make the technical assumption that the trajectories occur in the region $r > \frac{r_0}{2}$, condition which is satisfied in the short time interval Δ, if initially it is $r > \frac{r_0}{2} + P\Delta$ (note that $P\Delta \to 0$ if P diverges). This assumption is not essential, it is done only to avoid the singularity of the toroidal coordinates for $r = 0$. Actually for $r \leq \frac{r_0}{2}$ we could use cartesian coordinates, since the magnetic field is bounded, and the analysis follows well known results as [16], [17].
We fix a positive number $\gamma' > \gamma$. Hence we state:

Lemma 3. *Let $t' \in [0, T]$. The following hold:*

$$\textit{If} \qquad |r_1(t')\dot\alpha_1(t') - r_2(t')\dot\alpha_2(t')| \leq P^{\gamma'}$$

then

$$\sup_{t\in[t',t'+\Delta]} |r_1(t)\dot\alpha_1(t) - r_2(t)\dot\alpha_2(t)| \leq 2P^{\gamma'}. \tag{3.8}$$

$$\textit{If} \qquad |r_1(t')\dot\alpha_1(t') - r_2(t')\dot\alpha_2(t')| \geq P^{\gamma'}$$

then

$$\inf_{t\in[t',t'+\Delta]} |r_1(t)\dot\alpha_1(t) - r_2(t)\dot\alpha_2(t)| \geq \frac{1}{2}P^{\gamma'}. \tag{3.9}$$

Let us put $v^\perp = \sqrt{v_r^2 + v_\theta^2}$, denoting the corresponding quantity for the two characteristics as $v_i^\perp$, $i = 1, 2$.

Lemma 4. *Let $t' \in [0, T]$. The following hold:*
If

$$|v_1^\perp(t')| \leq \sqrt{P}$$

then

$$\sup_{t\in[t',t'+\Delta]} |v_1^\perp(t)| \leq 2\sqrt{P}. \tag{3.10}$$

If

$$|v_1^\perp(t')| \geq \sqrt{P}$$

then

$$\inf_{t\in[t',t'+\Delta]} |v_1^\perp(t)| \geq \frac{\sqrt{P}}{2}. \tag{3.11}$$

In the following Lemma it is stated the so-called *rectification property* of the characteristics. The proof is quite similar to the one given in Ref.s [3, 4], suitably adapted to our context.

Lemma 5. *Let $t' \in [0, T]$ and assume that*

$$|v_{1,\alpha}(t') - v_{2,\alpha}(t')| \geq hP^{\gamma'}, \quad \textit{for some } h \geq 1,$$

$$v_{1,\alpha} := r_1 \dot{\alpha}_1, \qquad v_{2,\alpha} := r_2 \dot{\alpha}_2.$$

Then, there exists $t_0 \in [t', t' + \Delta]$ such that

$$|X(t) - Y(t)| \geq h \frac{P^{\gamma'}}{8} |t - t_0|$$

for all $t \in [t', t' + \Delta]$.

Proof of Theorem 2. We will use in the sequel cartesian coordinates for volume elements and integrand functions, and toroidal coordinates for the parametrization of the region of integration.
We partition the interval $[0, T]$ by N intervals $[t_i, t_{i+1}]$ $i = 0, ..., N - 1$, with $t_0 = 0$, $t_N = T$ and $\frac{1}{2}\Delta \leq t_{i+1} - t_i \leq \Delta$. Hence it is:

$$\int_0^t E(X(s), s)\, ds = \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} E(X(s), s)\, ds. \tag{3.12}$$

For a fixed i we consider the time evolution of the system over the time interval $[t_i, t_{i+1}]$. For any $t \in [t_i, t_{i+1}]$ we set

$$(X(t), V(t)) := (X(t, t_i, x, v), V(t, t_i, x, v))$$

being

$$X(t_i) = x, \quad V(t_i) = v,$$

which corresponds to the characteristic (3.6), and $(Y(t), W(t))$, solution to system (2.7) such that

$$Y(t_i) = y, \quad W(t_i) = w,$$

corresponding to (3.7). Analogously we put

$$v_{1,\alpha}(t_i) = v_{1,\alpha}, \qquad v_1^{\perp}(t_i) = v_1^{\perp}, \qquad v_{2,\alpha}(t_i) = v_{2,\alpha}, \qquad v_2^{\perp}(t_i) = v_2^{\perp}.$$

By the invariance of f along the motion, the change of variables $(\bar{y}, \bar{w}) = (Y(t), W(t))$ and the Liouville theorem yield:

$$|E(X(t), t)| \leq \int \frac{f(\bar{y}, \bar{w}, t)}{|X(t) - \bar{y}|^2}\, d\bar{y} d\bar{w} = \int \frac{f(y, w, t_i)}{|X(t) - Y(t)|^2}\, dy dw. \tag{3.13}$$

We put

$$\mathcal{I}_m = \int_{S_m} \frac{f(y,w,t_i)}{|X(t)-Y(t)|^2}\, dydw \quad m=1,2,3$$

and

$$S_1 = \{(y,w) : |v_{1,\alpha} - v_{2,\alpha}| \le P^{\gamma'}\} \tag{3.14}$$

$$S_2 = \{(y,w) : |v_2^{\perp}| \le \sqrt{P}\} \tag{3.15}$$

$$S_3 = (S_1 \cup S_2)^c \tag{3.16}$$

Hence it is:

$$|E(X(t),t)| \le \sum_{m=1}^{3} \mathcal{I}_m. \tag{3.17}$$

Let us start by $\mathcal{I}_1$. By (3.8) it follows that if $(y,w) \in S_1$ then $|v_{1,\alpha}(t) - v_{2,\alpha}(t)| \le 2P^{\gamma'}$ for any $t \in [t_i, t_{i+1}]$. Hence setting $S_1' = \{(\bar{y},\bar{w}) : |\bar{v}_{2,\alpha} - v_{1,\alpha}(t)| \le 2P^{\gamma'}\}$ and χ for the characteristic function, by Hölder inequality and Lemma 1 we obtain:

$$\begin{aligned}
&\mathcal{I}_1 \le \int_{S_1'} \frac{f(\bar{y},\bar{w},t)}{|X(t)-\bar{y}|^2}\, d\bar{y}d\bar{w} \le \\
&\int_{|X(t)-\bar{y}|\le\epsilon} \frac{f(\bar{y},\bar{w},t)}{|X(t)-\bar{y}|^2}\chi(S_1')\, d\bar{y}d\bar{w} + \int_{|X(t)-\bar{y}|>\epsilon} \frac{\rho(\bar{y},t)}{|X(t)-\bar{y}|^2}\, d\bar{y} \le \\
&C_1\epsilon \int \chi(S_1')\, d\bar{w} \; + \left(\int \rho(\bar{y},t)^{\frac{5}{3}} d\bar{y}\right)^{\frac{3}{5}} \left(\int_{|X(t)-\bar{y}|>\epsilon} \frac{1}{|X(t)-\bar{y}|^5}\, d\bar{y}\right)^{\frac{2}{5}} \le \\
&C\epsilon P^{2+\gamma'} + C\epsilon^{-\frac{4}{5}}.
\end{aligned} \tag{3.18}$$

The minimum in ϵ of the right hand side is attained for $\epsilon = CP^{-\frac{10}{9}-\frac{5}{9}\gamma'}$, so we get:

$$\mathcal{I}_1 \le CP^{\frac{8}{9}+\frac{4}{9}\gamma'}. \tag{3.19}$$

To perform the integral over the set S_2 we observe that by (3.10) if $(y,w) \in S_2$ then for all $t \in [t_i, t_{i+1}]$ it is $|v_2^{\perp}(t)| \le 2\sqrt{P}$, hence by the same arguments used to obtain (3.19) we have:

$$\mathcal{I}_2 \le C\epsilon P^2 + C\epsilon^{-\frac{4}{5}} \le CP^{\frac{8}{9}}. \tag{3.20}$$

Now we consider the integral over S_3, for which we need to use some ideas from [4] in the case of infinite mass, in order to deal with the short time interval Δ. For $t \in [t_i, t_{i+1}]$ we introduce the sets $A_{h,k}$ and $B_{h,k}$, with $k = 0,1,2,...,m$ and $h = 1,2,...,m'$, defined in the following way:

$$\begin{aligned}
A_{h,k} = \big\{(y,w,t) :\; & hP^{\gamma'} \le |v_{1,\alpha} - v_{2,\alpha}| \le (h+1)P^{\gamma'}, \\
& \alpha_{k+1} < |v_2^{\perp}| \le \alpha_k,\; |X(t)-Y(t)| \le l_{h,k}\big\}
\end{aligned} \tag{3.21}$$

$$B_{h,k} = \big\{(y,w,t): \ hP^{\gamma'} < |v_{1,\alpha} - v_{2,\alpha}| \le (h+1)P^{\gamma'}, \\ \alpha_{k+1} < |v_2^{\perp}| \le \alpha_k, \ |X(t) - Y(t)| > l_{h,k}\big\} \tag{3.22}$$

where

$$\alpha_k = \frac{P}{2^k} \qquad l_{h,k} = \frac{2^{2k}}{hP^{1+\eta}}, \tag{3.23}$$

with $\eta > \gamma'$ to be fixed later. Since we are in S_3, it is immediately seen that

$$m \le \frac{3}{4}\log_2 P \quad m' \le \frac{2P}{P^{\gamma'}} - 1. \tag{3.24}$$

Consequently we put

$$\mathcal{I}_3 \le \sum_{h=1}^{m'}\sum_{k=0}^{m}\big(\mathcal{I}_3'(h,k) + \mathcal{I}_3''(h,k)\big) \tag{3.25}$$

being

$$\mathcal{I}_3'(h,k) = \int_{A_{h,k}} \frac{f(y,w,t_i)}{|X(t)-Y(t)|^2}\, dydw \tag{3.26}$$

and

$$\mathcal{I}_3''(h,k) = \int_{B_{h,k}} \frac{f(y,w,t_i)}{|X(t)-Y(t)|^2}\, dydw. \tag{3.27}$$

We start by $\mathcal{I}_3'(h,k)$. The same arguments used in the proof of Lemma 3 and Lemma 4, given in the Appendix, show that $\forall\,(y,w,t) \in A_{h,k}$ and $t \in [t_i, t_{i+1}]$ it is:

$$(h-1)P^{\gamma'} \le |v_{1,\alpha}(t) - v_{2,\alpha}(t)| \le (h+2)P^{\gamma'} \tag{3.28}$$

and

$$\frac{\alpha_{k+1}}{2} \le |v_2^{\perp}(t)| \le 2\alpha_k. \tag{3.29}$$

Hence, setting

$$A'_{h,k} = \big\{(\bar y,\bar w,t): \ hP^{\gamma'} \le |v_{1,\alpha}(t) - \bar v_{2,\alpha}| \le (h+1)P^{\gamma'}, \\ \frac{\alpha_{k+1}}{2} \le |\bar v_2^{\perp}| \le 2\alpha_k, \ |X(t) - \bar y| \le l_{h,k}\big\} \tag{3.30}$$

we have

$$\mathcal{I}_3'(h,k) \le \int_{A'_{h,k}} \frac{f(\bar y,\bar w,t)}{|X(t)-\bar y|^2}\, d\bar y d\bar w. \tag{3.31}$$

By the choice of the parameters α_k and $l_{h,k}$ made in (3.23) we have:

$$\mathcal{I}_3'(h,k) \le CC_0\, l_{h,k} \int_{A'_{h,k}} d\bar w \le C\, l_{h,k}\, \alpha_k^2 \int_{A'_{h,k}} d\bar w_1 \le \\ C\, l_{h,k}\, \alpha_k^2\, P^{\gamma'} \le C\, P^{1+\gamma'-\eta}. \tag{3.32}$$

Hence by (3.24)

$$\sum_{h=1}^{m'}\sum_{k=0}^{m}\mathcal{I}_3'(h,k) \le C\,P^{1+\gamma'-\eta}\sum_{k=0}^{m}\sum_{h=1}^{m'}\frac{1}{h} \le C\,P^{1+\gamma'-\eta}\,\log^2 P. \tag{3.33}$$

It remains to consider the set $B_{h,k}$, for which we need the rectification technique and the time average. By Lemma 5 there exists $t_0 \in [t_i, t_{i+1}]$ such that

$$|X(t)-Y(t)| \ge h\frac{P^{\gamma'}}{8}|t-t_0|. \tag{3.34}$$

Let

$$B'_{h,k} = \big\{(y,w) : (y,w,t) \in B_{h,k} \ \text{ for some } \ t \in [t_i, t_{i+1}]\big\}. \tag{3.35}$$

Then

$$\begin{aligned}&\int_{t_i}^{t_{i+1}} \mathcal{I}_3''(h,k)\;dt = \int_{t_i}^{t_{i+1}} dt \int_{B_{h,k}} \frac{f(y,w,t_i)}{|X(t)-Y(t)|^2}\,dydw \le \\ &\int_{B'_{h,k}} f(y,w,t_i)\left(\int_{t_i}^{t_{i+1}} \frac{\chi(B_{h,k})}{|X(t)-Y(t)|^2}\,dt\right)\,dydw.\end{aligned} \tag{3.36}$$

Putting $A = 4l_{h,k}/(hP^{\gamma'})$, by (3.34) we have:

$$\begin{aligned}&\int_{t_i}^{t_{i+1}} \frac{\chi(B_{h,k})}{|X(t)-Y(t)|^2}\,dt \le \int_{\{t:|t-t_0|\le A\}} \frac{\chi(B_{h,k})}{|X(t)-Y(t)|^2}\,dt\; + \\ &\int_{\{t:|t-t_0|>A\}} \frac{\chi(B_{h,k})}{|X(t)-Y(t)|^2}\,dt \le \\ &\int_{\{t:|t-t_0|\le A\}} \frac{1}{l_{h,k}^2}\,dt + \int_{\{t:|t-t_0|>A\}} \frac{64}{h^2P^{2\gamma'}|t-t_0|^2}\,dt \le \\ &\quad \frac{2A}{l_{h,k}^2} + \frac{C}{h^2P^{2\gamma'}}\int_A^\infty \frac{1}{t^2}\,dt = \frac{C}{hP^{\gamma'}l_{h,k}}.\end{aligned} \tag{3.37}$$

Thus by (3.36)

$$\begin{aligned}\int_{t_i}^{t_{i+1}} \mathcal{I}_3''(h,k)\,dt \le &\frac{C}{hP^{\gamma'}l_{h,k}}\int_{B'_{h,k}} f(y,w,t_i)\,dydw \\ \le &\frac{C}{hP^{\gamma'}l_{h,k}\,\alpha_{k+1}^2}\int_{B'_{h,k}} w^2 f(y,w,t_i)\,dydw,\end{aligned} \tag{3.38}$$

since, being $(y,w) \in B'_{h,k}$, then $|w| \ge \alpha_{k+1}$. Moreover the kinetic energy is bounded by the conservation of the energy (2.14), hence

$$\sum_{h=1}^{m'}\sum_{k=0}^{m}\int_{B'_{h,k}} w^2 f(y,w,t_i)\,dydw \le C\mathcal{E}(0), \tag{3.39}$$

and

$$\sum_{h=1}^{m'}\sum_{k=0}^{m}\int_{t_i}^{t_{i+1}} \mathcal{I}_3''(h,k)\,dt \le C\,P^{-\gamma'-1+\eta}. \tag{3.40}$$

By multiplying and dividing by Δ we have:

$$\sum_{h=1}^{m'}\sum_{k=0}^{m}\int_{t_i}^{t_{i+1}} \mathcal{I}_3''(h,k)\,dt \le C\,P^{1-\gamma'-\gamma+\eta}\Delta, \tag{3.41}$$

and remembering the constraints $\eta > \gamma'$, $\gamma' > \gamma$, in order to keep the exponent of P in (3.41) less than 1 (whose reason will be clear further) it is sufficient to take

$$\gamma' = 2\gamma \qquad \text{and} \qquad \eta \in (2\gamma, 3\gamma).$$

Now we are able to conclude the proof. By (3.17) it is:

$$\int_0^t |E(X(s),s)|ds \le \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\left(\sum_{m=1}^{3}\mathcal{I}_m\right)\,dt \tag{3.42}$$

and estimates (3.19), (3.20), (3.25), (3.33) and (3.41) give us:

$$\int_{t_i}^{t_{i+1}}\left(\sum_{m=1}^{3}\mathcal{I}_m\right)dt \le C\Delta\Big[P^{\frac{8}{9}+\frac{4}{9}\gamma'} + P^{\frac{8}{9}} + P^{1+\gamma'-\eta}\log^2 P + P^{1-\gamma'-\gamma+\eta}\Big]. \tag{3.43}$$

Now, with the choice stated above on the parameters η, γ', and taking $\gamma \in (0, \frac{1}{8})$, it follows:

$$\int_{t_i}^{t_{i+1}}\left(\sum_{m=1}^{3}\mathcal{I}_m\right)\,dt \le C\Delta P^q, \tag{3.44}$$

for some $q < 1$. In conclusion (3.42), (3.44), and the fact that $T \le N\Delta \le 2T$, imply

$$\int_0^t |E(X(s),s)|\,ds \le CTP^q, \qquad q < 1. \tag{3.45}$$

Estimate (3.45) allows to conclude the proof. Indeed from (2.10) and (3.45) it follows:

$$V^2(t) \le \mathcal{V}_0^2 + 2P\int_0^t |E(X(s),s)|\,ds \le \mathcal{V}_0^2 + CTP^{1+q} \tag{3.46}$$

and, by taking the supremum in the left hand side, we obtain

$$P^2 \le \mathcal{V}_0^2 + CTP^{1+q}$$

This implies $P(T) < \infty$ for any $T > 0$. □

Proof of Theorem 1. Using the previous result of the boundedness of $P(T)$ over an arbitrary time T, by standard methods we can obtain global existence and uniqueness of the solution to (2.7) (or (3.4)). It remains to prove the confinement of the plasma inside the torus in the time interval $[0, T]$. We consider the second equation of (3.4),

$$(R + r\cos\alpha)\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\cos\alpha - 2\,\dot{\alpha}\,\dot{\theta}\,r\sin\alpha = E_\theta - \frac{a'(r)\,\dot{r}}{R + r\cos\alpha}, \qquad (3.47)$$

which, after multiplying by $(R + r\cos\alpha)$, becomes (remembering (3.5)),

$$a'(r)\dot{r} = -(R{+}r\cos\alpha)^2\ddot{\theta} - 2v_r v_\theta\cos\alpha + 2v_\alpha v_\theta\sin\alpha + (R{+}r\cos\alpha)E_\theta. \quad (3.48)$$

Integrating in time (3.48), for the left hand side we have

$$\int_0^t a'(r(s))\frac{dr}{ds}\,ds = a(r(t)) - a(r(0)), \qquad (3.49)$$

while for the right hand side we get

$$\begin{aligned}
&\int_0^t \Big(-(R + r\cos\alpha)^2\ddot{\theta} - 2v_r v_\theta\cos\alpha + 2v_\alpha v_\theta\sin\alpha + (R + r\cos\alpha)E_\theta\Big)\,ds \\
&= -\Big[(R + r\cos\alpha)^2\dot{\theta}\Big]_0^t + \int_0^t \Big(2\dot{\theta}(R + r\cos\alpha)(\dot{r}\cos\alpha - r\dot{\alpha}\sin\alpha)\Big)\,ds \\
&\quad + \int_0^t (-2v_r v_\theta\cos\alpha + 2v_\alpha v_\theta\sin\alpha + (R + r\cos\alpha)E_\theta)\,ds. \qquad (3.50)
\end{aligned}$$

It is easily seen that (3.49) diverges to ∞ for $r \to r_0$ (border of the torus), while (3.50) stays finite since $P(T)$ and the electric field are bounded, as seen before. □

Appendix

Proof of Lemma 1. One has:

$$\rho(x,t) = \int f dv = \int_{|v|\le a} f\ dv + \int_{|v|>a} f\ dv \le CC_1 a^3 + \frac{1}{a^2}\int |v|^2 f dv.$$

Minimizing on a we get:

$$\rho(x,t) \le C\left(\int |v|^2 f dv\right)^{\frac{3}{5}}.$$

It follows from the conservation of the energy (2.14) that the kinetic energy is bounded, so that:

$$\left(\int (\rho(x,t))^{\frac{5}{3}} dx\right)^{\frac{3}{5}} \le C.$$

Proof of Lemma 2. One has:

$$|E(x,t)| \le \int \frac{\rho(y,t)}{|x-y|^2}dy = \int_{|x-y|\le\epsilon} \frac{\rho(y,t)}{|x-y|^2}dy + \int_{|x-y|>\epsilon} \frac{\rho(y,t)}{|x-y|^2}dy \le$$

$$C \sup_y \rho(y,t)\epsilon + \left(\int (\rho(y,t))^{\frac{5}{3}} dy\right)^{\frac{3}{5}} \left(\int_{|x-y|>\epsilon} \frac{1}{|x-y|^5}dy\right)^{\frac{2}{5}} \le$$

$$C \sup_y \rho(y,t)\epsilon + C\epsilon^{-\frac{4}{5}},$$

by Lemma 1. Minimizing in ϵ:

$$|E(x,t)| \le C\left(\sup_y \rho(y,t)\right)^{\frac{4}{9}}.$$

On the other side:

$$\sup_y \rho(y,t) \le \sup_y \int f(y,w,t)dw \le CP^3$$

so that we have:

$$|E(x,t)| \le CP^{\frac{4}{3}}.$$

Proof of Lemma 3. Considering the third equation of (3.4), we see that along this direction α the magnetic field is not influential. We then obtain

$$\frac{d}{dt}(\dot\alpha\, r) = -\dot\alpha\, \dot r - \frac{v_\theta^2}{R + r\cos\alpha}\sin\alpha + E_\alpha, \tag{3.51}$$

and integrating in time,

$$v_\alpha(t) - v_\alpha(t') = \int_{t'}^{t} \left(-\dot\alpha(s)\,\dot r(s) - \frac{v_\theta^2(s)}{R + r(s)\cos\alpha(s)}\sin\alpha(s) + E_\alpha(s)\right) ds. \tag{3.52}$$

We have $|\dot r| = |v_r| \le P$, $|v_\theta| \le P$, $|E_\alpha| \le C_2 P^{\frac{4}{3}}$, and for $\dot\alpha = \frac{1}{r}v_\alpha$ if we restrict to consider a motion occurring in the region $\frac{1}{2}r_0 \le r \le r_0$ (for $0 \le r \le \frac{1}{2}r_0$ we can switch to cartesian coordinates, since the magnetic field is bounded), we have $|\dot\alpha| \le 2P$. Hence

$$|v_\alpha(t)| \le |v_\alpha(t')| + \left(2 + \frac{1}{R - r_0} + C_2\right) P^2(t - t'), \tag{3.53}$$

therefore by Lemma 2 and (3.3) we get, for any $t \in [t', t' + \Delta]$:

$$|r_1(t)\dot\alpha_1(t) - r_2(t)\dot\alpha_2(t)| = |v_{1,\alpha}(t) - v_{2,\alpha}(t)| \le$$

$$|v_{1,\alpha}(t') - v_{2,\alpha}(t')| + 2\left(2 + \frac{1}{R - r_0} + C_2\right) P^2\Delta \le$$

$$P^{\gamma'} + \frac{1}{2} P^{\gamma} \leq 2P^{\gamma'}.$$

Analogously we prove the second statement:

$$|v_{1,\alpha}(t) - v_{2,\alpha}(t)| \geq |v_{1,\alpha}(t') - v_{2,\alpha}(t')| - \frac{1}{2C_3} P^2 \Delta \geq$$

$$P^{\gamma'} - \frac{1}{2} P^{\gamma} \geq \frac{1}{2} P^{\gamma'}.$$

Proof of Lemma 4. By (2.10) and the definition of B it is:

$$\frac{d}{dt}\left[v_1^{\perp}(t)\right]^2 = 2v_1^{\perp}(t) \cdot E_{\perp}(t), \tag{3.54}$$

with $E_{\perp} = \sqrt{E_r^2 + E_\theta^2}$. We prove the thesis by contradiction. Assume that there exists a time interval $[t^*, t^{**}] \subset [t', t' + \Delta)$ such that $|v_1^{\perp}(t^*)| = \sqrt{P}$, $|v_1^{\perp}(t^{**})| = 2\sqrt{P}$ and $\sqrt{P} < |v_1^{\perp}(t)| < 2\sqrt{P}$ $\forall t \in (t^*, t^{**})$. Then from (3.54) it follows, by Lemma 2 and (3.3):

$$\begin{aligned} |v_1^{\perp}(t^{**})|^2 \leq \; & |v_1^{\perp}(t^*)|^2 + 2\int_{t^*}^{t^{**}} ds\, |v_1^{\perp}(s)|\,|E_{\perp}(s)| \leq \\ & P + 4\sqrt{P}\int_{t^*}^{t^{**}} ds\, |E(s)| \leq \\ & P + 4\sqrt{P}\Delta C_2 P^{\frac{4}{3}} \leq P + P^{\gamma - \frac{1}{6}} < 2P. \end{aligned} \tag{3.55}$$

The contradiction proves the thesis. Now we prove (3.11). As before, assume that there exists a time interval $[t^*, t^{**}] \subset [t', t'+\Delta)$ such that $|v_1^{\perp}(t^*)| = \sqrt{P}$, $|v_1^{\perp}(t^{**})| = \frac{\sqrt{P}}{2}$ and $\frac{\sqrt{P}}{2} < |v_1^{\perp}(t)| < \sqrt{P}$ $\forall t \in (t^*, t^{**})$. Then from (3.54) it follows, by Lemma 2 and (3.3):

$$\begin{aligned} |v_1^{\perp}(t^{**})|^2 \geq |v_1^{\perp}(t^*)|^2 - 2\int_{t^*}^{t^{**}} ds\, |v_1^{\perp}(s)|\,|E_{\perp}(s)| \geq \\ P - 2\sqrt{P}\int_{t^*}^{t^{**}} ds\, |E(s)| \geq P - \frac{1}{2}P^{\gamma - \frac{1}{6}} > \frac{P}{2}. \end{aligned} \tag{3.56}$$

Hence also in this case the contradiction proves the thesis.

Proof of Lemma 5. Let $t_0 \in [t', t' + \Delta]$ be the time at which

$$\left| \int_{t'}^{t} [v_{1,\alpha}(s) - v_{2,\alpha}(s)]\, ds + \lambda(t') \right|$$

has the minimum value, where $\lambda(t') = r_1(t')\alpha_1(t') - r_2(t')\alpha_2(t')$.

We put

$$\Gamma(t) = \int_{t'}^{t} [v_{1,\alpha}(s) - v_{2,\alpha}(s)]\, ds + \lambda(t').$$

Moreover we define the function

$$\bar{\Gamma}(t) = \Gamma(t_0) + \dot{\Gamma}(t_0)(t - t_0).$$

Since the magnetic force does not act on the α-component (in toroidal coordinates) of the velocity, recalling (3.51) it is

$$\begin{aligned} \frac{d^2}{dt^2}\left(\Gamma(t) - \bar{\Gamma}(t)\right) = -\dot{\alpha}_1(t)\,\dot{r}_1(t) - \frac{[v_{1,\theta}(t)]^2 \sin\alpha_1(t)}{R + r_1(t)\cos\alpha_1(t)} + E_\alpha(X(t),t) + \\ \dot{\alpha}_2(t)\,\dot{r}_2(t) + \frac{[v_{2,\theta}(t)]^2 \sin\alpha_2(t)}{R + r_2(t)\cos\alpha_2(t)} - E_\alpha(Y(t),t), \\ \Gamma(t_0) = \bar{\Gamma}(t_0), \quad \dot{\Gamma}(t_0) = \dot{\bar{\Gamma}}(t_0), \end{aligned} \tag{3.57}$$

denoting by $E_\alpha(X(t),t)$ the α-component of the electric field acting on the characteristic 1, and $E_\alpha(Y(t),t)$ the α-component of the electric field acting on the characteristic 2. From (3.57) it follows,

$$\begin{aligned} &\Gamma(t) = \bar{\Gamma}(t) + \\ &\int_{t_0}^{t} ds \int_{t_0}^{s} d\tau \Big[-\dot{\alpha}_1(\tau)\,\dot{r}_1(\tau) - \frac{[v_{1,\theta}(\tau)]^2 \sin\alpha_1(\tau)}{R + r_1(\tau)\cos\alpha_1(\tau)} + E_\alpha(X(\tau),\tau) \\ &\qquad + \dot{\alpha}_2(\tau)\,\dot{r}_2(\tau) + \frac{[v_{2,\theta}(\tau)]^2 \sin\alpha_2(\tau)}{R + r_2(\tau)\cos\alpha_2(\tau)} - E_\alpha(Y(\tau),\tau)\Big] \end{aligned}$$

and reasoning as after (3.52) (i.e., supposing $r \geq \frac{r_0}{2}$),

$$\begin{aligned} &\int_{t_0}^{t} ds \int_{t_0}^{s} d\tau \left| -\dot{\alpha}_1(\tau)\,\dot{r}_1(\tau) - \frac{[v_{1,\theta}(\tau)]^2 \sin\alpha_1(\tau)}{R + r_1(\tau)\cos\alpha_1(\tau)} + E_\alpha(X(\tau),\tau) \right. \\ &\qquad \left. + \dot{\alpha}_2(\tau)\,\dot{r}_2(\tau) + \frac{[v_{2,\theta}(\tau)]^2 \sin\alpha_2(\tau)}{R + r_2(\tau)\cos\alpha_2(\tau)} - E_\alpha(Y(\tau),\tau) \right| \leq \\ &\qquad \frac{1}{2C_3} P^2 \frac{|t - t_0|^2}{2} \leq \frac{1}{4C_3} P^2 |t - t_0| \Delta \leq \frac{1}{4} P^\gamma |t - t_0|. \end{aligned}$$

Hence,

$$|\Gamma(t)| \geq |\bar{\Gamma}(t)| - \frac{P^\gamma}{4} |t - t_0|. \tag{3.58}$$

Now we have:

$$|\bar{\Gamma}(t)|^2 = |\Gamma(t_0)|^2 + 2\Gamma(t_0)\dot{\Gamma}(t_0)(t - t_0) + |\dot{\Gamma}(t_0)|^2 |t - t_0|^2.$$

We observe that $\Gamma(t_0)\dot{\Gamma}(t_0)(t-t_0) \geq 0$. Indeed, if $t_0 \in (t', t'+\Delta)$ then $\dot{\Gamma}(t_0) = 0$ while if $t_0 = t'$ or $t_0 = t'+\Delta$ the product $\Gamma(t_0)\dot{\Gamma}(t_0)(t-t_0) \geq 0$. Hence

$$|\bar{\Gamma}(t)|^2 \geq |\dot{\Gamma}(t_0)|^2 |t-t_0|^2.$$

By Lemma 3 (adapted to this context with a factor $h \geq 1$), since $t_0 \in [t', t'+\Delta]$ it is

$$|\dot{\Gamma}(t_0)| \geq h\frac{P^{\gamma'}}{2},$$

hence

$$|\bar{\Gamma}(t)| \geq h\frac{P^{\gamma'}}{2}|t-t_0|,$$

and since $\gamma' > \gamma$, by (3.58) we get

$$|\Gamma(t)| \geq h\frac{P^{\gamma'}}{4}|t-t_0|. \tag{3.59}$$

We finally achieve Lemma 5 noting that we can bound

$$\left| \int_{t'}^{t} [v_{1,\alpha}(s) - v_{2,\alpha}(s)]\, ds + \lambda(t') \right| \leq 2|X(t) - Y(t)|. \tag{3.60}$$

In fact the left hand side of (3.60) is the separation along the α-coordinate (length of arc, if it is identically $r_1 \equiv r_2$), and in the worst case ($r_1 \equiv r_2$), since we are looking at small lengths, the double of the chord is greater than the length of the corresponding arc, for angles smaller than π.

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